# 2-DIMENSIONAL LIE ALGEBRAS AND SEPARATRICES FOR VECTOR FIELDS ON $(\mathbb{C}^3, 0)$

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Abstract. We show that holomorphic vector fields on $(\mathbb{C}^3, 0)$ have separatrices provided that they are embedded in a rank 2 representation of a two-dimensional Lie algebra. As an application of this result we show, in particular, that the second jet of a holomorphic vector field defined on a compact complex manifold $M$ of dimension 3 cannot vanish at an isolated singular point provided that $M$ carries more than a single holomorphic vector field.

## 1. Introduction

Consider a compact complex manifold $M$ of dimension 3 and denote by $\mathrm{Aut}\,(M)$ the group of holomorphic diffeomorphisms of $M$. It is well-known that $\mathrm{Aut}\,(M)$ is a finite dimensional complex Lie group whose Lie algebra can be identified with $\mathfrak{X}\,(M)$, the space of all holomorphic vector fields defined on $M$; see [1]. In this paper, the following will be proved:

**Theorem A**. *Consider a compact complex manifold $M$ of dimension 3 and assume that the dimension of $\mathrm{Aut}\,(M)$ is at least 2. Let $Z$ be an element of $\mathfrak{X}\,(M)$ and suppose that $p \in M$ is an isolated singularity of $Z$. Then*

$$J^2(Z)\,(p) \neq 0$$

*i.e., the second jet of $Z$ at the point $p$ does not vanish.*

Note that $M$ is not assumed to be algebraic in the above theorem. In fact, if $M$ is algebraic, or more generally Kähler, then the statement holds even if the dimension of $\mathrm{Aut}\,(M)$ equals 1; see Remark 3.4.

Theorem A is the most elaborate result in this paper. In terms of importance, however, Theorem B below is a general local result playing the central role in the proof of Theorem A as well as in the proof of Theorem C below.

To state Theorem B, we consider two holomorphic vector fields $X$ and $Y$ defined on a neighborhood $U$ of $(0, 0, 0) \in \mathbb{C}^3$ and we assume that they are not linearly dependent at every point of $U$. In other words, the distribution spanned by $X$ and $Y$ has rank 2 away from a proper analytic subset of $U$. We also assume that $X$ and $Y$ generate a Lie algebra of dimension 2. In other words, either these vector fields commute or they satisfy the more general relation $[X, Y] = c\,Y$, where $c \in \mathbb{C}^*$ and where the brackets stand for the commutator of two vector fields. Then we have:

**Theorem B**. *Let $X$ and $Y$ be two holomorphic vector fields defined on a neighborhood $U$ of $(0, 0, 0) \in \mathbb{C}^3$ which are not linearly dependent on all of $U$. Suppose that $X$ and $Y$ vanish at the origin and that one of the following conditions holds:*

- $[X, Y] = 0$;

- $[X,Y] = cY$, *for some* $c \in \mathbb{C}^*$.

*Then there exists a germ of analytic curve* $\mathcal{C} \subset \mathbb{C}^3$ *passing through the origin and simultaneously invariant under* $X$ *and* $Y$.

As indicated, our technique to derive Theorem A from Theorem B also allows for other similar results and, in fact, the assumption that $M$ is compact is not fully indispensable in many cases. For example, suppose that $N$ is a Stein manifold of dimension 3 and suppose that $N$ is effectively acted upon by a finite dimensional Lie group $G$. Then the Lie algebra $\mathfrak{G}$ of $G$ embeds into the space $\mathfrak{X}_{\rm comp}(N)$ of *complete* holomorphic vector fields on $N$. The study of these complete holomorphic vector fields on Stein manifolds is a topic of interest having its roots in a classical work of Suzuki [29]. In this direction, our techniques yield:

**Theorem C**. *Let* $N$ *denote a Stein manifold of dimension* 3 *and consider a finite dimensional Lie algebra* $\mathfrak{G}$ *embedded in* $\mathfrak{X}_{\rm comp}(N)$ *(the space of complete holomorphic vector fields on* $N$*). Assume that the dimension of* $\mathfrak{G}$ *is at least* 2. *If* $Z$ *is an element of* $\mathfrak{G} \subseteq \mathfrak{X}(M)$ *possessing an isolated singular point* $p \in N$, *then the linear part of* $Z$ *at* $p$ *cannot vanish, i.e.* $p$ *is a non-degenerate singularity of* $Z$.

Having stated the main results obtained in this paper, it is convenient to place them in perspective with respect to previous works. We shall begin with Theorem B since it constitutes a novel ingredient leading, in particular, to the proof of Theorem A. For this, recall that a singular holomorphic foliation of *dimension* 1 on $(\mathbb{C}^n, 0)$ is nothing but the foliation induced by the local orbits of a holomorphic vector field having a singular set of codimension at least 2. In the case of foliations having dimension 1, a *separatrix* is a *germ of* analytic curve passing through the singular point and invariant under the foliation in question. In dimension 2, a remarkable theorem due to Camacho and Sad [5] asserts that every holomorphic foliation on $(\mathbb{C}^2, 0)$ possesses a separatrix; their paper then completes a classical work by Briot and Bouquet. Unfortunately, the existence of separatrices is no longer a general phenomenon once the dimension increases as shown by Gomez-Mont and Luengo in [9]. The paper [9] also contains examples of foliations without separatrix on a *singular surface*, an issue previously discussed in [4]. The examples provided in [9], however, include singular surfaces realized as hypersurfaces of $\mathbb{C}^3$ and foliations realized by holomorphic vector fields. A basic question motivated by Gomez-Mont and Luengo's examples and aiming at finding appropriate generalizations of Camacho-Sad theorem concerns the existence of separatrices for vector fields as those considered in Theorem B. This type of question also appears in the works of Stolovitch, Vey, and Zung about normal forms for abelian actions and invariant sets; see the survey [28] and its reference list. From this point of view, Theorem B is satisfactory for $(\mathbb{C}^3, 0)$. Finally, in the commutative case, Theorem B nicely complements the main result of [23] concerning the existence of separatrices for the *codimension* 1 *foliation* spanned by $X$ and $Y$. The reader will also note that the analogue of [23] in the case of affine actions is known to be false since the classical work of Jouanolou; see [15].

Concerning Theorem A, it essentially constitutes a partial answer to a question raised by Ghys long ago. This question is better formulated in the context of *semi-complete* vector fields; see [22] or Section 3. In fact, Ghys has asked whether or not an isolated singular point $p$ of a semi-complete vector field $Z$ always satisfies $J^2(Z)(p) \neq 0$. The answer is known to be affirmative in dimension 2. Whereas no counterexample is known in higher dimensions, it appears to exist a consensus that an affirmative answer to Ghys question, say in dimension 3, cannot be obtained without a fine analysis of singular points. This analysis should involve, in particular, theorems about reduction of singularities which, by themselves, are already fairly complicated in dimension 3. From this point of view, the advantage of Theorem A lies in the fact that it provides a shortcut to an affirmative statement, albeit this statement is slightly weaker than the original conjecture. Nonetheless, Ghys question was motivated by the potential applications of this type of result to problems about bounds for the dimension of the automorphism group of compact complex manifolds. As far as this type of application is targeted, the reader will note that Theorem A is satisfactory since the additional assumption on which it relies can be assumed to hold without loss of generality.

Finally, let us further comment on the relation of Theorem A and some well-known results concerning the automorphism groups of *algebraic manifolds*. If $M$ is algebraic and if $\mathrm{Aut}_0(M)$ denotes the connected component containing the identity of $\mathrm{Aut}\,(M)$, then $\mathrm{Aut}_0(M)$ is an algebraic group algebraically acting on $M$. Algebraic group actions have simple dynamics as follows, for example, from a classical theorem due to Rosenlicht [25]. From the very beginning, this observation imposes strong constraints on the structure of vector fields on $M$. To a good extent, the same principle of actions with simple dynamics holds for Kähler manifolds thanks to the work of Fujiki [7]. These facts have allowed significant progress in understanding vector fields on algebraic manifolds through classical works by Carrell, Lieberman and others. More recently, Hwang brought new ideas to the field and made important progress in several questions; see [13], [14]. For example, recalling that a singularity of a vector field is said to be *degenerate* if the first jet of the vector field vanishes at the singular point in question, in [14] it is proved that an algebraic manifold carrying a vector field having an isolated *degenerate singularity* must be rational. This Hwang's statement may be compared to our Theorem A and deserves further comments.

It is convenient to first consider the case of complex surfaces. It was shown in [8] that an arbitrary compact complex surface carrying a holomorphic vector field with a degenerate isolated singular point is a Hirzebruch surface. However, already in dimension 3, this type of statement no longer holds even from the birational point of view. Indeed, the $\mathrm{SL}\,(2,\mathbb{C})$-quasi homogeneous threefolds constructed by A. Guillot in [11] carry holomorphic vector fields with a degenerate isolated singular point. Yet, these threefolds do not admit any Kähler structure. The dynamics of the corresponding Halphen vector field is also described in [11] and it turns out to be fairly complicated. Therefore, the standard methods used in the algebraic/Kähler contexts no longer work for general complex manifolds of dimension 3 since complicated dynamics may genuinely appear in this context.

