EQUIVALENCE AND SEMI-COMPLETUDE OF FOLIATIONS

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Abstract. Holomorphic vector fields of strict Siegel type with an isolated singularity at the origin are considered. It is proved that, under suitable conditions always verified in dimension 3, the saturated of a transversal section to the separatrix associated to a certain eigenvalue together with the $n-1$ dimensional invariant subspace through the origin transverse to that separatrix contains a neighbourhood of the origin. As an application we prove that those vector fields admit a semi-complete representative. Another consequence is an extension of the Theorem of Mattei-Moussu, already obtained by Elizarov and Il'Yashenko by a different method.

1. Introduction

Let X be a holomorphic vector field in $\mathbb{C}^2$ with an isolated singularity at the origin and such that the eigenvalues of its linear part are both non zero. Then the foliation associated to X admits a semi-complete representative [10]. We can ask if this result is still valid for higher dimensions.

In $\mathbb{C}^2$, the Siegel Domain is a thin set (it has measure zero). Contrary to the $\mathbb{C}^2$ case, for $\mathbb{C}^n$, with $n \geq 3$, the interior of the Siegel Domain is non empty; thus the Siegel Domain represents an important set for the problem above. As the conclusions for the Poincaré Domain are easy, we will focus in the Siegel Domain.

Let $X : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, $n \geq 3$, be a holomorphic vector field, where $\lambda_1, \ldots, \lambda_n$ represent the eigenvalues of $DX(0)$, verifying:

a) the origin is an isolated singularity
b) X is of Siegel type (0 belongs to the convex hull of $\{\lambda_1, \ldots, \lambda_n\}$)
c) all eigenvalues of $DX(0)$ are non zero and there exists a straight line through the origin (of $\mathbb{C}$) separating λ_1 from the others eigenvalues in the complex plane
d) up to a change of coordinates, $X = \sum_{i=1}^{n} \lambda_i x_i (1 + f_i(x)) \partial/\partial x_i$, where $x = (x_1, \ldots, x_n)$ and $f_i(0) = 0$ for all i.

Up to multiplication by a constant we can assume that $\lambda_1 = 1$.

In this paper only vector fields with an isolated singularity at the origin are considered, even if not explicitly stated.

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It is important to remark that if $n = 3$ and X is a vector field with an isolated singularity at the origin and of strict Siegel type (the convex hull of $\{\lambda_1, \ldots, \lambda_n\}$ contains a neighbourhood of the origin), then c) and d) are immediately satisfied; in particular, there exists at least one eigenvalue λ_i such that the angle between λ_i and the other eigenvalues is greater than $\pi/2$ (the eigenvalues viewed as vectors in $\mathbb{R}^2$).

Let X, in $(\mathbb{C}^2, 0)$, be a holomorphic vector field with an isolated singularity at the origin and of "strict Siegel type" ($\lambda_1/\lambda_2 \in \mathbb{R}^-$, where λ_1 and λ_2 are the eigenvalues of $DX(0)$). Then X admits two separatrices. Let Σ be a transversal section to one of the separatrices and $\mathcal{F}$ the foliation associated to X. It is well known that the saturated of Σ by $\mathcal{F}$ together with the other separatrix contains a neighbourhood of the origin [8]. For higher dimension we obtain:

Proposition. *Let X be a holomorphic vector field verifying a), b), c) and d). The saturated of a transversal section to the separatrix tangent to the eigenspace associated to λ_1 at a point in the separatrix (sufficiently close to the origin), together with the invariant manifold transverse to that separatrix contains a neighbourhood of the origin.*

As an application we prove:

Theorem. *Let $\mathcal{F}$ be the foliation associated to a holomorphic vector field X verifying a), b), c) and d). Then $\mathcal{F}$ admits a semi-complete vector field, in a neighbourhood of the singularity, as its representative.*

In particular, any vector field in $(\mathbb{C}^3, 0)$ of strict Siegel type admits a semi-complete representative.

It is obvious that if X and Y, two holomorphic vector fields, are analytically equivalent in a neighbourhood of a singularity then the holonomy relatively to the separatrices, if they exist, are analytically conjugated (there exist holomorphic vector fields, defined in a neighbourhood of the origin of $\mathbb{C}^n$, $n \geq 3$, without separatrices [5, 7]).

Mattei and Moussu proved the reciprocal of this result for vector fields, in $(\mathbb{C}^2, 0)$, of strict Siegel type (if the eigenvalues of the linear part, at the singularity, are in the Poincaré Domain and do not verify any resonance relation between them, then both X and the holonomy are linearizable).

