Secondary Bifurcations in Systems with All-to-All Coupling. Part II.

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Abstract

In a recent paper Dias and Stewart (Secondary Bifurcations in Systems with All-to-All Coupling, *Proc. R. Soc. Lond. A* (2003) **459**, 1969-1986.) studied the existence, branching geometry, and stability of secondary branches of equilibria in all-to-all coupled systems of differential equations, that is, equations that are equivariant under the permutation action of the symmetric group $\mathbf{S}_N$. They consider the general cubic order truncation system of this type. Primary branches in such systems correspond to partitions of N into two parts p, q with $p+q=N$. Secondary branches correspond to partitions of N into three parts $a,\ b,\ c$ with $a+b+c=N$. They prove that when all of the a,b,c are different from $N/3$ secondary branches exist, and are (generically) globally unstable in the cubic-order system. In this work they realized that the cubic order system is too degenerate to provide secondary branches if $a=b=c$. In this paper we prove the existence and the branching geometry of secondary branches of equilibria with $\mathbf{S}_n \times \mathbf{S}_n \times \mathbf{S}_n$ symmetry, in systems of ordinary differential equations that commute with the permutation action of the symmetric group $\mathbf{S}_{3n}$ (action on $\mathbf{R}^{3n}$). Moreover, we prove that the solutions of the secondary branch are (generically) globally unstable in the fifth-order truncation of the system.

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1 Introduction

The original motivation for this work came from evolutionary biology. Cohen and Stewart [1] introduced a system of $\mathbf{S}_N$-equivariant ordinary differential equations (ODEs) that models sympatric speciation as a form of spontaneous symmetry-breaking in a system with $\mathbf{S}_N$-symmetry. Elmhirst [3, 4] studied the stability of the primary branches in such a model and also linked it to a biological specific model of speciation. Stewart *et al.* [7] made numerical studies of relatively concrete models. Here the population is aggregated into N discrete 'cells', with a vector x_j representing values of

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some phenotypic observable - the phenotype - the organisms form and behavior. If the initial population is monomorphic (single-species) then the system of ODEs representing the time-evolution of the phenotypes should be equivariant under the action of the symmetric group $\mathbf{S}_N$; that is, the model is an example of an *all-to-all coupled system.* Symmetry-breaking bifurcations of the system correspond to the splitting of the population into two or more distinct morphs (species).

Dias and Stewart [2] continue the study of the general cubic truncation of a centre manifold reduction of a system of that type, which takes the form

$$\frac{dx_i}{dt} = \lambda x_i + B(Nx_i^2 - \pi_2) + C(Nx_i^3 - \pi_3) + Dx_i\pi_2 \tag{1.1}$$

for $i = 1, \ldots, N$. Here $\lambda, B, C, D \in \mathbf{R}$ are parameters, $x_i \in \mathbf{R}$ for all i, the coordinates satisfy $x_1 + \cdots + x_N \equiv 0$, and

$$\pi_2 = x_1^2 + \cdots + x_N^2, \quad \pi_3 = x_1^3 + \cdots + x_N^3$$

Their study was motivated by numerical simulations showing jump bifurcations between primary branches. These jumps correspond to the loss of stability of the primary branches, see Stewart *et al.* [7]. Primary branches in such systems correspond to partitions of N into two parts p, q with $p + q = N$. Secondary branches correspond to partitions of N into three parts a, b, c with $a + b + c = N$. They remarked that the cubic-order system (1.1) is too degenerate to provide secondary branches in the case $a = b = c$. We focus our work in this case.

In this paper we study the general fifth order truncation of a centre-manifold reduction of a $\mathbf{S}_N$-equivariant system, which takes the form

$$\frac{dx}{dt} = G(x, \lambda) \tag{1.2}$$

where

$$\begin{aligned} G_i(x,\lambda) = \quad & \lambda x_i + B(Nx_i^2 - \pi_2) + C(Nx_i^3 - \pi_3) + Dx_i\pi_2 + \\ & E(Nx_i^4 - \pi_4) + F(Nx_i^2\pi_2 - \pi_2^2) + Gx_i\pi_3 + \\ & H(Nx_i^5 - \pi_5) + I(Nx_i^3\pi_2 - \pi_3\pi_2) + J(Nx_i^2\pi_3 - \pi_3\pi_2) + Lx_i\pi_4 + Mx_i\pi_2^2 \end{aligned} \tag{1.3}$$

for $i = 1, \ldots, N$. Here $\lambda, B, C, D, E, F, G, H, I, J, L, M \in \mathbf{R}$ are parameters, $x_i \in \mathbf{R}$ for all i, and as before the coordinates satisfy $x_1 + \cdots + x_N \equiv 0$. Also

$$\pi_i = x_1^i + \cdots + x_N^i$$

for $i = 2, \ldots, 5$.

The aim of this paper is to study the existence, branching geometry, and stability of secondary branches of equilibria with $\mathbf{S}_a \times \mathbf{S}_a \times \mathbf{S}_a$ symmetry of the system (1.2) where G is defined by (1.3). Thus we consider $N = 3a$.

In Section 2 we begin by reviewing some concepts related to equivariant bifurcation theory, and some results related to general $\mathbf{S}_N$-symmetric systems. In particular, we obtain the general fifth order truncation (1.2) of a general smooth $\mathbf{S}_N$-equivariant vector field posed on the $\mathbf{S}_N$-absolutely irreducible space

$$V_1 = \{x \in \mathbf{R}^n : x_1 + \cdots + x_N = 0\}$$

In section 3 we suppose $N = 3a$ and we look for secondary branches of steady-state solutions for the system (1.2) that are $\mathbf{S}_a \times \mathbf{S}_a \times \mathbf{S}_a$-symmetric obtained by bifurcation from a primary branch (with isotropy group (conjugate to) $\mathbf{S}_a \times \mathbf{S}_{2a}$). We prove in Theorem 3.1 the generic existence of a branch of solutions of (1.2) with that symmetry. Moreover, the existence of this branch of solutions does not depend on the fact that the vector field is truncated to the fifth order. Specifically, the restriction of a general $\mathbf{S}_{3a}$-equivariant system to the fixed-point subspace of $\Sigma = \mathbf{S}_a \times \mathbf{S}_a \times \mathbf{S}_a$ is $\mathbf{D}_3$-equivariant. The $\mathbf{D}_3$-singularity results of Golubitsky *et al.* [6] imply that the generic existence and stability of this branch of solutions depends only on certain nondegeneracy conditions on the coefficients of the fifth order truncation of a general smooth $\mathbf{S}_{3a}$-equivariant vector field posed on V_1 as above (where $N = 3a$). See Section 4. Furthermore, we describe in Theorem 4.3 the parameter regions of stability of the solutions with Σ-symmetry (in $\mathrm{Fix}(\Sigma)$). In Theorem 4.5 we prove that the solutions of the secondary branch are (generically) globally unstable for the fifth order truncation of the system.

2 Background

In this section we review some concepts related to equivariant bifurcation theory, and some results related to $\mathbf{S}_N$-symmetric systems. For a detailed discussion of the basics of equivariant bifurcation theory see Golubitsky *et al.* [6]. We summarise a few key points.

Consider a system of ODEs

$$\frac{dx}{dt} = G(x, \lambda) \tag{2.4}$$

where $x \in V = \mathbf{R}^N$, the vector field $G : V \times \mathbf{R} \to V$ is smooth, and $\lambda \in \mathbf{R}$ is a bifurcation parameter. Suppose that a compact Lie group Γ acts linearly (and without loss of generality orthogonally) on V. Recall that G *commutes* with the action of Γ (or it is Γ-*equivariant*) if

$$G(\gamma x, \lambda) = \gamma G(x, \lambda)$$

for all $\gamma \in \Gamma, x \in V$ and $\lambda \in \mathbf{R}$. Henceforth we assume G to be Γ-equivariant. The group

$$\Sigma_x = \{\gamma \in \Gamma \ : \ \gamma x = x\} \subseteq \Gamma$$

is the *isotropy subgroup* of $x \in V$. The *fixed-point space* of a subgroup $\Sigma \subseteq \Gamma$ is the subspace of V defined by

$$\mathrm{Fix}(\Sigma) = \{x \in V \ : \ \gamma x = x, \ \forall \, \gamma \in \Sigma\}$$

For any Γ-equivariant mapping G and any subgroup $\Sigma \subseteq \Gamma$ we have

$$G(\mathrm{Fix}(\Sigma) \times \mathbf{R}) \subseteq \mathrm{Fix}(\Sigma)$$

An isotropy subgroup of Γ is *axial* if it has a 1-dimensional fixed-point space. An equilibrium with axial isotropy is called an *axial equilibrium*, and a branch of axial equilibria is an *axial branch.*

A subspace $W \subseteq V$ is *absolutely irreducible* for Γ if the only matrices commuting with the action of Γ on W are the scalar multiples of the identity. Note that if W is absolutely irreducible for Γ then it is irreducible ([6] Lemma XII 3.3).

