From Singularity Theory to Finiteness of Walrasian Equilibria

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Abstract

The main result of this paper states that there exists an open and dense subset of the set of critical economies whose associated equilibria are finite in number. The proof rests on results and concepts from singularity theory.

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1 Introduction

We establish that a generic critical economy has a finite number of equilibria. By generic, we mean that this property holds for a residual, therefore dense, subset in the C^∞ compact-open topology. Moreover, this subset is open in the C^∞ Whitney topology. In fact, we show that there is a class of economies with a finite number of equilibria which not only contains all regular economies but even generic critical economies. This extension to non-regular economies adds to a long series of contributions on local isolation, determinacy, and other generic properties of the set of equilibrium prices, spawned by Debreu's seminal 1970 paper. It also extends results by Allen (1984) and Mas-Colell and Nachbar (1991) and proves conjectures found in the latter.

Debreu (1970) establishes that for all but a Lebesgue measure zero subset of endowments, Arrow-Debreu economies are regular and thus have locally isolated, indeed finitely many equilibria. Despite its fundamental nature, the full extension of this result to critical economies has so far been elusive. The proof of finiteness in a strong generic sense for both regular and critical economies would not only confirm heretofore unproven conjectures, but also further characterize the equilibrium manifold.

Allen (1984) and Mas-Colell and Nachbar (1991) make contributions in this direction. Allen (1984) works with economies parametrized on a compact manifold with boundary and establishes finiteness of equilibria for a residual subset of parameters in both the C^∞ Whitney and compact-open topologies. Her proof resorts to results in differential topology applied directly to smooth (C^∞) aggregate excess demand (henceforth, AED) functions rather than the deeper primitives of preferences and endowments.

Mas-Colell and Nachbar (1991), on the other hand, work directly with preferences and endowments and use no more than transversality to obtain countability (though not finiteness) of equilibria for a residual subset of utility functions of the first consumer in the C^∞ compact-open topology. They conjecture (pp. 402–403) that the results can be strengthened in two ways, namely that finiteness (rather than mere countability) should hold for open and dense (rather than merely residual) sets, but do point to the mathematical subtleties associated with the study of the set of zeros in the neighborhood of a critical zero.

A standard approach for a transversality-based argument is to perturb an economy (preferences and endowments) so as to transform its AED into a function with isolated zeros. The challenge of this type of exercise is to construct a sufficiently rich, yet tractable perturbation, and to relate the perturbed AED to the deeper primitives of preferences and endowments. Results recently established by Castro *et al.* (2010), allow us to do just this and perturb AED *directly*.[1] It is in this spirit that Castro and Dakhlia (2008) obtain finiteness by establishing that, generically, AED is Thom-Boardman stratified. While their approach has the advantage of being geometrically intuitive, their result ultimately only holds for analytic functions.

In this paper, we do not focus on regularity, since by definition it excludes critical economies, but rather on whether AED functions are of *Finite Singularity Type* (FST).[2] The FST concept is first defined by du Plessis in Gibson *et al.* (1976, Definition III, 2.7) and has been used strictly in the context of singularity theory, which may be why it has so far not drawn economists' attention. Although its formulation is straightforward, the concept is quite subtle. We apply it to the case of non-parametrized economies to convey the essential ideas in the clearest possible way. The application of our approach to explicitly parametrized economies is a natural, though non-trivial, problem to which we hope to come back in the future.

In short, our approach is thus as follows: (1) perturb AED directly (knowing that such a perturbation is equivalent to a perturbation of a single consumer's preferences); (2) establish that, generically (even when restricting to the space of critical maps), AED functions are of FST.

After a section establishing notation and some preliminary results, we proceed with two technical sections: Section 3 on differential topology and Section 4 on singularity theory. Section 5 contains the results that lead to the proof of finiteness of Walrasian equilibria.

2 Notation and preliminary results

Consider an economy with L commodities ($\ell = 1, \ldots, L$) and I agents ($i = 1, \ldots, I$). Let Ω be the non-negative orthant of $\mathbb{R}^L$ and let each agent i be defined by her endowment $\omega^i \in \Omega$ and her preferences $\succsim_i$, a complete order on Ω with the following "rationality" properties:

(P1) completeness and transitivity.

If $x \succsim_i y$ and $y \succsim_i x$, then x is indifferent to y and we write $x \sim_i y$. If $x \succsim_i y$ but not $x \sim_i y$, then x is strictly preferred to y and we write $x \succ_i y$. We call the partial preference order $\succ_i$ continuous if it satisfies:

(P2) continuity ($\{x : y \succ_i x\}$ and $\{y : y \succ_i x\}$ are open).

In addition, we shall assume strict monotonicity and strict convexity:

[1] By the same token, Allen's result can also be related to deeper economic primitives.

[2] To be precise, we require that all germs of the AED function be of finite singularity type. Note that regular maps are trivially of FST.

