The use of Kontorovich-Lebedev's transform in an analysis of regularized Schrödinger equation

S. Yakubovich * and N. Vieira†

Department of Mathematics
Faculty of Science
University of Porto
Rua do Campo Alegre 687
4169-007 Porto, Portugal
E-mail: {syakubov,nvieira}@fc.up.pt

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Abstract

In this paper we introduce a notion of Schrödinger's kernel to the familiar Kontorovich-Lebedev transform. In order to control its singularity at infinity, we will need to implement the so-called regularization procedure. Hence we will obtain a sequence of regularized kernels which converge to the original kernel when a regularization parameter tends to zero. We study differential properties of the regularized kernel and a solution for a certain type of regularized Schrödinger equations. A family of the regularized Weierstrass transforms is presented. Finally, we examine a pointwise convergence of this sequence of operators, when the regularization parameter tends to zero.

Keywords: Kontorovich-Lebedev transform, Schrödinger equation, heat kernel, regularization procedure, modified Bessel functions.

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1 Introduction

Time evolution problems are of extreme importance in mathematical physics. However, there is still a strong need to develop further special techniques to deal with these problems. A possible approach in the analysis of time evolution equations presumes the use of integral transforms and special functions. For instance, in [3] the authors appeal to a combination of Clifford analysis and integral transforms to study the time dependent Schrödinger equation in some specific manifolds. Another type of integrals transforms which can be applied to study this type of equations are index transforms, in particular, the Kontorovich-Lebedev transform. This transform has been used in many applications including, for instance, fluid mechanics, quantum and nano-optics and plasmonics. It proves to be an effective tool in solving the resulting differential equations when modeling optical or electronic response of such problems. It arises naturally when one deals with the method of separation of variables to solve boundary-value problems in terms of cylindrical coordinate system (for more details see [7]). An application of index transforms to examine Schrödinger's operator is connected to some techniques developed for elliptic equations, which were involved to study the heat operator (see [9]). However, we need to take into account that in many aspects the Schrödinger operator is substantially different from the heat operator. First of all the Galilean group is the invariance group associated to the Schrödinger equation, while the parabolic group is the invariance group that is associated to the heat equation (see [8]). Secondly, the Schrödinger equation is related to the Minkowski space-time metric, while the heat equation is linked to the parabolic space-time metric (see [8]). More important for us, under an analytical point of view, the singularity for large values of variable t, i.e. when $t \to +\infty$, of the Schrödinger kernel is not removable by standard methods. To overcome this problem we introduce a regularization procedure prior to the development of a hypoelliptic analysis (see [2], [3] and [8]).

The main goal of this paper is to show that index transforms can be applied in the analysis of certain type of regularized Schrödinger equation. More specifically, we will investigate differential properties of the regularized Schrödinger kernel related to the Kontorovich-Lebedev transform, which will be introduced below. Moreover, a pointwise convergence of a sequence of Weierstrass's integral operators, which are defined via the regularized kernel when a regularization parameter tends to zero, will be established.

2 Preliminaries

The Kontorovich-Lebedev transform is given by the formula [7], [13]

$$\mathcal{K}_{i\tau}[f] = \int_{\mathbb{R}_+} K_{i\tau}(x) f(x)\, dx, \tag{1}$$

where integral (1) converges with respect to the norm in $L_2(\mathbb{R}_+, \tau \sinh(\pi\tau)\, d\tau)$. The corresponding Parseval identity holds

$$\int_{\mathbb{R}_+} \tau \sinh(\pi\tau) |\mathcal{K}_{i\tau}[f]|^2\, d\tau = \frac{\pi^2}{2} \int_{\mathbb{R}_+} x |f(x)|^2\, dx \tag{2}$$

as well as the inversion formula

$$f(x) = \frac{2}{x\pi^2} \int_{\mathbb{R}_+} \tau \sinh(\pi\tau) K_{i\tau}(x) \mathcal{K}_{i\tau}[f]\, d\tau, \tag{3}$$

where integral (3) converges with respect to the norm in $L_2(\mathbb{R}_+, x\, dx)$. The kernel $K_{i\tau}(x)$ is the modified Bessel function of pure imaginary index $i\tau$, which is an eigenfunction of the following second order differential operator

$$\mathcal{A}_x = x^2 - x\frac{d}{dx}x\frac{d}{dx}, \tag{4}$$

i.e. we have

$$\mathcal{A}_x K_{i\tau}(x) = \tau^2 K_{i\tau}(x). \tag{5}$$

It has, in particular, the following integral representation (see [1])

$$K_\nu(z) = \frac{\sqrt{\pi} z^\nu}{2^\nu \Gamma\left(\nu + \frac{1}{2}\right)} \int_{\mathbb{R}_+} e^{-z\cosh(w)} \sinh^{2\nu}(w)\, dw, . \tag{6}$$

where $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(\nu) > -\frac{1}{2}$. Moreover, it verifies the following relation

$$K_{i\tau}(z) = \frac{\pi}{2\sinh(\pi\tau)} \left[I_{-i\tau}(z) - I_{i\tau}(z)\right] \tag{7}$$

and admits the asymptotic behavior

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} \left[1 + O\left(\frac{1}{z}\right)\right], \quad z \to +\infty \tag{8}$$

$$K_\nu(z) = O(z^{-|\mathrm{Re}(\nu)|}), \quad z \to 0 \tag{9}$$

$$K_0(z) = -\log(z) + O(1), \quad z \to 0. \tag{10}$$

When $\tau \to +\infty$ and $x > 0$, the modified Bessel function $K_{i\tau}(x)$ behaves as (see [13])

$$K_{i\tau}(x) = \sqrt{\frac{2\pi}{\tau}}\, e^{-\frac{\pi\tau}{2}}\ \sin\left(\tau\left(\log\left(\frac{2\tau}{x}\right)-1\right)+\frac{\pi}{4}+\frac{x^2}{4\tau}\right)\left(1+O\left(\frac{1}{\tau}\right)\right). \quad (11)$$

The convolution of the Kontorovich-Lebedev transform is defined accordingly [13, 14]

$$(f*h)(x) = \frac{1}{2x}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy}+\frac{yu}{x}\right)} f(u)h(y)\ du\ dy, \quad x > 0. \quad (12)$$

This operator (12) is well-defined in the Banach ring $L^{\alpha}(\mathbb{R}_+) \equiv L_1\left(\mathbb{R}_+, K_{\alpha}(x)\ dx\right)$, $\alpha \in \mathbb{R}_+$, i.e., the space of all summable functions $f : \mathbb{R}_+ \to \mathbb{C}$ with respect to the measure $K_{\alpha}(x)\ dx$ for which

$$||f||_{L^{\alpha}(\mathbb{R}_+)} = \int_{\mathbb{R}_+} |f(x)|K_{\alpha}(x)\ dx$$

is finite. The following embedding relations take place

$$\begin{aligned} L^{\alpha}(\mathbb{R}_+) \equiv L^{-\alpha}(\mathbb{R}_+), \quad L^{\alpha}(\mathbb{R}_+) \subseteq L^{\beta}(\mathbb{R}_+), \quad |\alpha| \geq |\beta| \geq 0, \quad \alpha,\beta \in \mathbb{R} \\ L^{\alpha}(\mathbb{R}) \supset L_p(\mathbb{R}_+, x\ dx), \quad 2 < p \leq +\infty, \quad |\alpha| < 1-\frac{2}{p}, \end{aligned}$$

where $L_p(\mathbb{R}_+, x\ dx)$ is a weighted space with the norm

$$\begin{aligned} ||f||_{L_p(\mathbb{R}_+,x\ dx)} &= \left(\int_{\mathbb{R}_+} |f(x)|^p x\ dx\right)^{\frac{1}{p}}, \quad 1 \leq p < +\infty, \\ ||f||_{L_\infty(\mathbb{R}_+,x\ dx)} &= \operatorname*{ess\ sup}_{x\in\mathbb{R}_+} |f(x)|. \end{aligned}$$

