Fundamental solutions of the fractional two-parameter telegraph equation

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Abstract

The present paper is intended to investigate a fractional telegraph equation of the form

$$-a\left(D^{\alpha+1}_{t_0^+}u\right)(t,x)+\left(D^{\beta+1}_{x_0^+}u\right)(t,x)-b\left(D^{\alpha}_{t_0^+}u\right)(t,x)-cu(t,x)=0,$$

with positive real parameters a, b, c. Here $D^{\alpha+1}_{t_0^+}$, $D^{\alpha}_{t_0^+}$,$D^{\beta+1}_{x_0^+}$ are operators of the Riemann- Liouville fractional derivative, where $0<\alpha\leq 1$ and $0<\beta\leq 1$. A symbolic operational form of the solutions in terms of the Mittag-Leffler functions is exhibited. Using Banach fixed point theorem, the existence and uniqueness of solutions is studied for this kind of fractional differential equations.

Keywords: Fractional telegraph equation, Riemann-Liouville fractional integrals and derivatives, generalized Mittag-Leffler function.
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1 Introduction

The fractional calculus is one of the most accurate tools to refine the description of natural phenomena. Fractional differential equations have attracted in the recent years a considerable interest due to their frequent appearance in various fields and their more accurate models of systems under consideration provided by fractional derivatives.

The fractional telegraph equation has been recently considered by many authors. Cascaval et al. [1] discussed the time-fractional telegraph equations, dealing with wellposedness and presenting a study of their asymptotic behavior by using the Riemann-Liouville approach. Orsingher and Beghin [4] discussed the time-fractional telegraph equation and telegraph processes with Brownian time, showing that some processes are governed by time-fractional telegraph equations. Chen et al. [2] examined and derived a solution of the time-fractional telegraph equation with three kinds of nonhomogeneous boundary conditions, by the method of separation of variables. Recently, in [7] the first author dealt with fractional generalization of the Laplace equation for rectangular domains which related to Riemann-Liouville's fractional derivatives.

These derivatives and the corresponding integral of order $\gamma > 0$ are

$$(D^{\gamma}_{a^+}v)(x) = \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\gamma)} \int_a^x \frac{v(t)}{(x-t)^{\gamma-n+1}} dt, \quad a, x > 0, n = [\gamma]+1, \tag{1}$$

$$(I^{\gamma}_{a^+}v)(x) = \frac{1}{\Gamma(\gamma)} \int_a^x \frac{v(t)}{(x-t)^{1-\gamma}} dt, \quad a, x > 0, \tag{2}$$

respectively, where $[\gamma]$ means the integer part of γ (see e.g. [6] , [8]).

In this paper, we consider the following class of fractional telegraph equation

$$-a\left(D^{\alpha+1}_{t_0^+}u\right)(t,x) + \left(D^{\beta+1}_{x_0^+}u\right)(t,x) - b\left(D^{\alpha}_{t_0^+}u\right)(t,x) - cu(t,x) = 0, \tag{3}$$

with a, b, c are parameters connected with resistance, inductance, capacitance and conductance of the cable, respectively. Here, we present a general operational approach to describe fundamental solutions of the fractional two-parameter telegraph equation (3). Operational solutions will be done in terms of the generalized Mittag-Leffler function $E_{\mu,\nu}(z)$ (see e.g. [7], [6], [8]) which, in turn, is defined in terms of the power series

$$E_{\mu,\nu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu n + \nu)}, \quad \mu > 0, \nu \in \mathbb{R}, z \in \mathbb{C}. \tag{4}$$

Particular simple cases are

$$E_{1,2}(z) = \frac{e^z - 1}{z}, \quad E_{2,2}(z) = \frac{\sinh(z^{1/2})}{z^{1/2}}.$$

We note that $E_{\mu,\nu}(z)$ is an entire function of order $1/\mu$ and type 1.

In the sequel we will prove the existence and uniqueness of solutions concerning equation (3).

2 Preliminaries

We start by recalling some definitions and facts from the theory of fractional differential operators.