(P3) strict monotonicity ($x \geq y$ ($x_\ell \geq y_\ell$, $\forall \ell = 1, \ldots, L$) and $x \neq y$ $\Rightarrow x \succ_i y$);

(P4) strict convexity ($x \sim_i y$ and $x \neq y \Rightarrow \forall \alpha \in (0,1)$, $\alpha x + (1-\alpha) y \succ_i x$).

As in Castro *et al.* (2010), let Ξ denote the space of all such preferences. Furthermore, denote $\mathcal{E} \equiv (\Xi \times \Omega)^I$ with typical element $e = (\succ^1, \omega^1, \ldots, \succ^I, \omega^I)$ the space of all L by I economies described by preferences and endowments.

Next, consider the set of smooth (i.e., C^∞) pure exchange economies with L commodities ($\ell = 1, \ldots, L$) and I ($i = 1, \ldots, I$) agents, generated by each element $e \in \mathcal{E}$ and described by aggregate excess demand (AED) functions z depending smoothly on prices and endowments $\omega \in \Omega \subseteq \mathbb{R}^L_+$. Normalize prices to lie in the $(L-1)$-dimensional unit simplex $\Delta \equiv \{p \in \mathbb{R}^L_+ \mid \sum_{\ell=1}^L p_\ell = 1\}$. Note that an *equilibrium price* is $p \in \Delta$ such that $z(p, \omega) = 0$. We write $z(p, \omega) = z(p)$, when ω is held fixed.

We further assume the standard Boundary Condition found in Debreu (1970) for at least one consumer (cf. Assumption A therein), in Allen (1984) and in Mas-Colell and Nachbar (1991). We state the Boundary Condition in terms of the AED as in Allen (1984), but the notion is equivalent to that used in Castro *et al.* (2010):

Definition 2.1. (Boundary Condition) An AED z fulfills the *boundary condition* (BC), if for every $\omega \in \Omega$ and every sequence $(p_n)_{n \in \mathbb{N}} \in \Delta$ converging to the boundary $\partial \Delta$ as $n \to \infty$, $\|z(p_n, \omega)\|$ is unbounded.

We denote by $\mathcal{Z} \subset C^\infty(\mathrm{int}(\Delta), \mathbb{R}^{L-1})$ the set of smooth AEDs $z : \mathrm{int}(\Delta) \to \mathbb{R}^{L-1}$, where $\mathrm{int}(\Delta)$ denotes the interior of Δ. We also introduce the subset $\mathcal{Z}^*$ of $\mathcal{Z}$ of AEDs satisfying (BC). The next result shows that in the present setting, (BC) is equivalent to properness. Hence, if we denote by $C^\infty_{\mathrm{pr}}(\mathrm{int}(\Delta), \mathbb{R}^{L-1})$ the subspace of proper maps, we have

$$\mathcal{Z}^* = \mathcal{Z} \cap C^\infty_{\mathrm{pr}}(\mathrm{int}(\Delta), \mathbb{R}^{L-1}).$$

Proposition 2.2. *Let $z \in \mathcal{Z}$. Then z satisfies (BC) if and only if it is proper.*

Proof. For proving that (BC) implies properness, we need to show that for any convergent sequence in the target, any sequence of pre-images in the source has a convergent subsequence. For a convergent sequence in $\mathbb{R}^{L-1}$, we take the corresponding sequence $(p_n)_{n \in \mathbb{N}}$ in $\mathrm{int}(\Delta)$. Since Δ is compact $(p_n)_{n \in \mathbb{N}}$ has a convergent subsequence, say $(p_{n_i})_{n_i \in \mathbb{N}}$. This subsequence converges to a point in $\mathrm{int}(\Delta)$. Otherwise, $(p_{n_i})_{n_i \in \mathbb{N}}$ would converge to the boundary of Δ and by (BC) the original sequence of images in $\mathbb{R}^{L-1}$ would be unbounded.

Conversely, suppose that $z \in \mathcal{Z}$ is proper but does not satisfy (BC). Then there is a sequence $(p_n)_{n\in\mathbb{N}} \in \operatorname{int}(\Delta)$ which converges to the boundary $\partial\Delta$ and such that $\|z(p_n)\| < M$ for some $M > 0$. By passing to a subsequence if necessary we can assume that $(z(p_n))_{n\in\mathbb{N}}$ is convergent. Since z is proper, $(p_n)_{n\in\mathbb{N}}$ has a convergent subsequence in $\operatorname{int}(\Delta)$, which is absurd. $\square$

The equivalence between AED functions in $\mathcal{Z}^*$ and the underlying preferences and endowments in $\mathcal{E}$ established in Castro *et al.* (2010, Theorem 4.6 and Corollary 4.7) allows us to work with AEDs directly and still obtain genericity results couched in the more fundamental primitives of preferences and endowment. More formally, using the current notation, the result states:

Theorem 2.3 (Castro *et al.* (2010))**.** *Let $z_0 \in \mathcal{Z}^*$ be the AED for a C^2 economy $e_0 \in \mathcal{E}$ with L goods and I agents characterized by C^2-preferences $\succ_0^i$ satisfying (P1)–(P4) and (BC) and endowments ω_0^i, $i = 1, \ldots, I$. An AED z is a perturbation of z_0 if and only if z is the AED for an economy $e \in \mathcal{E}$ with L goods and I agents such that the new preferences $\succ^1$ of the first agent are perturbations of $\succ_0^1$ and the new endowments ω^1 are perturbations of ω_0^1.*

3 Concepts from Differential Topology

This section recalls some basic concepts from differential topology (see, for example, Golubitsky and Guillemin (1973), Hirsch (1976) or du Plessis and Wall (1995)). Unless explicitly stated to the contrary, manifolds are assumed to have empty boundary.