The factorization property is true for the convolution (12) in terms of the Kontorovich-Lebedev transform in the space $L^{\alpha}(\mathbb{R}_+)$, namely

$$\mathcal{K}_{i\tau}[f*h] = \mathcal{K}_{i\tau}[f]\mathcal{K}_{i\tau}[h], \quad \tau \in \mathbb{R}_+. \quad (13)$$

This equality is based on the Macdonald formula [1]

$$K_{\nu}(x)K_{\nu}(y) = \frac{1}{2}\int_{\mathbb{R}_+} e^{-\frac{1}{2}\left(t\frac{x^2+y^2}{xy}+\frac{xy}{t}\right)} K_{\nu}(t)\ \frac{dt}{t}. \quad (14)$$

It is also proved in [14] and [13] that the Kontorovich-Lebedev transform is a bounded operator from $L^{\alpha}(\mathbb{R}_+)$ into the space of bounded continuous functions on $\mathbb{R}_+$ vanishing in the infinite. Furthermore, convolution (12) of two functions $f, h \in L^{\alpha}(\mathbb{R}_+)$ exists as a Lebesgue integral and belongs to $L^{\alpha}(\mathbb{R}_+)$. It satisfies the Young type inequality

$$||f * h||_{L^{\alpha}(\mathbb{R}_+)} \leq ||f||_{L^{\alpha}(\mathbb{R}_+)} ||h||_{L^{\alpha}(\mathbb{R}_+)}.$$

Further, let us recall some definitions and results regarding the space $S_2(\mathbb{R}_+)$ that will be used in the sequel (for more details see [12] and [11]).

Definition 2.1. *A function $f : \mathbb{R}_+ \to \mathbb{C}$ is said to be in $S_2(\mathbb{R}_+)$ if $f \in L_2\left(\mathbb{R}_+, \frac{dx}{x}\right)$ and $\mathcal{A}_x f \in L_2\left(\mathbb{R}_+, \frac{dx}{x}\right)$, where the operator $\mathcal{A}_x$ is defined by (4).*

The $k-$th iterate of the operator $\mathcal{A}_x$, $\mathcal{A}_x^k f \in L_2\left(\mathbb{R}_+, \frac{dx}{x}\right)$, $k \in \mathbb{N}_0$ means that there exists a function $v(x)$ in $L_2\left(\mathbb{R}_+, \frac{dx}{x}\right)$ denoted by $\mathcal{A}_x^k f$ such that for all $\phi \in \mathcal{D}(\mathbb{R}_+)$

$$\int_{\mathbb{R}_+} f(x) \mathcal{A}_x^k \phi \, \frac{dx}{x} = \int_{\mathbb{R}_+} v(x)\phi(x) \, \frac{dx}{x}.$$

It is proved that $S_2(\mathbb{R}_+)$ is a Banach space which with the norm

$$||f||_{S_2(\mathbb{R}_+)} = \left(\int_{\mathbb{R}_+} |f(x)|^2 \, \frac{dx}{x} + \int_{\mathbb{R}_+} |\mathcal{A}_x f|^2 \, \frac{dx}{x} \right)^{\frac{1}{2}}.$$

A characterization of $S_2(\mathbb{R}_+)$ can be given in terms of the Kontorovich-Lebedev transform (1). First we observe that $f \in L_2\left(\mathbb{R}_+, \frac{dx}{x}\right)$ means $\frac{f(x)}{x} \in L_2(\mathbb{R}_+, x \, dx)$. We have

Theorem 2.2. *Let $f \in L_2\left(\mathbb{R}_+, \frac{dx}{x}\right)$ with the Kontorovich-Lebedev transform $\mathcal{K}_{i\tau}\left[\frac{f(x)}{x}\right]$. Then $f \in S_2(\mathbb{R}_+)$ (i.e. $\mathcal{A}_x f \in L_2\left(\mathbb{R}_+, \frac{dx}{x}\right)$) if and only if $\tau \to \tau^2 \mathcal{K}_{i\tau}\left[\frac{f(x)}{x}\right]$ is in $L_2(\mathbb{R}_+, \tau \sinh(\pi\tau) \, d\tau)$. Moreover, $\mathcal{K}_{i\tau}\left[\frac{\mathcal{A}_x f}{x}\right] = \tau^2 \mathcal{K}_{i\tau}\left[\frac{f(x)}{x}\right]$ and therefore*

$$||f||^2_{S_2(\mathbb{R}_+)} = \frac{2}{\pi^2} \int_{\mathbb{R}_+} \tau \sinh(\pi\tau) \left| \mathcal{K}_{i\tau}\left[\frac{f(x)}{x}\right] \right|^2 (1 + \tau^4) \, d\tau.$$

One can extend the previous theorem for the iterates $\mathcal{A}^k$, $k \in \mathbb{N}_0$. Indeed, we have

Theorem 2.3. *The iterate $\mathcal{A}_x^k f \in L_2\left(\mathbb{R}_+, \frac{dx}{x}\right)$, $k \in \mathbb{N}_0$ if and only if $\tau \to \tau^{2k} \mathcal{K}_{i\tau}\left[\frac{f(x)}{x}\right]$ is in $L_2(\mathbb{R}_+, \tau \sinh(\pi\tau) \, d\tau)$. Moreover,*

$$\mathcal{K}_{i\tau}\left[\frac{\mathcal{A}_x^k f}{x}\right] = \tau^{2k} \mathcal{K}_{i\tau}\left[\frac{f(x)}{x}\right].$$

Finally in this section we recall some basic definitions regarding generalized functions (for more details see [15]). The space of test functions $\mathcal{D}(\mathbb{R}_+)$ consists of the $C^\infty(\mathbb{R}_+)-$functions that have compact support in $\mathbb{R}_+$. Let $\mathcal{D}'(\mathbb{R}_+)$ be the space dual to $\mathcal{D}(\mathbb{R}_+)$. A sequence of generalized functions in $\mathcal{D}'(\mathbb{R}_+)$ converges in $\mathcal{D}'(\mathbb{R}_+)$ if it converges on every element of $\mathcal{D}(\mathbb{R}_+)$.

3 The regularized Schrödinger kernel and its properties

Using ideas presented in [9] a natural extension of the heat kernel for the case of Schrödinger's equation will be the following integral

$$h_t(x,y) = \frac{2}{x\pi^2} \int_{\mathbb{R}_+} e^{-it\tau^2} \tau \sinh(\pi\tau) K_{i\tau}(x) K_{i\tau}(y)\, d\tau, \tag{15}$$

where $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+$ and $t > 0$. Integral (15) converges relatively and this will be verified by Lemma 5.1 below taking into account asymptotic behavior (11) and integration by parts. However, a differentiation by t under the integral sign in (15) is impossible since it drives us to a divergent integral. In order to overcome this problem, we need to regularize the Schrödinger kernel (15) involving a regularization parameter. We will study in the sequel a convergence of this family of kernels and the corresponding integral operators when the regularization parameter tends to zero.