Definition 2.1 *(see [6]) By* $AC^n([a,b]), n \in \mathbb{N}$*, one denotes the class of functions* $v(x)$*, which are continuously differentiable on the segment* $[a,b]$ *up to the order* $n-1$ *and* $v^{(n-1)}(x)$ *is absolutely continuous on* $[a,b]$*.*

It is known [6] that the class $AC^n([a,b]), n \in \mathbb{N}$ contains only functions represented in the form

$$v(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}\varphi(t)dt + \sum_{k=0}^{n-1} c_k(x-a)^k, \tag{5}$$

where $\varphi(t) \in L_1(a,b)$ and c_k are arbitrary constants. It is not difficult to find that $\varphi(t) = v^n(t)$, $c_k = v^{(k)}(a)/k!$. Moreover, if $v(x) \in AC^n([a,b])$, then fractional derivative (1) exists almost everywhere and can be represented by the formula

$$(D^\gamma_{a^+}v)(x) = \sum_{k=0}^{n-1} \frac{v^{(k)}(a)}{\Gamma(1+k-\gamma)}(x-a)^{k-\gamma} + \frac{1}{\Gamma(n-\gamma)}\int_a^x \frac{v^{(n)}(t)}{(x-t)^{\gamma-n+1}}dt, \tag{6}$$

$n = [\gamma]+1$.

Definition 2.2 *(see [6]) By* $I^\gamma_{a^+}(L_1)$ *denotes the class of functions* v *represented by a left-sided fractional integral (2) of a summable function, that is,* $v = I^\gamma_{a^+}\varphi$*,* $\varphi \in L_1(a,b)$*.*

This class of functions is described below.

Theorem 2.3 *(see [6]) A function* $v(x) \in I^\gamma_{a^+}(L_1)$*,* $\gamma > 0$ *if and only if* $(I^{n-\gamma}_{a^+}v)(x) \in AC^n([a,b])$*,* $n = [\gamma]+1$ *and* $(I^{n-\gamma}_{a^+}v)^{(k)}(a) = 0$*,* $k = 0, 1, \ldots, n-1$*.*

Definition 2.4 *(see [6]) One will say that a function* $v \in L_1(a,b)$ *has a summable fractional derivative* $(D^\gamma_{a^+}v)(x)$ *if* $(I^{n-\gamma}_{a^+}v)(x) \in AC^n([a,b])$*,* $n = [\gamma]+1$*.*

If $(D^\gamma_{a^+}v)(x) = (d/dx)^n(I^{n-\gamma}_{a^+}v)(x)$ exists in the ordinary sense, that is, $(I^{n-\gamma}_{a^+}v)(x)$ is differentiable in each point up to the order n, then $v(x)$ evidently admits the derivative $(D^\gamma_{a^+}v)(x)$ in the sense of definition (2.4).

So, if $v(x) \in I^\gamma_{a^+}(L_1)$, then $(I^\gamma_{a^+}D^\gamma_{a^+}v)(x) = v(x)$. Otherwise if v just admits a summable fractional derivative, then the composition of fractional operators (1) and (2) can be written in the form (see [6])

$$(I^\gamma_{a^+}D^\gamma_{a^+}v)(x) = v(x) - \sum_{k=0}^{n-1} \frac{(x-a)^{\gamma-k-1}}{\Gamma(\gamma-k)}(I^{n-\gamma}_{a^+}v)^{(n-k-1)}(a), \quad n = [\gamma]+1. \tag{7}$$

Nevertheless, we note that $(D^\gamma_{a^+}I^\gamma_{a^+}v)(x) = v(x)$ for any summable function v.

3 Fundamental solutions of the fractional two-parameter telegraph equation

In this section, we present a general operational approach to describe fundamental solutions of the fractional two-parameter telegraph equation (3). Moreover, its operational solutions will be written in terms of the generalized Mittag-Leffler function $E_{\mu,\nu}(z)$.

So, let us consider fractional telegraph equation (3).

Theorem 3.1 *Let $u(t,x) \in L_1(\Omega)$, $\Omega = [t_0, T_0] \times [x_0, X_0]$ and $u \in I^{\alpha+1}_{t_0^+}(L_1)$, $u \in I^{\beta+1}_{x_0^+}(L_1)$ by $t \in [t_0, T_0]$ and $x \in [x_0, X_0]$, respectively. Then the unique solution of (3) is zero.*

Proof: Since $u(t,x)$ belongs to classes $I^{\alpha+1}_{t_0^+}(L_1)$, $I^{\beta+1}_{x_0^+}(L_1)$ by $t \in [t_0, T_0]$ and $x \in [x_0, X_0]$, respectively, we take operator $I^{\alpha+1}_{t_0^+}$ from both sides of (3) to obtain

$$\begin{aligned} &-aI^{\alpha+1}_{t_0^+}\left(D^{\alpha+1}_{t_0^+}u\right)(t,x) + I^{\alpha+1}_{t_0^+}\left(D^{\beta+1}_{x_0^+}u\right)(t,x) - bI^{\alpha+1}_{t_0^+}\left(D^{\alpha}_{t_0^+}u\right)(t,x) \\ &-c\left(I^{\alpha+1}_{t_0^+}u\right)(t,x) = 0. \end{aligned} \tag{8}$$

