The Schrödinger equation on cylinders and the n-torus

R. S. Kraußhar[†] N. Vieira[*]

[†] Institute of Mathematics
University of Paderborn
Warburgerstr. 100
D-33098 Paderborn, Germany.
E-mail: soeren.krausshar@uni-paderborn.de

[*] Department of Mathematics
Faculty of Science
University of Porto
Rua do Campo Alegre
4169-007 Porto, Portugal
E-mail: nvieira@fc.up.pt

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Abstract

In this paper we study the solutions to the Schrödinger equation on some conformally flat cylinders and on the n-torus. First we apply an appropriate regularization procedure. Using the Clifford algebra calculus with an appropriate Witt basis, the solutions can be expressed as multiperiodic eigensolutions to the regularized parabolic-type Dirac operator. We study their fundamental properties, give representation formulas of all these solutions in terms of multiperiodic generalizations of the elliptic functions in the context of the regularized parabolic-type Dirac operator.

Furthermore, we also develop some integral representation formulas. In particular we set up a Green type formula for the solutions to the homogeneous regularized Schrödinger equation on cylinders and $n-$tori. Then we treat the inhomogeneous Schrödinger equation with prescribed boundary conditions in Lipschitz domains on these manifolds, and we present an L_p-decomposition where one of the components is the kernel of the first order differential operator that factorizes the cylindrical (resp. toroidal) Schrödinger operator. Finally, we study the behavior of our results in the limit case where the regularization parameter tends to zero.

Keywords: Schrödinger equation on manifolds, regularized parabolic-type Dirac operators, hypoelliptic equations, regularization, Hodge decomposition
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1 Introduction

Time evolution problems are of extreme importance in mathematical physics. However, there is still a strong need to develop further special techniques to deal with these problems, in particular when non-linearities are involved.

For stationary problems in Clifford analysis setting, the theory developed by K. Gürlebeck and W. Sprößig [8] and their upfollowing students, which is based on an orthogonal decomposition of the L_2-space into the subspace of the null-solutions to the corresponding Dirac operator and its complement, has been successfully applied to obtain new insightful structural results and new efficient solution algorithms of a wide range of elliptic equations. These include for instance the stationary Lamé-, Navier-Stokes-, Maxwell- and Schrödinger equations. See also [4], [7], [8], [12] or [15]. Indeed this kind of L_2-space decomposition (when applicable) represents one of the most central aspects of complex and hypercomplex analysis and turned out to be the key ingredient in the development of the treatment of these PDEs.

Unfortunately, there is no simple way to extend this theory directly to non-stationary problems. A first step in this direction has been made in [3] in which the authors treated the instationary Navier-Stokes equation over time-varying domains.

One of the main objectives of this paper is to obtain such a decomposition for the case of the time-dependent Schrödinger equation on a class of conformally flat cylinders and the n-torus with different spin structures.

In order to construct such a decomposition we will try to apply some of the techniques developed for the elliptic equations that were used to study the heat problem in the analysis of the Schrödinger problem. However, we need to take into account that in many aspects the Schrödinger operator is substantially different from the heat operator. First of all the Gallilean group is the invariance group associated to the first equation, while the parabolic group is the invariance group that is associated to the heat equation (see [16]). Secondly, the Schrödinger equation is related to the Minkowski space-time metric, while the heat equation is linked to the parabolic space-time metric (see [16]). More important for us, under an analytical point of view, the singularity $t = 0$ of the correspondent fundamental solutions is removable outside the origin in the second case but it is not removable in the case of Schrödinger operator. To overcome this problem we introduce a regularization procedure prior to the development of a hypoelliptic analysis (see [1], [5], [16] and [17]). For a fixed $\epsilon > 0$ we consider the regularized Schrödinger operator $\Delta + \mathbf{k}\partial_t$, where $\mathbf{k} = \frac{\epsilon + i}{\epsilon^2 + 1}$.

In the first part of this paper we construct the fundamental solution to the regularized Schödinger operator on a class of conformally flat n-dimensional cylinders and tori with different spin structures in terms of hypoelliptic multiperiodic generalizations of the Weierstraß $\wp$-function.

These geometric models belong to the most basic ones in modern quantum theory and quantum gravity and serve as useful toy models in cosmology.

After having introduced the basic geometric concepts we study the basic analytic properties of the generalized hypoelliptic Weierstraß $\wp$-function that we introduced.

In the particular case of the n-torus we prove that we can represent every solution to the regularized Schrödinger operator on the torus by a finite sum consisting of linear combinations of that particular generalized hypoelliptic Weierstraß $\wp$-function and of its partial derivatives. We get uniqueness up to an entire real analytic function that only depends on the time variable t.

After that, we set up some integral representation formulas for the null-solutions to the regularized Schrödinger operator on the different cylinders and the n-tori. In particular we develop a Green's integral formula for the solutions to the homogeneous regularized Schrödinger equation on these manifolds. Then we treat the inhomogeneous Schrödinger equation with prescribed boundary conditions in Lipschitz domains on these manifolds. Next we prove an L_p-decomposition where one of the components is the kernel of the first order differential operator that factorizes the cylindrical (resp. toroidal) regularized Schrödinger operator. Finally, we study the behavior of our results in the limit case $\epsilon \to 0$.

2 Preliminaries

2.1 Clifford algebras and hypoelliptic theory

We consider the n-dimensional vector space $\mathbb{R}^n$ endowed with an orthonormal basis $\{e_1, \cdots, e_n\}$.

We define the universal Clifford algebra $C\ell_{0,n}$ as the 2^n-dimensional associative algebra which preserves the multiplication rules $e_i e_j + e_j e_i = -2\delta_{i,j}$. A basis for $C\ell_{0,n}$ is given by the elements $e_0 = 1$ and $e_A = e_{h_1} \cdots e_{h_k}$, where $A = \{h_1, \ldots, h_k\} \subset M = \{1, \ldots, n\}$, for $1 \leq h_1 < \cdots < h_k \leq n$. Each element $x \in C\ell_{0,n}$ can be represented by $x = \sum_A x_A e_A$, $x_A \in \mathbb{R}$. The Clifford conjugation is defined by $\overline{e_j} = -e_j$ for all $j = 1, \ldots, n$ and we have $\overline{ab} = \overline{b}\overline{a}$.

We introduce the complexified Clifford algebra $\mathbb{C}_n$ as the tensor product

$$\mathbb{C} \otimes C\ell_{0,n} = \left\{ w = \sum_A z_A e_A,\ z_A \in \mathbb{C}, A \subset M \right\}$$

where the imaginary unit i interacts with the basis elements, that means $ie_j = e_j i$ for all $j = 1, \ldots, n$. Notice that for $a, b \in \mathbb{C}_n$ we only have $|ab| \leq 2^n |a||b|$. To avoid ambiguities with the Clifford conjugation we denote the complex conjugation mapping a complex scalar $a_A = a_{A0} + ia_{A1}$ onto $a_A = a_{A0} + -ia_{A1}$ by

$\sharp$. The complex conjugation leaves the elements e_j invariant, i.e. $e_j^\sharp = e_j$ for all $j = 1, \ldots, n$.

Next we introduce the Euclidean Dirac operator $D = \sum_{j=1}^n e_j \partial_{x_i}$. The latter one factorizes the n-dimensional Euclidean Laplacian, that is, $D^2 = -\Delta$. A $\mathbb{C}_n$-valued function that is defined on an open domain Ω, $u : \Omega \subset \mathbb{R}^n \mapsto \mathbb{C}_n$, is called *left-monogenic* if it satisfies $Du = 0$ on Ω (resp. *right-monogenic* if it satisfies $uD = 0$ on Ω).

A function $u : \Omega \mapsto \mathbb{C}_n$ has a representation $u = \sum_A u_A e_A$ with $\mathbb{C}$-valued components u_A. Properties such as continuity will be understood componentwisely. In the sequel we will use the short notation $L_p(\Omega)$, $C^k(\Omega)$, etc., instead of $L_p(\Omega, \mathbb{C}_n)$, $C^k(\Omega, \mathbb{C}_n)$ for the corresponding space. For more details on Clifford analysis, we refer the interested reader for instance to [6, 8].

The space $L_2(\Omega)$ can be endowed with the structure of a Hilbert $\mathbb{C}_n-$module by endowing it with the following inner product

$$< f, g > \ := \ \int_\Omega \overline{f(x,t)}^\sharp g(x,t)\, dxdt, \quad f,\, g \in L_2(\Omega).$$

Like in [3] we will imbed $\mathbb{R}^n$ into $\mathbb{R}^{n+2}$. For that purpose we add two new basis elements $\mathfrak{f}$ and $\mathfrak{f}^\dagger$ satisfying

$$\mathfrak{f}^2 = {\mathfrak{f}^\dagger}^2 = 0, \quad \mathfrak{f}\mathfrak{f}^\dagger + \mathfrak{f}^\dagger\mathfrak{f} = 1, \quad \mathfrak{f}e_j + e_j\mathfrak{f} = \mathfrak{f}^\dagger e_j + e_j \mathfrak{f}^\dagger = 0, j = 1, \cdots, n.$$

The extended basis is often called a Witt basis. This construction allows us to use a suitable factorization of the time evolution operators where only partial derivatives are used.