Under certain suitable genericity hypotheses for (2.4) steady-state bifurcation from a trivial equilibrium to axial equilibria for each axial subgroup of Γ is guaranteed by the Equivariant Branching Lemma of Vanderbauwhede and Cicogna ([6] Theorem XIII 3.3).

The Symmetric Group

Let the symmetric group $\Gamma = \mathbf{S}_N$ act on $V = \mathbf{R}^N$ by permutation of coordinates:

$$\rho(x_1, \ldots, x_N) = \left(x_{\rho^{-1}(1)}, \ldots, x_{\rho^{-1}(N)}\right) \quad (\rho \in \mathbf{S}_N)$$

The ring of the smooth $\mathbf{S}_N$-invariants over $\mathbf{R}$ is generated by

$$\pi_k = x_1^k + x_2^k + \cdots + x_N^k$$

where $k = 1, \ldots, N$. Denote by $[x_1^k] = [x_1^k, x_2^k, \ldots, x_N^k]^t$, for $k = 0, \ldots, N-1$. Then the module of the $\mathbf{S}_N$-equivariant smooth mappings from V to V are generated over the ring of the smooth $\mathbf{S}_N$-invariants by $[x_1^k]$ for $k = 0, \ldots, N-1$. For a detailed discussion see Golubitsky and Stewart [5] Chapter 1 Section 5.

In order to compute the isotropy subgroups Σ_x of $\mathbf{S}_N$ acting on $\mathbf{R}^N$, we partition $\{1, \ldots, N\}$ into disjoint blocks $B_1, \ldots, B_k$ with the property that $x_i = x_j$ if and only if i, j belong to the same block. Let $b_l = |B_l|$. Then

$$\Sigma_x = \mathbf{S}_{b_1} \times \cdots \times \mathbf{S}_{b_k}$$

where $\mathbf{S}_{b_l}$ is the symmetric group on the block B_l. Up to conjugacy, we may assume that

$$B_1 = \{1, \ldots, b_1\}, B_2 = \{b_1 + 1, \ldots, b_1 + b_2\}, \ldots, B_k = \{b_1 + b_2 + \cdots + b_{k-1} + 1, \ldots, N\}$$

where $b_1 \leq b_2 \leq \cdots \leq b_k$. Therefore, conjugacy classes of isotropy subgroups of $\mathbf{S}_N$ are in one-to-one correspondence with partitions of N into nonzero natural numbers arranged in ascending order. If Σ corresponds to a partition of N into k blocks, then dim Fix(Σ)= k.

We restrict the action of $\mathbf{S}_N$ onto the standard irreducible $\mathbf{R}^{N-1}$, that is,

$$V_1 = \{(x_1, x_2, \ldots, x_N) \in V : x_1 + x_2 + \cdots + x_N = 0\} \cong \mathbf{R}^{N-1}$$

Note that

$$V = \{(x, x, \ldots, x) : x \in \mathbf{R}\} \oplus V_1$$

where the action of $\mathbf{S}_N$ on $\{(x, x, \ldots, x) : x \in \mathbf{R}\}$ is trivial. Also the action of $\mathbf{S}_N$ on V_1 is absolutely irreducible.

The isotropy subgroups of $\mathbf{S}_N$ for the action on V_1 remain the same but the dimension of every fixed-point subspace is reduced by 1. In particular, the isotropy subgroups $\mathbf{S}_p \times \mathbf{S}_q$ where $p+q = N$ have one-dimensional fixed-point subspaces, that is, they are axial.

Suppose $G(x, \lambda)$ commutes with $\mathbf{S}_N$ on $\mathbf{R}^N \times \mathbf{R}$ where λ is the bifurcation parameter, and the Jacobian $(dG)_{(X_0, \lambda_0)}$ is singular where X_0 is a full symmetric equilibrium of (2.4). Then by the Equivariant Branching Lemma ([6] Theorem XIII 3.3), generically, there exist branches of equilibria of (2.4) bifurcating from X_0 at $\lambda = \lambda_0$ with isotropy subgroups $\mathbf{S}_p \times \mathbf{S}_q$. We call these the *primary branches.*

We obtain now the general form G of the center manifold reduction from the general $\mathbf{S}_N$-equivariant mapping G on V. As we have seen above, $G(x)$ has the following form:

$$G(x) = \sum_{k=0}^{N-1} p_k(\pi_1, \ldots, \pi_N)[x_1^k] \tag{2.5}$$

where p_k, for $k = 0, \ldots, N-1$, are smooth functions of the invariants π_k, for $k = 1, \ldots, N$.

From (2.5) we obtain the fifth order truncation of the Taylor expansion of G on V. By imposing the relation $\pi_1 = 0$ and then projecting the result onto V_1 we obtain (1.3). We show that this fifth order truncation captures the presence of a secondary branch of equilibria with symmetry $\mathbf{S}_a \times \mathbf{S}_a \times \mathbf{S}_a$ when $N = 3a$ and its stability.

3 Existence of Secondary Branches

Consider (1.2) where G is defined by (1.3) and suppose $N = 3a$ where a is a positive integer.

We look for secondary branches of equilibria of (1.2) with $\mathbf{S}_a \times \mathbf{S}_a \times \mathbf{S}_a$-symmetry. Any such secondary branch must live in the two-dimensional fixed-point subspace $\mathrm{Fix}(\mathbf{S}_a \times \mathbf{S}_a \times \mathbf{S}_a)$. Define

$$\Sigma_1 = \mathbf{S}_{\{1,\ldots,a\}} \times \mathbf{S}_{\{a+1\ldots,N\}}, \quad \Sigma_2 = \mathbf{S}_{\{1,\ldots,a,2a+1,\ldots,N\}} \times \mathbf{S}_{\{a+1,\ldots,2a\}}, \quad \Sigma_3 = \mathbf{S}_{\{1,\ldots,2a\}} \times \mathbf{S}_{\{2a+1,\ldots,N\}} \tag{3.6}$$

and

$$\Sigma = \mathbf{S}_a \times \mathbf{S}_a \times \mathbf{S}_a,$$

and let

$$\mathrm{Fix}(\Sigma) = \{(\underbrace{-x-y,\ldots}_{a};\underbrace{y,\ldots}_{a};\underbrace{x,\ldots}_{a}) : x, y \in \mathbf{R}\}$$

We restrict to $\mathrm{Fix}(\Sigma)$ the general $\mathbf{S}_N$-equivariant vector field G with components of degree less than or equal to five as in (1.3) obtaining the equations

$$\begin{cases} \dfrac{dx}{dt} = \lambda x + B(Nx^2 - \pi_2) + C(Nx^3 - \pi_3) + Dx\pi_2 + E(Nx^4 - \pi_4) + F(Nx^2\pi_2 - \pi_2^2) + \\ \qquad Gx\pi_3 + H(Nx^5 - \pi_5) + I(Nx^3\pi_2 - \pi_3\pi_2) + J(Nx^2\pi_3 - \pi_3\pi_2) + Lx\pi_4 + Mx\pi_2^2 \\ \\ \dfrac{dy}{dt} = \lambda y + B(Ny^2 - \pi_2) + C(Ny^3 - \pi_3) + Dy\pi_2 + E(Ny^4 - \pi_4) + F(Ny^2\pi_2 - \pi_2^2) + \\ \qquad Gy\pi_3 + H(Ny^5 - \pi_5) + I(Ny^3\pi_2 - \pi_3\pi_2) + J(Ny^2\pi_3 - \pi_3\pi_2) + Ly\pi_4 + My\pi_2^2 \end{cases} \tag{3.7}$$

where

$$\pi_i = a[(-x-y)^i + y^i + x^i)] \tag{3.8}$$

for $i = 2, 3, 4, 5$.