An extension, in a sense, of the Theorem of Mattei and Moussu is also proved, as a corollary of the proposition:

Theorem. *Let X and Y be two vector fields verifying a), b), c) and d). Denote by h_1^X and h_1^Y the holonomies of X and Y relatively to the separatrices of X and Y tangent to the eigenspace associated to the first eigenvalue, respectively. Then if h_1^X and h_1^Y are analytically conjugated, X and Y are analytically equivalent.*

This result has already been proved by Elizarov and Il'Yashenko [3]. However our proof is, in our opinion, simpler then theirs.

2. Vector fields of strict Siegel type

Let $X : U \subseteq \mathbb{C}^n \to \mathbb{C}^n$ and $Y : V \subseteq \mathbb{C}^n \to \mathbb{C}^n$ be holomorphic vector fields with a singularity at the origin. We say that X is analytically conjugated to Y in a neighbourhood of the origin if there exists a holomorphic diffeomorphism $H : V_1 \to U_1$, where $0 \in U_1 \subseteq U$, $0 \in V_1 \subseteq V$, such that $H(0) = 0$ and

$$Y = (DH)^{-1}(X \circ H).$$

We say that X and Y are analytically equivalent if X is analytically conjugated to fY, for some holomorphic function f verifying $f(0) \neq 0$.

The integral curves of any vector field X define a foliation of complex dimension 1. Two vector fields X and Y define the same foliation, in a neighbourhood of a point p, iff there exists a holomorphic function f, with $f(p) \neq 0$, such that $X = fY$.

Let $X : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a vector field, with an isolated singularity at the origin, and $\{\lambda_1, \dots, \lambda_n\}$ the set of eigenvalues of $DX(0)$. We say that X is of strict Siegel type if $0 \in \mathbb{C}$ belongs to the interior of the convex hull of the set $\{\lambda_1, \dots, \lambda_n\}$. In the particular case $n = 3$, $\lambda_i/\lambda_j \notin \mathbb{R}$ for every $i \neq j$.

Let X be a holomorphic vector field satisfying a), b), c) and d) and assume that it is written in its normal form

$$X = \sum_{i=1}^{n} \lambda_i x_i (1 + f_i(x)) \partial/\partial x_i \tag{1}$$

Thus the x_1-axis is the separatrix associated to the eigenvalues that can be separated from the others by a straight line through the origin: λ_1. From now on we assume that $\lambda_1 = 1$. Denote by $\mathfrak{X}_\mathfrak{n}$ the set of eigenvalues of type (1) verifying the conditions described above.

Let $\Sigma = \{(\varepsilon, x_2, \dots, x_n) : |x_i| \leq \varepsilon, \forall i = 2, \dots, n\}$ be a transversal section to the x_1-axis at the point $c(0)$, where $c : [0, 2\pi] \to \mathbb{C}^n$ is the curve defined by $c(\theta) = (\varepsilon e^{i\theta}, 0, \dots, 0)$ with $\varepsilon > 0$ sufficiently close to zero. There exists $0 < \delta < \varepsilon$ such that $\Sigma_\theta = \{(\varepsilon e^{i\theta}, x_2, \dots, x_n) : |x_i| \leq \delta, \forall i = 2, \dots, n\}$ is contained in the saturated of Σ (δ exists because $[0, 2\pi]$ is compact).

Let l be a straight line through the origin, in the complex plane, separating λ_1 from the others eigenvalues and L the part of its orthogonal straight line, through the origin, contained in the left half-plane with the vertex inclued. Suppose that $\alpha + i\beta$ is a directional vector of L, with $\alpha > 0$. Let $\bar{L}$ be the complex conjugate of L. We define the set

$$T = \{z \in \mathbb{C} : z = x + iy, x \in \bar{L}, -\pi < y \leq \pi\}$$

It is easy to verify that the image of T by the application $\phi(z) = \varepsilon e^z$ covers $\{z : |z| \leq \varepsilon\} \setminus \{0\}$. Moreover, this application is one-to-one.

Fixed $z \in T$ we consider the curve given by the line segment joining z and the intersection of the straight line parallel to $\bar{L}$ through z with the imaginary axis: $c_z(t) = z + \frac{1}{\alpha+i\beta}t$ (denote by t_z the instant that c_z intersects the imaginary axis).