Following [1], a partial differential operator is hypoelliptic if and only if its fundamental solution is a C^∞ function in $\mathbb{R}^n \times \mathbb{R}_0^+ \setminus \{(0,0)\}$.

2.2 Regularization of the Schrödinger operator

The following function e_- defined by

$$e_-(x,t) = i\ \frac{H(t)}{(4\pi i t)^{\frac{n}{2}}} \exp\left(-i\ \frac{|x|^2}{4t}\right)$$

is the fundamental solution to the Schrödinger operator (see [5]). It has a singularity in all the points of the hyperplane $t = 0$. This is different from the situation of deadling with the classical isolated point singularity for the hypoelliptic operators. Moreover, these singularities are not removable by standard calculations methods. This causes additional problems in the study of some integral operators or series that are constructed using these functions, because we cannot guarantee the convergence in the classical sense.

In order to overcome this problem we need to regularize the fundamental solution and the corresponding integral operators (see [16]). In the operators and correspondent fundamental solutions one has to substitute the imaginary unity by the parameter $\mathbf{k} = \frac{\epsilon + i}{\epsilon^2 + 1}$. Hence we obtain a family of operators and correspondent fundamental solutions, which are locally integrable in $\mathbb{R}^n \times \mathbb{R}_0^+ \setminus \{(0,0)\}$. Following [1], for each $\epsilon > 0$, the operator $(-\Delta - \mathbf{k}\partial_t)$ is a hypoelliptic operator. Its use and ensures the good behavior for the integral operators and its associated function series, as we shall see later on. The fundamental solution for this operator is given by

$$\begin{aligned} e_-^\epsilon(x,t) &= (\epsilon + i)e(x,(\epsilon+i)t) \\ &= (\epsilon+i)\ \frac{H(t)}{(4\pi(\epsilon+i)t)^{\frac{n}{2}}} \exp\left(-\ \frac{(\epsilon - i)|x|^2}{4(\epsilon^2+1)t}\right), \quad \epsilon > 0. \end{aligned}$$

This motivates us to consider the following regularized parabolic-type Dirac operator

Definition 2.1. *For a function $u \in W_p^a(\Omega)$, with $\Omega \subset \mathbb{R}^{n+1}$, $1 \leq p < +\infty$ and $a \in \mathbb{N}$, we define the forward/backward regularized parabolic-type Dirac operator as*

$$D_\pm^\epsilon u = (D + \mathfrak{f}\partial_t \pm \mathbf{k}\mathfrak{f}^\dagger)u, \tag{1}$$

where D stands for the usual (spatial) Euclidean Dirac operator that we introduced before.

This operator factorizes the correspondent forward/backward regularized Schrödinger operator, i. e.

$$(D_\pm^\epsilon)^2 u = (-\Delta \pm \mathbf{k}\partial_t)u. \tag{2}$$

This regularized operator satisfies that $D_\pm^\epsilon : W_p^1(\Omega) \to L_p(\Omega)$. For more details about this regularization procedure, we refer the reader to the preceding papers [5] and [18]. In all that follows in this paper we will restrict to consider without loss of generality the backward operator. In [5] we proved the following result:

Theorem 2.2. *For the sequence of parabolic-type Dirac operators D_-^ϵ, with $\epsilon > 0$, we have the following convergence*

$$\|D_- - D_-^\epsilon\|_{L_1(\Omega)} \to 0,$$

where $D_- = D + \mathfrak{f}\partial_t - i\mathfrak{f}^\dagger$, when $\epsilon \to 0$.

We now present the family of regularized fundamental solutions for this first order operator D_-^ϵ

Definition 2.3. *Given a fundamental solution $e_-^\epsilon = e_-^\epsilon(x,t)$ of the operator (2), then we have that the function $E_-^\epsilon(x,t) = D_-^\epsilon e_-^\epsilon(x,t)$ is a fundamental solution for the operator D_-^ϵ*

Easy calculations (see [5]) give

$$\begin{aligned} E_-^\epsilon(x,t) &= D_-^\epsilon e_-^\epsilon(x,t) \\ &= e_-^\epsilon(x,t)\left[\frac{-x}{2(\epsilon+i)t} + \mathfrak{f}\left(\frac{-n}{2t} + \frac{|x|^2}{4(\epsilon+i)t^2}\right) - \mathbf{k}\mathfrak{f}^\dagger\right]. \end{aligned} \tag{3}$$

In [5] the authors proved the following regularized Borel-Pompeiu type formula

$$\begin{aligned}&\int_{\partial\Omega} E_{-}^{\epsilon}(x-x_0,t-t_0)d\sigma_{x,t}\,u(x,t)\\ =\;& u(x_0,t_0)+\int_{\Omega} E_{-}^{\epsilon}(x-x_0,t-t_0)(D_{+}^{\epsilon}u)(x,t)dxdt,\;(x_0,t_0)\notin\partial\Omega\end{aligned}\quad . \qquad (4)$$

Here the surface element is given by the contraction of the homogeneous operator associated to D_{-}^{ϵ} with the volume element, i.e., $d\sigma_{x,t}=(D_x+\mathfrak{f}\partial_t)\rfloor dxdt$.

Moreover, when $u\in ker(D_{+}^{\epsilon})$ the authors presented the following regularized Cauchy type integral formula

$$\int_{\partial\Omega} E_{-}^{\epsilon}(x-x_0,t-t_0)d\sigma_{x,t}\;u(x,t)=u(x_0,t_0). \qquad (5)$$

For more details about the application of this regularization procedure to the Schrödinger operator in the context of Clifford analysis, see for instance [5] and [18].

3 Generalized hypoelliptic multiperiodic functions

In this section we construct for $k=1,\dots,n$ k-fold periodic regularized time-holomorphic solutions in the kernel of D_{-}^{ϵ}. For that we introduce the following definitions

Definition 3.1. *Let $\Omega\subseteq\mathbb{R}^{n+1}$ an open set. We say that a function $f:\Omega\to\mathbb{C}_n$ is regularized time-holomorphic in Ω if $D_{-}^{\epsilon}f(x,t)=0$ holds for all $(x,t)\in\Omega$.*

Definition 3.2. *Let $k\in\{1,\dots,n\}$. Suppose that $v_1,...,v_k$ are $\mathbb{R}$-linear independent vectors in $\mathbb{R}^n$. The lattice generated by these vectors will be denoted by $V=\mathbb{Z}v_1+...+\mathbb{Z}v_n$. Let $S\subset\mathbb{R}^{n+1}$ be a closed subset satisfying $S+v=S$, for all $v\in V$. We call a function $u:\mathbb{R}^{n+1}\setminus S\to\mathbb{C}_n$ which satisfies*

(1) $u(x+v,t)=u(x,t)$ for all $x\in V$.

(2) u is regularized time-holomorphic except in the points of S.

a k-fold periodic regularized time-holomorphic function.

3.1 The generalized regularized $\wp^{\epsilon}$-function for $\mathbf{D}_{-}^{\epsilon}$

In this subsection, we construct the simplest example of a non-trivial n-fold periodic regularized time-holomorphic function. The same construction can be applied to construct k-fold periodic functions with $k<n$.

First we recall the following relation concerning to the L_2-norm of the regularized fundamental solution E^ϵ_- (for more details see Theorem 3.4 for $p=2$ in [5]).

$$\begin{aligned} \|E^\epsilon_-\|_{L_2(\Omega)} &= \|D^\epsilon_- e^\epsilon_-\|_{L_2(\Omega)} \\ &\leq \|A_1\|_{L_2(\Omega)} + \|A_2\|_{L_2(\Omega)} + \|A_3\|_{L_2(\Omega)}, \end{aligned}$$

where

$$\begin{aligned} A_1(x,t) &= -e^\epsilon_-(x,t)\ \frac{x}{2(\epsilon+i)t} \\ A_2(x,t) &= e^\epsilon_-(x,t)\ \mathfrak{f}\left(\frac{-n}{2t} + \frac{|x|^2}{4(\epsilon+i)t^2}\right) \\ A_3(x,t) &= -e^\epsilon_-(x,t)\ \mathbf{k}\mathfrak{f}^\dagger. \end{aligned}$$

For simplicity, we restrict to consider the orthonormal lattice $V := \mathbb{Z}e_1 + \ldots + \mathbb{Z}e_n$. This lattice can be written as the union

$$V = \bigcup_{m=0}^{+\infty} V_m,$$

where

$$V_m := \{v \in V : \ \|v\|_{\max} = m\}.$$

We further consider the following subsets of this lattice

$$L_m := \{v \in \Omega : \ \|v\|_{\max} \leq m\}.$$

Obviously, the set L_m contains exactly $(2m+1)^n$ points. Hence, the cardinality of V_m is $\sharp\Omega_m = (2m+1)^n - (2m-1)^n$. The Euclidian distance between the set V_{m+1} and V_m has the value $d_m := \mathrm{dist}_2(V_{m+1}, V_m) = 1$. Now we are able to prove

Theorem 3.3. *For each $\epsilon > 0$ the series*

$$\wp^\epsilon(x,t) := \sum_{v\in V} E^\epsilon_-(x+v,t) \tag{6}$$

is normally convergent and represents a non-vanishing n-fold regularized periodic time-holomorphic function in $\mathbb{R}^{n+1}$ satisfying in each point of $\mathbb{R}^{n+1}\setminus S$ the regularized backward Schrödinger equation $(-\Delta - \mathbf{k}\partial_t)\wp^\epsilon = 0$, where S denotes the set of points $\{(x,t) \in \mathbb{R}^{n+1} \mid t = 0\}$. Notice that S is indeed invariant under translations of the period lattice lattice V as the latter one only induces shift in the spatial dimensions. In each point of S the function $\wp^\epsilon(x,t)$ has a singularity of order $\frac{n}{2}$.