Note that

$$\begin{aligned} \mathrm{Fix}(\Sigma_1) &= \{(\underbrace{-2x,\ldots}_{a};\underbrace{x,\ldots;x,\ldots}_{2a}) : x \in \mathbf{R}\} \\ \mathrm{Fix}(\Sigma_2) &= \{(\underbrace{x,\ldots}_{a};\underbrace{-2x,\ldots}_{a};\underbrace{x,\ldots}_{a}) : x \in \mathbf{R}\} \\ \mathrm{Fix}(\Sigma_3) &= \{(\underbrace{-\tfrac{1}{2}x,\ldots,-\tfrac{1}{2}x}_{2a};\underbrace{x,\ldots,x}_{a}) : x \in \mathbf{R}\} \end{aligned} \tag{3.9}$$

Observe that $\Sigma_1, \Sigma_2, \Sigma_3$ are the isotropy subgroups of $\mathbf{S}_N$ containing Σ. Moreover, they are axial subgroups and so by the Equivariant Branching Lemma generically there exist branches of equilibria of (1.2) (and so of (3.7)) with isotropy subgroups $\Sigma_1, \Sigma_2, \Sigma_3$. The solutions of equations (3.7) with Σ_1-symmetry satisfy $y = x$; those with Σ_2-symmetry satisfy $y = -2x$, and finally those with Σ_3-symmetry satisfy $y = -x/2$.

Equations (3.7) are equivariant under the quotient group $N(\Sigma)/\Sigma$ where $N(\Sigma)$ is the normalizer of Σ in $\mathbf{S}_N$. Since $\Sigma = \mathbf{S}_a \times \mathbf{S}_a \times \mathbf{S}_a \subseteq \mathbf{S}_N$ we have that $N(\Sigma)/\Sigma \cong \mathbf{D}_3$ where $\mathbf{D}_3$ is the dihedral group of order 6.

Theorem 3.1 *Suppose that $N = 3a$ and $\Sigma = \mathbf{S}_a \times \mathbf{S}_a \times \mathbf{S}_a$. Then equations (3.7) have a branch of equilibria with symmetry Σ. This is described by*

$$\begin{cases} \lambda + BN(x+y) + CN(x^2+y^2+xy) + D\pi_2 + \\ EN(x^3+y^3+xy^2+yx^2) + FN\pi_2(x+y) + G\pi_3 + \\ HN(x^4+x^3y+xy^3+x^2y^2+y^4) + IN\pi_2(x^2+y^2+xy) + JN\pi_3(x+y) + L\pi_4 + M\pi_2^2 = 0, \\ \\ B + (2aF+E)(x^2+y^2+xy) - (H+3aJ)(x^2y+xy^2) = 0, \end{cases} \tag{3.10}$$

where

$$\pi_i = a[(-x-y)^i + y^i + x^i],$$

for $i = 2, 3, 4$.

Proof: We look for steady-state solutions of equations (3.7), that is, solutions of

$$\begin{cases} \lambda x + B(Nx^2 - \pi_2) + C(Nx^3 - \pi_3) + Dx\pi_2 + E(Nx^4 - \pi_4) + F(Nx^2\pi_2 - \pi_2^2) + Gx\pi_3 + \\ H(Nx^5 - \pi_5) + I(Nx^3\pi_2 - \pi_3\pi_2) + J(Nx^2\pi_3 - \pi_3\pi_2) + Lx\pi_4 + Mx\pi_2^2 = 0 \\ \\ \lambda y + B(Ny^2 - \pi_2) + C(Ny^3 - \pi_3) + Dy\pi_2 + E(Ny^4 - \pi_4) + F(Ny^2\pi_2 - \pi_2^2) + Gy\pi_3 + \\ H(Ny^5 - \pi_5) + I(Ny^3\pi_2 - \pi_3\pi_2) + J(Ny^2\pi_3 - \pi_3\pi_2) + Ly\pi_4 + My\pi_2^2 = 0 \end{cases} \tag{3.11}$$

We distinguish the following two cases:
(1) Equilibria with $x = 0$: from (3.11) we have

$$\begin{cases} B\pi_2 + E\pi_4 + F\pi_2^2 = 0 \\ \\ \lambda y + B(Ny^2 - \pi_2) + CNy^3 + Dy\pi_2 + E(Ny^4 - \pi_4) + F(Ny^2\pi_2 - \pi_2^2) + \\ HNy^5 + INy^3\pi_2 + Ly\pi_4 + My\pi_2^2 = 0 \end{cases} \tag{3.12}$$

where $\pi_2 = 2ay^2$ and $\pi_4 = 2ay^4$. If $y = 0$ we have the trivial solution $(x, y, \lambda) = (0, 0, \lambda)$. If $y \neq 0$, then from (3.12) we get $(E + 2aF)y^2 = -B$. If $(-B)(E + 2aF) > 0$ we obtain

$$(x, y, \lambda) = \big(0, \pm\alpha, \mp aB\alpha - a(3C + 2D)\alpha^2 \mp a(E + 2aF)\alpha^3 - a(3H + 2L + 6aI + 4aM)\alpha^4\big)$$

where $\alpha = (-B)/(E+2aF)$. If $(-B)/(E+2aF) \le 0$ we obtain no new solutions.

(2) Equilibria with $x \neq 0$: the first equation of (3.11) implies that

$$\begin{aligned} \lambda = \quad & B\left(\tfrac{\pi_2}{x} - Nx\right) + C\left(\tfrac{\pi_3}{x} - Nx^2\right) - D\pi_2 + E\left(\tfrac{\pi_4}{x} - Nx^3\right) + F\left(\tfrac{\pi_2^2}{x} - Nx\pi_2\right) - G\pi_3 + \\ & H\left(\tfrac{\pi_5}{x} - Nx^4\right) + I\left(\tfrac{\pi_3\pi_2}{x} - Nx^2\pi_2\right) + J\left(\tfrac{\pi_3\pi_2}{x} - Nx\pi_3\right) - L\pi_4 - M\pi_2^2 \end{aligned} \tag{3.13}$$

and taking this in the second equation we obtain

$$\begin{aligned} (y-x)\big[B\pi_2 + C\pi_3 + E\pi_4 + F\pi_2^2 + H\pi_5 + I\pi_3\pi_2 + J\pi_3\pi_2 + xyN[B + (x+y)C + \\ (x^2+y^2+xy)E + F\pi_2 + H(x+y)(x^2+y^2) + I\pi_2(x+y) + J\pi_3\big] = 0 \end{aligned}$$

The zeros satisfying $y = x$ have Σ_1-symmetry. We now solve

$$\begin{aligned} B\pi_2 + C\pi_3 + E\pi_4 + F\pi_2^2 + H\pi_5 + I\pi_3\pi_2 + J\pi_3\pi_2 + xyN[B + (x+y)C + \\ (x^2+y^2+xy)E + F\pi_2 + H(x+y)(x^2+y^2) + I\pi_2(x+y) + J\pi_3] = 0 \end{aligned} \tag{3.14}$$

where $\pi_i = a[(-x-y)^i + y^i + x^i]$, for $i = 2,3,4,5$. Equation (3.14) is equivalent to

$$\begin{aligned} (x+2y)[B(2x+y) + 2aF(2x^3+y^3+3x^2y+3xy^2) + E(2x^3+y^3+3x^2y+3xy^2) - \\ H(xy^3+3x^2y^2+2x^3y) - 3aJ(xy^3+3x^2y^2+2x^3y)] = 0 \end{aligned}$$

and solutions with $x + 2y = 0$ have Σ_3-symmetry. Now from

$$\begin{aligned} B(2x+y) + 2aF(2x^3+y^3+3x^2y+3xy^2) + E(2x^3+y^3+3x^2y+3xy^2) - \\ H(xy^3+3x^2y^2+2x^3y) - 3aJ(xy^3+3x^2y^2+2x^3y) = 0 \end{aligned}$$

we get

$$(2x+y)\big[B + (2aF+E)(x^2+y^2+xy) - (H+3aJ)(x^2y+xy^2)\big] = 0 \tag{3.15}$$

Solutions such that $2x + y = 0$ have Σ_2-symmetry. The others satisfy

$$B + (2aF+E)(x^2+y^2+xy) - (H+3aJ)(x^2y+xy^2) = 0$$

and therefore we obtain a branch of equilibria with Σ-symmetry. Using (3.14), from (3.13) we have the first equation of (3.10).