Definition 3.1. *Let $t > 0$, $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+$. The following integral*

$$h_t^\epsilon(x,y) = \frac{2}{x\pi^2} \int_{\mathbb{R}_+} e^{-\mathbf{k}_\epsilon t\tau^2} \tau \sinh(\pi\tau) K_{i\tau}(x) K_{i\tau}(y)\, d\tau \tag{16}$$

with $\mathbf{k}_\epsilon = \frac{\epsilon+i}{\sqrt{\epsilon^2+1}}, \epsilon > 0$, is called the regularized Schrödinger kernel for the Kontorovich-Lebedev transform.

The differentiability with respect to t of the regularized kernel $h_t^\epsilon(x,y)$ is given by

Theorem 3.2. *For any $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+$ and $t > 0$, $h_t^\epsilon(x,y)$ is a infinitely differentiable function by t and it satisfies the estimate*

$$\left| \frac{\partial^m h_t^\epsilon(x,y)}{\partial t^m} \right| \leq \frac{\Gamma^{\frac{1}{4}}\left(4m + \frac{3}{2}\right)}{2^{m+1}\pi^{\frac{7}{8}}} \; \frac{e^{\frac{\pi^2}{4t\mathrm{Re}(\mathbf{k}_\epsilon)}}}{x\left(t\mathrm{Re}(\mathbf{k}_\epsilon)\right)^{m+\frac{1}{2}}} \; K_0^{\frac{1}{2}}\left(2\sqrt{x^2+y^2}\right), \tag{17}$$

where $\mathrm{Re}(\mathbf{k}_\epsilon) = \frac{\epsilon}{\sqrt{\epsilon^2+1}}$ is a real part of $\mathbf{k}_\epsilon$.

Proof. Appealing to the following inequality (c.f. [10]) for derivatives with respect to x of modified Bessel functions

$$\left|\frac{\partial^m K_{i\tau}(x)}{\partial x^m}\right| \le e^{-\delta\tau} K_m(x\cos\delta), \quad x, \tau > 0, \ \delta \in \left[0, \frac{\pi}{2}\right[, \ m = 0, 1, 2, \dots \tag{18}$$

we conclude that for $t > 0$ integral (16) and its derivatives of any order with respect to x and y converge absolutely and uniformly by $x \ge x_0 > 0$ and $y \ge y_0 > 0$. Therefore the regularized Schrödinger kernel (16) is infinitely differentiable by $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$. Similarly, we guarantee the differentiation with respect to $t > 0$ and get the formula

$$\frac{\partial^m h_t^\epsilon(x, y)}{\partial t^m} = \frac{2(-\mathbf{k}_\epsilon)^m}{x\pi^2} \int_{\mathbb{R}_+} e^{-\mathbf{k}_\epsilon t\tau^2} \tau^{2m+1} \sinh(\pi\tau) K_{i\tau}(x) K_{i\tau}(y) \, d\tau, \tag{19}$$

for $m = 0, 1, 2, \dots$. Further, applying the Schwarz inequality we deduce

$$\left|\frac{\partial^m h_t^\epsilon(x, y)}{\partial t^m}\right|$$
$$\le \frac{2}{x\pi^2} \left(\int_{\mathbb{R}_+} e^{-2t\tau^2 \mathrm{Re}(\mathbf{k}_\epsilon)} \tau^{4m+1} \sinh(\pi\tau) \, d\tau\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}_+} \tau \sinh(\pi\tau) \left|K_{i\tau}(x) K_{i\tau}(y)\right|^2 \, d\tau\right)^{\frac{1}{2}}. \tag{20}$$

The second integral in the right-hand side of (20) via (1), (2), Macdonald's formula (14) and relation (2.3.16.1) in [5] becomes

$$\int_{\mathbb{R}_+} \tau \sinh(\pi\tau) \left|K_{i\tau}(x) K_{i\tau}(y)\right|^2 \, d\tau = \frac{\pi^2}{4} K_0(2\sqrt{x^2 + y^2}). \tag{21}$$

For the first integral in (20) we get accordingly,

$$\int_{\mathbb{R}_+} e^{-2t\tau^2 \mathrm{Re}(\mathbf{k}_\epsilon)} \tau^{4m+1} \sinh(\pi\tau) \, d\tau$$
$$\le \frac{1}{2} \left(\int_{\mathbb{R}} e^{2\left(\pi\tau - t\tau^2 \mathrm{Re}(\mathbf{k}_\epsilon)\right)} \, d\tau\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}_+} e^{-2t\tau^2 \mathrm{Re}(\mathbf{k}_\epsilon)} \tau^{8m+1} \, d\tau^2\right)^{\frac{1}{2}}$$
$$= \frac{\pi^{\frac{1}{4}}}{2} \, \Gamma^{\frac{1}{2}} \left(4m + \frac{3}{2}\right) e^{\frac{\pi^2}{4t\mathrm{Re}(\mathbf{k}_\epsilon)}} \left(2t\mathrm{Re}(\mathbf{k}_\epsilon)\right)^{-(2m+1)}. \tag{22}$$

Substituting (21) and (22) in (20), we derive inequality (17). □

Now we will establish a connection between function h_t^ϵ and a Schrödinger type equation.

Theorem 3.3. *The function h_t^ϵ is a solution of regularized Schrödinger equations $(u = u(x,y,t), \Delta = \partial^2_{xx})$*

$$\left(-x^2\Delta - \mathbf{k}_\epsilon^{-1}\partial_t\right) u = \left(3x\partial_x - (x^2-1)\right) u \tag{23}$$

for each fixed $y \in \mathbb{R}_+$,

$$\left(-y^2\Delta - \mathbf{k}_\epsilon^{-1}\partial_t\right) u = \left(y\partial_y - y^2\right) u \tag{24}$$

for each fixed $x \in \mathbb{R}_+$, under the initial condition in the sense of distributions

$$\lim_{t\to 0} h_t^\epsilon(x,y) = \delta(x-y), \tag{25}$$

where δ is Dirac's delta function.

Proof. Taking into account (5) with absolute and uniform convergence by x and y of the integral (16) on any compact set of $\mathbb{R}_+ \times \mathbb{R}_+$, formula (19) can be rewritten in terms of the following differential equations

$$\frac{\partial^m h_t^\epsilon}{\partial t^m} = (-\mathbf{k}_\epsilon)^m \mathcal{A}_y^m h_t^\epsilon(x,y), \qquad \frac{\partial^m h_t^\epsilon}{\partial t^m} = \frac{(-\mathbf{k}_\epsilon)^m}{x} \mathcal{A}_x^m \left[x h_t^\epsilon(x,y)\right],$$

where $m = 0, 1, 2, \ldots$ and $\mathcal{A}_x^m$, $\mathcal{A}_y^m$ are m-th iterates of operator (4). In particular, if $m = 1$ we easily check, that h_t^ϵ is a solution of regularized Schröndinger type equations (23) and (24).