For each x_1, with $|x_1| \leq \varepsilon$, let $z = z(x_1) \in T$ be such that $\varepsilon e^z = x_1$. Let r_{x_1} be the logarithmic spiral curve $r_{x_1}(t) = (\varepsilon e^{c_z(t)}, 0, \dots, 0)$, $r_{x_1} : [0, t_z[\to \{x_i = 0, i \geq 2\}$. Denote now by r_x the lift of r_{x_1} to the leaf through the point $x = (x_1, \dots, x_n)$. The curve r_{x_1} is such that $r_{x_1}(0) = (x_1, 0, \dots, 0)$ and $|r_{x_1}(t_z)| = \varepsilon$. Consequently, r_x verifies $r_x(0) = x$ and $|p_1(r_x(t_z))| = \varepsilon$, where $p_i(x) = x_i$.

For simplicity in the notation, from now on let $v = \alpha + i\beta$ and $k_x = t_{z(x)}$.

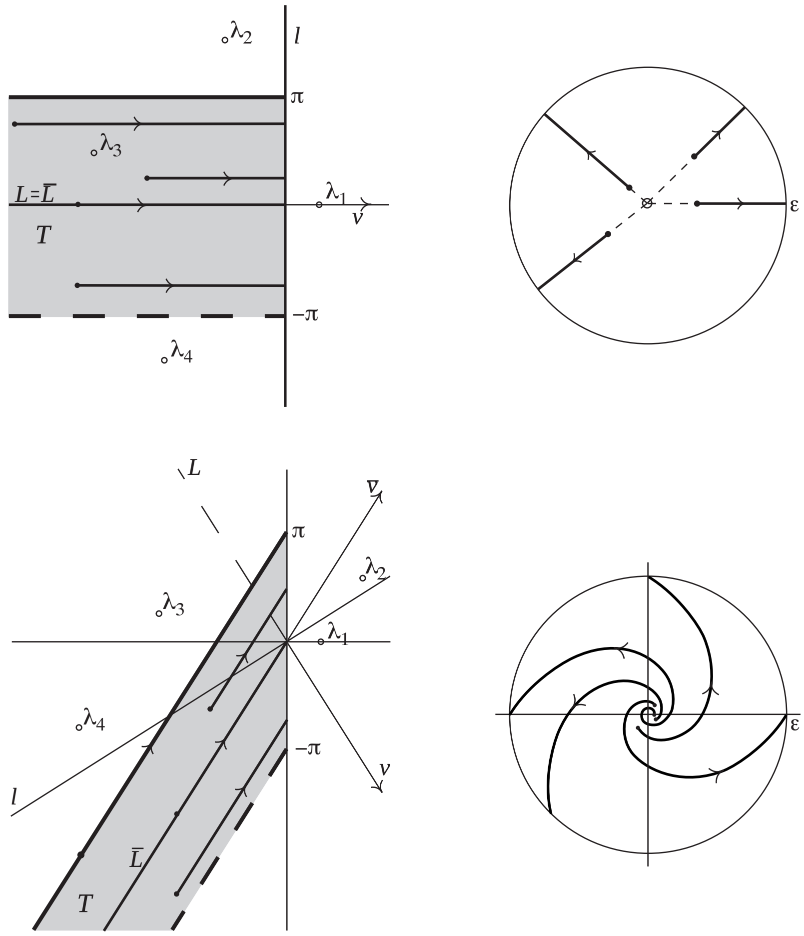

FIGURE 1

Lemma 1. *Let $X \in \mathfrak{X}_n$. Let V be the set of points x such that r_x intersects Σ_θ, for some θ. Then the set $V \cup \{x_1 = 0\}$ contains a neighbourhood of the origin.*

Proof. Denote by $\mathcal{F}_X$ the foliation associated to the vector field X. There exists a positive real number $\varepsilon < 1$ such that the projection p_1

is transverse to the leaves of $\mathcal{F}_X$ in a neighbourhood of the polydisc

$$P_{\varepsilon,\delta} = \{x \in \mathbb{C}^n : |x_1| \leq \varepsilon, |x_i| \leq \delta, i \geq 2\}$$

where δ is given as above.