Proof: To show the normal convergence of the series, let us consider an arbitrary compact subset $K \subset \mathbb{R}^{n+1} \backslash S$. Then there exists a positive $r \in \mathbb{R}$ such that all $(x,t) \in K$ satisfy $\|(x,t)\|_{\max} \leq \|(x,t)\|_{L_2(\Omega)} < r$. Suppose now that (x,t) is a point of K. Taking into account Theorem 3.4, for $p = 2$, in [5], we have

$$\sum_{m=[r]+1}^{+\infty} \sum_{v \in V_m} \|E_-^\epsilon(x+v,t)\|_{L_2(\Omega)} \leq$$

$$((2m+1)^{n+1} - (2m-1)^{n+1}) \left[\|A_1(x+v,t)\|_{L_2(\Omega)} + \|A_2(x+v,t)\|_{L_2(\Omega)} + \|A_3(x+v,t)\|_{L_2(\Omega)}\right],$$

where

$$\begin{aligned} A_1(x+v,t) &= -e_-^\epsilon(x+v,t)\, \frac{x+v}{2(\epsilon+i)t} \\ A_2(x+v,t) &= e_-^\epsilon(x+v,t)\, \mathfrak{f} \left(\frac{-n}{2t} + \frac{|x+v|^2}{4(\epsilon+i)t^2} \right) \\ A_3(x+v,t) &= -e_-^\epsilon(x+v,t)\, \mathbf{k}\mathfrak{f}^\dagger, \end{aligned}$$

Since the regularized fundamental solution E_-^ϵ is a function in L_2 (for more details see [5]) we can guarantee that the previous series is absolutely convergent. Hence the series (6), which can be written as

$$\wp_{0,\ldots,0}^\epsilon(x,t) := \sum_{m=0}^{+\infty} \sum_{v \in V_m} E_-^\epsilon(x+v,t) \tag{7}$$

converges normally on $\mathbb{R}^{n+1} \setminus V$. Since E_-^ϵ belongs to the kernel of D_-^ϵ in all $\mathbb{R}^n \setminus \{(0,0)\}$ and has a singularity of order $\frac{n}{2}$ at the origin and exponential decrease for $\|(x_1,...,x_n)\| \to +\infty$, $\|t\| \to 0$, with $(x_1,...,x_n,t) = (x,t)$. The series $\wp_{0,\ldots,0}^\epsilon(x,t)$ satisfies $D_-^\epsilon \wp_{0,\ldots,0}^\epsilon(x,t) = 0$ in each $(x,t) \in \mathbb{R}^{n+1} \setminus V$ and has a singularity of order $\frac{n}{2}$ in each point of S.

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3.2 Generalized elliptic functions of higher singularity order

Suppose that u is an n-fold periodic regularized time-holomorphic function with respect to the period lattce V and that it satisfies $D_-^\epsilon u = 0$ in $\mathbb{R}^{n+1} \setminus S$ where again S denotes the set of singularities. Let $\mathbf{m} := (m_1,...,m_n) \in \mathbb{N}_0^n$ be a multi index with length $|\mathbf{m}| := m_1 + ... + m_n$. Then the function $u_{\mathbf{m}}(x,t) = \dfrac{\partial^{|\mathbf{m}|} u(x,t)}{\partial x^{\mathbf{m}}}$ is also an n-fold periodic regularized time-holomorphic with respect to V and it also satisfies $D_-^\epsilon u_{\mathbf{m}}(x,t) = 0$. In particular, the functions $\wp_{\mathbf{m}}^\epsilon(x,t) := \frac{\partial^{|\mathbf{m}|} \wp_{\mathbf{m}}^\epsilon(x,t)}{\partial x^{\mathbf{m}}}$ are n-fold periodic and satisfy $D_-^\epsilon \wp_{\mathbf{m}}^\epsilon(x,t) = 0$ in each point of $\mathbb{R}^{n+1} \backslash S$. In each point of S they have a singularity of order $\frac{n}{2} - 1 + |\mathbf{m}|$.

In view of the translation invariance of the operator D^ϵ_-, we can construct n-fold periodic functions that have singularities in a given set of points $a_i + S$ of order N_i $(i = 1, ..., l)$ with $N_i \geq \frac{n}{2}$ by making the construction

$$\sum_{i=1}^{l} \wp^\epsilon_{\mathbf{N}_i}(x - a_i, t) b_i, \tag{8}$$

where $\mathbf{N}_i$ is a multi index of length N_i and where b_i are arbitrary elements from $\mathbb{C}_n$.

Owing to the compactness of the fundamental period cell, one can only construct regularized holomorphic elliptic functions with a finite number of isolated singularities. It is also possible to construct hypoelliptic functions with non-isolated singularities, as we shall mention below explicitly.

Notice that all constant functions $u \equiv C$, with $C \neq 0$, are not solutions of $D^\epsilon_- u = 0$.

Now, we present the main results of this section. They completely describe the set of n-fold regularized periodic time-holomorphic functions with non-essential singularities up to a space independent function $\phi = \phi(t)$.

Theorem 3.4. *Let $a_1, a_2, ..., a_p \in \mathbb{R}^{n+1} \setminus S$ be a finite set of points that are incongruent modulo V. Suppose that $u : \mathbb{R}^{n+1} \setminus \{a_1 + S, ..., a_p + S\} \to \mathbb{C}_n$ is an n-fold regularized periodic time-holomorphic function which has at most singularities at the points a_i of the order K_i. Then there exists complexified Clifford numbers $b_1, ..., b_p \in \mathbb{C}_n$ and a function $\phi(t)$ only depending on t such that*

$$u(x,t) = \sum_{i=1}^{p} \sum_{m=0}^{K_i - n} \sum_{m=m_1+...+m_n} [\wp^\epsilon_m(x - a_i, t) b_i] + \phi(t)$$

Proof: Since u is supposed to be regularized time-holomorphic with singularities at the points a_i of order K_i, its singular parts of the local Laurent series expansions are of the form $e^\epsilon_m(x - a_i, t) b_i$ in each point $a_i + S$, where

$$e^\epsilon_{\mathbf{m}}(y, s) := \frac{\partial^{|\mathbf{m}|}}{\partial y^{\mathbf{m}}} e^\epsilon(y, s) + \frac{\partial^{|\mathbf{m}|}}{\partial s^{\mathbf{m}}} e^\epsilon(y, s).$$

As a sum of n-fold periodic regularized time-holomorphic functions, the expression

$$g(x,t) = \sum_{i=1}^{p} \sum_{m=0}^{K_i - n} \sum_{m=m_1+...+m_n} [\wp^\epsilon_{\mathbf{m}}(x - a_i, t) b_i]$$

is also n-fold regularized periodic time-holomorphic and has also the same principal parts as u. Hence, the function $h := g - u$ is also an n-fold periodic regularized time-holomorphic holomorphic, but without singular parts, since these are canceled out. The function h is hence an entire time-holomorphic n-fold regularized periodic function.

Let us now fix $t = t_0$. Since $h|_{t=t_0}$ is n-fold regularized periodic time-holomorphic it takes all its values in the n-dimensional fundamental period cell with the edges $0, e_1, e_2, \ldots, e_n, e_1 + e_2, \ldots, e_1 + \cdots + e_n$ which is compact. Since $h|_{t=t_0}$ is continuous it must be bounded on that fundamental cell. As a consequence of the n-fold regularized periodicity, h must be a bounded function on the whole space $\mathbb{R}^n$. Since h is entire time-holomorphic on the extended space $\mathbb{R}^n \times \mathbb{R}^+$ we can obtain a Fischer decomposition of h (for more details see [19]), valid in whole space $\mathbb{R}^{n+1}$. Taking into account that the regularized fundamental solution (3) is unbounded relatively to the space coordinate, we conclude that the polynomials of the Fischer decomposition vanish identically. Hence $h|_{t=t_0} \equiv 0$ and the assertion is hereby proven.

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3.3 Hypoelliptic multiperiodic functions to the homogeneous regularized Schrödinger equation

From the n-fold regularized periodic time-holomorphic basic function $\wp^{\epsilon}_{0,\ldots,0}$ we can easily obtain n-fold regularized periodic solutions of the regularized Schrödinger operator $\Delta - \mathbf{k}\partial_t$. Let C_1, C_2 be some arbitrary complexified Clifford numbers from $\mathbb{C}_n$. Then the scalar part of the functions

$$\wp^{\epsilon}_{0,\ldots,0}(x,t)\ C_1 \qquad \text{and} \qquad \wp^{-\epsilon}_{0,\ldots,0}(x,t)\ C_2$$

as well as all its partial derivatives are $n-$fold periodic regularized time-holomorphic and satisfy the homogeneous regularized time-dependent Schrödinger equation $(\Delta - \mathbf{k}\partial_t)u = 0$ in the whole space $\mathbb{R}^{n+1} \setminus S$.