It remains to proof (25). Indeed, for any ϕ from the test function space $\mathcal{D}(\mathbb{R}_+)$

$$\lim_{t\to 0} \langle h_t^\epsilon(x,\cdot), \phi(\cdot)\rangle = \lim_{t\to 0} \frac{2}{x\pi^2} \int_{\mathbb{R}_+} e^{-\mathbf{k}_\epsilon t\tau^2} \tau \sinh(\pi\tau) K_{i\tau}(x) \mathcal{K}_{i\tau}[\phi]\, d\tau. \tag{26}$$

Taking into account relation (2.16.14.1) in [6] and the Parseval equality for the cosine Fourier transform [7], we get

$$\begin{aligned} &\frac{2}{x\pi^2} \int_{\mathbb{R}_+} e^{-\mathbf{k}_\epsilon t\tau^2} \tau \sinh(\pi\tau) K_{i\tau}(x) \mathcal{K}_{i\tau}[\phi]\, d\tau \\ &= \frac{\sqrt{2}}{x\pi\sqrt{\pi}} \int_{\mathbb{R}_+} e^{-\mathbf{k}_\epsilon t\tau^2} \tau \sinh\left(\frac{\pi\tau}{2}\right) K_{i\tau}(x) \int_{\mathbb{R}_+} \cos(\tau u) \mathcal{F}_c[\phi, \sinh(u)]\, du\, d\tau, \end{aligned} \tag{27}$$

where $\mathcal{F}_c[\phi, v]$ denotes the cosine Fourier transform of $\phi \in \mathcal{D}(\mathbb{R}_+)$. Hence, making use of differentiation, integration by parts, convolution properties and the Parseval equality for the Fourier transform, we obtain

$$\begin{aligned} &\frac{\sqrt{2}}{x\pi\sqrt{\pi}} \int_{\mathbb{R}_+} e^{-\mathbf{k}_\epsilon t\tau^2} \tau \sinh\left(\frac{\pi\tau}{2}\right) K_{i\tau}(x) \int_{\mathbb{R}_+} \cos(\tau u) \mathcal{F}_c[\phi, \sinh(u)]\, du\, d\tau \\ &= -\frac{\sqrt{2}}{x\pi\sqrt{\pi}} \int_{\mathbb{R}_+} e^{-\mathbf{k}_\epsilon t\tau^2} \sinh\left(\frac{\pi\tau}{2}\right) K_{i\tau}(x) \int_{\mathbb{R}_+} \sin(\tau u) \frac{\partial}{\partial u}[\mathcal{F}_c[\phi, \sinh(u)]]\, du\, d\tau \\ &= \frac{-1}{2ix\pi\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\mathbf{k}_\epsilon t\tau^2} \sinh\left(\frac{\pi\tau}{2}\right) K_{i\tau}(x) \int_{\mathbb{R}} e^{i\tau u} \frac{\partial}{\partial u}[\mathcal{F}_c[\phi, \sinh(u)]]\, du\, d\tau. \end{aligned} \tag{28}$$

By relations (2.5.36.1) and (2.5.54.6) in [5], the product $e^{-\mathbf{k}_\epsilon t\tau^2}\sinh\left(\frac{\pi\tau}{2}\right)K_{i\tau}(x)$ can be represented as the Fourier transform of a convolution, namely

$$
\begin{aligned}
& e^{-\mathbf{k}_\epsilon t\tau^2}\sinh\left(\frac{\pi\tau}{2}\right)K_{i\tau}(x) \\
&= \frac{i}{4t\sqrt{\pi\mathbf{k}_\epsilon t}}\int_{\mathbb{R}} ye^{-\frac{y^2}{4\mathbf{k}_\epsilon t}}e^{iy\tau}\int_{\mathbb{R}} e^{iy\tau}\int_y^{+\infty}\sin(x\sinh(v))\,dv\,dy\,dy \\
&= \frac{i}{4t\sqrt{\pi\mathbf{k}_\epsilon t}}\int_{\mathbb{R}} e^{iy\tau}\int_{\mathbb{R}} u\,e^{-\frac{u^2}{4\mathbf{k}_\epsilon t}}\int_{y-u}^{+\infty}\sin(x\sinh(v))\,dv\,du\,dy \\
&= \frac{i}{4t\sqrt{\pi\mathbf{k}_\epsilon t}}\int_{\mathbb{R}} e^{iy\tau}\int_{\mathbb{R}} e^{-\frac{u^2}{4\mathbf{k}_\epsilon t}}\sin(x\sinh(y-u))\,du\,dy,
\end{aligned}
$$

where the main branch of the square root of $\mathbf{k}_\epsilon$ is taken. Hence, (28) is equal to

$$
\begin{aligned}
& \frac{2}{x\pi^2}\int_{\mathbb{R}_+} e^{-\mathbf{k}_\epsilon t\tau^2}\tau\sinh(\pi\tau)K_{i\tau}(x)\mathcal{K}_{i\tau}[\phi]\,d\tau \\
&= -\frac{1}{2x\pi\sqrt{2\mathbf{k}_\epsilon t}}\int_{\mathbb{R}}\frac{d}{dy}[\mathcal{F}_c[\phi,\sinh(y)]]\int_{\mathbb{R}} e^{-\frac{u^2}{4\mathbf{k}_\epsilon t}}\sin(x\sinh(y-u))\,du\,dy \\
&= \frac{1}{2\pi\sqrt{2\mathbf{k}_\epsilon t}}\int_{\mathbb{R}}\mathcal{F}_c[\phi,\sinh(y)]\int_{\mathbb{R}} e^{-\frac{u^2}{4\mathbf{k}_\epsilon t}}\cos(x\sinh(y-u))\cosh(y-u)\,du\,dy \\
&= \frac{1}{\pi\sqrt{2}}\int_{\mathbb{R}}\mathcal{F}_c[\phi,\sinh(y)]\int_{\mathbb{R}} e^{-u^2}\cos(x\sinh(y-2u\sqrt{\mathbf{k}_\epsilon t}))\cosh(y-2u\sqrt{\mathbf{k}_\epsilon t})\,du\,dy.
\end{aligned}
\tag{29}
$$

But asymptotic properties at infinity of Fourier transforms of test functions allow us to pass to the limit when $t\to 0$ under the integral sign in the right-hand side of the latter equality in (29) via the Lebesgue dominated convergence theorem. Then after straightforward calculations and appealing to inversion formula for the cosine Fourier transform we derive

$$
\begin{aligned}
\lim_{t\to 0}\langle h_t^\epsilon(x,\cdot),\phi(\cdot)\rangle &= \frac{1}{\pi\sqrt{2}}\int_{\mathbb{R}}\mathcal{F}_c[\phi,\sinh(u)]\int_{\mathbb{R}} e^{-u^2}\cos(x\sinh(y))\cosh(y)\,du\,dy \\
&= \sqrt{\frac{2}{\pi}}\int_{\mathbb{R}_+}\cos(x\lambda)\mathcal{F}_c[\phi,\lambda]\,d\lambda = \phi(x),
\end{aligned}
$$

which yields (26). $\square$

Theorem 3.4. *The Kontorovich-Lebedev transform (1) by x of the kernel $h_t^\epsilon(x,y)$ is*

$$
\mathcal{K}_{i\tau}[h_t^\epsilon] = \int_{\mathbb{R}_+} K_{i\tau}(x)h_t^\epsilon(x,y)\,dx = e^{-\mathbf{k}_\epsilon t\tau^2}K_{i\tau}(y),
\tag{30}
$$

and by y

$$
\mathcal{K}_{i\tau}\left[\frac{h_t^\epsilon}{y}\right] = \int_{\mathbb{R}_+} K_{i\tau}(y)h_t^\epsilon(x,y)\,\frac{dy}{y} = e^{-\mathbf{k}_\epsilon t\tau^2}\frac{K_{i\tau}(x)}{x}.
\tag{31}
$$