Fix $x_1^0 \neq 0$ such that $|x_1^0| \leq \varepsilon$ and let z be such that $\varepsilon e^z = x_1^0$. The differential equation (1) restricted to $x_1 = \varepsilon e^{c_z(t)} = x_1^0 e^{\frac{t}{v}}$ is equivalent to the system of differential equations:

$$\begin{cases} \dfrac{dx_2}{dt} = \dfrac{\lambda_2}{v} x_2(1 + A_2(x_1^0 e^{\frac{t}{v}}, x_2, \dots, x_n)) \\ \vdots \\ \dfrac{dx_n}{dt} = \dfrac{\lambda_n}{v} x_n(1 + A_n(x_1^0 e^{\frac{t}{v}}, x_2, \dots, x_n)) \end{cases} \tag{2}$$

where A_i are holomorphic functions such that $A_i(0,0,0) = 0$, $\forall i \geq 2$. We take ε also in such a manner that in $P_{\varepsilon,\delta}$:

$$|A_i(x)| \leq \frac{\left|\Re\left(\dfrac{\lambda_i}{v}\right)\right|}{2\left|\dfrac{\lambda_i}{v}\right|}, \quad \forall i \geq 2 \tag{3}$$

Remark 1. As the eigenvalues $\lambda_2, \dots, \lambda_n$ are all in the side of the straight line l not containing the direction v, the angle between each eigenvalue and v is greater than $\pi/2$. Thus the real part of λ_i/v is negative.

Fix $x^0 \in P_{\varepsilon,\delta}$ such that $x_i^0 \neq 0$, $\forall i$. We are going to prove that the solution of the differential equation (2) verifies $x(t) \in P_{\varepsilon,\delta}$ for all $t \in [0, k_{x_1^0}]$. Remark that $p(r_{x^0}(t))$, where $p(x) = (x_2, \dots, x_n)$, is the solution of (2) with initial condition $(x_2(0), \dots, x_n(0)) = (x_2^0, \dots, x_n^0)$ and satisfies $|p_1(r_{x^0}(t))| = \varepsilon$ (i.e., intersects a transversal section of c) iff $t = k_{x_1^0}$. The case $x_i = 0$ for some $i \geq 2$ is analogous: remark that $\{x_i = 0\}$ are invariant manifolds of the foliation.

We have that (2) is equivalent to the system

$$\operatorname{Log}\left(\frac{x_i(t)}{x_i^0}\right) = \left(\frac{\lambda_i}{v} t + \int_0^t \frac{\lambda_i}{v} A_i(x_1^0 e^{\frac{s}{v}}, x_2(s) \dots, x_n(s)) ds\right)$$

As $\operatorname{Log}(w) = \log|w| + i \arg w$ we have:

$$\log\left|\frac{x_i(t)}{x_i^0}\right| = \Re\left(\frac{\lambda_i}{v} t + \int_0^t \frac{\lambda_i}{v} A_i(x_1^0 e^{\frac{s}{v}}, x_2(s), \dots, x_n(s)) ds\right)$$

and then

$$|x_i(t)| = |x_i^0| e^{\Re(\frac{\lambda_i}{v})t + \Re(\frac{\lambda_i}{v} \int_0^t A_i(x_1^0 e^{\frac{s}{v}}, x_2(s), \dots, x_n(s)) ds)}, \quad \forall i \geq 2$$

But as

$$\left|\Re\left(\frac{\lambda_i}{v}\int_0^t A_i(x_1^0 e^{\frac{s}{v}}, x_2(s), \dots, x_n(s))ds\right)\right|$$
$$\leq \left|\frac{\lambda_i}{v}\right| \int_0^t |A_i(x_1^0 e^{\frac{s}{v}}, x_2(s), \dots, x_n(s))|ds$$
$$\leq \left|\frac{\lambda_i}{v}\right| \frac{\left|\Re\left(\frac{\lambda_i}{v}\right)\right|}{2\left|\frac{\lambda_i}{v}\right|} t = -\Re\left(\frac{\lambda_i}{v}\right)\frac{t}{2}$$

we conclude that

$$\begin{aligned} |x_i(t)| &\leq |x_i^0| e^{\Re(\frac{\lambda_i}{v})t + |\Re(\frac{\lambda_i}{v}\int_0^t A(x_1^0 e^{\frac{s}{v}}, x_2(s), \dots, x_n(s))ds)|} \\ &\leq |x_i^0| e^{\Re(\frac{\lambda_i}{v})t - \Re(\frac{\lambda_i}{v})\frac{t}{2}} \\ &\leq |x_i^0| \quad (\leq \varepsilon) \end{aligned}$$

for all $t \in [0, k_{x_1^0}]$, as $\Re(\frac{\lambda_i}{v})$ is a negative real number. □

The condition d) is important to the proof of this lemma. The condition is pertinent as there exist vector fields, in $\mathbb{C}^3$, verifying a), b) and c) and not admitting any holomorphic manifold tangent to the manifold $\{z = 0\}$ [1].

Remark 2. If the angles between λ_1 and the remaining eigenvalues are greater than $\pi/2$ we can consider the imaginary axis as l. In this case for each $x \in P_{\varepsilon,\delta}$, r_x can be lifted along the radial directions.