We want to obtain a direct generalization of the Theorem 3.4 for $n-$fold periodic regularized time-holomorphic solutions of the homogeneous regularized Schrödinger equation. To this end we will consider the fundamental solution e^{ϵ}_{-}, which is the scalar part of the fundamental solution E^{ϵ}_{-}, and has a singularity of order $\frac{n}{2}$ at the origin.

Hence, the analogy of Theorem 3.4 in this context gets form

Theorem 3.5. *Let $a_1, a_2, ..., a_p \in \mathbb{R}^{n+1} \setminus S$ be a finite set of points that are incongruent modulo V. Suppose that $u : \mathbb{R}^{n+1} \setminus \{a_1 + S, ..., a_p + S\} \to \mathbb{C}_n$ is a $n-$fold periodic regularized time-holomorphic function in the kernel of the regularized Schrödinger operator which has at most singularities at the points a_i of the order K_i. Then there exists complexified Clifford numbers $b_1, ..., b_p \in \mathbb{C}_n$ and a regular function $\phi = \phi(t)$ such that*

$$u(x,t) = \sum_{i=1}^{p} \sum_{m=0}^{K_i-(n-1)} \sum_{m=m_1+\ldots+m_n} \left[\wp^{e^{\epsilon}}_{m}(x - a_i, t) b_i\right] + \phi(t),$$

where $\wp^{e^{\epsilon}}_{m}(x - a_i, t) b_i = \sum_{w \in V} e^{\epsilon}_{-}(x + w, t)$.

In the spirit of [9] one can adapt this formula to the case dealing with non-isolated singularities.

4 The regularized Schrödinger equation on the conformally flat cylinders and tori

4.1 Construction of spinor bundles and spinor sections

Let $\Omega_k := \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_k$ be a k-dimensional lattice where $k \in \{1, \ldots, n\}$. Again, to leave it simple we restrict to consider $v_i = e_i$ for $i = 1, \ldots, k$.

The space $\mathbb{R}^n$ is the universal covering space of the conformally flat manifolds by $\mathbb{R}^n/\Omega_k$ which shall be denoted by C_k. In the case $k = n$ we obtain a flat n-torus and in the case $n = 2, k = 1$ the classical three-dimensional flat infinite cylinder.

Consequently, there exists a well-defined projection map $p_k : \mathbb{R}^n \to C_k$. As in [10], we call an open subset $U \subset \mathbb{R}^n$ k-fold periodic if for each each $\mathbf{x} \in U$ the point $x + \omega$ again lies in U for every $\omega \in \Omega_k$. Then $p_k(U) =: U'$ is again an open subset on the manifold C_k. Suppose that $f : U \times \mathbb{R}^+ \to \mathbb{C}_n$ is a k-fold periodic function. Then the projection p_k induces a well-defined function $p_k(f) =: f' : U' \times \mathbb{R}^+ \to \mathbb{C}_n$ on C_k defined by $f((p_k^{-1}(\mathbf{x})'), t)$ for each $\mathbf{x}' \in U'$. The associated functions

$$\wp^{\epsilon}_{k;0,\ldots,0}(\mathbf{y} - \mathbf{x}; t) := \sum_{v \in \Omega_k} E^{\epsilon}_{-}(x + v, t),$$

and

$$P^{\epsilon}_{k;0,\ldots,0}(\mathbf{y} - \mathbf{x}; t) := \sum_{v \in \Omega_k} e^{\epsilon}_{-}(x + v, t)$$

induce functions $G^{\epsilon}_k(\mathbf{y}'-\mathbf{x}'; t)$ (resp. $H_k(\mathbf{y}'-\mathbf{x}'; t)$) on $C_k \times \mathbb{R}^+$ where $\mathbf{x}' := p_k(\mathbf{x})$ and $\mathbf{y}' := p_k(\mathbf{y})$. These functions are defined on $(C_k \times C_k) \times \mathbb{R}^+ \backslash diag(C_k \times C_k) \times \mathbb{R}^+$. The projection map p_k induces a projection of the operator D^{ϵ}_{-} to a differential operator ${D'}^{\epsilon}_{-}$ acting on differentiable functions on $C_k \times \mathbb{R}^+$. The operator ${D'}^{\epsilon}_{-}$ will be called the cylindrical (resp. toroidal) regularized parabolic-type Dirac operator. Its null solutions will be called cylindrical (resp. toroidal) regularized parabolic monogenic.

In the same way the projection map p_k induces a projection of the regularized Schrödinger operator $(\Delta_{\mathbf{x}} - \mathbf{k}\partial_t)$ to a second order operator $(\Delta_{\mathbf{x}'} - \mathbf{k}\partial_t)'$ which will be called the cylindrical (resp. toroidal) regularized Schrödinger operator. Its null-solutions are the solutions to the regularized Schrödinger equation on the manifold C_k.

More generally, as explained in [11], the decomposition of the lattice Ω_k into the direct sum of the sublattices $\Omega_l := \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_l$ and $\Omega_{k-l} := \mathbb{Z}e_{l+1} + \cdots + Ze_k$ gives rise to k conformally inequivalent different spinor bundles $E^{(l)}$ on $C_k \cong \mathbb{R}^n/\Omega_k$ by simply making the identification $(\mathbf{x}, X) \iff (x + \underline{m} + \underline{n}, (-1)^{m_1+\ldots+m_l} X)$ with $\mathbf{x} \in \mathbb{R}^n, X \in \mathbb{C}_n$. The projection p_k of the associated modifications of the hypoelliptic generalized $\wp$ function

$$\wp^{\epsilon}_{k;0,\cdots,l}(\mathbf{x}; t) := \sum_{\underline{m} \in \Omega_l} \sum_{\underline{n} \in \Omega_{k-l}} (-1)^{m_1+\cdots+m_l} E^{\epsilon}_{-;k}(\mathbf{x} + \underline{m} + \underline{n}; t)$$

then defines well defined regularized hypoelliptic spinor sections on the associated spinor bundles E^l of the C_k. The function $\wp^{\epsilon}_{k;0,\cdots,l}(x;t)$ satisfies

$$\wp^{\epsilon}_{k;0,\cdots,l}(x;t)(x+\omega;t) = (-1)^{m_1+\cdots+m_l}\wp^{\epsilon}_{k;0,\cdots,l}(x;t).$$

Its projection under p_k will be denoted by $G^{\epsilon}_{k,l}$. Similarly, the projection of the modified function $P^{\epsilon}_{k;0,\ldots,0;l}(x;t)$ defined by

$$P^{\epsilon}_{k;0,\ldots,0,l}(x;t) := \sum_{\underline{m}\in\Omega_l}\sum_{\underline{n}\in\Omega_{k-l}}(-1)^{m_1+\cdots+m_l}e^{\epsilon}_{-}(x+\underline{m}+\underline{n};t)$$

defines well defined spinor sections $H^{\epsilon}_{k,l}$ that are in the kernel of the associated spinorial regularized Schrödinger operator whose nullsolutions take values in the correspoding spinor bundles E^l of C_k. In the case where we take $l=n$ and where we make the trivial identification (x,X) with $(x+\omega,X)$ then we deal with the trivial bundle on C_k. In all what follows we restrict to formulate the results for the trivial bundle. For the other bundles the formulas can be adapted correspondingly, by substituting the sections G^{ϵ}_k by $G^{\epsilon}_{k,l}$ (resp. H^{ϵ}_k by $H^{\epsilon}_{k,l}$).

4.2 Integral representation formulas

Applying now the Borel-Pompeiu formula and Cauchy's integral formula for the regularized Dirac operator D^{ϵ}_{-} in the Euclidean spaces presented in Section 2, we obtain the following Cauchy integral formula for cylindrical (resp. toroidal) regularized time-holomorphic functions:

Theorem 4.1. *Suppose V' is a sub domain of a domain U' lying on $C_k\times\mathbb{R}^+$ and suppose that V' has a compact closure $cl(V')$. Assume further that $cl(V')\subset U'$ and that $p_k^{-1}(\partial V')$ is a Lipschitz surface. Let $f':U'\to\mathbb{C}_n$ be a cylindrical (resp. toroidal) left regularized time-holomorphic function. Then we have for each pair (y',t_0)*

$$f'(y',t_0) = \int\limits_{\partial V'} G^{\epsilon}_k(x'-y';t)(d_xp_kn(x;t))f'(x';t)dS(x';t), \qquad (9)$$

where d_xp_k is the derivative of p_k at x.

Suppose now that Σ is a sufficiently smooth hypersurface lying in $C_k\times\mathbb{R}^+$ and that U' is a domain whose boundary is Σ. Let u be an arbitrary $\mathbb{C}_n$ valued function belonging to $L_p(\Sigma)$. Then the integral

$$\int_{\Sigma} G^{\epsilon}_k(x'-y';t)(d_xp_kn(x;t))u(x';t)dS(x';t)$$

defines a cylindrical (resp. toroidal) regularized time-holomorphic function $f'(y';t)$ on U'. Notice that we only claim that u' belongs to L_p. The latter function does not necessarily need to have partial derivatives.

The cylindrical (resp. toroidal) regularized left time-holomorphic function f' lifts to a k-fold periodic regularized left time-holomorphic defined on the k-fold periodic open set $U = p_k^{-1}(U')$.

The projection map p_k also gives the following version of the Borel-Pompeiu formula for cylindrical (resp. toroidal) regularized time-holomorphic functions.