3.1 Germs and jets

In what follows let $f:\ N \to P$ be a C^∞ map between smooth manifolds of dimension n and p, respectively. Denote by $f^\wedge$ its germ at $x \in N$, that is, the equivalence class of maps $g:\ U \to P$ defined in some neighborhood $U \subset N$ of x, that agree with f in a (possibly smaller) neighborhood of x. The map f is also often called a representative of the germ $f^\wedge$. We often specify source and target values when referring to a germ by writing $f^\wedge:\ N, x \to P, y$ with $y = f(x)$.

The study of singularities makes ample use of the concept of k-jet which we may think of as the k^{th}-order Taylor polynomial of a map f at x. The formal definition may be found in Golubitsky and Guillemin (1973, Definition 2.1, p. 37).

Jet-space $J^k(N,P)$ is a smooth vector bundle over $N \times P$, therefore a manifold, and

$$\dim(J^k(N,P)) = n + p \begin{pmatrix} n+k \\ k \end{pmatrix}.$$

By taking coordinates, we can identify the fiber $J^k(N,P)_{x,y}$ with the space $J^k(n,p)$ of jets $\mathbb{R}^n \to \mathbb{R}^p$ with 0 source and target. Similarly, the projection on N defines a vector bundle $J^k(N,P) \to N$ and, for a smooth map $f\colon N \to P$, we can view its k-jet, $j^k f$, as a smooth section of this bundle.

3.2 Whitney and compact-open topologies

We shall use two standard topologies on $C^\infty(N,P)$, the C^l compact-open topology and the C^l Whitney topology ($0 \leq l \leq \infty$). Our reference for these concepts is du Plessis and Wall (1995, Section 3.4).

Recall that the C^k Whitney topology is, in general, finer than the C^k compact-open topology, in the sense that it has more open sets. However, if N is compact, the two topologies are equivalent. Note also that, in both cases, the C^l topologies are finer than the C^k topologies for $l \geq k$.

Remark 3.1. The jet map

$$j^k\colon C^\infty(N,P) \to C^\infty(N, J^k(N,P))$$

is continuous with respect to the C^{k+l}-topology on $C^\infty(N,P)$ and the C^l-topology on $C^\infty(N, J^k(N,P))$ for any $l \geq 0$.

3.3 Transversality

In order to have genericity results for the C^∞ compact-open topology as well as for the Whitney topology, in the setting of proper maps, we next prove a version of Thom Transversality in this setting.

Proposition 3.2. *Let N and P be smooth manifolds and let W be a closed submanifold of $J^k(N,P)$. Let*

$$T_{W,\mathrm{pr}} = \{f \in C^\infty_{\mathrm{pr}}(N,P) \mid j^k f \pitchfork W\}.$$

Let $1 \leq l \leq \infty$. Then the following holds:

(i) The subspace $T_{W,\mathrm{pr}}$ is an open dense subset of $C^\infty_{\mathrm{pr}}(N,P)$ in the Whitney C^{k+l} topology.

(ii) The subspace $T_{W,\mathrm{pr}}$ is a residual subset of $C^\infty_{\mathrm{pr}}(N,P)$ in the compact-open C^{k+l} topology.

Proof. (i) By Thom transversality (Golubitsky and Guillemin (1973, Theorem II, 4.9 and Corollary II, 4.10)) we know that

$$T_W = \{f \in C^\infty(N,P) \mid j^k f \pitchfork W\}.$$

is open and dense in $C^\infty(N,P)$ in the Whitney C^{k+l} topology. Intersecting both T_W and $C^\infty(N,P)$ with $C^\infty_{\mathrm{pr}}(N,P)$ openness is preserved. To prove that $T_{W,\mathrm{pr}}$ is dense, let $U \subset C^\infty_{\mathrm{pr}}(N,P)$ be a non-empty open set. By Hirsch (1976, Chap. 2, Theorem 1.5) $C^\infty_{\mathrm{pr}}(N,P)$ is open in $C^\infty(N,P)$. Therefore $T_{W,\mathrm{pr}}$ is open inside $C^\infty(N,P)$. Since $U \cap T_W \neq \emptyset$ and contained in $U \subset C^\infty_{\mathrm{pr}}(N,P)$, it follows that $U \cap T_{W,\mathrm{pr}}$ is non-empty.