Proof. First we easily observe that for fixed positive t, x, y the right-hand side of (30) and (31) belong to the space $L_2(\mathbb{R}_+, \tau \sinh(\pi\tau)\, d\tau)$. Therefore, by reciprocities (1) and (3) we can guarantee the validity of (30) and (31) with the convergence of integrals in L_2-sense. Nevertheless, via estimate (17) and the asymptotic behavior of the modified Bessel function (see expressions (8), (9) and (10)) we verify that integral (31) converges absolutely to the same limit. In order to examine the absolute convergence of integral (30) we will use (31) and the asymptotic behavior of $h_t^\epsilon(x, y)$ when $x \to 0$. To obtain, in turn, this asymptotic expansion, we start rewriting formula (16) in an equivalent form. In fact, taking into account relation (7) for modified Bessel functions and the parity of the integral, we get

$$h_t^\epsilon(x,y) = \frac{1}{xi\pi} \int_{\mathbb{R}} e^{-\mathbf{k}_\epsilon t\tau^2} \tau I_{-i\tau}(x) K_{i\tau}(y)\, dy, \tag{32}$$

where $I_\nu(z)$ is the modified Bessel function of the first kind represented by [1]

$$I_\nu(z) = \sum_{k=0}^{+\infty} \frac{\left(\frac{z}{2}\right)^{2k+\nu}}{k!\ \Gamma(k+\nu+1)}.$$

Substituting this series into (2.12), we have for $x \to 0$

$$h_t^\epsilon(x,y) \quad = \quad \frac{1}{x\pi i} \int_{\mathbb{R}} e^{-\mathbf{k}_\epsilon t\tau^2} \tau \frac{e^{-i\tau \log\left(\frac{x}{2}\right)}}{\Gamma(1-i\tau)} + \frac{1}{2\pi i} \int_{\mathbb{R}} e^{-\mathbf{k}_\epsilon t\tau^2} \tau \sum_{j=1}^{+\infty} \frac{\left(\frac{x}{2}\right)^{2j-i\tau-1}}{j!\ \Gamma(j-i\tau+1)} \tag{33}$$

In the meantime, by straightforward estimates we see that second term in (33) is $O(x)$, when $x \to 0$ and $t, y > 0$ are fixed. Considering the first term, we integrate by parts, eliminating the corresponding boundary terms. So, repeating this procedure n times we get

$$\frac{1}{x\pi i} \int_{\mathbb{R}} e^{-\mathbf{k}_\epsilon t\tau^2} \tau K_{i\tau}(y) \frac{e^{-i\tau \log\left(\frac{x}{2}\right)}}{\Gamma(1-i\tau)}\, d\tau = O\left(\frac{1}{x \log^n x}\right), \quad x \to 0,\ n \in \mathbb{N}.$$

Therefore, via inequality $|K_{i\tau}(x)| \leq K_0(x)$ and asymptotic formula (10) we take $n = 3, 4, \ldots$ and establish the absolute convergence of integral (30) to the same limit. $\square$

4 Regularized Weierstrass's type transform

The aim of this section is to investigate mapping properties and prove an inversion formula for Weierstrass's type integral operator, which is defined via the regularized Schrödinger kernel presented in the previous section.

Let us consider the following integral

$$(g_t^\epsilon f)(x) = \int_{\mathbb{R}_+} h_t^\epsilon(y,x) f(y)\ dy, \tag{34}$$

which we will call the regularized Weierstrass type transform. When $f \in S_2(\mathbb{R}_+)$ it convergent absolutely via Schwarz's inequality. Moreover, by Theorems 3.4 and 2.2 we find

$$\mathcal{K}_{i\tau}\left[\frac{(g_t^\epsilon f)(x)}{x}\right] = e^{-\mathbf{k}_\epsilon t\tau^2}\mathcal{K}_{i\tau}\left[\frac{f(x)}{x}\right] \tag{35}$$

and therefore, we can denote kernel (16) by the operator $e^{-\mathbf{k}_\epsilon t\mathcal{A}}$. Furthermore, it has $(g_t^\epsilon f)(x) = e^{-\mathbf{k}_\epsilon t\mathcal{A}}f$. So the action of $\mathcal{A}$ on the Kontorovich-Lebedev transform corresponds to a multiplication by τ^2, while the regularized Schrödinger kernel is a multiplication by $e^{-\mathbf{k}_\epsilon t\tau^2}$. Further, from Theorem 3.3 and Schwarz's inequality it follows that $(g_t^\epsilon f)(x)$ is an infinitely differentiable function of $x, t > 0$. It satisfies the regularized Schrödinger type equation

$$\frac{\partial(g_t^\epsilon)(x)}{\partial t} = -\mathbf{k}_\epsilon \mathcal{A}_x g_t^\epsilon f$$

for $t > 0$ with the initial condition

$$\lim_{t\to 0}(g_t^\epsilon f)(x) = f(x)$$

where $f \in L_2\left(\mathbb{R}_+, \frac{dx}{x}\right)$.

Theorem 4.1. *For each $t, \epsilon > 0$ integral transformation (34) is a bounded operator in $S_2(\mathbb{R}_+)$ and the following estimate holds*

$$||g_t^\epsilon||_{S_2(\mathbb{R}_+)} \leq ||f||_{S_2(\mathbb{R}_+)}.$$

Proof. Indeed, taking into account (2) and Theorem 3.4 we have

$$\begin{aligned}(g_t^\epsilon f)(x) &= \frac{2}{\pi^2}\int_{\mathbb{R}_+} \tau\sinh(\pi\tau)\mathcal{K}_{i\tau}[h_t^\epsilon(\cdot,x)]\mathcal{K}_{i\tau}\left[\frac{f(\cdot)}{\cdot}\right]\ d\tau \\ &= \frac{2}{\pi^2}\int_{\mathbb{R}_+} \tau\sinh(\pi\tau)e^{-\mathbf{k}_\epsilon t\tau^2}K_{i\tau}(x)\mathcal{K}_{i\tau}\left[\frac{f(\cdot)}{\cdot}\right]\ d\tau.\end{aligned}$$

Hence we observe immediately, that the latter integral and its derivatives are uniformly convergent for $x \geq x_0 > 0$. Using (5), we deduced

$$\mathcal{A}_x(g_t^\epsilon f) = \frac{2}{\pi^2}\int_{\mathbb{R}_+} \tau^3\sinh(\pi\tau)e^{-\mathbf{k}_\epsilon t\tau^2}K_{i\tau}(x)\mathcal{K}_{i\tau}\left[\frac{f(\cdot)}{\cdot}\right]\ d\tau,$$

and

$$
\begin{aligned}
||\mathcal{A}_x(g_t^\epsilon f)||^2_{L_2\left(\mathbb{R},\frac{dx}{x}\right)} &= \frac{2}{\pi^2}\int_{\mathbb{R}_+} \tau\sinh(\pi\tau)e^{-2\mathrm{Re}(\mathbf{k}_\epsilon)t\tau^2}\left|\mathcal{K}_{i\tau}\left[\frac{\mathcal{A}.f}{\cdot}\right]\right|^2 d\tau \\
&\le ||f||^2_{S_2(\mathbb{R}_+)} < \infty.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
||g_t^\epsilon f||^2_{L_2\left(\mathbb{R},\frac{dx}{x}\right)} &= \frac{2}{\pi^2}\int_{\mathbb{R}_+} \tau\sinh(\pi\tau)e^{-2\mathrm{Re}(\mathbf{k}_\epsilon)t\tau^2}\left|\mathcal{K}_{i\tau}\left[\frac{f(\cdot)}{\cdot}\right]\right|^2 d\tau \\
&\le ||f||^2_{S_2(\mathbb{R}_+)} < \infty.
\end{aligned}
$$

Therefore, by Theorem 2.2 we find

$$
||g_t^\epsilon f||^2_{L_2\left(\mathbb{R},\frac{dx}{x}\right)} = \frac{2}{\pi^2}\int_{\mathbb{R}_+} \tau\sinh(\pi\tau)e^{-2\mathrm{Re}(\mathbf{k}_\epsilon)t\tau^2}\left|\mathcal{K}_{i\tau}\left[\frac{f(\cdot)}{\cdot}\right]\right|^2 (1+\tau^4)\ d\tau \le ||f||^2_{S_2(\mathbb{R}_+)}.
$$

□

The next theorem will deal with an inversion formula for integral transformation (34).