Theorem 4.2. *Suppose that V' is a domain in $C_k \to \mathbb{R}^+$ with compact closure and strongly Lipschitz boundary. Suppose also that $\theta : cl(V') \to \mathbb{C}_n$ is a continuous function and that $\theta|_{V'}$ belongs to $C^1(V')$. Then for each pair $(y',t) \in V'$*

$$\begin{aligned}\theta(y';t_0) &= \Big(\int_{\partial V'} G_k^{\epsilon}(x'-y';t)(d_x p_k n(x;t))\theta(x';t)dS(x';t) \\ &\quad - \int_{V'} G_k^{\epsilon}(x'-y';t){D'}^{\epsilon}_{+}\theta(x';t)d\mu(x';t)\Big),\end{aligned}$$

where μ is the projection of volume Lebesgue measure on $\mathbb{R}^n \times \mathbb{R}^+$ onto $C_k \times \mathbb{R}^+$.

Let again U' be a sub domain of $C_k \times \mathbb{R}^+$ with compact closure and $\theta : U' \to \mathbb{C}_n$ be an L_p function with $1 < p < \infty$. Again, by adapting the results from [5] obtained for the Euclidean space we readily obtain that on C_k holds

$$[{D'}^{\epsilon}{}_{+;C_k}] \int_{U'} G_k^{\epsilon}(y'-x';t)\theta(x';t)d\mu(x';t) = \theta(y';t)$$

for each $(y',t) \in U'$.

Finally, using the k-fold periodic basic function $P^{\epsilon}_{k;0,\ldots,0}$ for the regularized Schrödinger operator, we obtain a Green's formula for solutions to the homogeneous regularized Schrödinger equation on C_k.

Theorem 4.3. *Suppose that $h : U' \to \mathbb{C}_n$ is a solution to the cylindrical (resp. toroidal) regularized Schrödinger operator on the domain $U' \subset C_k \times \mathbb{R}^+$. Suppose also that V' is a relatively compact subdomain of U' and that $cl(V') \subset U'$. Then provided the boundary of V' is sufficiently smooth*

$$\begin{aligned}h(y';t_0) &= \int_{\partial V'} (G_k^{\epsilon}(x'-y';t)(d_x p_k n(x;t))h(x';t) \\ &\quad + H_k^{\epsilon}(y'-x';t)(d_x p_k n(x;t)){D'}^{\epsilon}_{+;k}h(x';t))dS(x';t)\end{aligned}$$

for each $(y',t) \in V'$.

5 The inhomogeneous regularized Schrödinger equation on cylinders and tori

Throughout this section suppose that V' is a sub domain of an open subset $U' \subset C_k \times \mathbb{R}^+$ for $k = 1, \ldots, n$) and that the closure of V' has a strongly Lipschitz boundary $\partial V'$. Suppose that $f : V' \to \mathbb{C}_n$ is a function belonging to the Sobolev space $W_p^2(V')$, with $1 \leq p < +\infty$. Again let $[\Delta_{\mathbf{x}'} - \mathbf{k}\partial_t]'$ be the associated cylindrical (resp. toroidal) regularized Schrödinger operator.

The Borel-Pompeiu formula presented in Theorem 4.2 motivates us to introduce the following definition

Definition 5.1. *The cylindrical (resp. toroidal) regularized Teodorescu operator is defined from $W_p^l(V')$ to $W_{p+1}^l(V')$, with $1 \leq p < +\infty$, as*

$$[T_-^{C_k,\epsilon} f'(y';t_0)] = -\int\limits_{V'} G_k^\epsilon(x'-y';t) f'(x';t) dV'(x') dt$$

where x' and y' are distinct points from V'.

Notice that due to the exponentional decrease of the kernel function, the Teodorescu transform is always an L_2 bounded operator even if V' is an unbounded domain. Also from Theorem 4.2 we have

Definition 5.2. *The cylindrical (resp. toroidal) regularized Cauchy operator is defined from $W_{p-1}^l(\partial V')$ to $W_p^l(V') \cap \mathrm{Ker}({D'}_-^\epsilon)$, with $1 \leq p < +\infty$, as*

$$[F_-^{C_k,\epsilon} f'(y';t_0)] = \int\limits_{\partial V'} G_k^\epsilon(x'-y';t) n(x';t) d_x p_k(n(x;t)) f'(y';t) dS'(x';t).$$

Using the previous operators the Borel-Pompeiu formula presented in Theorem 4.2 can now be reformulated in the classical form

$$f' = F_-^{C_k,\epsilon} f' + T_-^{C_k,\epsilon} {D'}_-^\epsilon f',$$

as formulated for the Euclidean case in [8, 5] in the context of the elliptic operators. Adapting the arguments from [5] that were explicitly worked out for the Euclidean case, one can show that the following Hodge type decomposition holds for the space of the L_p functions over a domain V' of the manifold C_k

Theorem 5.3. *The space $L_p(\Omega)$, $1 \leq p < +\infty$ admits the following decomposition*

$$L_p(V') = \left(L_p(V') \cap \mathrm{Ker}({D'}_-^\epsilon)\right) \oplus {D'}_-^\epsilon \overset{\circ}{W}{}_p^1(V'), \tag{10}$$

for all $\epsilon > 0$, and we can define the following projectors

$$\begin{aligned} P_-^{C_k,\epsilon} : \quad & L_p(\Omega) && \to \mathrm{Ker} {D'}_-^\epsilon \cap L_p(V') \\ Q_-^{C_k,\epsilon} : \quad & L_p(\Omega) && \to {D'}_-^\epsilon \overset{\circ}{W}{}_p^1(V'), \end{aligned}$$

where $P_-^{C_k,\epsilon}$ is called cylindrical (rep. toroidal) regularized Bergman projector and $Q_-^{C_k,\epsilon} = I - P_-^{C_k,\epsilon}$ is called cylindrical (rep. toroidal) regularized Pompeiu projector .

For the particular case of $p = 2$ this decomposition is orthogonal and the space $\mathrm{Ker} {D'}^{\epsilon}_{-} \cap L_2(V')$ is a Banach space endowed with the L_2 inner product

$$\langle f', g' \rangle := \int_{V'} \overline{f(\mathbf{x}'; t)} g(\mathbf{x}'; t) dV(\mathbf{x}') dt.$$

Then, as a consequence of Cauchy's integral formula that we established in the previous section and Cauchy-Schwarz' equality we can show that this space has a continuous point evaluation and does hence possess a reproducing kernel $B(x', y'; t)$, satisfying

$$f'(y'; t_0) = \int\limits_{V'} B(x', y'; t) f(x'; t) dV(x') dt \quad \forall\, f' \in \mathrm{Ker} {D'}^{\epsilon}_{-} \cap L_2(V').$$

Let f be an arbitrary function from $L_2(V')$. Then the operator

$$[P_{-}^{C_k,\epsilon} f'(y'; t)] = \int\limits_{V'} B(x', y'; t) f(x'; t) dV(x') dt$$

correspondes to the projector presented in Theorem 5.3 for $p = 2$. With these operators we can represent in complete analogy to the Euclidean case treated in [3] the solutions to the inhomogeneous regularized Schrödinger equation on cylinders and tori. We establish

Theorem 5.4. *Let V' be a domain on the manifold C_k $(k = 1, \ldots, n)$ and $f \in W_p^2(V')$, with $1 \leq p < +\infty$. The the system*

$$(-\Delta_{\mathbf{x}'} - \mathbf{k}\partial_t)' u' = f' \quad \text{in } V' \tag{11}$$
$$u' = 0 \quad \text{at } \partial V' \tag{12}$$

has a unique solution $u \in W^2_{p+2,loc}(V')$ of the form

$$u' = T_{-}^{C_k,\epsilon} Q_{-}^{C_k,\epsilon} T_{-}^{C_k,\epsilon} f'. \tag{13}$$

Proof. To the proof one applies the factorization $({D'}^{\epsilon}_{-})^2 = (-\Delta_{\mathbf{x}'} - \mathbf{k}\partial_t)'$. Equation (11) thus can be written in the form

$$({D'}^{\epsilon}_{-})^2 u' = f'.$$

Now one applies the cylindrical (resp. toroidal) regularized Teodorescu transform $T_{-}^{C_k,\epsilon}$ to this equation which leads to

$$T_{-}^{C_k,\epsilon} ({D'}^{\epsilon}_{-}) [({D'}^{\epsilon}_{-}) u'] = T_{-}^{C_k,\epsilon} f'.$$

Next one applies the generalized Borel-Pompeiu's formula in the cylindrical (resp. toroidal) regularized version which leads to

$${D'}^{\epsilon}_{-} u' - F_{-}^{C_k,\epsilon} {D'}^{\epsilon}_{-} u' = T_{-}^{C_k,\epsilon} f'. \tag{14}$$

Now one applies the projector $Q_-^{C_k,\epsilon}$ to this equation which leads to

$$Q_-^{C_k,\epsilon} D'^{\epsilon}_- u' - Q_-^{C_k,\epsilon} F_-^{C_k,\epsilon} D'^{\epsilon}_- u' = Q_-^{C_k,\epsilon} T_-^{C_k,\epsilon} f'. \tag{15}$$