(ii) Cover N by a countable family $\{K_j\}_{j=1}^\infty$ of compact sets, such that each K_j is the image of a closed ball under some coordinate chart on N.

Let $\mathcal{T}_{W,j}$ be the set of maps into jet space which are transverse to W over K_j, i.e.,

$$\mathcal{T}_{W,j} = \{g \in C^\infty(N, J^k(N,P)) \mid g_{|K_j} \pitchfork W\}$$

and let $T_{W,\mathrm{pr},j}$ be the set of maps in $C^\infty_{\mathrm{pr}}(N,P)$ whose k-jet is transverse to W over K_j, i.e.,

$$\begin{aligned} T_{W,\mathrm{pr},j} &= \{f \in C^\infty_{\mathrm{pr}}(N,P) \mid j^k f_{|K_j} \pitchfork W\} \\ &= (j^k_{|C^\infty_{\mathrm{pr}}(N,P)})^{-1}(\mathcal{T}_{W,j}). \end{aligned}$$

Then we have

$$T_{W,\mathrm{pr}} = \bigcap_{j=1}^\infty T_{W,\mathrm{pr},j}. \tag{1}$$

It follows from Hirsch (1976, Theorem III, 2.1(b)) (which applies to manifolds with boundary and therefore to K_j) that $\mathcal{T}_{W,j}$ is open in the C^l compact-open topology on $C^\infty(N, J^k(N,P))$. Then Remark 3.1 gives that $T_{W,\mathrm{pr},j}$ is open in $C^\infty_{\mathrm{pr}}(N,P)$ in the C^{k+l} compact-open topology. Therefore, we see from (1) that $T_{W,\mathrm{pr}}$ is a countable intersection of open subsets in the C^{k+l} compact-open topology.

Finally, $T_{W,\mathrm{pr}}$ is dense in $C^\infty_{\mathrm{pr}}(N,P)$ in the C^l compact-open topology for any l, since by *(i)* it is dense in the finer C^∞ Whitney topology. We have seen that $T_{W,\mathrm{pr}}$ is a countable intersection of open subsets in the C^{k+l} compact-open topology, each of which must therefore be dense. It follows that $T_{W,\mathrm{pr}}$ is residual. □

Remark 3.3. A similar argument can be used to show that *(ii)* of Proposition 3.2 holds without restricting to proper maps.

4 Concepts from singularity theory

This section recalls the concepts from singularity theory necessary for our results. Our main references are Mather (1968), Gibson *et al.* (1976), Golubitsky and Guillemin (1973) and du Plessis and Wall (1995).

4.1 $\mathcal{K}$-Equivalence

As before we let $f: N \to P$ be a C^∞ map between smooth manifolds of dimension n and p, respectively. Denote by $f^\wedge: N, x \to P, y$ its germ at $x \in N$, with $y = f(x)$. A singularity of f (or of $f^\wedge$) is a point $x \in N$ for which the Jacobian of f fails to have maximal rank.

The mention of equivalence classes requires a notion of equivalence. Here, we are interested in $\mathcal{K}$- or contact-equivalence, as in Mather (1968, Definition 2.5, p. 138).

Recall that contact equivalence transforms the graph of a germ into the graph of equivalent germs.

We have the following useful result concerning the zeros of $\mathcal{K}$-equivalent map-germs.

Lemma 4.1. *Let $f^\wedge: N, x \to P, y$ be a map-germ. The set of zeros of $f^\wedge$ is preserved by $\mathcal{K}$-equivalence.*

Proof. Mather (1969, proof of theorem 2.1) shows that there exists a diffeomorphism-germ $h^\wedge: N, x \to N, x$ such that, if $f^\wedge, g^\wedge: N, x \to P, y$ are $\mathcal{K}$-equivalent map-germs, then

$$(f^\wedge \circ h^\wedge)^{-1}(0) = (g^\wedge)^{-1}(0).$$

□

Mather (1968) establishes necessary and sufficient conditions for a C^∞ map-germ to be finitely determined. A map-germ $f^\wedge: N, x \to P, y$ is finitely determined if there exists an integer k such that any germ $g^\wedge: N, x \to P, y$ with the same k-jet as $f^\wedge$ is equivalent to $f^\wedge$. We say the germ is k-determined to specify the order of the Taylor polynomial and, if there is a need for extra clarity, we may say $k - \mathcal{K}$-determined. Finally, a germ is said to be finitely $\mathcal{K}$-determined if it is $k - \mathcal{K}$-determined for some k.

4.2 Finite Singularity Type

For the concept of *finite singularity type* (FST) we refer to du Plessis and Wall (1995, Section 2.4) where several equivalent versions are given. We shall adopt the following:

Definition 4.2. A map-germ $f^\wedge : N, x \to P, y$ is of *finite singularity type* if and only if it is finitely $\mathcal{K}$-determined.

