Regular Synchrony Lattices for Product Coupled Cell Networks

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Abstract

There are several ways of constructing (bigger) networks from smaller networks. We consider here product networks using two types of product: the cartesian and the Kronecker (tensor). These two products, as graph operations, are quite different, implying that the associated coupled cell systems have distinct structures and so the kinds of dynamics expected to occur are difficult, if not impossible, to compare. Nevertheless, in this paper, we establish an inclusion relation between the lattices of synchrony subspaces for the cartesian and Kronecker products. Our main aim is to determine a relation between the lattices of synchrony subspaces for a product network and the component networks of the product. In this sense, we show how to obtain the lattice of regular synchrony subspaces for a product network from the lattices of synchrony subspaces for the component networks. Specifically, we prove that a tensor of subspaces is of synchrony for the product network if and only if the subspaces involved in the tensor are synchrony subspaces for the component networks of the product. We also show that, in general, there are (irregular) synchrony subspaces for the product network that are not described by the synchrony subspaces for the component networks, concluding that, in general, it is not possible to obtain the all synchrony lattice for the product network from the corresponding lattices for the component networks.

AMS classification scheme numbers: 34C15 37C10 06B23 05C76 05C50

Coupled cell systems are dynamical systems where the full system can be seen as a set of interacting smaller dynamical systems (the cells). The motivation for interpreting certain dynamical systems as coupled cell systems comes from many real life applications where it is advantageous to interpret the evolution in time of the system as determined by the dynamics of individual units and the links between these units. Properties encoding information of the cells and the interactions can be given by a network – a graph where the nodes represent the cells and the arrows the interactions between the cells. The product of networks is a natural process of constructing real networks from smaller ones. One ultimate goal is to derive how far we can describe the dynamics of the coupled cell systems associated with a (large) product network based on the dynamics of the coupled cell systems consistent with the structure of the component networks of the product. We have focused this dynamics question on the synchrony subspaces (flow-invariant

subspaces defined by equalities of certain cell coordinates), being our main aim to determine a relation between the synchrony subspaces for a product network and the component networks of the product. We show how to obtain a special set of synchrony subspaces for a product network from the sets of synchrony subspaces for the component networks - the synchrony subspaces that are tensors of synchrony subspaces for the component networks of the product. Moreover, we show that, in general, there are synchrony subspaces for the product network that are not described by the synchrony subspaces for the component networks, concluding so that, in general, it is not possible to obtain the all set of synchrony subspaces for the product network from the corresponding sets for the component networks.

1 Introduction

Coupled cell systems are dynamical systems where the full system can be seen as a set of interacting smaller dynamical systems (the *cells*). The motivation for interpreting certain dynamical systems as coupled cell systems comes from many real life applications where it is advantageous to interpret the evolution in time of the system as determined by the dynamics of individual units and the links between these units. Properties encoding information of the cells and the interactions can be given by a network. More precisely, a *network* architecture is a graph where the nodes represent the cells and the arrows the interactions between the cells. Cells that have the same phase space are denoted by the same symbol. Interactions that, from the dynamics point of view, have the same role are denoted by the same arrow type. The coupled cell systems associated with a network structure are the differential equations where the involved vector fields respect that structure. Observe that a network is then a directed graph where nodes and arrows can be distinguished, say using equivalence relations (on the set of nodes and on the set of arrows). See for example the approach of Stewart, Golubitsky *et al.* [22, 14, 13] or Field [11].

It is known that the network structure imposes restrictions at the dynamics that can occur for the associated coupled cell systems. See for example Aguiar *et al.* [2]. One important such restriction is the existence of *synchrony subspaces* – flow-invariant subspaces defined in terms of equalities of certain cell coordinates. By Stewart [21] (see also Aldis [5, 1]) the set of all synchrony subspaces for a network forms a complete lattice taking the relation of inclusion. Note that the intersection of two synchrony subspaces is again a synchrony subspace. The results of [22, 14] prove that a space is of synchrony (for any coupled cell system associated with the network) if and only if a certain combinatorial condition involving the network is satisfied, which can be translated in terms of the invariance of that space by the network adjacency matrices. An algebraic and algorithmic description of the lattice of synchrony subspaces for a network is given by Aguiar and Dias [3]. See also Kamei [16] and Kamei and Cock [17].

The product of networks is a natural process of constructing real networks from smaller ones. See for example Atay and Biyikoglu [8], Leskovec *et al.* [20]. From Graph Theory, it is known that there are several ways of forming from two digraphs a new digraph whose vertex set is the cartesian product of their vertex sets – these constructions are usually referred as 'products'. Two of such are the *cartesian product* and the *Kronecker (tensor) product* and are the ones we consider here. See Figures 1 and 2. One ultimate goal is to derive how far we can describe the dynamics of the coupled cell systems associated with a (large) product network based on the dynamics of the coupled cell systems consistent with the struture of the component networks of the product.