Using identity $\left(I^{\alpha+1}_{t_0^+}D^{\alpha+1}_{t_0^+}u\right)(t,x) = u(t,x)$ it becomes

$$-au(t,x) + I^{\alpha+1}_{t_0^+}\left(D^{\beta+1}_{x_0^+}u\right)(t,x) - b\left(I^{1}_{t_0^+}u\right)(t,x) - c\left(I^{\alpha+1}_{t_0^+}u\right)(t,x) = 0. \tag{9}$$

Applying operator $I^{\beta+1}_{x_0^+}$ from both sides of (9) and invoking Fubini's theorem we obtain

$$\begin{aligned} &-aI^{\beta+1}_{x_0^+}u(t,x) + I^{\alpha+1}_{t_0^+}I^{\beta+1}_{x_0^+}\left(D^{\beta+1}_{x_0^+}u\right)(t,x) - bI^{\beta+1}_{x_0^+}\left(I^{1}_{t_0^+}u\right)(t,x) \\ &-cI^{\beta+1}_{x_0^+}\left(I^{\alpha+1}_{t_0^+}u\right)(t,x) = 0. \end{aligned} \tag{10}$$

Using the identity $\left(I^{\beta+1}_{x_0^+}D^{\beta+1}_{x_0^+}u\right)(t,x) = u(t,x)$ it gives

$$\begin{aligned} &-a\left(I^{\beta+1}_{x_0^+}u\right)(t,x) + \left(I^{\alpha+1}_{t_0^+}u\right)(t,x) - b\left(I^{\beta+1}_{x_0^+}I^{1}_{t_0^+}u\right)(t,x) \\ &-c\left(I^{\beta+1}_{x_0^+}I^{\alpha+1}_{t_0^+}u\right)(t,x) = 0. \end{aligned} \tag{11}$$

Denoting by $U(t,s) = \int_{x_0}^{X_0} e^{-sy}u(t,y)dy$ the Laplace transform [9] by x of u and appealing to its operational properties, the latter equality becomes

$$
\begin{aligned}
&-as^{-\beta-1}U(t,s) + \frac{1}{\Gamma(\alpha+1)}\int_{t_0}^{t}(t-z)^{\alpha}U(z,s)dz - bs^{-\beta-1}\int_{t_0}^{t}U(z,s)dy \\
&-cs^{-\beta-1}\frac{1}{\Gamma(\alpha+1)}\int_{t_0}^{t}(t-z)^{\alpha}U(z,s)dz = 0, \qquad (12)
\end{aligned}
$$

which can be treated as a second kind homogeneous integral equation of the Volterra type

$$
U(t,s) - \int_{t_0}^{t}\left(\frac{s^{\beta+1}-c}{a\Gamma(\alpha+1)}(t-z)^{\alpha} - \frac{b}{a}\right)U(z,s)dz = 0. \qquad (13)
$$

But $U(t,s) \in L_1(t_0,T_0)$ via $U(t,s) \in I_{t_0^+}^{1+\alpha}(L_1)$. Therefore, as it is known (see, for example [7]) equation (13) has only a trivial solution in the class of summable functions. Cancelling the Laplace transform and using its uniqueness property in L_1 we have $u(t,x) = 0$.

■

Remark Taking the Laplace transform in variable t from both sides of (11) with $U(s,x) = \int_{t_0}^{T_0} e^{-sz}u(z,x)dz$, we come to the same conclusion for equation

$$
U(s,x) - \frac{as^{\alpha+1}+bs^{\alpha}+c}{\Gamma(\beta+1)}\int_{x_0}^{x}(x-y)^{\beta}U(s,y)dy = 0. \qquad (14)
$$

Now, we will prove the following Lemma.

Lemma 3.2 *Let $u(t,x) \in L_1(\Omega)$ admit a summable fractional derivative $\left(D_{x_0^+}^{\beta+1}u\right)(t,x)$ by $x \in [x_0,X_0]$ and belong to $I_{t_0^+}^{\alpha+1}(L_1)$ by $t \in [t_0,T_0]$. Then u is a solution of (3) if and only if U satisfies the Volterra integral equation*

$$
U(s,x) - \frac{as^{\alpha+1}+bs^{\alpha}+c}{\Gamma(\beta+1)}\int_{x_0}^{x}(x-y)^{\beta}U(s,y)dy = G(s,x),
$$

where $U(s,x) = \int_{t_0}^{T_0} e^{-sz}u(z,x)dz$,

$$
\begin{aligned}
G(s,x) &= \frac{(x-x_0)^{\beta}}{\Gamma(\beta+1)}G_0(s) + \frac{(x-x_0)^{\beta-1}}{\Gamma(\beta)}G_1(s), \qquad (15)\\
G_0(s) &= \int_{t_0}^{t} e^{-sz}g_0(z)dz \ \text{and}\ G_1(s) = \int_{t_0}^{t} e^{-sz}g_1(z)dz.
\end{aligned}
$$