Remark 3. If $\lambda_1 \neq 1$, $\bar{L}$ should be the reflection of L over the straight line through the origin and λ_1.

As an immediate consequence of Lemma 1 we can say:

Proposition 1. *Let X be a holomorphic vector field verifying* a), b), c) *and* d). *The saturated of a transversal section to the separatrix tangent to the eigenspace associated to λ_1 at a point in the separatrix (sufficiently close to the origin), toghether with the invariant manifold transverse to that separatrix contains a neighbourhood of the origin.*

Proof. By a linear change of coordinates X can be written in the form (1). Let $\Sigma_\theta = \{(\varepsilon e^{i\theta}, x_2, \dots, x_n) : |x_i| \leq \delta, \forall i \geq 2\}$ be the transversal sections defined before, which belong to the saturated of Σ, for all $\theta \in [0, 2\pi]$. Fix $x \in P_{\varepsilon,\delta}$ such that $x_1 \neq 0$ ($\{x_1 = 0\}$ is the invariant manifold transverse to the x_1-axis). By Lemma 1 the lift of r_x belongs to $P_{\varepsilon,\delta}$ and is such that $r_x(k_{x_1}) \in \Sigma_\theta$, for some θ. So x belongs to the saturated of Σ, i.e., of a transversal section to a small curve in the x_1-axis. □

The analogous of Mattei-Mossu's Theorem for higher dimension can be enunciated in the following way:

Theorem 1. *Let X and Y be two vector fields verifying a), b), c) and d). Denote by h_1^X and h_1^Y the holonomies of X and Y relatively to the separatrices of X and Y tangent to the eigenspace associated to the first eigenvalue, respectively. Then if h_1^X and h_1^Y are analytically conjugated, X and Y are analytically equivalent.*

Proof. Again, we can assume that X and Y are written in its normal form (1) with the first eigenvalue equal to 1 in both cases. We are going to construct a holomorphic diffeomorphism, in a neighbourhood of the origin, taking the leaves of X into the leaves of Y. Denote by $\mathcal{F}_X$ and $\mathcal{F}_Y$ the foliations associated to the vector fields X and Y, respectively.

Let $P_{\varepsilon,\delta}$, l and L as in the proof of Lemma 1, where $\varepsilon < 1$ is such that (3) is satisfied in $P_{\varepsilon,\delta}$ and the projection p_1 is transverse to all leaves of $\mathcal{F}_X \cap P_{\varepsilon,\delta}$ and of $\mathcal{F}_Y \cap P_{\varepsilon,\delta}$, excluding the leaves contained in the invariant manifold $\{x_1 = 0\}$.

Let $(\lambda_1, \dots, \lambda_n)$ and $(\beta_1, \dots, \beta_n)$ denote the eigenvalues of $DX(0)$ and $DY(0)$, respectively. As h_1^X and h_1^Y are analytically conjugated and $\lambda_1 = 1 = \beta_1$, we have that $\lambda_i = \beta_i$, $\forall i$.

Let c be the curve defined before ($c(\theta) = (\varepsilon e^{i\theta}, 0, \dots, 0)$) and Σ_θ^X and Σ_θ^Y the "vertical" transversal sections to the separatrices $\{x_i = 0, i \geq 2\}$ of X and Y, respectively, at the point $c(\theta)$.

Let $h_0 : \Sigma_0^X \to \Sigma_0^Y$ be the analytical conjugacy between h_1^X and h_1^Y. Denote by $l_\theta : \Sigma_0^X \to \Sigma_\theta^X$ and $\bar{l}_\theta : \Sigma_0^Y \to \Sigma_\theta^Y$ the aplications obtained by lifting the curve c to the leaves of $\mathcal{F}_X$ and $\mathcal{F}_Y$, respectively. We define the holomorphic diffeomorphism $h_t = \bar{l}_t \circ h_0 \circ l_t^{-1}$. The conjugacy relation $\bar{l}_{2\pi} \circ h_0 = h_0 \circ l_{2\pi}$ ($l_{2\pi}$ and $\bar{l}_{2\pi}$ are the holomomies) together with $h_{2\pi} = \bar{l}_{2\pi} \circ h_0 \circ l_{2\pi}^{-1}$ implies that $h_{2\pi} = h_0$. So we established a diffeomorphism along all transversal sections to the x_1-axis at any point of the curve c.