Another issue concerning birational equivalence has to do with the fact that holomorphic vector fields are not invariant by birational maps. Thus, for example, knowing that a manifold is rational does not immediately yield information on the dimension of its automorphism group. In fact, there are Hirzebruch surfaces with automorphism groups of arbitrarily large dimension. Along similar lines, the mentioned result of Hwang [14] does not imply that the second jet at an isolated singular point of a vector field on an algebraic manifold cannot vanish. An alternative to overcome these difficulties consists of looking at meromorphic *algebras of semi-complete vector fields*; see [11], [21]. The advantage of this notion lies in the fact that the semi-complete character of meromorphic vector fields is invariant under birational equivalence. In dimension 2, this point of view already appears in [12] where a rather detailed *birational theory* of semi-complete meromorphic vector fields was developed. For the study of these algebras of semi-complete vector fields, however, most of the previously mentioned algebraic methods are no longer available. Moreover, this problem is harder than the original problem of describing algebras of holomorphic vector fields on, say algebraic, manifolds. This additional difficulty was to be expected since most of the corresponding results would have consequences even in the case of transcendental manifolds. A concrete example illustrating these additional difficulties is again obtained from the work of Guillot in [11]: on $\mathbb{C}P(3)$ there is a semi-complete Lie algebra of meromorphic vector fields which is isomorphic to the Lie algebra of $\mathrm{SL}\,(2, \mathbb{C})$ and which cannot be made complete on any Kähler manifold of dimension 3. The fact that this Lie algebra cannot be made complete on any Kähler manifold was pointed out in [24] and the argument is as follows: the blow-up of the singular point associated to the Halphen vector field leads to a foliation having hyperbolic leaves on the exceptional divisor. If the initial manifold had a Kähler structure, a result by Brunella in [3] would imply that the leaves in question ought to be parabolic, hence yielding a contradiction. Still concerning the above mentioned Lie algebra of semi-complete meromorphic vector fields on $\mathbb{C}P(3)$, the Halphen vector field provides a distinguished vector field in this Lie algebra which exhibits a complicated dynamics that is not compatible with the action of algebraic group. Hence, summarizing this paragraph, it can be said that the study of Lie algebras of semi-complete vector fields is harder than the study of Lie algebras that can be made complete through birational transformations because both classical and recent algebraic techniques [13], [14] are not fully adequate to deal with the former. In this direction, an additional advantage of our methods lies in the fact that they only require the Lie algebras of vector fields to be semi-complete though this point will not be emphasized in the paper so as to make it shorter. In any event, at least as far as algebraic/Kähler manifolds are concerned, it seems likely that new progress on basic questions can be made by incorporating the techniques discussed in this paper to the well-established literature.

The discussion conducted in the paper is rather elementary and relies on well-known results. Besides fairly standard facts concerning singular spaces, we also use Malgrange's celebrated theorem in [17], a recent work by Guillot [10] concerning certain "singular" Kato surfaces equipped with vector fields, some basic knowledge of complex surfaces, and Milnor's fibration theorem along with some related material.

## 2. Codimension 1 foliations and invariant curves

Recall that a singular *holomorphic foliation of dimension* 1 on $(\mathbb{C}^3, 0)$ is, by definition, given by the local orbits of a holomorphic vector field $Z$ whose singular set $\mathrm{Sing}\,(Z)$ has codimension 2 or greater. Similarly, a *codimension* 1 *holomorphic foliation* on $(\mathbb{C}^3, 0)$ is associated with the distribution obtained through the kernel of a holomorphic 1-form $\Omega = \alpha\, dx + \beta\, dy + \gamma\, dz$ satisfying the Frobenius condition $\Omega \wedge d\Omega = 0$ and having a singular set $\mathrm{Sing}\,(\Omega)$ of codimension at least 2. In terms of notation, foliations of dimension 1 will typically be denoted by "$\mathcal{F}$" whereas "$\mathcal{D}$" will stand for codimension 1 foliations.

Henceforth we shall consider the setting of Theorem B. Hence there are holomorphic vector fields $X$ and $Y$ defined on a neighborhood $U$ of $(0, 0, 0) \in \mathbb{C}^3$ which are not parallel at every point $p$ in $U$. These vector fields are assumed either to commute or to satisfy the equation $[X, Y] = cY$, for some $c \in \mathbb{C}^*$ where $[X, Y]$ stands for the commutator of the two vector fields in question. In particular, $X$ and $Y$ span a singular codimension 1 foliation denoted by $\mathcal{D}$ whose singular set $\mathrm{Sing}\,(\mathcal{D})$ has codimension at least 2.

To begin our approach to Theorem B, note that $\mathrm{Sing}\,(\mathcal{D})$ is clearly invariant under both $X$ and $Y$. Thus the statement of Theorem B is immediately verified provided that the analytic set $\mathrm{Sing}\,(\mathcal{D})$ has dimension 1. Therefore, throughout this section, we shall assume that $\mathrm{Sing}\,(\mathcal{D})$ is reduced to the origin if not empty. In other words, the foliation $\mathcal{D}$ is either regular or it has an isolated singular point at the origin. Now Malgrange's theorem in [17] ensures that $\mathcal{D}$ is given by the level surfaces of some holomorphic function $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$. Summarizing, in order to prove Theorem B, we can assume without loss of generality that the following lemma holds:

**Lemma 2.1.** *The codimension* 1 *foliation* $\mathcal{D}$ *admits a non-constant holomorphic first integral* $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ *(which is, in fact, a submersion away from* $0 \in \mathbb{C}$*) so that the leaves of* $\mathcal{D}$ *coincide with the level sets of* $f$*. Furthermore, the foliation* $\mathcal{D}$ *is either regular or it has an isolated singular point at the origin.* □

To abridge notation, an analytic set of dimension 2 will be called an analytic surface (or simply a surface if no misunderstood is possible). Similarly an analytic set of dimension 1 will be called an analytic curve (or simply a curve).

Next consider the germ of analytic surface $S$ given by $S = f^{-1}(0)$ which is clearly invariant under both $X$ and $Y$. This surface can be assumed to be *irreducible* otherwise two irreducible components of it would intersect each other over a curve invariant under $X$ and $Y$ and hence satisfying the requirements of Theorem B. In any event, the subsequent discussion makes sense for every irreducible component $S$ of $f^{-1}(0)$. In the sequel, the restrictions of $X$ and $Y$ to $S$ will be respectively denoted by $X_{|S}$ and by $Y_{|S}$. Now, we have:

**Lemma 2.2.** *Suppose that the restrictions* $X_{|S}$ *and* $Y_{|S}$ *of* $X$ *and* $Y$ *to* $S$ *are not everywhere parallel. Then there exists a germ of analytic curve* $\mathcal{C} \subset S$ *invariant under both* $X$ *and* $Y$*.*

*Proof.* Consider first the *singular set* $\mathrm{Sing}\,(S) \subset \mathbb{C}^3$ of $S$ when $S$ is *viewed as an analytic surface in* $\mathbb{C}^3$. Apart from being possibly empty, $\mathrm{Sing}\,(S)$ is an analytic set whose dimension does not exceed 1. Furthermore $\mathrm{Sing}\,(S)$ is naturally invariant under both $X$ and $Y$. Hence, if the dimension of $\mathrm{Sing}\,(S)$ equals 1, the statement of the lemma follows at once.

Therefore, to prove the lemma, we can assume that either $\mathrm{Sing}\,(S)$ is empty or it is reduced to the origin. In either case, let us consider the set $\mathrm{Tang}\,(X_{|S}, Y_{|S})$ consisting of those points $q \in S$ such that $X(q)$ and $Y(q)$ are linearly dependent. By assumption, $\mathrm{Tang}\,(X_{|S}, Y_{|S})$ contains the origin so that it is not empty. Similarly, this set is a proper analytic subset of $S$.

Moreover $\mathrm{Tang}\,(X_{|S}, Y_{|S})$ is invariant under both $X$ and $Y$ since $X$ and $Y$ generates a Lie algebra of dimension 2. Therefore, if this set happens to be an analytic curve, the statement of the lemma results at once. Thus, summarizing what precedes, we may assume the following holds:

- The surface $S$ is either smooth or it has a unique singular point at the origin;
- The *tangency set* $\mathrm{Tang}\,(X_{|S}, Y_{|S})$ is reduced to the origin.