Proof: Indeed, under conditions of the lemma and returning to (10) we derive

$$-a\left(I^{\beta+1}_{x_0^+}u\right)(t,x)+I^{\alpha+1}_{t_0^+}\left[u(t,x)-\frac{(x-x_0)^\beta}{\Gamma(\beta+1)}g_0(t)-\frac{(x-x_0)^{\beta-1}}{\Gamma(\beta)}g_1(t)\right]$$
$$-b\left(I^{\beta+1}_{x_0^+}I^{1}_{t_0^+}u\right)(t,x)-c\left(I^{\beta+1}_{x_0^+}I^{\alpha+1}_{t_0^+}u\right)(t,x)=0, \tag{16}$$

where $g_0(t)=\left(D^{\beta}_{x_0^+}u\right)(t,x_0)$ and $g_1(t)=\left(I^{1-\beta}_{x_0^+}u\right)(t,x_0)$.

After application of the Laplace transform to both sides of (16) by t, we come out with

$$-\frac{a}{\Gamma(\beta+1)}\int_{x_0}^{x}(x-y)^\beta U(s,y)dy+s^{-\alpha-1}\left[U(s,x)-\frac{(x-x_0)^\beta}{\Gamma(\beta+1)}G_0(s)\right.$$
$$\left.-\frac{(x-x_0)^{\beta-1}}{\Gamma(\beta)}G_1(s)\right]-b\frac{s^{-1}}{\Gamma(\beta+1)}\int_{x_0}^{x}(x-y)^\beta U(s,y)dy$$
$$-c\frac{s^{-\alpha-1}}{\Gamma(\beta+1)}\int_{x_0}^{x}(x-y)^\beta U(s,y)dy=0. \tag{17}$$

Further, this can be rewritten as

$$U(s,x)-\frac{as^{\alpha+1}+bs^\alpha+c}{\Gamma(\beta+1)}\int_{x_0}^{x}(x-y)^\beta U(s,y)dy=G(s,x), \tag{18}$$

where

$$G(s,x)=\frac{(x-x_0)^\beta}{\Gamma(\beta+1)}G_0(s)+\frac{(x-x_0)^{\beta-1}}{\Gamma(\beta)}G_1(s),$$

$G_0(s)=\int_{t_0}^{t}e^{-sz}g_0(z)dz$ and $G_1(s)=\int_{t_0}^{t}e^{-sz}g_1(z)dz$.

■

Lemma 3.3 *Let $u(t,x)\in L_1(\Omega)$ admit a summable fractional derivative $\left(D^{\alpha+1}_{t_0^+}u\right)(t,x)$ by $t\in[t_0,T_0]$ and belong to $I^{\beta+1}_{x_0^+}(L_1)$ by $x\in[x_0,X_0]$. Then u is a solution of (3) if and only if U satisfies the Volterra integral equation*

$$U(t,s)-\int_{t_0}^{t}\left(\frac{s^{\beta+1}-c}{a\Gamma(\alpha+1)}(t-z)^\alpha-\frac{b}{a}\right)U(z,s)dz=F(t,s),$$

where $U(t,s)=\int_{x_0}^{X_0}e^{-sy}u(t,y)dy$,

$$F(t,s)=\frac{(t-t_0)^\alpha}{\Gamma(\alpha+1)}\left(F_0(s)-\frac{b}{a}F_1(s)\right)+\frac{(t-t_0)^{\alpha-1}}{\Gamma(\alpha)}F_1(s),$$

$$F_0(s)=\int_{x_0}^{x}e^{-sy}f_0(y)dy \text{ and } F_1(s)=\int_{x_0}^{x}e^{-sy}f_1(y)dy.$$