The characterization of FST is done best by resorting to auxiliary subsets of jet-space. We shall not give the definition of these sets as it would require further non-trivial concepts from singularity theory but, instead, use their properties which are given, at the local level, in du Plessis and Wall (1995, p. 30). We know then that, at the local level, there exist algebraic sets $W^k(n,p) \subseteq J^k(n,p)$ satisfying the following properties:

1. $W^k(n,p)$ is closed;
2. if there exists k such that $j^k f(x) \notin W^k(n,p)$ then $f^\wedge$ is $k-\mathcal{K}$-determined;
3. $\mathrm{codim}_{J^k(n,p)} W^k(n,p) \longrightarrow \infty$ as $k \to \infty$.[3]

From property 2 it follows that if there exists k such that $j^k f^\wedge(x) \notin W^k(n,p)$ then $f^\wedge$, the germ of f at x, is of FST.

At global level, we consider $C^\infty(N,P)$. We have a jet-bundle $J^k(N,P) \to N \times P$ with fiber over $(x,y) \in N \times P$

$$J^k(N,P)_{(x,y)} \simeq J^k(n,p).$$

We have a subbundle

$$W^k(N,P) \subseteq J^k(N,P) \tag{2}$$

defined by $W^k(n,p) \subseteq J^k(n,p)$. By property 1, $W^k(N,P) \subseteq J^k(N,P)$ is closed and by property 3, since $\mathrm{codim}_{J^k(N,P)} W^k(N,P) = \mathrm{codim}_{J^k(n,p)} W^k(n,p)$,

$$\lim_{k\to\infty} \mathrm{codim}_{J^k(N,P)} W^k(N,P) = \infty.$$

From the definition of the $W^k(N,P)$ (du Plessis and Wall (1995, p. 30)) one has that $W^k(N,P) \supseteq W^{k+1}(N,P) \supseteq \cdots$. We can therefore state the following:

Lemma 4.3. *If there exists k such that $j^k f(x) \notin W^k(N,P)$ for all $x \in N$ then $f^\wedge$ is of FST for all $x \in N$.*

Proof. The result follows from local property 2. □

We note that the hypothesis in Lemma 4.3 is weaker than demanding global FST (see du Plessis and Wall (1995) for the definition) but stronger than asking for $f^\wedge$ to be of FST at every point of N.

[3]This property goes back to Tougeron (1972, Lemma VII, 5.3).

Remark 4.4. Even for polynomial maps, the concept of FST is fairly subtle, and to check whether a given map germ is of FST, one resorts to the methods of Bruce *et al.* (1987). Thus, for example, the germ at $(0,0)$ of the map

$$f(x,y) = (xy, x^2)$$

is not of FST (as also follows from Theorem 5.6 below), while the germ at $(0,0)$ of the map

$$g(x,y) = (xy, x^2 + y^4)$$

is of FST (and indeed has $(0,0)$ as an isolated zero).

5 Results

5.1 Genericity of FST-maps

We can now state and prove the first main result, establishing that FST is a generic (valid in an open and dense set) property of maps (cf. du Plessis and Wall (1995, Section 2.4)).

Theorem 5.1. *The set*

$$C^\infty_{FST} = \{f \in C^\infty(N,P) \mid f^\wedge \textit{ is of FST for all } x\}$$

contains an open and dense subset in the Whitney C^∞ topology which is residual (therefore, dense) in the compact-open C^∞ topology.

Proof. Let k_0 be such that $\dim N < \operatorname{codim} W^k(N,P)$ for $k \geq k_0$. Then, by Golubitsky and Guillemin (1973, Proposition II, 4.2), we have that

$$j^k f(N) \cap W^k(N,P) = \emptyset,$$

if and only if

$$j^k f \pitchfork W^k(N,P).$$

Let $S_k = \{f \in C^\infty(N,P) \mid j^k f \pitchfork W^k(N,P)\}$. Thom Transversality (Golubitsky and Guillemin (1973, Theorem II,4.9 and Corollary II,4.10)) implies that $S_k \subseteq C^\infty(N,P)$ is open and dense in the Whitney C^∞ topology provided $k \geq k_0$. By Lemma 4.3, if $f \in S_k$, then $f^\wedge$ is of FST for all $x \in N$, so $f \in C^\infty_{FST}$. We have then proved that

$$S_k \subseteq C^\infty_{FST}.$$

Because when a map is k-determined, it is also $(k+1)$-determined, we have

$$C^\infty_{FST} \supseteq \bigcup_{k \geq k_0} S_k.$$

The union of open and dense sets is itself an open and dense set.

Finally, to obtain the result for the compact-open topology, we apply the same argument, but now using Proposition 3.2 instead of the Thom Transversality Theorem. □

Remark 5.2. In the present case (when the dimensions of N and P are the same) this approach is not applicable to showing genericity of just regular maps. Indeed, $f \in C^\infty(N,P)$ is regular if and only if $j^1 f(N)$ avoids the subspace W of $J^1(N,P)$ given by the vanishing of the determinant of the Jacobian matrix. But the codimension of W is one, which is never greater than the dimension of N. Therefore, Golubitsky and Guillemin (1973, Proposition II,4.2) cannot be applied.