□

4 Stability of the Secondary Branches

In this section we study the stability of the solutions of the secondary branch obtained in Theorem 3.1. As before we assume that $\Sigma = \mathbf{S}_a \times \mathbf{S}_a \times \mathbf{S}_a$ where $N = 3a$.

4.1 Stability in Fix(Σ)

As we mentioned before, equations (1.2) restricted to Fix(Σ) are equivariant under the quotient group $N(\Sigma)/\Sigma \cong \mathbf{D}_3$. That is, equations (3.7) are $\mathbf{D}_3$-equivariant. We start by analyzing the stability in Fix(Σ) of the solutions X_0 with symmetry Σ obtained in Theorem 3.1 and by showing that the Σ-branch bifurcates from the appropriate primary branches (when an eigenvalue changes sign). For that we reduce the equations (3.7) to a $\mathbf{D}_3$-equivariant bifurcation problem in two state variables, where $\mathbf{D}_3$ acts by its standard representation on $\mathbf{R}^2 \equiv \mathbf{C}$.

$\mathbf{D}_3$-Equivariant Bifurcation Problem

We begin with a brief description of the characterization of a $\mathbf{D}_3$-equivariant bifurcation problem obtained by Golubitsky *et al.* [6] (Chapter XIII Section 5, Chapter XIV Section 4, and Chapter XV Section 3).

Consider the standard action of $\mathbf{D}_3$ on $\mathbf{C} \equiv \mathbf{R}^2$ generated by

$$kz = \overline{z}, \qquad \xi z = e^{2\pi i/3} z \tag{4.16}$$

where $\xi = 2\pi/3$ and $\mathbf{D}_3 = \langle k, \xi \rangle$. Up to conjugacy, the only isotropy subgroup of $\mathbf{D}_3$ with one-dimensional fixed-point subspace is $\mathbf{Z}_2(k) = \{1, k\}$.

If $g : \mathbf{C} \times \mathbf{R} \to \mathbf{C}$ is smooth and commutes with this action of $\mathbf{D}_3$ on $\mathbf{C}$ then

$$g(z, \lambda) = p(u, v, \lambda) z + q(u, v, \lambda) \overline{z}^2 \tag{4.17}$$

where $u = z\overline{z}$, $v = z^3 + \overline{z}^3$ and $p, q : \mathbf{R}^3 \to \mathbf{R}$ are smooth functions. In order g to be a bifurcation problem, the linearization of (4.17) at $(z, \lambda) = (0, 0)$ must be zero and so $p(0, 0, 0) = 0$. Moreover, the genericity hypothesis of the Equivariant Branching Lemma [6] requires $p_\lambda(0, 0, 0) \neq 0$. A second nondegeneracy hypothesis,

$$q(0, 0, 0) \neq 0 \tag{4.18}$$

implies that generically the only (local) solution branches to $g = 0$ obtained by bifurcation from $(z, \lambda) = (0, 0)$ are those obtained using the Equivariant Branching Lemma, that is, those that have $\mathbf{Z}_2(k)$-symmetry or conjugate.

Note that there is a nontrivial $\mathbf{D}_3$-equivariant quadratic $\overline{z}^2$, and z and $\overline{z}^2$ are collinear only when $\mathrm{Im}(z^3) = 0$. If $\mathrm{Im}(z^3) \neq 0$ then solving $g = 0$ is equivalent to solving $p = q = 0$. Thus, under the genericity hypothesis (4.18) it is not possible to find solutions to (4.17) near the origin in this case. Moreover, Theorem [6] XIII 4.4 implies that generically the branch of $\mathbf{Z}_2(k)$ solutions is unstable. Therefore, in order to find asymptotically stable solutions to a $\mathbf{D}_3$-equivariant bifurcation problem by a local analysis, we must consider the degeneracy hypothesis $q(0, 0, 0) = 0$ and apply unfolding theory.

We state a normal form for the degenerate $\mathbf{D}_3$-equivariant bifurcation problem for which $q(0, 0, 0) = 0$. We follow Golubitsy *et al.* [6] Chapter XIV, Section 4. We begin by specifying the lower order terms in p and q as follows:

$$\begin{aligned} p(u, v, \lambda) &= \tilde{A}u + \tilde{B}v + \tilde{\alpha}\lambda + \cdots \\ q(u, v, \lambda) &= \tilde{C}u + \tilde{D}v + \tilde{\beta}\lambda + \cdots \end{aligned} \tag{4.19}$$

We call any $\mathbf{D}_3$-equivariant bifurcation problem g satisfying $p(0,0,0) = 0 = q(0,0,0)$ *nondegenerate* if

$$\tilde{\alpha} \neq 0, \quad \tilde{A} \neq 0, \quad \tilde{\alpha}\tilde{C} - \tilde{\beta}\tilde{A} \neq 0, \quad \tilde{A}\tilde{D} - \tilde{B}\tilde{C} \neq 0 \tag{4.20}$$

Theorem 4.1 [6] *Let g be a $\mathbf{D}_3$-equivariant bifurcation problem. Assume that $p(0,0,0) = 0 = q(0,0,0)$ and that g is nondegenerate. Then g is $\mathbf{D}_3$-equivalent to the normal form*

$$h(z,\lambda) = (\epsilon u + \delta\lambda)z + (\sigma u + mv)\overline{z}^2 \tag{4.21}$$

where $\epsilon = \mathrm{sgn}\ \tilde{\mathrm{A}}$, $\delta = \mathrm{sgn}\ \tilde{\alpha}$, $\sigma = \mathrm{sgn}\ (\tilde{\alpha}\tilde{\mathrm{C}} - \tilde{\beta}\tilde{\mathrm{A}})\mathrm{sgn}\ \tilde{\alpha}$, *and*

$$m = \mathrm{sgn}\ (\tilde{\mathrm{A}}) \frac{(\tilde{\mathrm{A}}\tilde{\mathrm{D}} - \tilde{\mathrm{B}}\tilde{\mathrm{C}})\tilde{\alpha}^2}{(\tilde{\alpha}\tilde{\mathrm{C}} - \tilde{\beta}\tilde{\mathrm{A}})^2}$$

Proof: See Golubitsky *et al.* [6], Theorem XIV 4.4. □

The next theorem states a universal $\mathbf{D}_3$-unfolding for the $\mathbf{D}_3$-normal form of Theorem 4.1 above.

Theorem 4.2 [6] *The $\mathbf{D}_3$-normal form $h(z,\lambda) = (\epsilon u + \delta\lambda)z + (\sigma u + mv)\overline{z}^2$ where $\epsilon, \delta, \sigma = \pm 1$ and $m \neq 0$, obtained in Theorem 4.1, has $\mathbf{D}_3$-codimension* 2 *and modality* 1. *A universal unfolding of h is*

$$H(z,\lambda,\mu,\alpha) = (\epsilon u + \delta\lambda)z + (\sigma u + \mu v + \alpha)\overline{z}^2 \tag{4.22}$$

where (μ, α) varies near $(m, 0)$.

Proof: See Golubitsky *et al.* [6], Theorem XV 3.3 (b). □

We present in Figure 1 (a) and (b), respectively, the bifurcation diagrams for the problems $\dot{z} = -h(z,\lambda)$ and $\dot{z} = -H(z,\lambda,\mu,\alpha)$ where

$$h(z,\lambda) = (u - \lambda)z + (u + mv)\overline{z}^2 \tag{4.23}$$

and

$$H(z,\lambda,\mu,\alpha) = (u - \lambda)z + (u + \mu v + \alpha)\overline{z}^2 \tag{4.24}$$

where $\mu \sim m$ and $\alpha \sim 0$. These figures appear in [6] (Figure XV 4.1 (b) and Figure XV 4.2 (c)) with opposite signs for the eigenvalues since the authors consider the eigenvalues of $(dh)_{(z,\lambda)}$ and $(dH)_{(z,\lambda)}$, and we show in Figure 1 the signs of the eigenvalues of $-(dh)_{(z,\lambda)}$ and $-(dH)_{(z,\lambda)}$.