Proof: In fact, similarly to the previous proof we deduce

$$
\begin{aligned}
&-aU(t,s)+\frac{s^{\beta+1}}{\Gamma(\alpha+1)}\int_{t_0}^{t}(t-z)^{\alpha}U(z,s)dz\\
&-b\int_{t_0}^{t}U(z,s)dz-\frac{c}{\Gamma(\alpha+1)}\int_{t_0}^{t}(t-z)^{\alpha}U(z,s)dz\\
&=-a\frac{(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}F_0(s)-a\frac{(t-t_0)^{\alpha-1}}{\Gamma(\alpha)}F_1(s)+b\frac{(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}F_1(s),
\end{aligned}
\tag{19}
$$

i.e.,

$$
U(t,s)-\int_{t_0}^{t}\left(\frac{s^{\beta+1}-c}{a\Gamma(\alpha+1)}(t-z)^{\alpha}-\frac{b}{a}\right)U(z,s)dz=F(t,s)
\tag{20}
$$

where

$$
\begin{aligned}
F(t,s)&=\frac{(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}\left(F_0(s)-\frac{b}{a}F_1(s)\right)+\frac{(t-t_0)^{\alpha-1}}{\Gamma(\alpha)}F_1(s),\\
F_0(s)&=\int_{x_0}^{x}e^{-sy}f_0(y)dy, F_1(s)=\int_{x_0}^{x}e^{-sy}f_1(y)dy,
\end{aligned}
\tag{21}
$$

with $f_0(x)=\left(D^{\alpha}_{t_0^+}u\right)(t_0,x)$ and $f_1(x)=\left(I^{1-\alpha}_{t_0^+}u\right)(t_0,x)$.

∎

Theorem 3.4 *Under conditions of Lemma (3.2) a solution of (18) can be written in terms of the Mittag-Leffler functions.*

Proof: Following [7] we see that the unique solution of (18) in a class of summable functions involves as the kernel the generalized Mittag-Leffler function (4), namely

$$
\begin{aligned}
U(s,x) &= G(s,x)+(as^{\alpha+1}+bs^{\alpha}+c)\int_{x_0}^{x}(x-y)^{\beta}\\
&\times E_{\alpha+1,\alpha+1}\left((as^{\alpha+1}+bs^{\alpha}+c)(x-y)^{\beta+1}\right)G(s,y)dy,
\end{aligned}
\tag{22}
$$

where the corresponding change of the order of integration and summation are motivated by estimates below. Precisely, we obtain

$$\begin{aligned}
&\left|(as^{\alpha+1}+bs^{\alpha}+c)\int_{x_0}^{x}(x-y)^{\beta}E_{\beta+1,\beta+1}\left((as^{\alpha+1}+bs^{\alpha}+c)(x-y)^{\beta+1}\right)G(s,y)dy\right| \\
&\le |G_0(s)|\sum_{n=0}^{+\infty}\frac{(a|s|^{\alpha+1}+b|s|^{\alpha}+c)^{n+1}}{\Gamma((n+1)(\beta+1))\Gamma(\beta+1)}\int_{x_0}^{x}(x-y)^{(\beta+1)n+\beta}(y-x_0)^{\beta}dy \\
&+|G_1(s)|\sum_{n=0}^{+\infty}\frac{(a|s|^{\alpha+1}+b|s|^{\alpha}+c)^{n+1}}{\Gamma((n+1)(\beta+1))\Gamma(\beta)}\int_{x_0}^{x}(x-y)^{(\beta+1)n+\beta}(y-x_0)^{\beta-1}dy \\
&\le (a|s|^{\alpha+1}+b|s|^{\alpha}+c)(X_0-x_0)^{2\beta} \\
&\times\Bigg[(X_0-x_0)E_{1+\beta,2(\beta+1)}\left(\left(a|s|^{\alpha+1}+b|s|^{\alpha}+c\right)(X_0-x_0)^{1+\beta}\right)\int_{t_0}^{t}e^{-sz}|g_0(z)|dz \\
&+E_{1+\beta,2\beta+1}\left((a|s|^{\alpha+1}+b|s|^{\alpha}+c)(X_0-x_0)^{1+\beta}\right)\int_{t_0}^{t}e^{-sz}|g_1(z)|dz\Bigg]<+\infty.
\end{aligned} \tag{23}$$

Furthermore, minding

$$\int_{x_0}^{x}(x-y)^{(1+\beta)n+\beta}(y-x_0)^{\beta}dy=\frac{(x-x_0)^{(1+\beta)n+2\beta+1}\Gamma(1+\beta)\Gamma((1+\beta)n+1+\beta)}{\Gamma((1+\beta)n+2(1+\beta))},$$

$$\int_{x_0}^{x}(x-y)^{(1+\beta)n+\beta}(y-x_0)^{\beta-1}dy=\frac{(y-y_0)^{(1+\beta)n+2\beta}\Gamma(\beta)\Gamma((1+\beta)n+1+\beta)}{\Gamma((1+\beta)n+2\beta+1)},$$

formula (22) can be rewritten in terms of the Mittag-Leffler functions as

$$\begin{aligned}
U(s,x) &= G_0(s)(x-x_0)^{\beta}E_{1+\beta,1+\beta}\left((s^{\alpha+1}+bs^{\alpha}+c)(x-x_0)^{1+\beta}\right) \\
&+ F_1(s)(t-t_0)^{\alpha-1}E_{1+\alpha,\alpha}\left((s^{\alpha+1}+bs^{\alpha}+c)(x-x_0)^{1+\beta}\right).
\end{aligned} \tag{24}$$