Corollary 5.3. *The set*

$$C^\infty_{FST,\mathrm{pr}} = \{f \in C^\infty_{\mathrm{pr}}(N,P) \mid f^\wedge \textit{ is of FST for all } x\}$$

contains an open and dense subset of $C^\infty_{\mathrm{pr}}(N,P)$ *in the Whitney* C^∞ *topology which is residual (therefore, dense) in the compact-open* C^∞ *topology.*

Proof. Since the subset of proper maps is open in the Whitney topology (Hirsch (1976, Chapter 2, Theorem 1.5, p. 38)) and $C^\infty_{FST,\mathrm{pr}} = C^\infty_{FST} \cap C^\infty_{\mathrm{pr}}(N,P)$ this is immediate from Theorem 5.1. □

One may (and, in fact, we shall do so in Section 5.3) ask whether it is possible to perturb a given map germ to being of FST without changing its 1-jet. In other words, whether density of FST maps still holds when restricting to maps with fixed 1-jet. This is a relevant point for applications and the answer is affirmative as we show next.

Theorem 5.4. *Let* $f_0 \colon N \to P$*. There exists* $f \colon N \to P$ *such that* $f^\wedge$ *is of FST for all* $x \in N$ *arbitrarily close (both in the Whitney and compact-open* C^∞ *topologies) to* f_0 *and such that* $j^1 f = j^1 f_0$*. If* f_0 *is proper, then* f *can be taken to be proper.*

Proof. Consider the projection

$$\pi \colon J^k(N,P) \to (J^k/J^1)(N,P)$$

where $(J^k/J^1)(N,P)$ is the quotient bundle corresponding to jets of degree k whose linear and constant parts vanish. Denote by $\tilde{j}^k g$ the projection of $j^k g$ in $(J^k/J^1)(N,P)$.

Let $W^k(N,P) \subset J^k(N,P)$ be as defined in (2) and let

$$\tilde{W}^k(N,P) = \pi(W^k(N,P)) \subset (J^k/J^1)(N,P).$$

Then

$$\lim_{k\to\infty} \operatorname{codim}_{(J^k/J^1)(N,P)} \tilde{W}^k(N,P) = \infty. \tag{3}$$

Fix f_0. It suffices to prove the result for the Whitney topology, since it is finer than the compact-open topology. Proceed as in the proof of density of Golubitsky and Guillemin (1973, Theorem II, 4.9) but take the space of perturbations B' to be the space of polynomial maps $\mathbb{R}^n \to \mathbb{R}^p$ of degree k with vanishing linear and constant terms. Note that $g_0 = f_0$. Furthermore, this new B' corresponds exactly to the fibers of $(J^k/J^1)(N,P)$.

The remaining part of the proof goes through to show that there exists a neighborhood $B \subset B'$ of $0 \in B'$ such that

$$\{b \in B \mid \tilde{j}^k g_b \pitchfork \tilde{W}^k(N,P)\}$$

is dense in B. Because of (3), for k sufficiently large, $\tilde{j}^k g_b \pitchfork \tilde{W}^k(N,P)$ means that

$$\tilde{j}^k g_b \cap \tilde{W}^k(N,P) = \emptyset.$$

In other words, g_b is a representative of a germ of FST (it is in fact k-determined, cf. local property 2 in Section 4.2).

The last statement follows because the subset of proper maps is open in the Whitney topology (Hirsch (1976, Chapter 2, Theorem 1.5, p. 38)). □

Remark 5.5. The previous theorem holds if we change $j^1 f = j^1 f_0$ to $j^r f = j^r f_0$ for any fixed r. The proof is analogous.

5.2 Local Isolation of Zeros

In order to relate FST to the set of zeros of a map, we need to consider both critical and regular zeros. This is because, from the point of view of applications to the study of AEDs, we are interested in the case when N and P are equidimensional ($n = p$).

In fact, if $n < p$ then all zeros are critical and we can refer to du Plessis and Wall (1995, p. 30) to conclude that the set of zeros is finite, provided f is such that $f^\wedge$ is of FST at all points: they show that $\Sigma(f) \cap f^{-1}(y)$ ($y \in P$)

is finite, where $\Sigma(f)$ is the set of all critical points. The case $n > p$ is of no interest since no zeros, critical or regular, are isolated.

In the equidimensional case, we can still refer to du Plessis and Wall (1995, p. 30) to get finiteness of the critical zeros. However, even though regular zeros are isolated, this does not rule out the existence of a sequence of regular zeros converging to a critical one (and this is the reason why Mas-Colell and Nachbar (1991, Corollary 3) only obtain countability, rather than finiteness, of equilibria). As we shall show, in the present context of FST maps, this situation cannot occur.

In view of Lemma 4.1, it is enough to state and prove the result for polynomial maps since $f^\wedge$ being of FST implies that $f^\wedge$ is $\mathcal{K}$-equivalent to its k-jet, a polynomial of degree k. This reduction is crucial, since it allows for the application of the Curve Selection Lemma in the proof below.