By Lemma 1, $r_x^X(k_{x_1}) \in \Sigma_\theta^X$ and $r_x^Y(k_{x_1}) \in \Sigma_\theta^Y$, for some θ, for every x in $P_{\varepsilon,\delta}$. Thus the diffeomorphism h_0 can also be transported along the spiral curves r_x^X and r_x^Y.

We have established an analytically conjugacy, Φ, between the foliations $\mathcal{F}_X \setminus \{x_1 = 0\}$ and $\mathcal{F}_Y \setminus \{x_1 = 0\}$ in a neighbourhood of the origin. We have to prove that this conjugacy can be extended to the invariant manifold $\{x_1 = 0\}$. As Φ is holomorphic in $P_{\varepsilon,\delta} \setminus \{x_1 = 0\}$ and $\{x_1 = 0\}$ is a thin set, it is sufficient to prove that Φ is bounded [6]. For this, we have to follow the constraction of Φ.

From now on we use the variables x to $\mathcal{F}_X$ and y to $\mathcal{F}_Y$. Fix $x^0 \in P_{\varepsilon,\delta}$. Taking $x_1(t) = x_1^0 e^{\frac{t}{v}}$, $r_{x^0}^X(t)$ is such that

$$\begin{aligned}\frac{d}{dt}|x_i(t)|^2 &= 2\Re(\bar{x_i(t)} x_i'(t)) \\ &= 2\Re\left(\frac{\lambda_i}{v}\right) |x_i(t)|^2 \Re(1 + A_i(x_1^0 e^{\frac{t}{v}}, y(t), z(t)))\end{aligned}$$

In this way, there is a constant $b > 0$ such that

$$\frac{d}{dt}\log|x_i(t)|^2 \leq 2\Re\left(\frac{\lambda_i}{v}\right)(1+b) \Longrightarrow |x_i(k_{x_1^0})| \leq |x_i^0|e^{\Re(\frac{\lambda_i}{v})(1+b)k_{x_1^0}}$$

Let $y^0 = h_{\arg(x(k_{x_1^0}))}(x(k_{x_1^0}))$. By construction $\Phi(x^0) = s_{y^0}(k_{x_1^0})$, where s_{y^0} is the lift of $s_{y_1^0} = (y_1^0 e^{-\frac{t}{v}}, 0, \ldots, 0)$ to the leaf through y^0. Thus, we obtain:

$$|y_i(t)| \leq |y_i^0|e^{-\Re(\frac{\lambda_i}{v})(1+b)k_{x_1^0}}, \quad i \geq 2$$

and so, if q is such that $|y_i^0| \leq q|x_i(k_{x_1^0})|$, for all $i \geq 2$, we obtain:

$$|y_i(k_{x_1^0})| \leq q|x_i(t_{x_1^0})|e^{-k_{x_1^0}\Re(\frac{\lambda_i}{v})(1+b)} \leq qe^{\Re(\frac{\lambda_i}{v})(1+b)k_{x_1^0}}e^{-\Re(\frac{\alpha_2}{v})(1+b)k_{x_1^0}}|x_i^0| = q|x_i^0|$$

We conclued that Φ^{-1} and, consequently, Φ are bounded. Thus Φ admits a holomorphic extension $\tilde{\Phi}$ to the invariant manifold $\{x_1 = 0\}$. As $\tilde{\Phi}$ has a holomorphic inverse map (the inverse map is constructed taking now the leaves of Y into the leaves of X in a similar way), $\tilde{\Phi}$ is a diffeomorphism in a neighbourhood of the origin.

It remains to prove that $\tilde{\Phi}$ takes the leaves of $\mathcal{F}_X|_{\{x_1=0\}}$ into the leaves of $\mathcal{F}_Y|_{\{x_1=0\}}$.

Let $Z = D\tilde{\Phi}(X \circ \tilde{\Phi})$. As $\mathcal{F}_Y|_{P_{\varepsilon,\delta}\setminus\{x_1=0\}}$ coincides with $\mathcal{F}_Z|_{P_{\varepsilon,\delta}\setminus\{x_1=0\}}$, there exists a holomorphic function f, defined in $P_{\varepsilon,\delta} \setminus \{x_1 = 0\}$, such that $fY = Z$. In particular $f = \frac{Z_2}{Y_2}$, where Y_2 (Z_2) is the second component of Y (Z). As $Y_2 = x_2(\beta_2 + \ldots)$, where dots means terms of order greater or equal to 1, and both Y_2 and Z_2 are holomorphic, f can be holomorphically extended to $U \setminus \{x_1 = 0, x_2 = 0\}$. Finally, as $\{x_1 = 0, x_2 = 0\}$ is a set of complex codimension greater than 1, f admitts a holomorphic extension, $\tilde{f}$, to $\{x_1 = 0, x_2 = 0\}$ [6, pag. 31] and this extension verifies $\tilde{f}Y = Z$ in U. Thus X and Y are analytically equivalent. □

3. Semi-completude of vector fields of strict Siegel type

The definition of a semi-complete vector field relatively to a (relatively compact) open set U was introduced in [9].