■

Theorem 3.5 *Under conditions of Lemma (3.3) a solution of (20) can be presented in the resolvent form.*

Proof: Indeed by operational method for direct and inverse Laplace transforms (see [5]), a solution of (20) can be expressed in the form

$$U(t,s)=F(t,s)-\int_{t_0}^{t}R_s(t-z)F(z,s)dz, \tag{25}$$

where the resolvent $R_s(t)$ is defined by

$$R_s(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tilde{R}_s(p)e^{pt}dp,$$

and

$$\tilde{R}_s(p) = \frac{\tilde{K}_s(p)}{1+\tilde{K}_s(p)}, \quad \tilde{K}_s(p) = \int_0^\infty K_s(t)e^{-pt}dt,$$

$$K_s(t) = \left(\frac{s^{\beta+1}-c}{a\Gamma(\alpha+1)}(t-z)^\alpha - \frac{b}{a}\right).$$

■

4 Existence and Uniqueness of Solutions

This section will be developed to the existence and uniqueness of solutions for equation (3) employing the Banach fixed point theorem.

We begin introducing some notations and results for further consideration.

Let $I = [a,b]$ $(a < b, a, b \in \mathbb{R})$ and $m \in \mathbb{N}_0$. Denoting by C^m a usual space of functions v which are m times continuously differentiable on I with the norm

$$\|v\|_{C^m} = \sum_{k=0}^{m} \|v^{(k)}\|_C = \sum_{k=0}^{m} \max_{x\in\Omega} |v^{(k)}(x)|,$$

$m \in \mathbb{N}_0$.

In particular, for $m = 0$, $C^0(I) \equiv C(I)$ is the space of continuous functions v on I with the norm $\|v\|_C = \max\limits_{x\in I} |v(x)|$.

For $0 \leq \gamma < 1$, denoting by $C^\gamma(I)$ a weighted space of functions v for $x \in (a,b]$ such that $(x-a)^{1+\gamma}v(x) \in C[a,b]$ and

$$\|v\|_{C^\gamma} = \|(x-a)^{1+\gamma}v(x)\|_C.$$

Theorem 4.1 *(see [6]) Let $\gamma \geq 0$ and $v(x) \in AC^n([a,b])$, $n = [\gamma]+1$. Then $D^\gamma_{a^+}v$ exists almost everywhere and may be represented in the form*

$$D^\gamma_{a^+}v = \sum_{k=0}^{n-1} \frac{v^{(k)}(a)}{\Gamma(1+k-\gamma)}(x-a)^{k-\gamma} + \frac{1}{\Gamma(n-\gamma)} \int_a^x \frac{v^{(n)}(t)}{(x-t)^{\gamma-n+1}}dt. \tag{26}$$

Theorem 4.2 *(see [3]) Let $\gamma \geq 0$ and $n = [\gamma]+1$. If $v(x) \in AC^n([a,b])$, then the Caputo fractional derivative ${}^C D^\gamma_{a^+}v$ exists almost everywhere on $[a,b]$, and if $\gamma \notin \mathbb{N}_0$, ${}^C D^\gamma_{a^+}v$ is represented by*

$${}^C D^\gamma_{a^+}v = \frac{1}{\Gamma(n-\gamma)} \int_a^x \frac{v^{(n)}(t)}{(x-t)^{\gamma-n+1}}dt := (I^{n-\gamma}_{a^+}D^n v)(x), \tag{27}$$

where $D = d/dx$.

Remark If $\gamma \notin \mathbb{N}_0$ and $n = [\gamma] + 1$, then

$$\left| I_{a^+}^{n-\gamma} D^n v)(x) \right| \leq \frac{\|v^{(n)}\|_C}{|\Gamma(n-\gamma)|(n-\gamma+1)} (x-a)^{n-\gamma}. \tag{28}$$