Since $F_-^{C_k,\epsilon} D'^{\epsilon}_- u' \in \mathrm{Ker} D'^{\epsilon}_-$ one has $Q_-^{C_k,\epsilon} F_-^{C_k,\epsilon} D'^{\epsilon}_- u' = 0$. Therefore, equation (15) is equivalent to

$$Q_-^{C_k,\epsilon} D'^{\epsilon}_- u' = Q_-^{C_k,\epsilon} T_-^{C_k,\epsilon} f'.$$

Next we again apply the cylindrical (resp. toroidal) regularized Teodorescu transform to this equation which leads to

$$T_-^{C_k,\epsilon} Q_-^{C_k,\epsilon} D'^{\epsilon}_- u' = T_-^{C_k,\epsilon} Q_-^{C_k,\epsilon} T_-^{C_k,\epsilon} f'.$$

Applying the specific mapping properties of these operators and again Borel-Pompeiu's formula, then the left hand-side of this equation simplifies to u' so that we finally obtain that

$$u' = T_-^{C_k,\epsilon} Q_-^{C_k,\epsilon} T_-^{C_k,\epsilon} f'.$$

The assertion now follows. □

By adapting the standard techniques from [8] to the setting of this paper we have the following generalization of the previous result

Theorem 5.5. *Let V' be a domain on the manifold C_k $(k = 1, \ldots, n)$, $f' \in W_p^2(V')$ and $g' \in W^2_{p+3/2}(\partial V')$, with $1 \leq p < +\infty$. Then the system*

$$(\Delta_{\mathbf{x}'} - \mathbf{k}\partial_t)' u' = f' \quad in\ V' \tag{16}$$

$$u' = g' \quad at\ \partial V' \tag{17}$$

has a solution $u \in W^2_{p+2,loc}(V')$ of the form

$$u' = F_-^{C_k,\epsilon} g' + T_-^{C_k,\epsilon} P_-^{C_k,\epsilon} D'^{\epsilon}_- h' + T_-^{C_k,\epsilon} Q_-^{C_k,\epsilon} T_-^{C_k,\epsilon} f'. \tag{18}$$

where h' is the unique $W^2_{p+2}(V')$ extension of g'.

Remark: Again, as in [8] we can represent the cylindrical (resp. toroidal) regularized Bergman projector in terms of algebraic expressions involving only the cylindrical (resp. toroidal) regularized Cauchy and Teodorescu transform, viz

$$P_-^{C_k,\epsilon} = F_-^{C_k,\epsilon}(tr T_-^{C_k,\epsilon} F_-^{C_k,\epsilon})^{-1} tr T_-^{C_k,\epsilon},$$

where tr is the usual trace operator. This formula allows us to represent the solutions to the inhomogeneous cylindrical (resp. toroidal) regularized Schrödinger equation in terms of the singular integral operators that we introduced in the previous section.

6 The limit case $\epsilon \to 0^+$

The aim of this section is to extend the results presented in the previous section to the original operators D'_- and $(-\Delta_{\mathbf{x}'} - i\partial_t)'$. In order to proceed in this direction we start by recalling the following result from [5]

Theorem 6.1. *For all* $1 \leq p < +\infty$, *we have the following weak convergence, in* $W_p^{-\frac{n}{2}-1}(V')$,

$$\left\langle E_-^\epsilon, \varphi \right\rangle \quad \to \quad \left\langle E_-, \varphi \right\rangle, \quad \varphi \in W_p^{\frac{n}{2}+1}(V'),$$

when $\epsilon \to 0$. *Here*

$$E_-(x,t) = e_-(x,t) \left[\frac{-x}{2it} + \mathfrak{f} \left(\frac{-n}{2t} + \frac{|x|^2}{4it^2} \right) - i\mathfrak{f}^\dagger \right].$$

This theorem implies the following corrolary

Corolary 6.2. *For all* $1 \leq p < +\infty$, *we have the following weak convergence, in* $W_p^{-\frac{n}{2}-1}(V')$,

$$\left\langle G_k^\epsilon, \varphi \right\rangle \quad \to \quad \left\langle G_k, \varphi \right\rangle, \quad \varphi \in W_p^{\frac{n}{2}+1}(V'),$$

when $\epsilon \to 0$. *Here* G_k *is the projection under* p_k *of*

$$\wp_{k;0,\cdots,l}(\mathbf{x};t) := \sum_{\underline{m} \in \Omega_l} \sum_{\underline{n} \in \Omega_{k-l}} (-1)^{m_1+\cdots+m_l} E_{-;k}(\mathbf{x} + \underline{m} + \underline{n}; t).$$

On the basis of these results we are in position to study the convergence of the family of operators $T_-^{C_k,\epsilon}$ and projectors $Q_-^{C_k,\epsilon}$ to the cylindrical (resp. toroidal) Teodorescu operator and the cylindrical (resp. toroidal) Bergaman projector associated to the cylindrical (resp. toroidal) Schrödinger operator defined as

$$[T_-^{C_k} f'(y';t_0)] \quad = \quad -\int_{V'} G_k(x'-y';t) f'(x';t) dV'(x') dt$$

$$\begin{aligned} Q_-^{C_k} &= I - P_-^{C_k} \\ &= I - \int_{V'} B(x',y';t) f(x';t) dV(x') dt, \end{aligned}$$

where $B(x',y';t)$ is a reproducing kernel, which satisfies

$$f'(y';t_0) = \int_{V'} B(x',y';t) f(x';t) dV(x') dt, \quad \forall\, f' \in \mathrm{Ker} D'_- \cap L_2(V').$$

Theorem 6.3. *The family of cylindrical (resp. toroidal) regularized Teodorescu operators* $T_-^{C_k,\epsilon}$ *converges weakly to* $T_-^{C_k}$ *in* $W_p^{\frac{n}{2}+1}(V')$*, for all* $1 \leq p < +\infty$.

Proof. Let $u \in L_p(V')$. From the previous theorem we may infer that we have for every $\varphi \in W_p^{\frac{n}{2}+1}(\Omega)$,

$$\begin{aligned}
\lim_{\epsilon\to 0^+} \left|\left\langle \left(T_-^{C_k,\epsilon} - T_-^{C_k}\right) u,\, \varphi\right\rangle\right| &= \lim_{\epsilon\to 0^+} \left|\langle (G_k^\epsilon - G_k) * u,\, \varphi\rangle\right| \\
&= \left|\left\langle \lim_{\epsilon\to 0^+} \left(G_k^\epsilon - G_k\right),\, u * \varphi\right\rangle\right| \\
&= 0
\end{aligned}$$

□

Theorem 6.4. *The family of projectors* $Q_-^{C_k,\epsilon}$ *is a fundamental family in* $W_p^{-\frac{n}{2}-1}(\Omega)$*, for all* $1 \leq p < +\infty$.

Proof. Let us start with the proof of the convergence. Consider $u \in L_p(\Omega)$ and $\varphi \in W_p^{\frac{n}{2}+1}(\Omega)$, where $1 \leq p < +\infty$. Since for all $\epsilon > 0$, $(Q_-^{C_k,\epsilon})^2 = Q_-^{C_k,\epsilon}$ and $Q_-^{C_k,\epsilon}\left(P_-^{C_k,\epsilon} u\right) = 0$, we have for any $\epsilon_1, \epsilon_2 > 0$

$$\begin{aligned}
\left|\langle Q_-^{C_k,\epsilon_1} u - Q_-^{C_k,\epsilon_2} u,\, \varphi\rangle\right| &= \left|\langle Q_-^{C_k,\epsilon_1}(P_-^{C_k,\epsilon_1} u + Q_-^{C_k,\epsilon_1} u) - Q_-^{C_k,\epsilon_2}(P_-^{C_k,\epsilon_1} u + Q_-^{C_k,\epsilon_1} u),\, \varphi\rangle\right| \\
&= \left|\langle Q_-^{C_k,\epsilon_1} u - Q_-^{C_k,\epsilon_2} P_-^{C_k,\epsilon_2} u - Q_-^{C_k,\epsilon_2} Q_-^{C_k,\epsilon_1} u,\, \varphi\rangle\right| \\
&\leq \underbrace{\left|\langle Q_-^{C_k,\epsilon_2} P_-^{C_k,\epsilon_1} u,\, \varphi\rangle\right|}_{(K)} + \underbrace{\left|\langle (I - Q_-^{C_k,\epsilon_2}) Q_-^{C_k,\epsilon_1} u,\, \varphi\rangle\right|}_{(L)}.
\end{aligned}$$

For $P_-^{C_k,\epsilon} : L_p(\Omega) \to \mathrm{Ker} D'^{\epsilon}_- \cap L_p(V')$ the projectors defined previously, we have for the term (K)

$$\begin{aligned}
\left|\langle Q_-^{C_k,\epsilon_2} P_-^{C_k,\epsilon_1} u,\, \varphi\rangle\right| &= \left|\langle Q_-^{C_k,\epsilon_2}(F_-^{C_k,\epsilon_1} P_-^{C_k,\epsilon_1} - Q_-^{C_k,\epsilon_2} F^{\epsilon_2}) P_-^{C_k,\epsilon_1}\rangle\right| \\
&= \left|\langle Q_-^{C_k,\epsilon_2}(I - T_-^{C_k,\epsilon_1} D'^{\epsilon_1}_- - (I - T_-^{C_k,\epsilon_2} D'^{\epsilon_2}_-)) P_-^{C_k,\epsilon_1} u,\, \varphi\rangle\right| \\
&= \left|\langle Q_-^{C_k,\epsilon_2}(T_-^{C_k,\epsilon_1} D'^{\epsilon_1}_- - T_-^{C_k,\epsilon_2} D'^{\epsilon_2}_-) P_-^{C_k,\epsilon_1} u,\, \varphi\rangle\right| \\
&= \left|\langle Q_-^{C_k,\epsilon_2}(T_-^{C_k,\epsilon_1}(D'^{\epsilon_1}_- - D'^{\epsilon_2}_-) + (T_-^{C_k,\epsilon_1} - T_-^{C_k,\epsilon_2}) D'^{\epsilon_2}_-) P_-^{C_k,\epsilon_1} u,\, \varphi\rangle\right|
\end{aligned}$$