Theorem 5.6. *Let M and P be manifolds of the same dimension and let $f : M \to P$ be a polynomial map. Assume $f^\wedge$ is of FST for all $x \in Z = f^{-1}(0)$. Then the critical zeros of f are isolated in M.*

Proof. We proceed by contradiction. Let p be a critical zero, that is, $p \in \Sigma(f)$. Assume that for all neighborhoods V of p in M there exists a regular zero x of f in V. The Curve Selection Lemma (see Milnor, 1968, Lemma 3.1) implies that there exists an analytic curve $\gamma\colon [0, \epsilon[\to M$ with $\gamma(0) = p$ and such that $\gamma(t)$ is a regular zero for $t > 0$. But, since γ is a curve of zeros of f, it is contained in $\Sigma(f)$. This contradicts the construction of γ. We can therefore assert that there exists a neighborhood V of p in M such that

$$V \cap \Sigma(f) \cap f^{-1}(0) = V \cap f^{-1}(0).$$

Because $f^\wedge$ is of FST at all points, $\Sigma(f)$ must be finite and $V \cap f^{-1}(0)$ contains only finitely many zeros. Hence, points in $\Sigma(f) \cap f^{-1}(0)$ are locally isolated. □

Corollary 5.7. *Let N and P be smooth manifolds of the same dimension and let $f\colon N \to P$ be a smooth map such that the germ $f^\wedge$ is of FST for all $x \in N$. Then the zeros of f are locally isolated. If, moreover, f is proper then the set of its zeros is finite.*

Proof. By Lemma 4.1 we may assume that $f^\wedge$ is polynomial and of FST in a neighborhood M of each point in N. Applying the result of du Plessis and Wall (1995, p. 30) cited above and Theorem 5.6 in such a neighborhood around each critical zero, we see that all critical zeros of f are isolated in N. We also know that the regular zeros are locally isolated since $\dim N = \dim P$. Finally, when f is proper, $f^{-1}(0)$ is compact, and local isolation implies finiteness. □

5.3 Finiteness of Walrasian equilibria

In this section we use the genericity of FST map germs to prove our main results on finiteness of equilibria. The concept of FST is essential here since, in the present setup of AEDs, we are in the equidimensional case where there is no corresponding genericity result for regular economies (see Remark 5.2).

In order to apply the results in the previous section to AEDs, we want to think of $N = \mathrm{int}(\Delta)$ and $P = \mathbb{R}^{L-1}$. These are smooth manifolds but $\mathrm{int}(\Delta)$ is a non-compact manifold and we therefore need Proposition 2.2 to prove finiteness of Walrasian equilibria.

The following Proposition is now an immediate consequence of Corollary 5.7.

Proposition 5.8. *Let $z \in \mathcal{Z}^*$ be such that $z^\wedge$ is of FST for all $p \in \mathrm{int}(\Delta)$. Then z has finitely many zeros.*

The following result is essential for restricting the genericity properties to the subspace $\mathcal{Z}^* \subseteq C^\infty_{\mathrm{pr}}(\mathrm{int}(\Delta), \mathbb{R}^{L-1})$.

Proposition 5.9. *The subspace $\mathcal{Z}^*$ of $C^\infty_{\mathrm{pr}}(\mathrm{int}(\Delta), \mathbb{R}^{L-1})$ is dense in the C^k compact-open topology for any $0 \leq k \leq \infty$.*

Proof. All references to C^k topologies in this proof refer to the compact-open topology. It is sufficient to prove the result for $k = \infty$ since the C^∞ topology is finer than all other C^k topologies. Moreover, we shall actually show that $\mathcal{Z}$ is dense in $C^\infty(\mathrm{int}(\Delta), \mathbb{R}^{L-1})$ from which the result is immediate.

Let $\emptyset \neq U \subseteq C^\infty(\mathrm{int}(\Delta), \mathbb{R}^{L-1})$ be open in the C^∞ topology. Let $\emptyset \neq V \subset U$ be open in the C^k topology.[4] By definition of the C^k topology

$$V = V^k \cap C^\infty(\mathrm{int}(\Delta), \mathbb{R}^{L-1})$$

for some $V^k \subseteq C^\infty(\mathrm{int}(\Delta), J^k(\mathrm{int}(\Delta), \mathbb{R}^{L-1}))$ which is open in the C^0 topology on the latter space. Hence, by definition of the C^0 topology, we may assume that

$$V^k = \bigcap_{i=1}^{n} A(K_i, U_i) \neq \emptyset,$$

where $K_i \subseteq \mathrm{int}(\Delta)$ is compact, $U_i \subseteq J^k(\mathrm{int}(\Delta), \mathbb{R}^{L-1})$ is open and

$$A(K_i, U_i) = \{g \ : \ g(K_i) \subseteq U_i\}.$$

[4] Such k and V exist because, given a basis for a topology, any open subset is the union of basic subsets.