Recalling equation (18) and cancelling the Laplace transform we obtain

$$\begin{aligned} &u(t,x) - a\left(I_{x_0^+}^{\beta+1} I_{t_0^+}^{-1-\alpha} u\right)(t,x) - b\left(I_{x_0^+}^{\beta+1} I_{t_0^+}^{-\alpha} u\right)(t,x) \\ &-c\left(I_{x_0^+}^{\beta+1} u\right)(t,x) = g(t,x), \end{aligned}$$

which can be rewritten as

$$\begin{aligned} &u(t,x) - a\left(I_{x_0^+}^{\beta+1} D_{t_0^+}^{\alpha+1} u\right)(t,x) - b\left(I_{x_0^+}^{\beta+1} D_{t_0^+}^{\alpha} u\right)(t,x) \\ &-c\left(I_{x_0^+}^{\beta+1} u\right)(t,x) = g(t,x), \end{aligned} \tag{29}$$

where

$$g(t,x) = \frac{(x-x_0)^\beta}{\Gamma(\beta+1)} g_0(t) + \frac{(x-x_0)^{\beta-1}}{\Gamma(\beta)} g_1(t),$$

with $0 < \alpha < 1$, $0 < \beta < 1$.

In order to prove the existence and uniqueness of solution for equation (3) under conditions of Lemma 3.2, it is sufficient to prove the existence of a unique solution of (29).

Now, we will establish an auxilliarly result.

Lemma 4.3 *The fractional integration operator $I_{x_0^+}^{\gamma}$ of order γ with $\gamma \in \mathbb{R}^+$ forms a map from $C[x_0, X_0]$ to itself for each $t \in [t_0, T_0]$, and we have the estimate*

$$\left\| I_{x_0^+}^{\gamma} u \right\|_{C[x_0,X_0]} \leq \frac{(X_0 - x_0)^{\gamma+1}}{\Gamma(\gamma+1)} \|u\|_{C[x_0,X_0]}.$$

Proof: First we prove that, if $u(t,x) \in C[x_0, X_0]$ then $(I_{x_0^+}^{\gamma} u)(t,x) \in C[x_0, X_0]$. In fact, for any $x \in [x_0, X_0]$ and $\Delta x > 0$, $x + \Delta x \leq X_0$ we have

$$\begin{aligned}
&\left|(I^{\gamma}_{x_0^+}u)(t, x+\Delta x) - (I^{\gamma}_{x_0^+}u)(t,x)\right| \\
&= \frac{1}{\Gamma(\gamma)}\left|\int_{x_0}^{x+\Delta x}(x+\Delta x - z)^{\gamma-1}u(t,z)dz - \int_{x_0}^{x}(x-z)^{\gamma-1}u(t,z)dz\right| \\
&\leq \frac{1}{\Gamma(\gamma)}\left[\left|\int_{x_0}^{x} u(t,z)\left((x+\Delta x - z)^{\gamma-1} - (x-z)^{\gamma-1}\right)dz\right|\right. \\
&+ \left.\left|\int_{x}^{x+\Delta x}(x+\Delta x - z)^{\gamma-1}u(t,z)dz\right|\right] \\
&\leq \frac{\|u(t,x)\|_{C[x_0,X_0]}}{(\gamma)\Gamma(\gamma)}\left[((x+\Delta x - x_0)^{\gamma} - (x-x_0)^{\gamma}) + 2(\Delta x)^{\gamma}\right] \\
&\leq \frac{\|u(t,x)\|_{C[x_0,X_0]}}{\Gamma(\gamma+1)}\left[((x+\Delta x - x_0)^{\gamma} - (x-x_0)^{\gamma}) + 2(\Delta x)^{\gamma}\right].
\end{aligned} \tag{30}$$

Therefore, when $\Delta x \to 0^+$ we have

$$\left|(I^{\gamma}_{x_0^+}u)(t, x+\Delta x) - (I^{\gamma}_{x_0^+}u)(t,x)\right| \to 0.$$

Similarly it is valid when $\Delta x \to 0^-$. Hence, $I^{\gamma}_{x_0^+}u \in C[x_0, X_0]$. Consequently,

$$\begin{aligned}
\left\|I^{\gamma}_{x_0^+}u\right\|_{C[x_0,X_0]} &= \max_{x\in[x_0,X_0]}\left|\frac{1}{\Gamma(\gamma)}\int_{x_0}^{x}(x-z)^{\gamma-1}u(t,z)dz\right| \\
&\leq \frac{\|u\|_{C[x_0,X_0]}}{\Gamma(\gamma)}\int_{x_0}^{x}(x-z)^{\gamma-1}dz \\
&\leq \frac{(X_0-x_0)^{\gamma+1}}{(\gamma)\Gamma(\gamma)}\|u\|_{C[x_0,X_0]} \\
&\leq \frac{(X_0-x_0)^{\gamma+1}}{\Gamma(\gamma+1)}\|u\|_{C[x_0,X_0]}.
\end{aligned} \tag{31}$$