To complete the proof of the lemma, it suffices to check that the above conditions cannot simultaneously be satisfied. For this, note that these conditions imply that the tangent sheaf to $S$ is locally free. Owing to the main result in [26], it follows that $S$ is smooth. However, since $S$ is smooth, the vector fields $X_{|S}$ and $Y_{|S}$ can be identified with vector fields defined around the origin of $\mathbb{C}^2$ and linearly dependent at the origin. Clearly, these two vector fields must remain linearly dependent over some analytic curve unless they are linearly dependent on a full neighborhood of $(0,0) \in \mathbb{C}^2$. The lemma is proved. $\square$

In view of the preceding, in order to prove Theorem B, we can assume without loss of generality that the vector fields $X$ and $Y$ are parallel at every point of $S$. In other words, the vector fields $X_{|S}$ and $Y_{|S}$ are everywhere parallel and thus they induce a unique singular holomorphic foliation on $S$ which will be denoted by $\mathcal{F}_S$. Theorem B is now reduced to showing that the foliation $\mathcal{F}_S$ defined on the analytic surface $S$ admits a separatrix through the origin. Even though this is not necessary, we may then assume that $S$ is not smooth, otherwise the existence of the mentioned separatrix is provided by the main result of [5]. In particular, Theorem B holds if the codimension 1 foliation $\mathcal{D}$ is, indeed, non-singular. In fact, in more accurate terms, Theorem B is now a consequence of the following result:

**Theorem 2.3.** *Suppose that $X$ and $Y$ are as in Theorem B and that they span a codimension 1 foliation $\mathcal{D}$ having an isolated singularity at $(0,0,0) \in \mathbb{C}^3$. Let $S$ be an analytic surface containing the origin and invariant under $\mathcal{D}$ and assume that neither $X$ nor $Y$ vanishes identically on $S$. Finally, assume also that $X$ and $Y$ are everywhere parallel on $S$. Then there is a germ of analytic curve $\mathcal{C}$ passing through the origin which is invariant under $X$ and $Y$.*

Theorem 2.3 is eminently a two-dimensional result valid for singular spaces; its proof begins by observing that the origin must be the unique zero of both $X$ and $Y$ on $S$, otherwise the zero-set of $X$ is invariant by $Y$ (and conversely) which immediately yields the desired invariant curve. Another useful observation is provided by Lemma 2.4 below which is stated in a slightly more general setting. Let $\overline{S} \subset \mathbb{C}^3$ be a germ of analytic surface with an isolated singular point at $(0,0,0) \in \mathbb{C}^3$. Given a singular holomorphic foliation $\mathcal{F}$ on $\overline{S}$, by a *holomorphic first integral* $h$ for $\mathcal{F}$ it is meant a holomorphic function defined on $\overline{S} \setminus \{(0,0,0)\}$ which is constant on the leaves of $\mathcal{F}$. With this terminology, we state:

**Lemma 2.4.** *Given $\overline{S}$ and $\mathcal{F}$ as above, assume that $\mathcal{F}$ admits a non-constant holomorphic first integral $h$. Then $\mathcal{F}$ has a separatrix.*

*Proof.* The argument is standard. For terminology and further detail, the reader is referred to [20], pages 210-212. First we claim that $h$ is *weakly holomorphic* on all of $\overline{S}$ meaning that $h$ is holomorphic on the regular part of $\overline{S}$ and is bounded on a neighborhood of the isolated singular point of $\overline{S}$ (identified with the origin). To check the claim, note that $h$ is assumed to be holomorphic, and therefore weakly holomorphic, on $\overline{S} \setminus \{(0,0,0)\}$. Since the origin has codimension 2 in $\overline{S}$, it follows that $h$ is, in fact, weakly holomorphic on all of $\overline{S}$. On the other hand, $\overline{S}$ is also a normal analytic surface since $\overline{S}$ is contained in $\mathbb{C}^3$ and has isolated singular points. Now, the fact that $\overline{S}$ is normal ensures that $h$ has a holomorphic extension $H$ defined on a neighborhood of $(0,0,0) \in \mathbb{C}^3$.

To finish the proof of the lemma consider the holomorphic function $H : (\mathbb{C}^3, 0) \to \mathbb{C}$ and set $H(0,0,0) = \lambda$. Next, note that the intersection between $\overline{S}$ and the analytic surface $H^{-1}(\lambda)$ is a certain analytic curve $\mathcal{C}$ passing through the origin and contained in $S$. Since the restriction $h$ of $H$ to $\overline{S}$ is a first integral for $\mathcal{F}$, it follows that $\mathcal{C}$ is invariant by $\mathcal{F}$ so that $\mathcal{C}$ constitutes a separatrix for this foliation. $\square$

Given a non-identically zero holomorphic vector field defined on an open set $U$ of some complex manifold, the *order of* $Z$ *at a point* $p \in U$, $\mathrm{ord}_p(Z)$, is simply the order of the first non-identically zero jet of $Z$ at $p$. In particular $\mathrm{ord}_p(Z) = 0$ if and only if $Z$ is regular at $p$.

Going back to the statement of Theorem 2.3, let us now prove a rather useful lemma.

**Lemma 2.5.** *Let $X$, $Y$, and $S$ be as in Theorem 2.3 and denote by $\mathcal{F}_S$ the foliation induced by $X$ and $Y$ on $S$. Then the restrictions $X_{|S}$ and $Y_{|S}$ of $X$ and $Y$ to $S$ must commute.*

*Proof.* We assume aiming at a contradiction that $[X_{|S}, Y_{|S}] = cY_{|S}$ for some $c \in \mathbb{C}^*$. Recall also that $X_{|S}$ and $Y_{|S}$ have a unique zero which coincides with the singular point of $S$. Since $X_{|S}$ and $Y_{|S}$ are everywhere parallel and $X_{|S}$ does not vanish on the regular part of $S$, we have $Y_{|S} = hX_{|S}$ for some holomorphic function $h$ defined on $S \setminus \{(0,0,0)\}$. The argument employed in the proof of Lemma 2.4 then implies that $h$ admits a holomorphic extension $H$ defined on a neighborhood of the origin in $\mathbb{C}^3$. We can assume that $H(0,0,0) \neq 0$ since otherwise the set $H^{-1}(0)$ would intersect $S$ on a analytic curve contained in the zero-set of $Y_{|S}$ which contradicts the assumption that $Y_{|S}$ has isolated zeros.

In the sequel, we assume without loss of generality that $H(0,0,0) = 1$. Consider a (minimal) resolution $\widetilde{S}$ of $S \subset \mathbb{C}^3$. The vector fields $X_{|S}$ and $Y_{|S}$ admit holomorphic lifts $\widetilde{X}_{|S}$ and $\widetilde{Y}_{|S}$ to $\widetilde{S}$, see for example [10]. Similarly, the function $H$ induces the holomorphic function $\widetilde{H} : \widetilde{S} \to \mathbb{C}$ given by $\widetilde{H} = H \circ \Pi$ where $\Pi$ stands for the resolution map $\Pi : \widetilde{S} \to S$. In particular, $\widetilde{H}$ is constant equal to 1 on the exceptional divisor $\Pi^{-1}(0,0,0)$.

We claim that there is a point $p \in \Pi^{-1}(0,0,0)$ at which the order $\mathrm{ord}_p(\widetilde{X}_{|S})$ of $\widetilde{X}_{|S}$ is at least 1. To check the claim, recall that $\mathrm{ord}_p(\widetilde{X}_{|S}) = 0$ if and only if $\widetilde{X}_{|S}$ is regular at $p$. Thus, if the claim is false, the vector field $\widetilde{X}_{|S}$ must be regular on all of the exceptional divisor $\Pi^{-1}(0,0,0)$. Assume that this is the case and consider an irreducible component $D$ of $\Pi^{-1}(0,0,0)$. The curve $D$ must be invariant under $X$ otherwise its self-intersection would be nonnegative what is impossible. This means that $X$ is tangent to $D$ and, since $X$ has no singular points in $\Pi^{-1}(0,0,0)$, $D$ must be an elliptic curve and regular leaf for the foliation associated to $X$. The index formula of [5], or simply the standard Bott connection, implies that the self-intersection of $D$ still vanishes and this yields the final contradiction.

Let then $p \in \Pi^{-1}(0,0,0)$ be such that $\mathrm{ord}_p(\widetilde{X}_{|S}) \geq 1$. Since $\widetilde{H}(p) = 1$, we have $\mathrm{ord}_p(\widetilde{X}_{|S}) = \mathrm{ord}_p(\widetilde{Y}_{|S})$. Now, suppose first that $\mathrm{ord}_p(\widetilde{X}_{|S}) \geq 2$. Then the order $\mathrm{ord}_p([\widetilde{X}_{|S}, \widetilde{Y}_{|S}])$ of $[\widetilde{X}_{|S}, \widetilde{Y}_{|S}]$ at $p$ satisfies

$$\mathrm{ord}_p([\widetilde{X}_{|S}, \widetilde{Y}_{|S}]) \geq 2\,\mathrm{ord}_p(\widetilde{X}_{|S}) - 1 > \mathrm{ord}_p(\widetilde{X}_{|S}) = \mathrm{ord}_p(\widetilde{Y}_{|S})$$

so that the equation $[\widetilde{X}_{|S}, \widetilde{Y}_{|S}] = c\widetilde{Y}_{|S}$ with $c \in \mathbb{C}^*$ cannot hold. Therefore it only remains to check the possibility of having $\mathrm{ord}_p(\widetilde{X}_{|S}) = 1$. For this, note again that $\widetilde{Y}_{|S} = \widetilde{H}\widetilde{X}_{|S}$ so that the first jet of $\widetilde{X}_{|S}$ and of $\widetilde{Y}_{|S}$ coincide at $p$ (recall that $\widetilde{H}(p) = 1$). Thus the linear part of the commutator $[\widetilde{X}_{|S}, \widetilde{Y}_{|S}]$ must still vanish and this ensures that the order of $[\widetilde{X}_{|S}, \widetilde{Y}_{|S}]$ at $p$ is again strictly larger than the order of $\widetilde{Y}_{|S}$. This contradicts the equation $[\widetilde{X}_{|S}, \widetilde{Y}_{|S}] = c\widetilde{Y}_{|S}$ and completes the proof of the lemma. $\square$

The next lemma is an application of Lemmas 2.4 and 2.5.