Since we are assuming that the trivial solution is asymptotically stable when $\lambda < 0$, we consider $\delta = -1$ in (4.22). Moreover, by transforming $g(z,\lambda)$ to $-g(-z,\lambda)$ we may fix another choice of signs since this change of coordinates preserves the asymptotic stability of solutions. We fix $\sigma = 1$. Finally, we show only the case $\epsilon = 1$ so that the $\mathbf{Z}_2$-branch of steady-state solutions is supercritical. Fixing $\delta = -1, \sigma = 1$ and $\epsilon = 1$ in (4.21) and (4.22) we obtain (4.23) and (4.24).

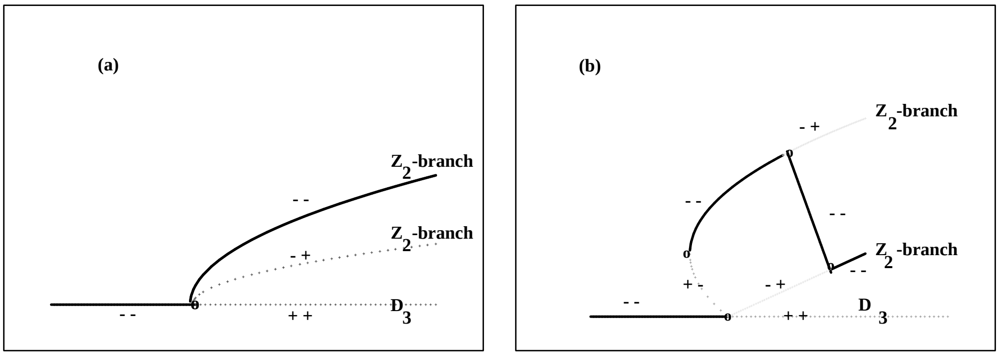

Figure 1: (a) Unperturbed $\mathbf{D}_3$-symmetric bifurcation diagram for $\dot{z} = -h(z,\lambda)$, where h is the normal form $h(z,\lambda) = (u-\lambda)z + (u+mv)\overline{z}^2$ (Figure [6] XV 4.1 (b)). (b) Bifurcation diagram for $\dot{z} = -H(z,\lambda)$, where H is defined by $H(z,\lambda,\mu,\alpha) = (u-\lambda)z + (u+\mu v+\alpha)\overline{z}^2$, with $\alpha < 0$ and $\mu > 0$ (Figure [6] XV 4.2 (c)).

Identification with the D_3-Equivariant Bifurcation Problem

Equations (3.7) are equivariant under the group $N(\Sigma)/\Sigma \cong \mathbf{D}_3$. In order to apply Theorem 4.2 we choose coordinates in $\mathrm{Fix}(\Sigma)$ such that the action of $N(\Sigma)/\Sigma$ on $\mathrm{Fix}(\Sigma)$ is given by (4.16).

Recall that $\mathrm{Fix}(\Sigma) = \{(-x-y,\ldots;y,\ldots;x,\ldots,x) : x,y \in \mathbf{R}\}$. Denote by

$$B_1 = (-1,\ldots,-1;0,\ldots,0;1,\ldots,1), \quad B_2 = (-1,\ldots,-1;1,\ldots,1;0,\ldots,0)$$

and note that $B = (B_1, B_2)$ is a basis for $\mathrm{Fix}(\Sigma)$. Denote by $(x,y)^t_B$ the coordinates vector of $(-x-y,\ldots;y,\ldots;x,\ldots)$ in the basis B and recall that the equations (1.2) restricted to $\mathrm{Fix}(\Sigma)$ in these new coordinates are given by (3.7).

We have $N(\Sigma)/\Sigma \cong \mathbf{D}_3 \cong \mathbf{S}_3$. Moreover

$$N(\Sigma)/\Sigma = \langle (12)\Sigma, (123)\Sigma \rangle$$

where we consider the action of the elements of $N(\Sigma)/\Sigma$ on $\mathrm{Fix}(\Sigma)$ defined by:

$$(12)\Sigma \cdot (-x-y,\ldots;y,\ldots;x,\ldots) \stackrel{\text{def}}{=} (y,\ldots;-x-y,\ldots;x,\ldots)$$

$$(123)\Sigma \cdot (-x-y,\ldots;y,\ldots;x,\ldots) \stackrel{\text{def}}{=} (x,\ldots;-x-y,\ldots;y,\ldots)$$

In matrix notation we have

$$(12)\Sigma \cdot \begin{pmatrix} x \\ y \end{pmatrix}_B = M \begin{pmatrix} x \\ y \end{pmatrix}_B$$

where $M = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$ has eigenvalues $-1, 1$. Also denoting $A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ we have

$$(123)\Sigma \cdot \begin{pmatrix} x \\ y \end{pmatrix}_B = A \begin{pmatrix} x \\ y \end{pmatrix}_B$$

where A has no real eigenvalues (it has the complex conjugate eigenvalues $e^{i\frac{2\pi}{3}}, e^{-i\frac{2\pi}{3}}$).

It follows that

$$b = \left(-\frac{2\sqrt{3}}{3} B_1 + \frac{\sqrt{3}}{3} B_2, B_2 \right)$$

is another basis for $\mathrm{Fix}(\Sigma)$ such that if $(X,Y)^t_b$ denote coordinates on this basis, then in these coordinates we have

$$(12)\Sigma \cdot \begin{pmatrix} X \\ Y \end{pmatrix}_b = \begin{pmatrix} X \\ -Y \end{pmatrix}_b$$

$$(123)\Sigma \cdot \begin{pmatrix} X \\ Y \end{pmatrix}_b = \begin{pmatrix} \cos\frac{2\pi}{3}X - \sin\frac{2\pi}{3}Y \\ \sin\frac{2\pi}{3}X + \cos\frac{2\pi}{3}Y \end{pmatrix}_b$$

(Recall (4.16)). We relate now the coordinates $(X,Y)^t_b$ and $(x,y)^t_B$. Consider the matrix representing the change of basis from b to B:

$$P = \begin{pmatrix} -\frac{2\sqrt{3}}{3} & 0 \\ \frac{\sqrt{3}}{3} & 1 \end{pmatrix}$$

It follows that the matrix representing the change of basis from B to b is

$$P^{-1} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & 0 \\ \frac{1}{2} & 1 \end{pmatrix}$$

Therefore we have the following relations

$$\begin{pmatrix} X \\ Y \end{pmatrix}_b = P^{-1} \begin{pmatrix} x \\ y \end{pmatrix}_B = \begin{pmatrix} -\frac{\sqrt{3}}{2}x \\ \frac{1}{2}x + y \end{pmatrix} \tag{4.25}$$

and

$$\begin{pmatrix} x \\ y \end{pmatrix}_B = P \begin{pmatrix} X \\ Y \end{pmatrix}_b = \begin{pmatrix} -\frac{2\sqrt{3}}{3}X \\ \frac{\sqrt{3}}{3}X + Y \end{pmatrix}$$

We proceed by writing equations (3.7) in the coordinates $(X,Y)^t_b$. From (4.25) we have

$$\begin{cases} \dot{X} = -\frac{\sqrt{3}}{2}\dot{x} \\ \dot{Y} = \frac{1}{2}\dot{x} + \dot{y} \end{cases}$$

and then equations (3.7) in X, Y are

$$\begin{aligned} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = {} & \lambda \begin{pmatrix} X \\ Y \end{pmatrix} + \left[-\tfrac{\sqrt{3}}{3}BN - \tfrac{\sqrt{3}}{3}\left(E + \tfrac{2}{3}FN\right)N(X^2+Y^2) \right] \begin{pmatrix} X^2 - Y^2 \\ -2XY \end{pmatrix} + \\ & \left[N(C + \tfrac{2}{3}D)(X^2+Y^2) - \tfrac{\sqrt{3}}{9}N\left(E+G\right)2X(X^2-3Y^2) \right] \begin{pmatrix} X \\ Y \end{pmatrix} + \\ & \tfrac{1}{9}HN \left[2X(X^2-3Y^2) \begin{pmatrix} X^2 - Y^2 \\ -2XY \end{pmatrix} + 9(X^2+Y^2)^2 \begin{pmatrix} X \\ Y \end{pmatrix} \right] + \\ & \tfrac{1}{9}JN^2 2X(X^2-3Y^2) \begin{pmatrix} X^2 - Y^2 \\ -2XY \end{pmatrix} + \tfrac{2}{3}N(IN + L + \tfrac{2}{3}MN)(X^2+Y^2)^2 \begin{pmatrix} X \\ Y \end{pmatrix} \end{aligned} \tag{4.26}$$