From applying Theorem 2.2 and Theorem 6.3 we may deduce the weak convergence of (K), in $W_p^{-\frac{n}{2}-1}(\Omega)$ for all $1 \leq p < +\infty$, of the right hand side of the last expression to zero. Finally, since $Q_-^{C_k,\epsilon_1} u \in D'_-\left(\overset{\circ}{W}{}_p^1(\Omega)\right)$, there exists $g \in \overset{\circ}{W}{}_p^1(\Omega)$ such that $u = D'^{\epsilon}_{-} g$. Therefore, (L) becomes

$$\begin{aligned}
\left|\langle (I - Q_-^{C_k,\epsilon_2}) Q_-^{C_k,\epsilon_1} u,\, \varphi\rangle\right| &= \left|\langle (I - Q_-^{C_k,\epsilon_2}) D'^{\epsilon}_{-} g,\, \varphi\rangle\right| \\
&= \left|\langle D'^{\epsilon_1}_{-} g - Q_-^{C_k,\epsilon_2} D'^{\epsilon_1}_{-} g + D'^{\epsilon_2}_{-} g - D'^{\epsilon_2}_{-} g\, \varphi\rangle\right| \\
&= \left|\langle Q_-^{C_k,\epsilon_2}(D'^{\epsilon}_{-} g - D'_{-} g) + (D'_{-} g - D'^{\epsilon}_{-} g),\, \varphi\rangle\right| \\
&= \left|\langle (D'_{-} g - D'^{\epsilon}_{-} g)(I - Q_-^{C_k,\epsilon_1}),\, \varphi\rangle\right|.
\end{aligned}$$

Once again, by Theorem 2.2 we conclude that the latter expression tends to zero when $\epsilon \to 0$. □

Now it remains to prove that $Q_-^{C_k}$ is idempotent. Hereby, we have

$$(Q_-^{C_k})^2 = \lim_{\epsilon \to 0} (Q_-^{C_k,\epsilon})^2 = \lim_{\epsilon \to 0} Q_-^{C_k,\epsilon} = Q_-.$$

Theorem 6.5. *For any given $f \in L_p(\Omega)$, consider the solutions (u^{ϵ}) for the problem*

$$\begin{aligned}
(-\Delta_{\mathbf{x}'} - \mathbf{k}\partial_t)' u'^{\epsilon} &= f' \quad in\ V' && (19) \\
u'^{\epsilon} &= 0 \quad at\ \partial V' && (20)
\end{aligned}$$

for each $\epsilon > 0$.

Then, the family of those solutions (u'^{ϵ}) is a fundamental family in $W_p^{-\frac{n}{p}-1}(\Omega)$, for all $1 \leq p < +\infty$.

Moreover, $(D'^{\epsilon}_{-} u'^{\epsilon})$ is a fundamental family in $W_p^{-\frac{n}{p}-1}(\Omega)$.

Proof. Let us consider $\varphi \in W_p^{\frac{n}{2}+1}(\Omega)$, $f \in L_p(\Omega)$ and a family of functions (u'^{ϵ}), such that $u'^{\epsilon} \in D'^{\epsilon}_{-}(\Omega)$ with $\epsilon > 0$, and $\epsilon_1, \epsilon_2 > 0$. Since the elements of the family are solutions of the problem (19), we have that $u'^{\epsilon} = T_-^{C_k,\epsilon} Q_-^{C_k,\epsilon} T_-^{C_k,\epsilon} f$ (for more details about this assertion see [5]). Then

$$\begin{aligned}
\left|\langle u'^{\epsilon_1} - u'^{\epsilon_2},\, \varphi\rangle\right| &= \left|\left\langle T_-^{C_k,\epsilon_1} Q_-^{C_k,\epsilon_1} T_-^{C_k,\epsilon_1} f - T_-^{C_k,\epsilon_2} Q_-^{C_k,\epsilon_2} T_-^{C_k,\epsilon_2} f,\, \varphi\right\rangle\right| \\
&= \left|\left\langle \left(T_-^{C_k,\epsilon_1} Q_-^{C_k,\epsilon_1} T_-^{C_k,\epsilon_1} - T_-^{C_k,\epsilon_2} Q_-^{C_k,\epsilon_2} T_-^{C_k,\epsilon_2}\right) f,\, \varphi\right\rangle\right| \\
&\leq \left|\left\langle \left(T_-^{C_k,\epsilon_1} Q_-^{C_k,\epsilon_1} \left(T_-^{C_k,\epsilon_1} - T_-^{C_k,\epsilon_2}\right)\right) f,\, \varphi\right\rangle\right| \\
&\quad + \left|\left\langle \left(\left(T_-^{C_k,\epsilon_1} - T_-^{C_k,\epsilon_2}\right) Q_-^{C_k,\epsilon_2} T_-^{C_k,\epsilon_2}\right) f,\, \varphi\right\rangle\right| \\
&\quad + \left|\left\langle \left(T_-^{C_k,\epsilon_1} \left(Q_-^{C_k,\epsilon_1} - Q_-^{C_k,\epsilon_2}\right) T_-^{C_k,\epsilon_2}\right) f,\, \varphi\right\rangle\right|.
\end{aligned}$$

By Theorem 6.3 and Theorem 6.4 we conclude that the right hand side of the last inequality tends to zero when $\epsilon_1, \epsilon_2 \to 0$.

Theorem 6.5 now guarantees that there exists a function $f \in L_p(\Omega)$ such that

$$D'^{\epsilon_1}_- u'^{\epsilon_1} = Q^{C_k,\epsilon_1}_- T^{C_k,\epsilon_1}_- f \quad \text{and} \quad D'^{\epsilon_2}_- u'^{\epsilon_2} = Q^{C_k,\epsilon_2}_- T^{C_k,\epsilon_2}_- f.$$

This in turn implies that

$$\begin{aligned} \left|\left\langle \left(Q^{C_k,\epsilon_1}_- T^{C_k,\epsilon_1}_- - Q^{C_k,\epsilon_2}_- T^{C_k,\epsilon_2}_-\right) f,\, \varphi\right\rangle\right| \quad &\leq \quad \left|\left\langle \left(Q^{C_k,\epsilon_1}_- \left(T^{C_k,\epsilon_1}_- - T^{C_k,\epsilon_2}_-\right)\right) f,\, \varphi\right\rangle\right| \\ &\quad + \left|\left\langle \left(\left(Q^{C_k,\epsilon_1}_- - Q^{C_k,\epsilon_2}_-\right) T^{C_k,\epsilon_2}_-\right) f,\, \varphi\right\rangle\right|. \end{aligned}$$

By Theorem 6.4 and Theorem 6.5 we conclude that the right hand side of the previous expression converges weakly to zero when $|\epsilon_1 - \epsilon_2| \to 0$, in $W_p^{-\frac{n}{2}-1}(\Omega)$, for all $1 \leq p < +\infty$. □

This result can be refined. By $u'_2 \in W_p^{-\frac{n}{2}-1}(V')$ we denote the function limit of the Cauchy family that we studied. Again Theorem 6.5 implies the existence of functions $f \in L_p(V')$ that satisfy

$$(-\Delta - i\partial_t)' u'_2 = f \quad \text{and} \quad (-\Delta - i\partial_t)' {u'}^{\epsilon}_2 = f,$$

with $u'_2|_\Gamma = 0 = {u'}^{\epsilon}_2|_\Gamma$.

Since the inverse operator $(-\Delta - i\partial_t)^{-1}$ exists and since the latter one is unique, cf. [17], we have that $(-\Delta - i\partial_t)'^{-1}$ also exists and it is unique, too. Hence, we can establish the following equality

$$u'_2 - {u'}^{\epsilon}_2 = (-\Delta - i\partial_t)'^{-1} \left((-\Delta - \mathbf{k}\partial_t)' - (-\Delta - i\partial_t)'\right) {u'}^{\epsilon}_2,$$

which implies that

$$||u'_2 - {u'}^{\epsilon}_2||_{L_p(V')} = ||(-\Delta - i\partial_t)'^{-1}||_{L_p(V')} \; ||(-\Delta - \mathbf{k}\partial_t)' - (-\Delta - i\partial_t)'||_{L_p(V')} \; ||{u'}^{\epsilon}_2||_{L_p(V')}.$$

Since $||(-\Delta - \mathbf{k}\partial_t)' - (-\Delta - i\partial_t)'||_{L_1(V')}$ converges to zero when $\epsilon \to 0$, we may conclude that the right hand side of the last expression also converges to zero. This fact implies that $u'_2 \in L_p(V')$.