**Lemma 2.6.** *Let $X$, $Y$, $S$, and $\mathcal{F}_S$ be as in Theorem 2.3. If the restrictions $X_{|S}$ and $Y_{|S}$ of $X$ and $Y$ to $S$ do not differ by a multiplicative constant, then $\mathcal{F}_S$ admits a separatrix.*

*Proof.* As mentioned, the vector fields $X_{|S}$ and $Y_{|S}$ can be assumed to have an isolated zero at $(0,0,0) \subset S$. Moreover Lemma 2.5 ensures that $[X_{|S}, Y_{|S}] = 0$. In particular, the function $h$ defined on $S \setminus \{(0,0,0)\}$ by means of the equation $Y_{|S} = hX_{|S}$ must be a first integral of $\mathcal{F}_S$. Hence Lemma 2.4 ensures that $\mathcal{F}_S$ admits a separatrix unless $h$ is constant. The lemma is proved. $\square$

We are now ready to prove Theorem 2.3.

*Proof of Theorem 2.3.* We assume aiming at a contradiction that no germ of curve $\mathcal{C} \subset S$ passing through the origin is simultaneously invariant under $X$ and $Y$. Since $X$ and $Y$ are everywhere parallel on $S$, the last assertion simply means that no germ of curve $\mathcal{C} \subset S$ through the origin is invariant by either $X$ or $Y$ (individually). Next consider the vector fields $X_{|S}$ and $Y_{|S}$ obtained as restrictions to $S$ of $X$ and $Y$, respectively. As already mentioned, the singular set of $S$ must consist of isolated points. Moreover this singular set contains the zero sets of both $X$ and $Y$.

Next note that Lemma 2.5 implies that the commutator $[X_{|S}, Y_{|S}]$ vanishes identically on $S$, i.e. the commutator of the initial vector fields $[X, Y]$ equals zero on the surface $S$. Since the zero sets of $X$ and $Y$ are constituted by isolated points, there follows that the equation $[X, Y] = cY$ cannot hold for $c \neq 0$. In other words, only the abelian case in which $[X, Y] = 0$ needs to be considered in the remainder of the proof. In addition, owing to Lemma 2.6, there also follows that the vector field $Y_{|S}$ must be a constant multiple of $X_{|S}$.

In the sequel we denote by $\mathfrak{G}$ the abelian Lie algebra generated by $X$ and $Y$. Also consider $c_0 \in \mathbb{C}$ such that $Y_{|S} = c_0 X_{|S}$ and set $Z_1 = Y - c_0 X$. Clearly $Z_1$ belongs to $\mathfrak{G}$ and $Z_1$ does not vanish identically on a neighborhood of the origin since $X$ and $Y$ are linearly independent at generic points. In fact, $Z_1$ and $X$ are linearly independent at generic points. However, by construction, the vector field $Z_1$ equals zero on the surface $S = f^{-1}(0)$, where $f$ is the first integral of $\mathcal{D}$ mentioned in Lemma 2.1. Thus, there is $k_1 \in \mathbb{N}^*$ such that $Y_1 = Z_1/f^{k_1}$ is a holomorphic vector field defined on a neighborhood of $(0,0,0) \in \mathbb{C}^3$ which does not vanish identically on $S$ (recall that $S$ is assumed to be irreducible). Moreover, $Y_1$ is still tangent to the foliation $\mathcal{D}$. In particular $Y_1$ leaves $S$ invariant.

*Claim.* The vector field $Y_1$ commutes with $X$ and with $Y$. Furthermore the restriction $Y_{1|S}$ of $Y_1$ to $S$ is still a constant multiple of $X_{|S}$.

*Proof of the Claim.* Since $f$ is a first integral of both $X$ and $Y$, the fact that $[X, Y_1] = 0$ (resp. $[Y, Y_1] = 0$) follows from observing that $f^{k_1} Y_1$ lies in the abelian Lie algebra $\mathfrak{G}$. Furthermore $Y_1$ is still linearly independent with $X$ at generic points since so is $Z_1$. There follows that the abelian Lie algebra $\mathfrak{G}_1$ generated by $X$ and by $Y_1$ is isomorphic to $\mathfrak{G}$ and still spans the same foliation $\mathcal{D}$.

Next recall that no germ of curve $\mathcal{C}_1 \subset S$ through the origin can be invariant under $X$. Hence Lemma 2.2 ensures that $X_{|S}$ and $Y_{1|S}$ are again everywhere parallel. In turn, Lemma 2.6 implies that $Y_{1|S}$ is a constant multiple of $X_{|S}$. The claim is proved. $\square$

Setting $Y_{1|S} = c_1 X_{|S}$, the vector field $Z_2 = Y_1 - c_1 X$ lies in $\mathfrak{G}_1$ and is linearly independent with $X$ at generic points since $Y_1$ is so (in particular $Z_2$ does not vanish identically on a neighborhood of $(0,0,0) \in \mathbb{C}^3$). Yet this vector field vanishes identically over $S$ so that we obtain $Z_2 = Y_1 - c_1 X = f^{k_2} Y_2$ where $Y_2$ is a holomorphic vector field defined around $(0,0,0) \in \mathbb{C}^3$ which does not vanish identically on $S$. Moreover, $Y_2$ is again tangent to the foliation $\mathcal{D}$ and leaves $S$ invariant. Again the vector field $Y_2$ commutes with $X$. The abelian lie algebra generated generated by $X$ and $Y_2$ is then denoted by $\mathfrak{G}_2$. Since $Z_2$ is linearly independent with $X$ at generic points, so is $Y_2$. In other words, the Lie algebra $\mathfrak{G}_2$ spans the foliation $\mathcal{D}$ as well. At this point, the combination of Lemmas 2.2 and 2.6 ensures again that ($Y_2$ and $X$ commute and that) the *restriction* $Y_{2|S}$ *of* $Y_2$ *to* $S$ *is a constant multiple of* $X_{|S}$.

Summarizing what precedes, we have

$$Y = c_0 X + f^{k_1} Y_1 = c_0 X + c_1 f^{k_1} X + f^{k_1+k_2} Y_2 \,.$$

We can continue the argument by induction. There is $c_2 \in \mathbb{C}^*$ such that $Y_{2|S} = c_2 X_{|S}$. The vector field $Z_3 = Y_2 - c_2 X$ lies in $\mathfrak{G}_2$ and, in turn, can be written as $Z_2 = f^{k_3} Y_3$ where $Y_3$ is a holomorphic vector field which does not vanish identically on $S$. Furthermore $Y_3$ is tangent to $\mathcal{D}$ and is linearly independent with $X$ at generic points. Thus the preceding argument can be applied to $Y_3$ as well. Therefore the restriction $Y_{3|S}$ of $Y_3$ to $S$ is a constant multiple of $X_{|S}$. In other words, we have

$$Y = c_0 X + c_1 f^{k_1} X + c_2 f^{k_1+k_2} X + f^{k_1+k_2+k_3} Y_3 \,.$$

By continuing this procedure we shall eventually find a vector field $Y_n$ as above whose restriction to $S$ will not be a constant multiple of $X$ since $X$ and $Y$ are linearly independent at generic points on a neighborhood of $(0,0,0) \in \mathbb{C}^3$. A contradiction proving Theorem 2.3 will then immediately arise. □

*Proof of Theorem B.* It follows from Theorem 2.3 combined to the discussion conducted in the beginning of this section. We summarize the argument for the convenience of the reader.

Consider holomorphic vector fields $X$ and $Y$ as in Theorem B. We assume aiming at a contradiction that there is no (germ of) analytic curve $\mathcal{C} \subset \mathbb{C}^3$ containing the origin and simultaneously invariant under $X$ and $Y$. In particular, this implies that the codimension 1 foliation $\mathcal{D}$ spanned by $X$ and $Y$ possesses an isolated singular point at the origin. Furthermore this foliation is given by the level surfaces of a holomorphic function $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$. Denote by $S$ an irreducible component of $f^{-1}(0)$ and consider the restrictions $X_{|S}$ and $Y_{|S}$ of $X$ and $Y$ to $S$.

Since we are assuming that Theorem B does not hold, it follows from Lemma 2.2 that $X_{|S}$ and $Y_{|S}$ are parallel at every point of $S$. Note that these two vector fields cannot simultaneously be identically zero for otherwise the statement is obvious. If none of these vector fields vanish identically then Theorem B follows from Theorem 2.3. Therefore there remains only one possibility not yet covered by our discussion which corresponds to the situation where one of the vector fields vanishes identically over $S$ while the other has an isolated zero at the origin (identified with the isolated singular point of $S$). This case can however be dealt with by arguing as in the proof of Theorem 2.3. This is as follows. To fix notation, suppose that $Y_{|S}$ vanishes identically. Then, up to dividing $Y$ by a suitable power of the first integral $f$, we shall obtain a vector field $\overline{Y}$ whose restriction to $S$ is no longer identically zero. Again the Lie algebra generated by $X$ and by $\overline{Y}$ is isomorphic to the Lie algebra generated by $X$ and by $Y$. Thus the procedure described in the proof of Theorem 2.3

to produce a separatrix for the foliation induced on $S$ can be started with the vector fields $X$ and $\overline{Y}$. This completes the proof of Theorem B. □

## 3. Lie groups actions and holomorphic vector fields

All vector fields considered in the remainder of this paper are assumed not to vanish identically unless otherwise stated.