Identifying $z = X + iY$ in (4.26) yields the equation

$$
\begin{aligned}
&\dot{z} - \left[\lambda + \left(C + \tfrac{2}{3}D\right) Nu - \tfrac{\sqrt{3}}{9}\left(E+G\right) Nv + \tfrac{2}{3}N\left(\tfrac{3}{2}H + IN + L + \tfrac{2}{3}MN\right) u^2\right] z \\
&- \left[-\tfrac{\sqrt{3}}{3}BN - \tfrac{\sqrt{3}}{3}\left(E + \tfrac{2}{3}FN\right) Nu + \tfrac{1}{9}(H + JN)Nv\right] \overline{z}^2 = 0
\end{aligned}
\tag{4.27}
$$

where $u = z\overline{z}$ and $v = z^3 + \overline{z}^3$. We can now state the following theorem:

Theorem 4.3 *Consider the equations (4.27) where*

$$
B < 0, \quad 3C + 2D < 0, \quad 3E + 2FN > 0, \quad (3C + 2D)(H + JN) - (E + G)(3E + 2FN) > 0 \tag{4.28}
$$

Then for small enough $B \neq 0$ equations (4.27) have a branch of stable steady-state solutions with trivial isotropy (for the $\mathbf{D}_3$-problem) that bifurcates from the branch of steady-state solutions with $\mathbf{Z}_2(k)$-symmetry.

Proof: Writing (4.27) as $\dot{z} + g(z, \lambda) = 0$ where $g(z, \lambda) = p(u, v, \lambda)z + q(u, v, \lambda)\overline{z}^2$, we have

$$
\begin{aligned}
p(u, v, \lambda) &= -\lambda + \beta_1 u + \beta_2 v + \beta_3 u^2, \\
q(u, v, \lambda) &= \beta_4 + \beta_5 u + \beta_6 v,
\end{aligned}
\tag{4.29}
$$

where $\beta_1 = -\left(C + \frac{2}{3}D\right) N, \beta_2 = \frac{\sqrt{3}}{9}(E + G)N, \beta_3 = -\frac{2}{3}\left(\frac{3}{2}H + IN + L + \frac{2}{3}MN\right) N, \beta_4 = \frac{\sqrt{3}}{3}BN,$ $\beta_5 = \frac{\sqrt{3}}{3}\left(E + \frac{2}{3}FN\right) N, \beta_6 = -\frac{1}{9}(H + JN)N.$

Note that $p(0, 0, 0) = 0$ and $p_\lambda(0, 0, 0) \neq 0$. Moreover

$$
q(0, 0, 0) = \beta_4 = \frac{\sqrt{3}}{3}BN
$$

and $q(0, 0, 0) = 0$ if and only if $B = 0$.

Let $B = 0$ and recall (4.19) and (4.20). Comparing (4.29) with (4.19) we obtain

$$
\tilde{\alpha} = -1, \qquad \tilde{A} = \beta_1, \qquad \tilde{\alpha}\tilde{C} - \tilde{\beta}\tilde{A} = -\beta_5, \qquad \tilde{A}\tilde{D} - \tilde{B}\tilde{C} = \beta_1\beta_6 - \beta_2\beta_5
$$

By Theorem 4.1, g (with $B = 0$) is $\mathbf{D}_3$-equivalent to (4.21) if g is nondegenerate. That is, if

$$
\beta_1 \neq 0, \quad \beta_5 \neq 0, \quad \beta_1\beta_6 - \beta_2\beta_5 \neq 0
$$

where

$$
\epsilon = \operatorname{sgn} \beta_1, \quad \delta = -1, \quad \sigma = \operatorname{sgn} \beta_5, \quad m = \operatorname{sgn}(\beta_1)\frac{\beta_1\beta_6 - \beta_2\beta_5}{\beta_5^2}
$$

Moreover, a universal unfolding of (4.21) is (4.22):

$$
H(z, \lambda, \mu, \alpha) = (\epsilon u + \delta\lambda)z + (\sigma u + \mu v + \alpha)\overline{z}^2
$$

where (μ, α) varies near $(m, 0)$.

Suppose now the conditions (4.28). Then it follows that

$$
\epsilon = 1, \quad \delta = -1, \quad \sigma = 1,
$$

and (μ, α) varies near $(m, 0)$ where

$$\text{sgn } m = \text{sgn } (\beta_1\beta_6 - \beta_2\beta_5) = 1$$

and

$$H(z, \lambda, \mu, \alpha) = (u - \lambda)z + (u + \mu v + \alpha)\overline{z}^2$$

The bifurcation diagram for $\dot{z} + H(z, \lambda, \mu, \alpha) = 0$ appears in Figure 1(b) for $\alpha < 0$ (and $\mu > 0$). Note that there is a secondary branch of stable steady-state solutions with trivial symmetry bifurcating from the branch of steady-state solutions with $\mathbf{Z}_2(k)$-symmetry.

Thus there is a branch of steady-state solutions with trivial symmetry of $\dot{z} + g(z, \lambda) = 0$ that satisfy

$$p(u, v, \lambda) = -\lambda + \beta_1 u + \beta_2 v + \beta_3 u^2 = 0,$$

$$q(u, v, \lambda) = \beta_4 + \beta_5 u + \beta_6 v = 0$$

provided $B < 0$ and small enough.

Observe that for small $\beta_4 < 0$ (thus $B < 0$ and small enough) and $\beta_5 > 0$ the solutions of (4.30) (near the origin) form a circlelike curve in the (x, y)-plane of radius approximately $\sqrt{|\beta_4|/\beta_5}$. This is exactly true for $\beta_6 = 0$, and approximately true for $\beta_6 \neq 0$. It follows that in the (x, y, λ)-space this curve intersects the $y = 0$ plane at two points (x^-, λ^-) and (x^+, λ^+) where $x^- < 0 < x^+$ that correspond to intersection points of the branch with trivial isotropy (for the $\mathbf{D}_3$-problem) and solutions with isotropy $\mathbf{Z}_2$.

Moreover the stability is determined by

$$\begin{aligned}
\text{tr } \left((dg)_{(z,\lambda)}\right) \quad &= 2(up_u + \tfrac{v}{2}(3p_v + q_u) + 3q_v u^2) \\
&= 2(\beta_1 u + 2\beta_3 u^2 + \tfrac{v}{2}(3\beta_2 + \beta_5) + 3\beta_6 u^2) \\
\\
\det \left((dg)_{(z,\lambda)}\right) \quad &= 3(p_v q_u - p_u q_v)(z^3 - \overline{z}^3)^2 \\
&= 12(\text{Im } (z^3))^2(\beta_1\beta_6 - \beta_2\beta_5 + 2\beta_3\beta_6 u)
\end{aligned}$$

and so the solutions (near the origin) are stable since from (4.28) we have $\beta_1 > 0$ and $\beta_1\beta_6 - \beta_2\beta_5 > 0$. The same conclusion can be derived from the fact that $\mathbf{D}_3$-equivalence preserves the asymptotic stability of the solutions with trivial symmetry ([6] Chapter XV, Section 4).

□

4.2 Intersection between the Primary and the Secondary Branches

Let $N = 3a$ and write the second equation in (3.10) as

$$\nu + \beta(x^2 + y^2 + xy) + \gamma(x^2 y + xy^2) = 0 \tag{4.30}$$

where

$$\nu = B, \quad \beta = E + \tfrac{2}{3}FN, \quad \gamma = -(H + JN) \tag{4.31}$$

Assuming the conditions of Theorem 4.3 we have $\beta > 0$ and $\nu < 0$.