Theorem 4.2. *Let* $t,\epsilon>0, f\in S_2(\mathbb{R}_+)$ *and* $e^{t\tau^2}\mathcal{K}_{i\tau}\left[\frac{(g_t^\epsilon f)(x)}{x}\right]\in L_2(\mathbb{R}_+,\tau\sinh(\pi\tau)\ d\tau)$. *Then for almost all* $x>0$ *an inversion formula*

$$
f(x) = e^{\mathbf{k}_\epsilon t\mathcal{A}_x} g_t^\epsilon \tag{36}
$$

holds.

Proof. Indeed, from (35) we derive

$$
\mathcal{K}_{i\tau}\left[\frac{f(x)}{x}\right] = e^{\mathbf{k}_\epsilon t\tau^2}\mathcal{K}_{i\tau}\left[\frac{(g_t^\epsilon f)(x)}{x}\right],\quad x>0, \tag{37}
$$

where the right -hand side of the last expression is from $L_2(\mathbb{R}_+,\tau\sinh(\pi\tau)\ d\tau)$ via conditions of the theorem. Then

$$
\tau^{2k}\mathcal{K}_{i\tau}\left[\frac{(g_t^\epsilon)(x)}{x}\right]\in L_2(\mathbb{R}_+,\tau\sinh(\pi\tau)\ d\tau),
$$

for any $k\in\mathbb{N}_0$. Taking into account Theorem 2.3 we write

$$
\begin{aligned}
F_n^\epsilon(t,\tau) &= \sum_{m=0}^{n}\frac{(\mathbf{k}_\epsilon t)^m}{m!}\tau^{2m}\mathcal{K}_{i\tau}\left[\frac{(g_t^\epsilon f)(x)}{x}\right] \\
&= \sum_{m=0}^{n}\frac{(\mathbf{k}_\epsilon t)^m}{m!}\mathcal{K}_{i\tau}\left[\frac{\mathcal{A}_x^m(g_t^\epsilon f)}{x}\right] \\
&= \mathcal{K}_{i\tau}\left[\frac{P_n(\mathbf{k}_\epsilon t\mathcal{A}_x)(g_t^\epsilon f)}{x}\right],
\end{aligned}
$$

where $P_n(z)$ is the nth Taylor polynomial of the exponential function e^z. From the inversion formula of the Kontorovich-Lebedev transform (3) we get

$$P_n(\mathbf{k}_\epsilon t\mathcal{A}_x)(g_t^\epsilon f)(x) = \frac{2}{\pi^2}\int_{\mathbb{R}_+} \tau\sinh(\pi\tau)K_{i\tau}(x)P_n(\mathbf{k}_\epsilon t\tau^2)\mathcal{K}_{i\tau}\left[\frac{(g_t^\epsilon f)(x)}{x}\right]\, d\tau, \quad n\in\mathbb{N}_0,$$

where the last integral converges absolutely for any n since

$$e^{t\tau^2}\mathcal{K}_{i\tau}\left[\frac{(g_t^\epsilon f)(x)}{x}\right] \in L_2(\mathbb{R}_+, \tau\sinh(\pi\tau)\, d\tau).$$

On the other hand we have from (37)

$$f(x) = \lim_{T\to+\infty}\frac{2}{\pi^2}\int_0^T \tau\sinh(\pi\tau)K_{i\tau}(x)e^{\mathbf{k}_\epsilon t\tau^2}\mathcal{K}_{i\tau}\left[\frac{(g_t^\epsilon f)(\cdot)}{\cdot}\right]\, d\tau$$

and Parseval equality (2) yields

$$\int_{\mathbb{R}_+}|f(x)-P_n(\mathbf{k}_\epsilon t\mathcal{A}_x)(g_t^\epsilon f)|^2\frac{dx}{x}$$
$$= \frac{2}{\pi^2}\int_{\mathbb{R}_+}\tau\sinh(\pi\tau)\left|e^{\mathbf{k}_\epsilon t\tau^2}-P_n(\mathbf{k}_\epsilon t\tau^2)\right|^2\left|\mathcal{K}_{i\tau}\left[\frac{(g_t^\epsilon f)(\cdot)}{\cdot}\right]\right|^2 d\tau. \tag{38}$$

Since $\left|e^{\mathbf{k}_\epsilon t\tau^2}-P_n(\mathbf{k}_\epsilon t\tau^2)\right| < 2e^{t\tau^2}$ and tends to zero when $n\to\infty$ for each $t,\tau,\epsilon>0$, by Lebesgue dominated convergence theorem we obtain that the left hand-side of the latter equality vanishes as well. Therefore with a convergence by the norm in $L_2\left(\mathbb{R}_+, \frac{dx}{x}\right)$ we arrive at the inversion formula of g_t^ϵ which can be written in the symbolic form (36). However, since (38) is true for some subsequence P_{n_k} when the convergence is pointwise, we have equality (36) for almost all $x>0$. $\square$

5 The limit case $\epsilon\to 0^+$

Here we will deal with the behavior of the regularized Schrödinger kernel $h_t^\epsilon(x,y)$ and Weierstrass's type transform $(g_t^\epsilon f)(x)$ when the parameter ϵ goes to zero.

Lemma 5.1. *The following integral*

$$\int_T^\infty\left(e^{-\mathbf{k}_\epsilon t\tau^2}-e^{-it\tau^2}\right)\tau\sinh(\pi\tau)K_{i\tau}(x)K_{i\tau}(y)\, d\tau, \quad x,y>0, \tag{39}$$

where $T>0$ is fixed and sufficiently big, is uniformly convergent by $\epsilon\in[0,\epsilon_0]$.