Consider a compact complex manifold $M$ and let $\mathrm{Aut}\,(M)$ denote the group of holomorphic diffeomorphisms of $M$. It is well known that $\mathrm{Aut}\,(M)$ is a finite-dimensional complex Lie group whose action on $M$ is faithful and holomorphic. Moreover, the Lie algebra of $\mathrm{Aut}\,(M)$ can be identified to the Lie algebra $\mathfrak{X}\,(M)$ formed by all holomorphic vector fields defined on $M$, see for example [1].

Since $M$ is compact, every holomorphic vector field $X$ defined on all of $M$ is *complete* in the sense that it gives rise to an action of $(\mathbb{C}, +)$ on $M$. In particular, the restriction of $X$ to every open set $U \subset M$ is *semi-complete* on $U$, according to the definition given in [22] which is recalled below for the convenience of the reader.

**Definition 3.1.** *A holomorphic vector field $X$ on a complex manifold $N$ is called semi-complete if for every $p \in N$ there exists a connected domain $U_p \subset \mathbb{C}$ with $0 \in U_p$ and a holomorphic map $\phi_p : U_p \to N$ satisfying the following conditions:*

- *$\phi_p(0) = p$ and $d\phi_p/dt|_{t=t_0} = X(\phi_p(t_0))$.*
- *For every sequence $\{t_i\} \subset U_p$ such that $\lim_{i\to\infty} t_i \in \partial U_p$ the sequence $\{\phi_p(t_i)\}$ escapes from every compact subset of $N$.*

If $X$ is semi-complete on $U$ and $V \subset U$ is another open set, then the restriction of $X$ to $V$ is semi-complete as well. Hence, there is a well-defined notion of *semi-complete singularity*. Furthermore, if a singularity of a vector field happens *not to be* semi-complete, then this singularity cannot be realized as singularity of a vector field defined on a compact complex manifold. Keeping this principle in mind, a fundamental result implicitly formulated in [22] reads as follows.

**Lemma 3.2.** *Assume that $Y$ is a holomorphic vector field with isolated singular points and defined around the origin of $\mathbb{C}^n$. Suppose that the second jet $J^2(Y)\,(0, \ldots, 0)$ of $Y$ at the origin vanishes and that $Y$ is tangent to some irreducible analytic curve $\mathcal{C}$ passing through the origin. Then the germ of $Y$ is not semi-complete.*

*Proof.* It suffices to sketch the elementary argument. The restriction $Y_{\mathcal{C}}$ of $Y$ to $\mathcal{C}$ does not vanish identically since the origin is an isolated singularity of $Y$. Clearly it suffices to show that $Y_{\mathcal{C}}$ is not semi-complete. For this, recall that the curve $\mathcal{C}$ admits an irreducible Puiseux parameterization $\mathcal{P} : (\mathbb{C}, 0) \to (\mathcal{C}, 0)$. In particular, the pull-back $\mathcal{P}^*Y_{\mathcal{C}}$ is a (non-identically zero) holomorphic vector field defined on a neighborhood of $0 \in \mathbb{C}$. The proof of the lemma is then reduced to checking that $\mathcal{P}^*Y_{\mathcal{C}}$ is not semi-complete on any neighborhood of $0 \in \mathbb{C}$.

Since the second jet of $Y$ vanishes at the origin of $\mathbb{C}^n$, it is straightforward to check that the second jet of $\mathcal{P}^*Y_{\mathcal{C}}$ vanishes at $0 \in \mathbb{C}$. Hence $\mathcal{P}^*Y_{\mathcal{C}}$ is locally given by $z^s f(z)\partial/\partial z$ for some $s \geq 3$ and a holomorphic function $f$ satisfying $f(0) \neq 0$. Now, assuming that $\mathcal{P}^*Y_{\mathcal{C}} = z^s f(z)\partial/\partial z$ is semi-complete, it follows that so are the vector fields $Y_n = \Lambda_n^*(z^s f(z)\partial/\partial z)$ where $\Lambda_n : \mathbb{C} \to \mathbb{C}$ is given by $\Lambda_n(z) = z/n$. Also every constant multiple of a semi-complete vector field is again semi-complete. Putting together these two remarks, the vector field

$z^s\partial/\partial z$ is the uniform limit of a sequence of semi-complete vector fields which is obtained by suitably renormalizing the sequence $\{Y_n\}$. There follows that $z^s\partial/\partial z$ must be semi-complete as well, see for example [8]. However the solution $\phi$ of the differential equation corresponding to $z^s\partial/\partial z$ is given by

$$\phi(T) = \frac{z_0}{\sqrt[s-1]{1 - Tz_0^{s-1}(s-1)}}$$

where $\phi(0) = z_0$. Since $s \geq 3$, this solution is multivalued so that $z^s\partial/\partial z$ is never semi-complete on a neighborhood of $0 \in \mathbb{C}$. The lemma follows. □

Let then $M$ be as in Theorem A. In particular, the dimension of $\mathfrak{X}(M)$ is at least 2. Hereafter, we assume aiming at a contradiction that $Y \in \mathfrak{X}(M)$ is a holomorphic vector field possessing an isolated singular point $p \in M$ where the second jet of $Y$ vanishes (notation: $J^2(Y)(p) = 0$). The idea is then to exploit Theorem B and Lemma 3.2 to derive a contradiction implying Theorem A. In this section, we shall content ourselves with establishing a slightly weaker statement, namely:

**Theorem 3.3.** *Assume $Y \in \mathfrak{X}(M)$ is a holomorphic vector field possessing an isolated singular point $p \in M$ where its second jet vanishes. Then $Y$ admits a non-constant meromorphic first integral $f$ defined on $M$. Moreover, the Lie algebra $\mathfrak{X}(M)$ contains a vector field $X$ such that $X = fY$.*

In the next section, we shall study in detail the exceptional situation described in Theorem 3.3 so as to exclude its existence. Theorem A will then automatically follow.

In the remainder of this section, we shall focus on the proof of Theorem 3.3.

For every non-identically zero vector field $Z \in \mathfrak{X}(M)$, denote by $\mathrm{ord}_p(Z)$ the order of $Z$ at $p \in M$, see Section 2. We then set $k = \mathrm{ord}_p(Y)$ so that, by assumption, we have $k \geq 3$. Since the vector space $\mathfrak{X}(M)$ has finite dimension, the subset of $\mathbb{N}$ formed by those integers that are realized as the order at $p$ of some vector field $Z \in \mathfrak{X}(M)$ is finite. The (attained) supremum of this set will be denoted by $m \in \mathbb{N}^*$. Clearly $m \geq k \geq 3$. Now let us consider two different possibilities for $m$ and $k$.

A. We have $m \geq k$ and, if $m = k$, then there is a vector field $X \in \mathfrak{X}(M)$ which is not a constant multiple of $Y$ and satisfies $\mathrm{ord}_p(X) = m = k$.
B. We have $m = k$ and every vector field $Z \in \mathfrak{X}(M)$ satisfying $\mathrm{ord}_p(Z) = m = k$ is a constant multiple of $Y$.

Case A. To prove Theorem 3.3 in this case, we proceed as follows. Fix a vector field $X \in \mathfrak{X}(M)$, which is not a constant multiple of $Y$, and whose order at $p$ equals $m$. Note that the commutator $[X, Y]$ must vanish identically since, otherwise, its order at $p$ is at least $m + k - 1 \geq m + 2 > m$ which contradicts the definition of $m$.

*Proof of Theorem 3.3 in* Case A. According to Lemma 3.2, the vector field $Y$ cannot admit a local separatrix through $p$. Since we have $[X, Y] = 0$, Theorem B implies that $X$ and $Y$ must be everywhere parallel so that they define the same singular holomorphic foliation $\mathcal{F}$ on $M$. Also we can write $X = fY$ for a certain meromorphic function $f$ defined on $M$. This meromorphic function is by assumption non-constant. Furthermore, as already seen, the restriction of $f$ to a leaf of $\mathcal{F}$ must be constant since $[X, Y] = 0$. The proof of Theorem 3.3 in Case A is then completed. □

Case B. Consider a vector field $X \in \mathfrak{X}(M)$ which is not a constant multiple of $Y$. The existence of $X$ is ensured by the fact that the dimension of $\mathfrak{X}(M)$ is at least 2.

*Proof of Theorem 3.3 in* Case B. Consider the order $\mathrm{ord}_p(X)$ of $X$ at the point $p \in M$. Assume first that $\mathrm{ord}_p(X) \geq 2$. Then $X$ and $Y$ must commute since otherwise the order of $[X, Y]$ is at least $k - 1 + \mathrm{ord}_p(X) \geq k + 1$ which contradicts the fact that $m = k$. Since Lemma 3.2 ensures that $Y$ cannot have a separatrix through $p$, it follows from Theorem B that $X$ is everywhere parallel to $Y$. Now the argument used to prove Theorem 3.3 in Case A can also be applied to the present situation to complete the proof of the theorem. In fact, the mentioned argument shows that whenever $\mathfrak{X}(M)$ contains a vector field commuting with $Y$, the statement of Theorem 3.3 must hold.

Suppose now that $\mathrm{ord}_p(X) \leq 1$. As mentioned above, we can assume without loss of generality that $X$ does not commute with $Y$. Consider then $Z = [X, Y] \in \mathfrak{X}(M)$. The order $\mathrm{ord}_p(Z)$ of $Z$ at $p$ is at least $k - 1 \geq 2$ which, in turn, ensures that $[Z, Y] = 0$. Otherwise the order of $[Z, Y]$ would be at least $k + k - 1 - 1 \geq k + 1$ ($k \geq 3$) which contradicts the assumption that $m$ is the greatest integer realized as the order at $p$ of a vector field in $\mathfrak{X}(M)$. Since $[Z, Y] = 0$, Theorem 3.3 will again follow unless $Z$ is a constant multiple of $Y$. Hence, we only need to discuss this last possibility.