The Σ-branch intersects the Σ_1-branch if $y = x$. Using this in (4.30) we get

$$\nu + 3\beta x^2 + 2\gamma x^3 = 0 \tag{4.32}$$

Note that if we assume $\beta >> |\gamma|, |\nu|$, then if

$$p(x) = \nu + 3\beta x^2 + 2\gamma x^3$$

we have

$$p'(x) = 6x(\gamma x + \beta)$$

and so $p(x)$ has three real roots, say, x_1^-, x_1^+, x_1, where $x_1^- < 0 < x_1^+$, and $x_1 > -\frac{\beta}{\gamma}$ if $\gamma < 0$, or $x_1 < -\frac{\beta}{\gamma}$ if $\gamma > 0$. Thus the Σ-branch intersects the Σ_1-branch at three points with the x-coordinate x_1^-, x_1^+, x_1. In the (x, λ)-plane, we denote by (x_1^-, λ_1^-) and (x_1^+, λ_1^+) the intersections between the two branches in Figure 1(b). Similarly, we have intersections in (x, λ)-plane that we denote by $(x_2^-, \lambda_2^-), (x_2^+, \lambda_2^+)$ and $(x_3^-, \lambda_3^-), (x_3^+, \lambda_3^+)$ between the Σ-branch and the Σ_2-branch and Σ_3-branch, respectively.

4.3 Secondary Branches: Full Stability

Given an equilibrium $X_0 = (x_0, \lambda_0)$ of (1.2), in the Σ-branch obtained in Theorem 3.1, in order to analyze the stability of this solution, we need to compute the eigenvalues of the Jacobian $(dG)_{(x_0,\lambda_0)}$. We use now the decomposition of V_1 into isotypic components for the action of Σ to block-diagonalize the Jacobian on V_1. We have

$$V_1 = \mathrm{Fix}(\Sigma) \oplus U_1 \oplus U_2 \oplus U_3$$

where

$$\begin{aligned} U_1 &= \{(x_1, \ldots, x_a; 0, \ldots, 0; 0, \ldots, 0) \in V_1 : x_1 + \cdots + x_a = 0\} \\ U_2 &= \{(0, \ldots, 0; x_{a+1}, \ldots, x_{2a}; 0, \ldots, 0) \in V_1 : x_{a+1} + \cdots + x_{2a} = 0\} \\ U_3 &= \{(0, \ldots, 0; 0, \ldots, 0; x_{2a+1}, \ldots, x_{3a}) \in V_1 : x_{2a+1} + \cdots + x_{3a} = 0\} \end{aligned}$$

The action of Σ is absolutely irreducible on each isotypic component U_i, for $i = 1, 2, 3$ and trivial on $\mathrm{Fix}(\Sigma)$. Moreover, dim $U_i = a - 1$. Thus $(dG)_{X_0}$, when restricted to each of the U_i, has a real eigenvalue λ_i with multiplicity $a - 1$. Since $(dG)_{X_0}$ commutes with Σ,

$$(dG)_{X_0} = \begin{pmatrix} C_1 & C_2 & C_3 \\ C_4 & C_5 & C_6 \\ C_7 & C_8 & C_9 \end{pmatrix} \tag{4.33}$$

where the blocks correspond to the isotypic decomposition and C_1, C_5, C_9 commute with S_a.

If we write a square matrix M of order a with rows $l_1, \ldots, l_a$, and if M commutes with S_a, then

$$M = \begin{pmatrix} l_1 \\ (12) \cdot l_1 \\ \vdots \\ (1a) \cdot l_1 \end{pmatrix}$$

where if $l_1 = (m_1, \ldots, m_a)$, then $(1i) \cdot l_1 = (m_i, m_2, \ldots, m_{i-1}, m_1, m_{i+1}, \ldots, m_a)$. Moreover, l_1 is invariant under S_{a-1} in the last $a - 1$ entries. Thus, l_1 is of type $(m_1, m_2, \ldots, m_2)$. Applying this to C_1, C_5, C_9 we get

$$C_i = \begin{pmatrix} a_i & & b_i \\ & \ddots & \\ b_i & & a_i \end{pmatrix} \tag{4.34}$$

for $i = 1, 5, 9$, where

$$a_1 = (\partial G_1/\partial x_1)_{X_0}, \quad a_5 = (\partial G_{a+1}/\partial x_{a+1})_{X_0}, \quad a_9 = (\partial G_{2a+1}/\partial x_{2a+1})_{X_0}$$

$$b_1 = (\partial G_1/\partial x_2)_{X_0}, \quad b_5 = (\partial G_{a+1}/\partial x_{a+2})_{X_0}, \quad b_9 = (\partial G_{2a+1}/\partial x_{2a+2})_{X_0}$$

The other symmetry restrictions on the C_i, for $i \neq 1, 5, 9$, imply that the rest of the matrices each have one identical entry. From this we obtain basis for each U_i composed by eigenvectors of $(dG)_{X_0}$:

$$\begin{array}{rrl}
U_1: & \nu_1 = & (1, -1, 0, \ldots, 0; 0, \ldots, 0; 0, \ldots, 0)^T \\
& \nu_2 = & (0, 1, -1, 0, \ldots, 0; 0, \ldots, 0; 0, \ldots, 0)^T \\
& \vdots & \\
& \nu_{a-1} = & (0, \ldots, 0, 1, -1; 0, \ldots, 0; 0, \ldots, 0)^T \\
U_2: & \psi_1 = & (0, \ldots, 0; 1, -1, 0, \ldots, 0; 0, \ldots, 0)^T \\
& \psi_2 = & (0, \ldots, 0; 0, 1, -1, \ldots, 0; 0, \ldots, 0)^T \\
& \vdots & \\
& \psi_{a-1} = & (0, \ldots, 0; 0, \ldots, 0, 1, -1; 0, \ldots, 0)^T \\
U_3: & \phi_1 = & (0, \ldots, 0; 0, \ldots, 0; 1, -1, 0, \ldots, 0)^T \\
& \phi_2 = & (0, \ldots, 0; 0, \ldots, 0; 0, 1, -1, 0, \ldots, 0)^T \\
& \vdots & \\
& \phi_{a-1} = & (0, \ldots, 0; 0, \ldots, 0; 0, \ldots, 0, 1, -1)^T
\end{array}$$

Moreover the eigenvalue associated with ν_i is

$$\lambda_1 = a_1 - b_1 = (\partial G_1/\partial x_1)_{X_0} - (\partial G_1/\partial x_2)_{X_0}$$

the one associated with ψ_i is

$$\lambda_2 = a_5 - b_5 = (\partial G_{a+1}/\partial x_{a+1})_{X_0} - (\partial G_{a+1}/\partial x_{a+2})_{X_0}$$

and the one associated with ϕ_i is

$$\lambda_3 = a_9 - b_9 = (\partial G_{2a+1}/\partial x_{2a+1})_{X_0} - (\partial G_{2a+1}/\partial x_{2a+2})_{X_0}$$

The branching conditions for Σ of Theorem 3.1 yield:

$$
\begin{aligned}
\lambda_1 = \ & -3BN(x+y) + CN(2x^2+2y^2+5xy)+ \\
& EN(-5x^3-5y^3-13xy^2-13yx^2) - 2FN^2(x+y)(x^2+y^2+xy)+ \\
& HN(4y^2+7xy+4x^2)(x^2+y^2+3xy) + \tfrac{2}{3}IN^2(2x^4+9x^2y^2+2y^4+7x^3y+7xy^3)+ \\
& 3JN^2xy(x^2+y^2+2xy) \\
\\
\lambda_2 = \ & BN(y-x) + CN(y-x)(2y+x)+ \\
& EN(y-x)(x^2+3y^2+2xy) + \tfrac{2}{3}FN^2(y-x)(x^2+y^2+xy)- \\
& HN(x^4+x^3y+xy^3+x^2y^2-4y^4) + \tfrac{2}{3}IN^2(-x^4+2y^4-2x^3y+2xy^3)+ \\
& 3JN^2(x^3y-xy^3) \\
\\
\lambda_3 = \ & BN(x-y) + CN(x-y)(2x+y)+ \\
& EN(x-y)(y^2+3x^2+2xy) + \tfrac{2}{3}FN^2(x-y)(x^2+y^2+xy)- \\
& HN(x^3y+xy^3+x^2y^2+y^4-4x^4) + \tfrac{2}{3}IN^2(2x^4-y^4+x^3y-2xy^3)+ \\
& 3JN^2(xy^3-x^3y)
\end{aligned}
\tag{4.35}
$$

where x and y are related by the second equation of (3.10). Using (4.35) and (3.10) we obtain:

Lemma 4.4 *Let X_0 be an equilibrium of (3.7) in the Σ-branch obtained in Theorem 3.1. Then the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of $(dG)_{X_0}$ are*

$$
\begin{aligned}
\lambda_1 &= N(x+2y)(2x+y)S_2(x,-x-y) \\
\lambda_2 &= N(x+2y)(y-x)S_2(x,y) \\
\lambda_3 &= N(x-y)(2x+y)S_2(y,x)
\end{aligned}
\tag{4.36}
$$

where

$$
S_2(x,y) = C + Ey + \tfrac{2}{3}IN(x^2+y^2+xy) + H(x^2+2y^2+xy) \tag{4.37}
$$

and x and y are as in the second equation of (3.10):

$$
B + \left(\frac{2}{3}FN + E\right)(x^2+y^2+xy) - (H+JN)(x^2y+xy^2) = 0
$$

Remark 4.5 Suppose X_0 corresponds to a solution of the primary branch with Σ_1-symmetry. Note that the isotypic decomposition of V_1 for the action of Σ_1 is

$$
V_1 = W_0 \oplus W_1 \oplus W_2
$$

where

$$
\begin{aligned}
W_0 &= \mathrm{Fix}(\Sigma_1) = \{(-2x,\ldots;x,\ldots;x,\ldots) : x \in \mathbf{R}\} \\
W_1 &= \{(x_1,\ldots,x_a;0,\ldots,0) \in V_1 : x_1+\cdots+x_a = 0\} \\
W_2 &= \{(0,\ldots,0;x_{a+1},\ldots,x_{3a}) \in V_1 : x_{a+1}+\cdots+x_{3a} = 0\}
\end{aligned}
$$

The action of Σ_1 is absolutely irreducible on each W_1, W_2 and trivial on W_0. It follows then that the Jacobian $(dG)_{X_0}$ has (at most) three distinct real eigenvalues, μ_j, one for each W_j, with multiplicity $\dim W_j$.

The stability in $\mathrm{Fix}(\Sigma)$ for the solution with Σ_1-symmetry is determined by the eigenvalue μ_0 associated with $W_0 = \mathrm{Fix}(\Sigma_1)$ and μ_2 since $\mathrm{Fix}(\Sigma) \bigcap W_2 \neq \{0\}$.

Suppose now that X_0 corresponds to a solution of the Σ-branch and of the Σ_1-branch. Then the eigenvalue μ_2 is zero and it is associated with the eigenspace W_2. Moreover, $U_2 \subseteq W_2$ and

$U_3 \subseteq W_2$. Therefore X_0 is a zero of λ_2 and λ_3, and we have the factor $y - x$ in the expressions for λ_2 and λ_3 that appear in (4.36). Similarly, we justify the factors $x + 2y$ and $2x + y$ in those expressions. $\diamond$

Theorem 4.6 *Assume the conditions of Theorem 4.3 and let X_0 be an equilibrium of the secondary branch of steady-state solutions with symmetry Σ obtained in Theorem 3.1 and guaranteed by Theorem 4.3 for negative values of B sufficiently small. Consider the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of $(dG)_{X_0}$ as in Lemma 4.4 and suppose parameters values $B, C, \ldots, H$ such that*

$$S_2(x, y) \neq 0$$

for all x, y such that

$$B + \left(\frac{2}{3}FN + E\right)(x^2 + y^2 + xy) - (H + JN)(x^2 y + xy^2) = 0$$

and

$$x_1^- < x < x_1^+$$

Then the solutions of the secondary branch are unstable.

Proof: Theorem 3.1 and Theorem 4.3 prove the existence of the secondary branch of equilibria of (3.7) for B negative and small enough obtained by bifurcation from the primary branches with Σ_i-symmetry for $i = 1, 2, 3$. Recall from Section 4.2 that each primary branch with Σ_i-symmetry intersects the Σ-branch at two points, with x-coordinate denoted by x_i^- and x_i^+ and where $x_i^- < x < x_i^+$. Let X_0 be an equilibrium of (3.7) in the Σ-branch and consider the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of $(dG)_{X_0}$ as in Lemma 4.4 (defining the stability of X_0 at the isotypic components for the action of Σ).

The solutions of (4.30) with $x = 0$ in the (x, y)-plane are the points $\left(0, \pm\sqrt{-\nu/\beta}\right)$, where $\nu = B$ and $\beta = E + \frac{2}{3}FN$, where from the conditions of Theorem 4.3 we have $\nu < 0$ and $\beta > 0$. Computing $\lambda_1, \lambda_2, \lambda_3$ for these points we obtain

$$\lambda_1 = 2Ny^2 S_2(0, -y)$$

$$\lambda_2 = 2Ny^2 S_2(0, y)$$

$$\lambda_3 = -Ny^2 S_2(y, 0)$$

where $y = \pm\sqrt{-\nu/\beta}$. It follows then that the signs of $\lambda_1, \lambda_2, \lambda_3$ are determined by the signs of $S_2(0, -y), S_2(0, y)$, and $S_2(y, 0)$, respectively.

Assuming $S_2(x, y) \neq 0$ along the Σ-branch we have

$$\operatorname{sgn}\,(S_2(y, 0)) = \operatorname{sgn}\,(S_2(0, y))$$

Thus we have that λ_2, λ_3 have opposite signs at the points of the Σ-branch $(0, \pm\sqrt{-\nu/\beta})$. Therefore these points are unstable equilibrium points that lie in the Σ-branch. The same conclusion holds for the other equilibrium points in the Σ-branch (not corresponding to the intersection points with the primary branches with Σ_i-symmetry, for $i = 1, 2, 3$).

$\square$

Suppose parameter values $B, C, \ldots, H$ such that there is an equilibrium $X_0 = (x_0, y_0, \lambda_0)$ of the secondary branch with symmetry Σ obtained in Theorem 3.1 such that

$$S_2(x_0, y_0) = 0$$

Generically, we can assume that X_0 is not an intersection point between the Σ-branch and one of the Σ_i-branches, for $i = 1, 2, 3$. This situation corresponds to a tertiary bifurcation at $\lambda = \lambda_0$ from the secondary branch and implies the change of the sign of one of the eigenvalues determining the stability of the steady-state solutions of the Σ-branch near X_0.

Denote by

$$\begin{array}{l} \mathcal{R}_1 = \{(x,y) \in \mathbf{R}^2 : -\frac{1}{2}x < y < x\} \\ \mathcal{R}_2 = \{(x,y) \in \mathbf{R}^2 : x < y < -2x\} \\ \mathcal{R}_6 = \{(x,y) \in \mathbf{R}^2 : -2x < y < -\frac{1}{2}x\} \end{array}$$

Assume that $(x_0, y_0) \in \mathcal{R}_1$. (The other cases are addressed in a similar way.) Then the eigenvalue λ_2 determining the stability of the equilibrium points in the Σ-branch and in region $\mathcal{R}_1$ changes sign.

We can find steady-state solutions in the Σ-branch of the region $\mathcal{R}_1$ and their $\mathbf{D}_3$-orbit points in region $\mathcal{R}_2$, close to (x_1^+, y_1^+) (one of the intersections between the Σ-branch and the Σ_1-branch). By symmetry these points in both regions have the same stability. Moreover, there are equilibria in those conditions such that the functions $S_2(x, -x-y), S_2(x, y)$ and $S_2(y, x)$ do not vary their signs. In this case, the signs of two of the eigenvalues (λ_2 and λ_3) determining the stability (outside $\mathrm{Fix}(\Sigma)$) of those equilibria in $\mathcal{R}_1$ have opposite signs from those in their $\mathbf{D}_3$-orbits in region $\mathcal{R}_2$. The same reasoning applies to steady-state solutions of the Σ-branch close to the point (x_3^+, y_3^+) and their orbits by $\mathbf{D}_3$ in the regions $\mathcal{R}_1$ and $\mathcal{R}_6$. It follows then that no stability is possible for the Σ-branch solutions where $S_2(x, y) \neq 0$ in the $\mathcal{R}_1$-region (and so in the other regions of the plane excluding the points of the Σ-branch where $S_2(x, y) = 0$ and the intersection points with the Σ_i-branches).

From Theorem 4.6 and the above discussion we conclude that the solutions of the secondary branch are (generically) globally unstable in the fifth-order truncation of the system.

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