Proof. Taking into account asymptotic expansion (11) we have when $\tau \to \infty$

$$
\begin{aligned}
&K_{i\tau}(x)K_{i\tau}(y) \\
&= \frac{\pi}{\tau}e^{-\pi\tau}\left[\cos\left(\tau\log\left(\frac{y}{x}\right)\right) + \sin\left(\tau\left(\log\left(\frac{4\tau^2}{xy}\right) - 2\right) + \frac{x^2}{2\tau}\right)\right]\left(1 + O\left(\frac{1}{\tau}\right)\right).
\end{aligned}
$$

Hence integral (39) becomes

$$
\begin{aligned}
&O\left(\int_T^{+\infty}\left(e^{-\mathbf{k}_\epsilon t\tau^2} - e^{-it\tau^2}\right)\left[\cos\left(\tau\log\left(\frac{y}{x}\right)\right) + \sin\left(\tau\left(\log\left(\frac{4\tau^2}{xy}\right) - 2\right)\right)\right]\ d\tau\right) \\
&+ O\left(\int_T^{+\infty}\left(e^{-\mathbf{k}_\epsilon t\tau^2} - e^{-it\tau^2}\right)\left[\cos\left(\tau\log\left(\frac{y}{x}\right)\right) + \sin\left(\tau\left(\log\left(\frac{4\tau^2}{xy}\right) - 2\right)\right)\right]\ \frac{d\tau}{\tau}\right) \\
&= \underbrace{O\left(\int_T^{+\infty} e^{-\mathbf{k}_\epsilon t\tau^2}\ \cos\left(\tau\log\left(\frac{y}{x}\right)\right)\ d\tau\right)}_{\mathbf{I}} + \underbrace{O\left(\int_T^{+\infty} e^{-\mathbf{k}_\epsilon t\tau^2}\ \sin\left(\tau\log\left(\frac{4\tau^2}{xy}\right) - 2\tau\right)\ d\tau\right)}_{\mathbf{II}} \\
&+ \underbrace{O\left(\int_T^{+\infty} e^{-it\tau^2}\ \cos\left(\tau\log\left(\frac{y}{x}\right)\right)\ d\tau\right)}_{\mathbf{III}} + \underbrace{O\left(\int_T^{+\infty} e^{-it\tau^2}\ \sin\left(\tau\left(\log\left(\frac{4\tau^2}{xy}\right) - 2\right)\right)\ d\tau\right)}_{\mathbf{IV}} \\
&+ \underbrace{O\left(\int_T^{+\infty} e^{-\mathbf{k}_\epsilon t\tau^2}\ \cos\left(\tau\log\left(\frac{y}{x}\right)\right)\ \frac{d\tau}{\tau}\right)}_{\mathbf{V}} + \underbrace{O\left(\int_T^{+\infty} e^{-\mathbf{k}_\epsilon t\tau^2}\ \sin\left(\tau\log\left(\frac{4\tau^2}{xy}\right) - 2\tau\right)\ \frac{d\tau}{\tau}\right)}_{\mathbf{VI}} \\
&+ \underbrace{O\left(\int_T^{+\infty} e^{-it\tau^2}\ \cos\left(\tau\log\left(\frac{y}{x}\right)\right)\ \frac{d\tau}{\tau}\right)}_{\mathbf{VII}} + \underbrace{O\left(\int_T^{+\infty} e^{-it\tau^2}\ \sin\left(\tau\left(\log\left(\frac{4\tau^2}{xy}\right) - 2\right)\right)\ \frac{d\tau}{\tau}\right)}_{\mathbf{VIII}}.
\end{aligned} \tag{40}
$$

We examine the convergence of integral **I** (in a similar way we proceed to integrals **II**, **III**, **IV**, **V**, **VI**, **VII**, **VIII**). Precisely, making the substitution $\tau^2 = v$ and integrating by parts, one gets

$$
\begin{aligned}
&\frac{1}{2}\int_{T^2}^{+\infty} e^{-\mathbf{k}_\epsilon tv}\ \cos\left(\sqrt{v}\log\left(\frac{y}{x}\right)\right)\ \frac{dv}{\sqrt{v}} = \frac{e^{-\mathbf{k}_\epsilon tT^2}}{2t\mathbf{k}_\epsilon T}\cos\left(T\log\left(\frac{y}{x}\right)\right) \\
&\quad - \frac{\log\left(\frac{y}{x}\right)}{4\mathbf{k}_\epsilon t}\underbrace{\int_{T^2}^{+\infty} e^{-\mathbf{k}_\epsilon tv}\sin\left(\sqrt{v}\log\left(\frac{y}{x}\right)\right)\ \frac{dv}{v}}_{\mathbf{IX}} - \frac{1}{4\mathbf{k}_\epsilon t}\underbrace{\int_{T^2}^{+\infty} e^{-\mathbf{k}_\epsilon tv}\cos\left(\sqrt{v}\log\left(\frac{y}{x}\right)\right)\ \frac{dv}{v^{\frac{3}{2}}}}_{\mathbf{X}}.
\end{aligned} \tag{41}
$$

Regarding integral **X** one, we can evidently guarantee its absolutely and uniformly convergence by $\epsilon \in [0, \epsilon_0]$. Concerning integral **IX** after integration by parts again we find

$$\int_{T^2}^{+\infty} e^{-\mathbf{k}_\epsilon tv} \sin\left(\sqrt{v}\log\left(\frac{y}{x}\right)\right) \frac{dv}{4v} = -\frac{e^{-\mathbf{k}_\epsilon tT^2}}{4t\mathbf{k}_\epsilon T^2} \sin\left(T\log\left(\frac{y}{x}\right)\right)$$
$$+ \frac{1}{8t\mathbf{k}_\epsilon} \underbrace{\int_{T^2}^{+\infty} e^{-\mathbf{k}_\epsilon tv} \cos\left(\sqrt{v}\log\left(\frac{y}{x}\right)\right) \frac{dv}{v^{\frac{3}{2}}}}_{\mathbf{XI}} - \frac{1}{4t\mathbf{k}_\epsilon} \underbrace{\int_{T^2}^{+\infty} e^{-\mathbf{k}_\epsilon tv} \sin\left(\sqrt{v}\log\left(\frac{y}{x}\right)\right) \frac{dv}{v^2}}_{\mathbf{XII}}.$$

Since the latter integrals **XI** and **XII** converge absolutely and uniformly convergent by $\epsilon \in [0, \epsilon_0]$ and $|\mathbf{k}_\epsilon| = 1$ we establish in the same manner uniform convergence of other mentioned integrals and complete the proof of lemma 5.1. □

We will show now that the regularized Schrödinger kernel $h_t^\epsilon(x,y)$ converges weakly in $\mathcal{D}'(\mathbb{R}_+)$.

Theorem 5.2. *For each $x, t > 0$ it has*

$$(g_t^\epsilon\ \phi)(x) \to (g_t\ \phi)(x), \quad \phi \in \mathcal{D}(\mathbb{R}_+),$$

when $\epsilon \to 0^+$, where $(g_t\ \phi)(x)$ is the Weierstrass type transform associated with the kernel $h_t(x,y)$ (see (15)*).*

Proof. In fact,

$$|(g_t^\epsilon\ \phi)(x) - (g_t\ \phi)(x)| = \left|\int_{\mathrm{supp}\phi} \left(h_t^\epsilon(y,x) - h(y,x)\right)\phi(y)\ dy\right|$$
$$= \left|\int_{\mathrm{supp}\phi} \frac{2}{y\pi^2} \int_{\mathbb{R}_+} \left(e^{-\mathbf{k}_\epsilon t\tau^2} - e^{-it\tau^2}\right) \tau\sinh(\pi\tau) K_{i\tau}(x) K_{i\tau}(y)\ d\tau\ \phi(y)\ dy\right|. \quad (42)$$