Therefore we assume that $Z = [X, Y] = cY$ for some $c \in \mathbb{C}^*$. To obtain a contradiction finishing the proof of Theorem 3.3 it suffices to check that $X$ and $Y$ cannot be everywhere parallel. In fact, if $X$ and $Y$ are linearly independent at generic points of $M$, then Theorem B implies the existence of a separatrix for $Y$ through $p$ which is impossible. Finally, to check that $X$ and $Y$ have to be linearly independent at generic points, suppose for a contradiction that they were parallel at every point of $M$. Then we can again consider a meromorphic function $f$ such that $X = fY$. However this function $f$ must be holomorphic around $p$ since $p$ is an isolated singularity of $Y$ and $X$ is holomorphic. From this it follows that the order of $X$ at $p$ is greater than or equal to the order of $Y$ at $p$. A contradiction immediately arises since the former is bounded by 1 while the latter is at least 3. The proof of the theorem is completed. □

**Remark 3.4.** The purpose of this remark is to substantiate the claim made in the introduction according to which the assumption $\dim(\mathrm{Aut}(M)) \geq 2$ is superfluous if $M$ is algebraic/Kähler. In fact, suppose that $\dim(\mathrm{Aut}(M)) = 1$ and let $X$ be the unique holomorphic vector field on $M$. The flow of $X$ can be identified with $\mathrm{Aut}_0(M)$. Next assume also that $M$ is algebraic. Then the identification of $\mathrm{Aut}_0(M)$ implies that the flow of $\mathbb{C}$ yields an algebraic action of $\mathbb{C}$ on $M$. According to Rosenlicht classical theorem [25], the vector field $X$ must admit 2 independent meromorphic first integrals given by the existence of geometric quotient for the action in question. From this it becomes clear that every singular point of $X$ must admit a separatrix. The conclusion then follows again from Lemma 3.2.

In the case where $M$ is only supposed to be Kähler, the same argument applies up to substituting Rosenlicht theorem by Fujiki's construction in [7].

Let us close this section with the proof of Theorem C.

*Proof of Theorem C.* Suppose for a contradiction that $Y$ is a complete holomorphic vector field on $N$ having an isolated singular point $p \in N$ at which its linear part is equal to zero. First we are going to show that the desired contradiction arises as soon as the vector field

$Y$ has a separatrix $\mathcal{C}$ passing through $p$. For this suppose that $\mathcal{C}$ is a separatrix for $Y$ at $p$ as indicated. Naturally we can assume that $\mathcal{C}$ is irreducible so that an irreducible Puiseux parametrization $\mathcal{P} : (\mathbb{C}, 0) \to (\mathcal{C}, p)$ for $\mathcal{C}$ can be chosen. As in the proof of Lemma 3.2, consider the one-dimensional vector field $\mathcal{P}^* Y_{\mathcal{C}}$ defined around $0 \in \mathbb{C}$. Since the linear part of $Y$ at $p$ equals zero, there follows that the linear part of $\mathcal{P}^* Y_{\mathcal{C}}$ at $0 \in \mathbb{C}$ is also equal to zero. Furthermore, since the vector field $\mathcal{P}^* Y_{\mathcal{C}}$ must be semi-complete, its quadratic part at $0 \in \mathbb{C}$ is necessarily different from zero. From this we also conclude that $\mathcal{C}$ is smooth at $p$. To derive a contradiction, we now proceed as follows. Let $\mathcal{C}_Y$ denote the saturated of $\mathcal{C}$ by the flow of $Y$. In other words, $\mathcal{C}_Y$ is the global orbit of $Y$ passing through a point $\mathcal{P}(\epsilon)$, for some small $\epsilon \neq 0$, union with the singular point $p$ itself. Clearly $\mathcal{C}$ is an immersed, smooth Riemann surface (possibly non-compact) in $N$. Also, by assumption, the restriction of $Y$ to $\mathcal{C}_Y$ endows this Riemann surface with a non-trivial complete holomorphic vector field. Furthermore, the vector field in question has a quadratic singular point (identified with $p$). On the other hand, there follows from the standard Riemann uniformization that the only Riemann surface admitting a non-trivial complete holomorphic vector field exhibiting a quadratic singular point is the Riemann sphere. Therefore $\mathcal{C}_Y$ must be the Riemann sphere and, in particular, compact. This is impossible since $N$ is Stein.

Summarizing what precedes, in order to obtain a contradiction proving Theorem C, we can assume without loss of generality that $Y$ does not have a separatrix $\mathcal{C}$ at $p$. Now, by repeating the argument used in the proof of Theorem 3.3, we still conclude the existence of a complete vector field $X$ on $N$ having the form $X = fY$ for some non-constant meromorphic first integral $f$ of $X, Y$. Moreover $f$ must be holomorphic on a neighborhood of $p$ since $Y$ has an isolated singularity at $p$. Letting $f(p) = \alpha \in \mathbb{C}$, consider the possibly singular surface $f^{-1}(\alpha)$ containing $p$ and invariant by $X$. Since this surface may be disconnected so that we denote by $D$ a connected irreducible component of $f^{-1}(\alpha)$ containing $p$. To finish the proof of Theorem C, it suffices to check that the restriction $Y_D$ of $Y$ to $D$ possesses a local separatrix at $p$. Note that, by construction, $D$ is a possibly open singular surface equipped with a complete vector field given by the restriction of $Y$. Moreover, if this vector field has no separatrix at the singular point $p$, then we are in conditions to apply the main theorem in [10] (see Lemma 4.1 for further details). This theorem enables to conclude that $D$ is compact from the fact that the restriction of $Y$ to this surface is complete. The compactness of $D$, however, clearly contradicts the assumption that $N$ is Stein. The proof of Theorem C is finished. □

## 4. Proof of Theorem A

In this section we shall derive Theorem A from Theorem 3.3. To do this, we shall study the case of exception described in Theorem 3.3 with a view to show that the corresponding situation can never be produced. The discussion below relies heavily on the recent work of Guillot [10].

In what follows, we assume that $M$ is a compact complex manifold of dimension 3 carrying a holomorphic vector field $Y$ which has an isolated singularity $p \in M$ at which its second jet vanishes. In other words, we have $J^2(Y)(p) = 0$. We also assume the existence of another *holomorphic vector field* $X$ on $M$ having the form $X = fY$ where $f$ is a non-constant meromorphic first integral of both $X$ and $Y$. Since $X$ is holomorphic, it follows that the divisor of poles of $f$ must be contained in the divisor of zeros of $Y$ which is therefore non-empty. In any event, we know already that $f$ is holomorphic on a neighborhood of $p$. Set $f(p) = \lambda \in \mathbb{C}$ and consider the compact (possibly singular) surface $S \subset M$ given by $S = f^{-1}(\lambda)$. Denote by $Y_{|S}$ (resp. $X_{|S}$) the restriction of $Y$ (resp. $X$) to $S$.

**Lemma 4.1.** *The surface $S$ has a unique singular point at $p$ and its minimal resolution $\widetilde{S}$ is a Kato surface. Furthermore the vector field $Y_{|S}$ has no singular point other than $p \in M$.*

*Proof.* Clearly $Y_{|S}$ is complete on $S$ so that it is also semi-complete on a neighborhood of $p \in S$. Since $p$ is an isolated singular point of $Y$ (and hence of $Y_{|S}$) where the second jet of $Y$ equals zero, it follows from Lemma 3.2 that the foliation on $S$ associated with $Y_{|S}$ has no separatrix at $p$. This is exactly the context of Guillot's work [10]. In fact, he shows that the minimal resolution of $S$ is a Kato surface. Furthermore the Kato surface $\widetilde{S}$ carries a unique holomorphic vector field (represented by the corresponding transform of $Y_{|S}$) and this vector field is regular away from the exceptional divisor. This implies our lemma. □

In view of the preceding, we shall call $S$ *a singular Kato surface*. It is understood that a singular Kato surface has a unique singular point and admits an ordinary Kato surface as minimal resolution.

Recall that the pole divisor of $f$ is contained in the zero-divisor of $Y$. Since $Y_{|S}$ has a unique singular point (the isolated singularity $p$), it follows that the zero-divisor of $Y$ does not intersect $S$. Therefore the meromorphic function $f$ is actually holomorphic on a neighborhood of $S$ in $M$. Thus, restricted to a neighborhood $U$ of $S$, $f$ induces a structure of singular fibration in $U$ with connected fibers. In fact, the following holds:

**Lemma 4.2.** *The function $f$ induces a proper holomorphic map $F : U \to \mathbb{D} \subset \mathbb{C}$, where $\mathbb{D}$ denote the unit disc of $\mathbb{C}$, satisfying the following conditions:*

1. *For every $z \in D \setminus \{0\}$ the fiber $F^{-1}(z)$ over $z$ is a connected smooth complex surface.*
2. *The (connected) fiber $F^{-1}(0)$ is a singular Kato surface whose singular point is denoted by $p \in U \subset M$.*
3. *The vector field $Y$ is tangent to the fibers of $F$. Furthermore $p$ is the only singular point of $Y$ in $U$.*

*Proof.* The first item is immediate up to reducing the neighborhood $U$ of $S$. In fact, $F$ defines a proper fibration whose set of critical values is a proper analytic subset of $\mathbb{D}$. Therefore up to reducing $U$ (or $\mathbb{D}$), we can assume without loss of generality that $0 \in \mathbb{D} \subset \mathbb{C}$ is the only critical value of $F$ which implies item (1). In turn, item (2) follows from Lemma 4.1.