■

Theorem 4.4 *Integral equation (29) has a unique solution whenever* $0 < \xi < 1$ *and where*

$$\begin{aligned}
\xi &= \frac{(X_0-x_0)^{\beta+2}}{\Gamma(\beta+2)}\left[a\left(\frac{(T_0-t_0)^{\gamma-\alpha}}{|\Gamma(-\alpha)|} + \frac{(T_0-t_0)^{\gamma+1-\alpha}}{\Gamma(1-\alpha)} + \frac{(T_0-t_0)^{\gamma+2-\alpha}}{\Gamma(1-\alpha)(2-\alpha)}\right)\right. \\
&+ \left. b\left(\frac{(T_0-t_0)^{\gamma+1-\alpha}}{\Gamma(1-\alpha)} + \frac{(T_0-t_0)^{\gamma+2-\alpha}}{\Gamma(1-\alpha)(2-\alpha)}\right) + c(T_0-t_0)^{\gamma+1}\right],
\end{aligned}$$

with $\gamma > \alpha$.

Proof: We denote by X the Banach space

$$X = \{u : u(.,x) \in C([x_0, X_0]), u(t,.) \in C^2([t_0, T_0])\}$$

and by Y the Banach space

$$Y = \{u : u(.,x) \in C([x_0, X_0]), u(t,.) \in C^\gamma([t_0, T_0])\}.$$

Next, we put $T : X \to Y$,

$$(Tu)(t,x) = a\left(I^{\beta+1}_{x_0^+} D^{\alpha+1}_{t_0^+} u\right)(t,x) + b\left(I^{\beta+1}_{x_0^+} D^{\alpha}_{t_0^+} u\right)(t,x) + c\left(I^{\beta+1}_{x_0^+} u\right)(t,x) + g(t,x).$$

Hence, we rewrite equation (29) in the form

$$u(t,x) = (Tu)(t,x).$$

Calling definitions of $C^\gamma-$norm, C^2-norm and $C-$norm and taking into account (26), (27) and (28), we have

$$\begin{aligned}
\|Tu_1 - Tu_2\|_Y &= \|aI^{\beta+1}_{x_0^+} D^{\alpha+1}_{t_0^+}(u_1-u_2) + bI^{\beta+1}_{x_0^+} D^{\alpha}_{t_0^+}(u_1-u_2) + cI^{\beta+1}_{x_0^+}(u_1-u_2)\|_Y \\
&\leq \frac{(X_0-x_0)^{\beta+2}}{\Gamma(\beta+2)} \Big[a\|(t-t_0)^{1+\gamma} D^{\alpha+1}_{t_0^+}(u_1-u_2)\|_Y \\
&+ b\|(t-t_0)^{\gamma+1} D^{\alpha}_{t_0^+}(u_1-u_2)\|_Y + c\|(t-t_0)^{\gamma+1}(u_1-u_2)\|_Y\Big] \\
&\leq \frac{(X_0-x_0)^{\beta+2}}{\Gamma(\beta+2)} \Bigg[a\left(\frac{(T_0-t_0)^{\gamma-\alpha}}{|\Gamma(-\alpha)|} + \frac{(T_0-t_0)^{\gamma+1-\alpha}}{\Gamma(1-\alpha)}\right. \\
&+ \left.\frac{(T_0-t_0)^{\gamma+2-\alpha}}{\Gamma(1-\alpha)(2-\alpha)}\right) + b\left(\frac{(T_0-t_0)^{\gamma+1-\alpha}}{\Gamma(1-\alpha)} + \frac{(T_0-t_0)^{\gamma+2-\alpha}}{\Gamma(1-\alpha)(2-\alpha)}\right) \\
&+ c(T_0-t_0)^{\gamma+1}\Bigg] \|u_1-u_2\|_X \\
&= \xi\|u_1-u_2\|_X, \qquad (32)
\end{aligned}$$

where we let

$$\begin{aligned}
\xi &= \frac{(X_0-x_0)^{\beta+2}}{\Gamma(\beta+2)} \left[a\left(\frac{(T_0-t_0)^{\gamma-\alpha}}{|\Gamma(-\alpha)|} + \frac{(T_0-t_0)^{\gamma+1-\alpha}}{\Gamma(1-\alpha)} + \frac{(T_0-t_0)^{\gamma+2-\alpha}}{\Gamma(1-\alpha)(2-\alpha)}\right)\right. \\
&+ \left. b\left(\frac{(T_0-t_0)^{\gamma+1-\alpha}}{\Gamma(1-\alpha)} + \frac{(T_0-t_0)^{\gamma+2-\alpha}}{\Gamma(1-\alpha)(2-\alpha)}\right) + c(T_0-t_0)^{\gamma+1}\right],
\end{aligned}$$

and $\gamma > \alpha$.

∎

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