Fixing a big $T > 0$, one can divide the previous integral in two integrals

$$\left|\int_{\mathrm{supp}\phi} \frac{2}{y\pi^2} \int_{\mathbb{R}_+} \left(e^{-\mathbf{k}_\epsilon t\tau^2} - e^{-it\tau^2}\right) \tau\sinh(\pi\tau) K_{i\tau}(x) K_{i\tau}(y)\ d\tau\ \phi(y)\ dy\right|$$
$$= \left|\int_{\mathrm{supp}\phi} \frac{2}{y\pi^2} \left[\underbrace{\int_0^T \left(e^{-\mathbf{k}_\epsilon t\tau^2} - e^{-it\tau^2}\right) \tau\sinh(\pi\tau) K_{i\tau}(x) K_{i\tau}(y)\ d\tau}_{\mathbf{IX}}\right.\right.$$
$$\left.\left. + \underbrace{\int_T^{+\infty} \left(e^{-\mathbf{k}_\epsilon t\tau^2} - e^{-it\tau^2}\right) \tau\sinh(\pi\tau) K_{i\tau}(x) K_{i\tau}(y)\ d\tau}_{\mathbf{X}}\right] \phi(y)\ dy\right|.$$

But integral **IX** is uniformly convergent by $\epsilon \in [0, \epsilon_0]$ as a proper integral. On the other side, Lemma 5.1 yields that integral **X** converges uniformly as well on this interval. Consequently, one can pass to the limit when ϵ tends to zero under the integral sign, proving the theorem. $\square$

Finally, we establish a pointwise convergence of a family of Weierstrass's transforms of integrable functions. We have

$$(g_t f)(x) = \int_{\mathbb{R}_+} h_t(x,y) f(y)\, dy; \quad f \in L_1\left(\mathbb{R}_+, y^{-\frac{\pi}{t_0}}\, dy\right), \tag{43}$$

where $t_0 > 0$, is given by

Theorem 5.3. *Let $f \in L_1\left(\mathbb{R}_+, y^{-\frac{\pi}{t_0}} dy\right)$, $t_0 > 0$. Then $(g_t^\epsilon f)(x)$ converges pointwisely to $(g_t f)(x)$, when $\epsilon \to 0$.*

Proof. Indeed, taking into account relations (2.16.52.8) and (2.5.57.1) in [6] [5], respectively, and Parseval's equality for sine Fourier transform after making an elementary substitution we write the kernel $h_t^\epsilon(x,y)$ in the form

$$\begin{aligned} h_t^\epsilon(x,y) &= \frac{y e^{\frac{\pi^2}{4\mathbf{k}_\epsilon t}}}{\pi\sqrt{\pi \mathbf{k}_\epsilon t}} \int_1^\infty e^{-(\operatorname{arccosh}^2(u)/(4\mathbf{k}_\epsilon t))} \frac{K_1\left(\sqrt{x^2+y^2+2xyu}\right)}{\sqrt{x^2+y^2+2xyu}} \\ &\times \frac{\left(u+\sqrt{u^2-1}\right)^{\frac{i\pi}{\mathbf{k}_\epsilon t}} - 1}{\left(u+\sqrt{u^2-1}\right)^{\frac{i\pi}{2\mathbf{k}_\epsilon t}}}\, du. \end{aligned} \tag{44}$$

Hence

$$\begin{aligned} |h_t^\epsilon(x,y)| &\le \frac{2y e^{\frac{\pi^2}{4t_0}}}{\pi\sqrt{\pi t_0}} \int_1^\infty \frac{K_1\left(\sqrt{x^2+y^2+2xyu}\right)}{\sqrt{x^2+y^2+2xyu}} \left(u+\sqrt{u^2-1}\right)^{\pi/t_0} du \\ &\le \frac{e^{\frac{\pi^2}{4t_0}}}{x\pi(\pi t_0)^{1/2}(xy)^{\pi/t_0}} \int_0^\infty K_1\left(\sqrt{u}\right) u^{\frac{\pi}{t_0}-\frac{1}{2}}\, du. \end{aligned}$$

Taking into consideration relation (2.16.2.2) in [6] we obtain the uniform estimate with respect to $\epsilon \in [0, \epsilon_0]$

$$|h_t^\epsilon(x,y)| \le \frac{2^{\frac{2\pi}{t_0}} e^{\frac{\pi^2}{4t_0}}}{x\pi(\pi t_0)^{1/2}(xy)^{\pi/t_0}} \Gamma\left(\frac{\pi}{t_0}+1\right)\Gamma\left(\frac{\pi}{t_0}\right). \tag{45}$$

Consequently, for all $x > 0$

$$\left|\int_{\mathbb{R}_+} h_t^\epsilon(x,y) f(y) dy\right| \le \int_{\mathbb{R}_+} |h_t^\epsilon(x,y) f(y)|\, dy < \frac{C}{x^{\pi/t_0+1}} \int_{\mathbb{R}_+} |f(y)| y^{-\frac{\pi}{t_0}}\, dy < \infty.$$

Hence with the aid of the Lebesgue dominated convergence theorem and Lemma 5.1 we immediately complete the proof of Theorem 5.3. $\square$

References

[1] Abramowitz, M. and Stegun, I.A., *Handbook of Mathematical Functions*, Dover, New York, 1972.

[2] Cerejeiras, P. and Vieira, N., *Regularization of the non-stationary Schrödinger operator*, Math. Meth. in Appl. Sc., **32** No.4, (2009), 535-555.

[3] Kraußhar, R.S. and Vieira, N., *The Schrödinger equation on cylinders and the n-torus*, J. Evol. Equ., On-line First, DOI: 10.1007/s00028-010-0089-4.

[4] Lieb, E.H. and Loss, M., *Analysis*, Graduate Studies in Math. - Vol 14, American Math. Soc., Providence, Rhode Island, 2001.

[5] Prudnikov, A.P., Brychkov, Yu.A. and Marichev, O.I., *Integrals and Series, Volume 1: Elementary Functions*, Gordon and Breach, 1986.

[6] Prudnikov, A.P., Brychkov, Yu.A. and Marichev, O.I., *Integrals and Series, Volume 2: Special Functions*, Gordon and Breach, 1986.

[7] Sneddon, I.N., *The use of integral transforms*, McGraw-Hill, New York, 1972.

[8] Tao, T., *Nonlinear dispersive equations. Local and global analysis*, CBMS Regional Conference Series in Mathematics, **Vol. 106**, American mathematical Society: Providence, RI, 2006.

[9] Yakubovich, S., *The heat kernel and Heisenberg inequalities related to the Kontorovich-Lebedev transform*, Commun. Pure. Appl. Anal., **10** No.2, (2011), 745-760.

[10] Yakubovich, S., *A class of polynomials and discrete transformations associated with the Kontorovich-Lebedev operators*, Integral Transforms and Special Functions, **20** No.7, (2009), 551-567.

[11] Yakubovich, S., *On a testing-function space for distributions associated with the Kontorovich-Lebedev transform*, Collect. Math., **57** No.3, (2006), 279-293.

[12] Yakubovich, S., *The Kontorovich-Lebedev transformation on Sobolev type spaces*, Sarajevo J. of Mathematics, **1** No. 14, (2005), 211-234.

[13] Yakubovich, S., *Index transforms*, World Scientific Publishing Company, Singapore, New Yersey, London and Hong Kong, 1996.

[14] Yakubovich, S. and Luchko, Yu.F., *The hypergeometric approach to integral transforms and convolutions*, Kluwers Ser. Math. and Appl.: **Vol. 287**, Dordrecht, Boston, London, 1994.

[15] Zemanian, A.H., *Generalized Integral Transform*, McGray Hill: New York, 1965.