Naturally the statement in item (3) is also to be understood up to reducing $U$. To check it, consider a neighborhood $V \subset U$ of $p$ containing no other singular point of $Y$. The existence of $V$ is nothing but the assumption that $p$ is an isolated singular point for $Y$. We also know that $Y_{|S}$ has no singular point on the compact set $S \setminus V$; see Lemma 4.1. Thus there is a neighborhood $W \subset M$ of $S \setminus V$ on which $Y$ is regular. It then suffices to choose $U$ contained in the union $V \cup W$. The proof of the lemma is completed. □

The next step consists of further detailing the structure of the fiber $F^{-1}(0)$. First recall that the general construction of Kato surfaces can be summarized as follows. Consider a non-singular surface $\widehat{S}$ along with a divisor $\widehat{D}$ which can be collapsed to yield a neighborhood of the origin in $\mathbb{C}^2$. In other words, $(\widehat{S}, \widehat{D})$ is obtained by means of finitely many blow-ups sitting above the origin of $\mathbb{C}^2$. In particular, to the pair $(\widehat{S}, \widehat{D})$ it is associated the contraction map $\widehat{\pi} : (\widehat{S}, \widehat{D}) \to (\mathbb{C}^2, 0)$. Next consider a point $q \in \widehat{D}$ which, for our purposes, can be chosen as a regular point of $\widehat{D}$. Suppose also that we are given a local holomorphic diffeomorphism $\widehat{\sigma} : (\mathbb{C}^2, 0) \to (\widehat{S}, q)$. The pair $(\widehat{\pi}, \widehat{\sigma})$ is said to be the *Kato data* of the surface. To obtain a Kato surface from the pair $(\widehat{\pi}, \widehat{\sigma})$, we choose $\varepsilon > 0$ sufficiently small and the manifold with boundary $\widehat{N}$ given by

$$\widehat{N} = \widehat{\pi}^{-1}(B^4_\varepsilon \cup \Delta^3_\varepsilon) \setminus \widehat{\sigma}(B^4_\varepsilon)$$

where $B^4_\varepsilon \subset \mathbb{C}^2$ stands for the open ball around the origin of radius $\varepsilon$ and where $\Delta^3_\varepsilon = \partial B^4_\varepsilon$ is the boundary of $B^4_\varepsilon$. Naturally the superscripts 3 and 4 are intended to reminding us of the corresponding real dimensions and they might be useful to avoid confusion in the subsequent discussion. The boundary of $\widehat{N}$ has two connected components $\widehat{\pi}^{-1}(\Delta^3_\varepsilon)$ and $\widehat{\sigma}(\Delta^3_\varepsilon)$ which can be identified to each other by means of $\widehat{\sigma} \circ \widehat{\pi}$. The quotient of this identification is a *Kato surface* which can be made minimal up to contracting all exceptional curves.

Recall that $S = F^{-1}(0)$ has a unique singular point $p$ and that $S$ is equipped with a holomorphic vector field $Y_{|S}$ having no zeros away from $p$. Moreover, $S$ contains no germ of analytic curve passing through $p$ and invariant under $Y_{|S}$. Denoting by $\widetilde{S}$ the minimal (good) resolution of $S$, the main result of [10] can now be rephrased in our context as follows.

- $\widetilde{S}$ is a Kato surface and comes equipped with a natural resolution map $\pi_S : \widetilde{S} \to S$ which is a diffeomorphism away from $\widetilde{D} = \pi^{-1}(p) \subset \widetilde{S}$.
- The vector field $Y_{|S}$ lifts to a holomorphic vector field $\widetilde{Y}_{|S}$ on $\widetilde{S}$ whose associated foliation is denoted by $\widetilde{\mathcal{F}}_{|S}$ (note that $\widetilde{Y}_{|S}$ possesses a curve of zeros).
- The irreducible components of the divisor $\widetilde{D} = \pi^{-1}(p)$ are all rational curves and they are invariant under $\widetilde{\mathcal{F}}_{|S}$.
- The dual graph of $\widetilde{D} = \pi^{-1}(p)$ contains a unique cycle (and it is not reduced to this cycle). Furthermore, the only vertices of this graph having valence greater than 2 belong to the cycle and their valences are exactly 3 (see Figure 1).

As mentioned, we shall say that $S = F^{-1}(0)$ is a *singular Kato surface* meaning that $S$ and its minimal good resolution $\widetilde{S}$ satisfy all of the above conditions.

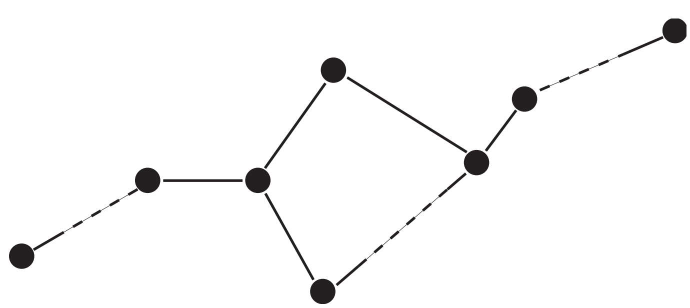

FIGURE 1. Divisor $\widetilde{D}$

On the other hand, there also follows from Lemma 4.2 that the fiber $F^{-1}(t)$ over $t \in \mathbb{D}^*$ is a complex surface equipped with a non-singular holomorphic vector fields $Y_{|F^{-1}(t)}$. According to Mizuhara [19], see also [2], the fiber $F^{-1}(t)$ belongs to the following list:

(1) A complex torus $\mathbb{C}^2/\Lambda$;
(2) A flat holomorphic fiber bundle over an elliptic curve;
(3) An elliptic surface without singular fibers or with singular fibers of type $mI_0$ only. In other words, the singular fibers are elliptic curves with finite multiplicity;
(4) A Hopf surface;
(5) A positive Inoue surface.

Additional information on each of these surfaces will be given as they become necessary. Recall that the proof of Theorem A is reduced to showing that a fibration $F : U \to \mathbb{D} \subset \mathbb{C}$ satisfying all of the preceding conditions cannot exist. More precisely, assuming that $S = F^{-1}(0)$ is a singular Kato surface, we are going to show that the regular fibers $F^{-1}(t)$, $t \neq 0$, cannot belong to the Mizuhara's list (1)—(5) of surfaces equipped with non-singular vector fields. This will be done by way of contradiction. Thus we assume aiming at a contradiction that $F : U \to \mathbb{D} \subset \mathbb{C}$ is such that $S = F^{-1}(0)$ is a singular Kato surface and that the remaining fibers $F^{-1}(t)$ are one of the surfaces in Mizuhara's list.

To begin with, consider the fibration $F : U \to \mathbb{D}$ given by Lemma 4.2 and recall that $F$ has connected fibers (by construction). Denote by $m \in \mathbb{N}^*$ the multiplicity of the singular fiber $S = F^{-1}(0)$.

Next consider a small open ball $B^6_\varepsilon \subset U$ about $p \in S = F^{-1}(0)$ whose boundary will be denoted by $\Delta^5_\varepsilon$. The ball $B^6_\varepsilon$ can be chosen so that its boundary $\Delta^5_\varepsilon$ intersects $S$ transversely. We set $S_\varepsilon = B^6_\varepsilon \cap S$ and $S_c = S \setminus (B^6_\varepsilon \cup \Delta^5_\varepsilon)$. Thus both sets $S_\varepsilon$ and $S_c$ are open in $S$ and they have the same boundary which coincides with $S \cap \Delta^5_\varepsilon$. Now, we also set $S_\Delta = S \cap \Delta^5_\varepsilon = \partial S_\varepsilon = \partial S_c$.

Next note that $S_\Delta = S \cap \Delta^5_\varepsilon$ is a real 3-dimensional manifold whose first Betti number equals 1 as it follows from the fact that the minimal resolution $\widetilde{S}$ is a Kato surface (cf. the structure of the divisor $\widetilde{D}$).

Now consider a regular fiber $F_t = F^{-1}(t)$, $t \neq 0$, of $F : U \to \mathbb{D} \subset \mathbb{C}$. Modulo choosing $t$ very small, we set $F_{t,\varepsilon} = B^6_\varepsilon \cap F_t$. The resulting situation is similar to the classical context of Milnor's fibration theorem [18]. The only difference arises from the multiplicity $m$ of the singular fiber $S = F^{-1}(0)$ which, in the local setting of Milnor theorem, is always equal to 1. The effect of the (global) multiplicity $m$ of $S = F^{-1}(0)$ translates in the local context of Milnor as follows. The surface $F_{t,\varepsilon}$ has $m$ components $F^{(1)}_{t,\varepsilon}, \ldots, F^{(m)}_{t,\varepsilon}$ and each of them is diffeomorphic to the Milnor fiber associated with the singular point in question. In particular, each of the surfaces $F^{(1)}_{t,\varepsilon}, \ldots, F^{(m)}_{t,\varepsilon}$ is connected and simply connected; see [18].