Let $\varepsilon > 0$ be such that the trimmed simplex $\Delta_\varepsilon = \{p \in \Delta \ : \ p_i \geq \varepsilon\}$ contains the compact set $K = \bigcup_{i=1}^n K_i$ and let $f \in V$. By the Sonnenschein–Mantel–Debreu Theorem (Debreu 1974, Theorem, p. 16) there exists $z \in \mathcal{Z}$ such that $z_{|\Delta_\varepsilon} = f_{|\Delta_\varepsilon}$. This implies that $z \in V$. □

We have seen in Corollary 5.3 that the set of proper maps whose germs are of FST at every point is generic in the set of all smooth proper maps, meaning that it is residual for the C^∞ compact-open topology and contains an open and dense set in the C^∞ Whitney topology. Therefore, Propositions 5.8 and 5.9 imply the following main result, where *generic* is to be understood in the above sense.

Theorem 5.10. *The number of all equilibria, regular and critical, is finite in the economies defined by generic (for the C^∞ compact-open topology) AEDs in $\mathcal{Z}^*$.*

Note that our result is stronger than that of Mas-Colell and Nachbar (1991, Corollary 3) in that we obtain finiteness, rather than countability, of equilibria for the generic economy.

Furthermore, we can restrict attention to the subspace of critical economies to show that any critical economy can be perturbed to become of FST, whilst remaining critical. In other words, FST is a stable property even among critical economies. This extends the result of Allen (1984) to critical economies.

Theorem 5.11. *The number of equilibria of an economy defined by a generic (for the C^∞ compact-open topology) critical AED is finite.*

Proof. The subset of FST critical economies in $\mathcal{Z}^*$ is residual inside the set of all critical economies in $\mathcal{Z}^*$. In fact density follows as a corollary of Theorem 5.4 and Proposition 5.9 while residuality follows from Corollary 5.3. □

With respect to the C^∞ Whitney topology, it is clear from our analysis that we obtain finiteness of equilibria for economies (even critical ones) defined by an *open* subset of AEDs in $\mathcal{Z}^*$. However, we do not obtain density for lack of a result analogous to Proposition 5.9 for the Whitney topology. A missing link for this is an extension of the Sonnenschein–Mantel–Debreu Theorem to the whole open simplex $\text{int}(\Delta)$. Nevertheless, we have the following (note that the open sets mentioned are dense in the compact-open topology).

Proposition 5.12. *The number of equilibria, regular and critical, is finite in the economies defined by an open (for the C^∞ Whitney topology) subset of AEDs in $\mathcal{Z}^*$. Furthermore, the number of equilibria of an economy defined by an open (for the C^∞ Whitney topology) subset of critical AEDs is finite.*

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References

[1] Allen, B. (1984): On the finiteness of the equilibrium price set, CARESS Working Paper #85–03, University of Pennsylvania.

[2] Bruce, J.W., A.A. du Plessis and C.T.C. Wall (1987): Determinacy and unipotency, Invent. Math. 88 (1987), no. 3, 521–554.

[3] Castro, S.B.S.D., S. Dakhlia, and P. Gothen (2010): Direct perturbations of aggregate excess demand functions, Journal of Mathematical Economics 46, 562–571.

[4] Castro, S.B.S.D. and S. Dakhlia, (2008): Thom-Boardman stratification of aggregate excess demand and finiteness of equilibria, SSRN Working Paper 1156100.

[5] Debreu, G. (1970): Economies with a finite set of equilibria, Econometrica 38, 387–392.

[6] Debreu, G. (1974): Excess Demand Functions, Journal of Mathematical Economics 1, 15–21.

[7] du Plessis, A.A. and C.T.C. Wall (1995): The Geometry of Topological Stability, OUP.

[8] Gibson, C.G., K. Wirthmüller, A.A. du Plessis and E.J.N. Looijenga (1976): Topological Stability of Smooth Mappings, LNM 552.

[9] Golubitsky, M. and V. Guillemin (1973): Stable Mappings and their Singularities, Springer Verlag, New York.

[10] Hirsch, M.W. (1976): Differential topology, Graduate Texts in Mathematics, No. 33, Springer-Verlag, New York-Heidelberg.

[11] Mas-Colell, A (1985), The Theory of General Economic Equilibrium: a Differentiable Approach, Cambridge University Press.

[12] Mas-Colell, A. and J.H. Nachbar (1991): On the finiteness of the number of critical equilibria, with an application to random selections, Journal of Mathematical Economics 20, 397–409.

[13] Mather, J. (1968): Stability of C^∞ maps, III. Finitely determined mapgerms, *Publ. Math. IHES*, **35**,127–156.

[14] Mather, J. (1969): Stability of C^∞ maps, IV. Classification of stable germs by R-algebras, *Publ. Math. IHES*, **37**,223–248.

[15] Mather, J. (1973): On Boardman Singularities, in *Dynamical Systems*, M. Peixoto (ed.), 233–248, Academic Press, New York.

[16] Milnor, J. (1968): Singular points of complex hypersurfaces, Princeton University Press.

[17] Tougeron, J.C. (1972): Idéaux de fonctions différentiables, Springer.