Moreover, we can guarantee

(i) For any two elements ${u'}^{\epsilon_1}_2$ and ${u'}^{\epsilon_2}_2$ of the fundamental family studied in Theorem 6.4 and Theorem 6.5 we can find functions ${g'}^{\epsilon_1}_2, {g'}^{\epsilon_2}_2 \in \overset{\circ}{W}{}^1_p(V')$ that satisfy

$${u'}^{\epsilon_1}_2 = D'^{\epsilon_1}_- {g'}^{\epsilon_1}_2 \quad \text{and} \quad {u'}^{\epsilon_2}_2 = D'^{\epsilon_2}_- {g'}^{\epsilon_2}_2$$

and

$$\begin{aligned} ||D'^{\epsilon_2}_-({g'}^{\epsilon_1}_2 - g^{\epsilon_2}_2)||_{L_p(V')} \quad &= \quad ||D'^{\epsilon_2}_- {g'}^{\epsilon_1}_2 - D'^{\epsilon_1}_- {g'}^{\epsilon_1}_2 + D'^{\epsilon_1}_- {g'}^{\epsilon_2}_2 - D'^{\epsilon_2}_- {g'}^{\epsilon_2}_2||_{L_p(V')} \\ &\leq \quad ||\left(D'^{\epsilon_2}_- - D'^{\epsilon_1}_-\right) {g'}^{\epsilon_1}_2||_{L_p(V')} \\ &\qquad + ||{u'}^{\epsilon_1}_2 - {u'}^{\epsilon_2}_2||_{L_p(V')}. \end{aligned}$$

As a consequence of Theorem 2.2, Theorem 6.5 and the above mentioned considerations, we may readily infer that the right hand side of the previous expression converges to zero, when $|\epsilon_1 - \epsilon_2| \to 0$, i.e.

$$||D'^{\epsilon_2}_{-}(g'^{\epsilon_1}_{2} - g^{\epsilon_2}_{2})||_{L_p(V')} \to 0, \quad \text{when } |\epsilon_1 - \epsilon_2| \to 0.$$

Since $||D'^{\epsilon}_{-}|| \to ||D'_{-}|| < \infty$, when $\epsilon \to 0$, we conclude that $g' \to g'^{\epsilon_2}_{2} + C$, when $|\epsilon_1 - \epsilon_2| \to 0$ and $\epsilon_1, \epsilon_2 \to 0$, where $C \in \ker(D'_{-})$.

Under these conditions we showed that for each $u' \in L_p(V')$ there exists a function $v' \in \overset{\circ}{W}{}^1_p(V')$ such that $u' = D'_{-}v'$.

(ii) Suppose that there exist two functions $g'_1, g'_2 \in \overset{\circ}{W}{}^1_p(V')$, such that

$$u' = D'_{-}g'_1 \quad \text{and} \quad u' = D'_{-}g'_2,$$

is satisfied for the same function $u' \in L_p(V')$. Then we have

$$\begin{aligned}(-\Delta - i\partial_t)'g'_1 = (-\Delta - i\partial_t)'g'_2 \quad &\Leftrightarrow \quad g'_1 = (-\Delta - i\partial_t)'^{-1}(-\Delta - i\partial_t)'g'_2 \\ &\Leftrightarrow \quad g'_1 = g'_2.\end{aligned}$$

The assertion is hereby proven.

Theorem 6.6. *For each $u' \in L_p(V')$, the family of $P'^{C_k,\epsilon}_{-}u$ converges to $\hat{u'}$ in $\ker(D'^{\epsilon}_{-}) \cap L_p(V')$, for all $\epsilon > 0$ and $1 \leq p < +\infty$.*

Proof. The proof is made in three steps: First let us consider a function $\varphi \in W_p^{\frac{n}{2}+1}(V')$, a function $u \in L_p(V')$, and a family of functions (u'^{ϵ}_1), where $u'^{\epsilon}_1 \in \ker(D'^{\epsilon}_{-}) \cap L_p(V')$ with $\epsilon > 0$, with $1 \leq p < +\infty$.

Let ϵ_1, $\epsilon_2 > 0$. In view of the decomposition (10) we have for $u'^{\epsilon_1}_1$, $u'^{\epsilon_2}_1 \in \ker(D'^{\epsilon_1}_{-})$, $\ker(D'^{\epsilon_2}_{-})$

$$\begin{aligned}\left|\langle u'^{\epsilon_1}_1 - u'^{\epsilon_2}_1, \varphi\rangle\right| &= \left|\langle (u - u^{\epsilon_1}_2) - (u - u'^{\epsilon_2}_2), \varphi\rangle\right| \\ &\leq \left|\langle u'^{\epsilon_2}_2 - u'^{\epsilon_1}_2, \varphi\rangle\right|,\end{aligned}$$

where $u'^{\epsilon_1}_2$ and $u'^{\epsilon_1}_2$ are elements of the fundamental family (u'^{ϵ}_2), where $u'^{\epsilon}_2 \in D'^{\epsilon}_{-}(\overset{\circ}{W}{}^1_p(V'))$ for $\epsilon > 0$. By Theorem 6.5 we conclude that the right hand side of the last expression converges weakly to zero, in $W_p^{-\frac{n}{2}-1}(V')$, when $|\epsilon_1 - \epsilon_2| \to 0$.

This proves that (P^{ϵ}_{-}) is a fundamental family in $W_p^{-\frac{n}{2}-1}(V')$.

Moreover, using the techniques and arguments presented for the family $D'^{\epsilon}_{-}u'^{\epsilon}$, with $\epsilon > 0$, after Theorem 6.5, we can refine our conclusion. Consequently we can prove that the function limit is in $L_p(V')$.

Finally, let us denote by u'_1 the function limit of this fundamental family. For a given $\varphi \in W_p^{\frac{n}{2}+1}(V')$, with $1 \leq p < +\infty$, we have

$$\begin{aligned}|\langle D'_{-}u'_1, \varphi\rangle| &= \left|\langle D'_{-}u'_1 - D'^{\epsilon}_{-}u'^{\epsilon}_1, \varphi\rangle\right| \\ &\leq \left|\langle D'_{-}(u'_1 - u'^{\epsilon}_1), \varphi\rangle\right| + \left|\langle (D'_{-} - D'^{\epsilon}_{-})u'^{\epsilon}_1(x,t), \varphi\rangle\right|.\end{aligned}$$

Theorem 2.2 and Theorem 6.6 guarantee that the first and second term of the right hand side of the last expression converges to 0 when $\epsilon \to 0$. ☐

Summarizing, for each $u' \in L_p(V')$, we have $u' = P_-^{C_k,\epsilon}u' + Q_-^{C_k,\epsilon}u'$. Also, we proved that

$$\begin{array}{rcl} Q_-^{C_k,\epsilon}u' & \to & Q_-^{C_k}u' \\ (Q_-^{C_k})^2u' & = & Q_-^{C_k}u', \end{array}$$

which implies that $Q_-^{C_k}$ is a projector and that we can define a projector $P_-^{C_k}$ by

$$P_-^{C_k}u' \quad = \quad u' - Q_-^{C_k}u',$$

with $P_-^{C_k}u' \in \ker(D'_-) \cap L_p(V')$.

As a consequence, we have the following Hodge-type decomposition

Theorem 6.7. *For $1 \leq p < +\infty$, the following decomposition*

$$L_p(V') \quad = \quad (L_p(V') \cap \ker(D'_-)) \oplus D'_-(\overset{\circ}{W}{}_p^1(V'))$$.

holds.

Moreover, we can define the following projectors

$$\begin{array}{rl} P_-^{C_k}: \quad L_p(V') & \to L_p(V') \cap ker(D'_-) \\ Q_-^{C_k}: \quad L_p(V') & \to D'_-\left(\overset{\circ}{W}{}_p^1(V')\right), \end{array}$$

where $P_-^{C_k}$ and $Q_-^{C_k}$ are called cylindrical (resp. toroidal) Schrödinger-Bergman projectors.

From the previous result we have this two immediate applications

Theorem 6.8. *Let V' be a domain on the manifold C_k $(k = 1, \ldots, n)$ and $f \in W_p^2(V')$, with $1 \leq p < +\infty$. The the system*

$$\begin{array}{rcl} (-\Delta_{\mathbf{x}'} - i\partial_t)'u' & = & f' \quad in\ V' \\ u' & = & 0 \quad at\ \partial V' \end{array}$$

has a unique solution $u \in W_{p+2,loc}^2(V')$ of the form

$$u' = T_-^{C_k}Q_-^{C_k}T_-^{C_k}f'.$$

Theorem 6.9. *Let V' be a domain on the manifold C_k $(k = 1, \ldots, n)$, $f' \in W_p^2(V')$ and $g' \in W_{p+3/2}^2(\partial V')$, with $1 \leq p < +\infty$. Then the system*

$$\begin{array}{rcl} (-\Delta_{\mathbf{x}'} - i\partial_t)'u' & = & f' \quad in\ V' \\ u' & = & g' \quad at\ \partial V' \end{array}$$

has a solution $u \in W_{p+2,loc}^2(V')$ of the form

$$u' = F_-^{C_k}g' + T_-^{C_k}P_-^{C_k}D'^{\epsilon}_-h' + T_-^{C_k}Q_-^{C_k}T_-^{C_k}f'.$$

where h' is the unique $W_{p+2}^2(V')$ extension of g'.

Remark: All the results presented in this section can be deduced for H_k.

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