Next we denote by $F_{t,\Delta}$ the intersection of $F_t$ with $\Delta^5_\varepsilon$. Again $F_{t,\Delta}$ consists of the union of its connected components $F^{(1)}_{t,\Delta}, \ldots, F^{(m)}_{t,\Delta}$ where $F^{(i)}_{t,\Delta} = F^{(i)}_{t,\varepsilon} \cap \Delta^5_\varepsilon$, $i = 1, \ldots, m$. Clearly each component $F^{(i)}_{t,\Delta}$ is diffeomorphic to the link $S_\Delta$.

Finally $F_{t,c}$ will denote the open set of $F_t$ given by $F_{t,c} = F_t \setminus (F_{t,\varepsilon} \cup F_{t,\Delta})$. The set $F_{t,c}$ is a connected open surface with $m$ boundary components given by $F^{(i)}_{t,\Delta}$, for $i = 1, \ldots, m$.

**Lemma 4.3.** *The first Betti number of the fiber $F_t = F^{-1}(t)$ is at most one provided that $t \neq 0$ is small enough.*

*Proof.* First we recall that the first Betti number of a Kato surface is equal to 1. Now, by considering the fibration induced by $F$ as a foliation, it is clear that the holonomy associated to the leaf $S \setminus \{p\}$ is finite. Thus, the manifold with boundary $F_{t,c} \cup F_{t,\Delta}$ is a finite covering of $S_c \cup S_\Delta$.

Now recall that $F_{t,\Delta}$ is constituted by the $m$ components $F^{(1)}_{t,\Delta}, \ldots, F^{(m)}_{t,\Delta}$. To recover the fiber $F^{-1}(t)$, we shall glue to each boundary component $F^{(i)}_{t,\Delta}$ the surface $F^{(i)}_{t,\varepsilon}$. In performing this construction, note that the preceding implies that the first Betti number of $F^{(i)}_{t,\Delta}$ equals the first Betti number of $S_\Delta$, namely 1. Similarly, the first Betti number of $F_{t,c}$ equals the first Betti number of $S_c$. Since, in turn, $F^{(i)}_{t,\varepsilon}$ is simply connected, the lemma follows from a simple application of Mayer-Vietoris argument. $\square$

Lemma 4.3 implies that the fibers $F_t$, $t \neq 0$, cannot be as in items (1)—(3) of Mizuhara's list. Thus, in order to finish the proof of Theorem A, we just need to investigate the case where the fibers $F_t$ are Hopf surfaces and the case where they are positive Inoue surfaces.

First recall that a *Hopf surface* is a surface obtained as the quotient of $\mathbb{C}^2 \setminus \{(0,0)\}$ by a free action of a discrete group. Similarly, a *positive Inoue surface* is obtained as the quotient of $\mathbb{D} \times \mathbb{C}$ by a discrete, cocompact group of automorphisms having the form $(w, z) \mapsto (\gamma(w), z + a(w))$ where $\mathbb{D}$ denotes the hyperbolic disc, $\gamma$ is an isometry of $\mathbb{D}$, and $a$ is a holomorphic function on $\mathbb{D}$. The reader will note that all these surfaces have second Betti number equal to zero.

*Proof of Theorem A.* Consider again the fiber $F_t$ and its subsets $F_{t,\varepsilon}$, $F_{t,\Delta}$, and $F_{t,c}$. Recall that $F_{t,\varepsilon} = F^{(1)}_{t,\varepsilon} \cup \ldots \cup F^{(m)}_{t,\varepsilon}$ where each component $F^{(i)}_{t,\varepsilon}$ can be identified with the Milnor fiber. Similarly we have $F^{(1)}_{t,\Delta} \cup \ldots \cup F^{(m)}_{t,\Delta}$ where each $F^{(i)}_{t,\Delta}$ can be identified with the link $S_\Delta$.

Next note that the second homology group of $F^{(i)}_{t,\varepsilon}$ is generated by the *vanishing cycles* and it is free abelian on at least one generator; see [18]. Since the second Betti number of $F_t$ vanishes, the Mayer-Vietoris sequence yields

$$H_3(F_t) \longrightarrow H_2(F_{t,\Delta}) \stackrel{(i_*, j_*)}{\longrightarrow} H_2(F_{t,\varepsilon}) \oplus H_2(F_{t,c}) \longrightarrow 0\,.$$

Note that the second Betti number of $F^{(i)}_{t,\Delta}$ is one by Poincaré duality. Thus the rank of $H_2(F_{t,\Delta})$ is $m$. Similarly, the rank of $H_2(F_{t,\varepsilon}) \oplus H_2(F_{t,c})$ is at least $m$ since $F_{t,\varepsilon}$ consists of $m$ connected components $F^{(1)}_{t,\varepsilon}, \ldots, F^{(m)}_{t,\varepsilon}$ pairwise diffeomorphic and having second Betti number strictly positive. On the other hand, the homomorphism $(i_*, j_*)$ is onto $H_2(F_{t,\epsilon}) \oplus H_2(F_{t,c})$ which means that the rank of $H_2(F_{t,\varepsilon}) \oplus H_2(F_{t,c})$ must be exactly $m$. In other words, the second Betti number of $F^{(1)}_{t,\varepsilon}$ must be 1. Equivalently, restricting our attention to the local context on a neighborhood of the singular point $p$, the *Milnor number* of the singular surface $F^{-1}(0)$ equals 1; see [18].

In the remainder of the proof we shall focus attention on a neighborhood of the singular point $p$. In other words, we shall work in the context of Milnor's theorem [18] and therefore we can ignore the effect of multiple fibers, i.e. in this local context the multiplicity of $F^{-1}(0)$

identified with $F^{-1}(0) \cap B^6_\varepsilon$ is necessarily equal to one. With this in mind, to derive a final contradiction proving Theorem A, it suffices to check that the Milnor number $\mu$ associated with the singular point $p \in S = F^{-1}(0)$ must be strictly greater than one. A short argument in this direction requires Laufer's formula in [16] stating that

$$1 + \mu = \mathcal{E}_{\rm top}\,(\widetilde{D}) + K.K + 12 \dim_{\mathbb{C}} H^1(\widetilde{S}, \mathcal{O})\,, \tag{1}$$

where $\mathcal{E}_{\rm top}\,(\widetilde{D})$ stands for the topological Euler characteristic of the exceptional divisor $\widetilde{D}$ and where $K.K$ is the self-intersection of the canonical class in the resolution $\widetilde{S}$ of $S$. Albeit Laufer's formula above is a local statement, so that both $K.K$ and $\dim_{\mathbb{C}} H^1(\widetilde{S}, \mathcal{O})$ should be understood in the local sense, there follows from the general construction of singular Kato surfaces that this local data coincides with the global data of the entire fiber $S = F^{-1}(0)$ along with its global resolution $\widetilde{S}$ (corresponding to an ordinary Kato surface). Alternatively, Noether and Riemann-Roch formulas as used below remain valid in the local context as explained, for example, in [27], page 122. In any event, the numeric values of $K.K$ and of $\dim_{\mathbb{C}} H^1(\widetilde{S}, \mathcal{O})$ can be computed with the help of the compact Kato surface $\widetilde{S}$. To carry out these computations, first recall that the *holomorphic Euler characteristic* $\mathcal{X}(\widetilde{S})$ of $\widetilde{S}$ is defined by $\mathcal{X}(\widetilde{S}) = 1 - \dim_{\mathbb{C}} H^1(\widetilde{S}, \mathcal{O}) + \dim_{\mathbb{C}} H^2(\widetilde{S}, \mathcal{O})$. In turn, Noether formula yields

$$K.K = 12\,\mathcal{X}\,(\widetilde{S}) - \mathcal{E}\,(\widetilde{S})$$

where $\mathcal{E}\,(\widetilde{S})$ is the usual Euler characteristic of the real four dimensional manifold $\widetilde{S}$; see [6], pages 8 and 9. There follows that

$$K.K + 12 \dim_{\mathbb{C}} H^1(\widetilde{S}, \mathcal{O}) = 12 + 12 \dim_{\mathbb{C}} H^2(\widetilde{S}, \mathcal{O}) - \mathcal{E}\,(\widetilde{S})\,.$$

Combined to Formula (1), the preceding equation yields

$$1 + \mu = 12 + 12 \dim_{\mathbb{C}} H^2(\widetilde{S}, \mathcal{O}) + \mathcal{E}_{\rm top}\,(\widetilde{D}) - \mathcal{E}\,(\widetilde{S})\,.$$

Therefore to finish the proof of Theorem A it is enough to check that $\mathcal{E}_{\rm top}\,(\widetilde{D}) \geq \mathcal{E}\,(\widetilde{S})$. This is however easy: since the first Betti number of $\widetilde{S}$ equals 1, Poincaré duality ensures that $\mathcal{E}\,(\widetilde{S})$ is equal to the second Betti number of $\widetilde{S}$. In turn, this second Betti number is nothing but the number of irreducible components in the exceptional divisor $\widetilde{D}$; see [10]. Again, the description of the exceptional divisor $\widetilde{D}$ shows that $\mathcal{E}_{\rm top}(\widetilde{D})$ also equals the number of irreducible components in $\widetilde{D}$. The proof of Theorem A is completed. □

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