Definition 1. Let X be a holomorphic vector field defined in complex manifold M and $U \subseteq M$ an open subset of M. We say that X is semi-complete relatively to U if there exists a holomorphic application

$$\Phi : \Omega \subseteq \mathbb{C} \times U \to U$$

where Ω is an open set containing $\{0\} \times U$ such that

a) $\Phi(0, x) = x \qquad \forall x \in M$
b) $X(x) = \frac{d}{dT}|_{T=0}\Phi(T, x)$
c) $\Phi(T_1+T_2, x) = \Phi(T_2, \Phi(T_1, x))$, when the two members are defined

d) $(T_i, x) \in \Omega$ and $(T_i, x) \to \partial\Omega \quad \Rightarrow \quad \Phi(T_i, x) \to \partial U$

We call Φ the semi-complete flow associated to the vector field X.

We say that X is complete if there is a holomorphic application $\Phi : \mathbb{C} \times M \to M$ satisfying a), b) and c).

Semi-complete vector fields are essentially the local version of complete vector fields, where blow-up on finite time cannot occur:

Proposition 2. [9] *Let X be a complete holomorphic vector field on a complex manifold M. The restriction of X to any connected (relatively compact) open set U $(U \subseteq M)$ is a semi-complete vector field relatively to U.*

Therefore, if a holomorphic vector field in an open set U is not semi-complete it cannot be extended to a compact manifold containing U.

To each one of its orbits (leaves), L, we can associate a holomorphic differential 1-form, denoted by dT_L, such that $dT_L(X) = 1$. A sufficient conditions for a vector field to be semi-complete in an open set U is presented in [10]. The regular orbits of a vector field X $(X \not\equiv 0)$ are Riemann surfaces.

Proposition 3. [10] *Let X be a holomorphic vector field defined in a neighbourhood U of the origin of $\mathbb{C}^n$. Suppose that for all regular orbits L of X and every $c : [0, 1] \to L$ such that $c(0) \neq c(1)$ the integral of dT_L over c is non zero. Then the vector field X is semi-complete in U.*

The foliations, in complex manifolds of dimension 2, associated to semi-complete vector fields with an isolated singularity are completely characterized [4, 10]. In particular it is proved that if $X : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, with an isolated singularity at the origin, is such that the eigenvalues of $DX(0)$ are non zero, there exists a holomorphic function f, with $f(0) \neq 0$ such that fX is semi-complete.

Let $X : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a holomorphic vector field with an isolated singularity at the origin. If the eigenvalues of $DX(0)$ are in the Poincaré Domain and do not have resonances between themselves, X is analytically conjugated to its linear part and, consequently, X is a semi-complete vector field.

Assume now that X is in the Poincaré Domain but there exist resonance relations between the eigenvalues of its linear part, at the isolated singularity. It is well known that we can rearrange the variables so that the Dulac's normal form of X can be written in the form:

$$\sum_{i=1}^{n} (\lambda_i x_i + p_i(x_1, \dots, x_{i-1}))\partial/\partial x_i$$

where p_i is a polynomial. This vector field is obviously semi-complete: we integrate $\dot{x_1} = \lambda_1 x_1$ and replace its solution in the equation $\dot{x_2} = \lambda_1 x_2 + p(x_1)$, which is now a linear non-autonomous holomorphic ordinary differential equation. Proceeding in the same way for the other equations and remarking that, in the Poincaré Domain the Dulac's normal form of a holomorphic vector field X is analytically conjugated to X, the conclusion follows.

Assume now that $X : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is a holomophic vector field of Siegel type, with an isolated singularity at the origin and verifying c) and d). Thus it can be written in the form (1):

$$Z = \sum_{i=1}^{n} \lambda_i x_i (1 + f_i(x)) \partial/\partial x_i$$

Denote by $\mathcal{F}$ the foliation defined by Z. The vector fields Z and Y, where Y is given by

$$Y = \lambda_1 x \partial/\partial x_1 + \sum_{i=2}^{n} \lambda_i x_i (1 + f_i(x))(1 + f_1(x))^{-1} \partial/\partial x_i$$

represent the same foliation in a neighbourhood of the origin. In this way, each foliation associated to vector fields as above can be represented by a vector field of the type:

$$\lambda_1 x \partial/\partial x_1 + \sum_{i=2}^{n} \lambda_i x_i (1 + g_i(x)) \partial/\partial x_i$$

where g_i are holomorphic functions in a neighbourhood of the origin satisfying $g_i(0) = 0$.

The study of foliations admitting a semi-complete representative of "strict Siegel type" in $\mathbb{C}^2$ is based in the following fact: the lift of the radial curves $x_1(t) = Ae^t$ through points in a certain neighbourhood of the origin remain in the same neighbourhood [10].

In a sense, Proposition 1 is a generalization, for higher dimensions and under certain conditions, of this result: spiral curves are considered instead of radial curves. So, the proof of our theorem (classification) is similar to the one in [10].

Theorem 2. *Let $\mathcal{F}$ be the foliation associated to a holomorphic vector field X verifying a), b), c) and d). Then $\mathcal{F}$ admits a semi-complete vector field, in a neighbourhood of the singularity, as its representative.*

Proof. Let $\mathcal{F}$ be the foliation associated to a vector field X satisfying a), b), c) and d). X is analytically equivalent, in a neighbourhood of the singularity, to the vector field

$$Y = x_1 \partial/\partial x_1 + \sum_{i=2}^{n} \lambda_i x_i (1 + g_i(x))$$

where g_i are holomorphic functions in a neighbourhood of the origin with $g_i(0) = 0$ for all i. Denote by $\mathcal{F}_Y$ the foliation associated to Y.

There exists a neighbourhood of the origin in which the projection p_1 is transverse to the leaves of $\mathcal{F}_Y$ except to the set $\{x_1 = 0\}$. Let $P_{\varepsilon,\delta}$ be a polydisc as in Lemma 1, contained in that neighbourhood.

Let L be a regular leaf of $\mathcal{F}_Y$ and $c_L : [0,1] \to L$ an open curve such that $\int_{c_L} dT_L = 0$. Denote $p_1(c_L)$ by c. Then

$$0 = \int_{c_L} dT_L = \int_c \frac{dx_1}{x_1}$$

As the 1-dimensional vector field $x\partial/\partial x$ is semi-complete, c is a closed curve, homotopic to a point in $\{x_i = 0, i \geq 2\} \setminus \{0\}$. We proceed now as in [10].

Let $S = \{(ke^{i\theta}, 0, \dots, 0) : \theta \in [0, 2\pi], 0 \leq k \leq \varepsilon\} \subseteq \{x_i = 0, i \geq 2\}$. The curve c is contained in S.

Take a homotopy, following the spiral directions defined in the last section, between c and a curve $\bar{c}$ contained in the set $\{(x_1, 0, \dots, 0) : |x_1| = |c(0)|\} \subseteq S$. In the proof of lemma 1 it is implicit that this homotopy can be lifted, by p_1, to L. So we have a homotopy between c_L and $\bar{c_L}$, where $\bar{c_L}$ is the lift, by p_1, of $\bar{c}$ to L. As c_L is an open curve, p_1 is transverse to the leaves of $\mathcal{F}_Y$ and the homotopy is made along the spiral directions, $\bar{c_L}$ is also an open curve. More specifically, $\bar{c_L}(0) = c_L(0)$ and $\bar{c_L}(1) = c_L(1)$.

The curve $\bar{c}$ is also a closed curve homotopic to a point. Consider the homotopy $H(s,t)$ between $\bar{c}$ and the constant curve $\bar{c}(0) = \bar{c}(1)$ such that $H(s,0) = \bar{c}(0)$ and $H(s,1) = \bar{c}(1)$ for every $s \in [0,1]$. This homotopy can be lifted, by p_1, to a homotopy between $\bar{c_L}$ and a curve c_0 that projects into a point. As p_1 is transverse to the leaves of $\mathcal{F}_Y$, c_0 must be constant. But such $c_0(0) = \bar{c_L}(0) \neq \bar{c_L}(1) = c_0(1)$. Contradiction.

So c_L is closed and, consequently, Y is semi-complete in some neighbourhood of the origin.

As X and Y are analytically equivalent, Y is analytically conjugated to fX, for some holomorphic function f with $f(0) \neq 0$. Thus $\mathcal{F}$ admits a semi-complete vector field as representative. □

As we said before, any vector field in $(\mathbb{C}^3, 0)$ with an isolated singularity at the origin and of strict Siegel type satisfies c) and d); therefore admits a semi-complete representative.

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