It is known that coupled cell systems can support robust heteroclinic cycles, and one way of understanding the way that occurs, is by exploring the lattice of synchrony subspaces for the underlying network structure associated with the coupled cell systems. See for example Aguiar *et al.* [2], where robust heteroclinic cycles appear naturally in coupled cell systems that have no symmetry, yet, they have structure consistent with networks determining the existence of synchrony subspaces where heteroclinic connections occur in a robust way. That is, they persist under perturbations of the vector fields as long these mantain the admissibility under the networks involved. The recent work of Ashwin and Postlethwaite [7] gives two explicit methods for design of coupled cell systems that realize a given graph as a heteroclinic network. Another possible way of constructing coupled cell systems with heteroclinic networks is by making the product of coupled cell systems where it is known the existence of robust heteroclinic cycles. See Ashwin and Field [6] for a specific example with nine cells. Identifying the synchrony subspaces that arise from the component networks of the product is the first step if we intend to construct heteroclinic networks in product networks from smaller networks having heteroclinic behaviour.

The main aim of the present work is to establish a relation between the lattices of synchrony subspaces for a product network and the component networks of the product. See Aguiar *et al.* [4], for an analogous study for the join and coalescence network operations. Regardless of the synchrony subspaces for the product component networks determine the existence of certain synchrony subspaces for the product network, as we prove in this paper in a rigorous way, there may also exist, as we show, 'new' synchrony subspaces – new in the sense that they emerge because a product network is, in principle, a richer digraph, where for example, symmetries can arise even though the component networks of the product have no symmetry. Moreover, despite the definitions of cartesian product and Kronecker (tensor) product networks are quite different, we show that there is an inclusion relation between the lattices of synchrony subspaces for these products.

Framework of the paper

In Section 3, we present the definitions of cartesian product network and Kronecker product network of identical-edge networks. Given any two identical-edge networks $\mathcal{N}_1$ and $\mathcal{N}_2$, the cartesian product network $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ is the network where each node in the network $\mathcal{N}_1$ is replaced by a copy of the network $\mathcal{N}_2$. More precisely, assume $\mathbf{P}_{ij}$ is the jth cell in the copy of the network $\mathcal{N}_2$ that replaces the ith cell of the network $\mathcal{N}_1$. Then we have an arrow from cell $\mathbf{P}_{ij}$ to cell $\mathbf{P}_{lj}$ if and only if there is an arrow from the ith cell to the lth cell in the network $\mathcal{N}_1$; there is an arrow from cell $\mathbf{P}_{ij}$ to cell $\mathbf{P}_{il}$ if and only if there is an arrow from the jth cell to the lth cell in the network $\mathcal{N}_2$. (See Definition 3.1.) The Kronecker product network $\mathcal{N}_1 \otimes \mathcal{N}_2$ is the network with adjacency matrix $A_1 \otimes A_2$ if A_i is the adjacency matrix of the network $\mathcal{N}_i$, for $i = 1, 2$. (See Definition 3.6.) See for example the cartesian product network $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ and the Kronecker product $\mathcal{N}_1 \otimes \mathcal{N}_2$ in Figures 1, 2, respectively, for the 2-cell network $\mathcal{N}_1$ and the 3-cell network $\mathcal{N}_2$ presented there.

In Section 4.2, we establish an inclusion relation between the lattices of synchrony subspaces for the cartesian and Kronecker products by showing that any synchrony subspace for the cartesian product network $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ is a synchrony subspace for the Kronecker product network

$\mathcal{N}_1 \otimes \mathcal{N}_2$. See Proposition 4.5.

Our main results are Theorem 5.10 and Theorem 6.5, in Section 5 and Section 6, respectively, where we prove that the collection of all regular synchrony subspaces for a (cartesian or Kronecker) product network is a lattice. By a *regular subspace* we mean a subspace of the product space given as a tensor product of a subspace of the phase space for $\mathcal{N}_1$ with a subspace of the phase space for $\mathcal{N}_2$. Moreover, we prove that the regular synchrony lattices for $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ and $\mathcal{N}_1 \otimes \mathcal{N}_2$ coincide and are given by the tensor product of the synchrony lattices for $\mathcal{N}_1$ and $\mathcal{N}_2$.

We also show that, in general, the regular synchrony lattice and the synchrony lattice for a (cartesian or Kronecker) product network do not coincide. Specifically, we show examples of product networks having irregular synchrony subspaces which are not a sum of regular synchrony subspaces. Thus, we can conclude that, the synchrony lattice for the (cartesian or Kronecker) product network is not, in general, completely determined by the synchrony lattices for the component networks of the product. Yet, the regular synchrony lattice is.

2 Coupled Cell Networks

Following Stewart, Golubitsky *et al.* [22, 14, 13], a *coupled cell network* is a directed graph (digraph) whose nodes represent the cells and the directed arrows (or edges or arcs) the couplings. Note that we can have self loops, or multiarrows (parallel arcs) – arrows with the same head and tail cells.

Equivalence relations on the set of nodes and on the set of arrows can be defined symbolizing the following:
(a) Two nodes are in the same *cell equivalence class* if they represent individual dynamics with the same state space.
(b) Two arrows are in the same *arrow equivalence class* if they represent couplings of the same type.

The following consistency condition is assumed: if two arrows are of the same type then the corresponding head cells are in the same cell equivalence class and the same holds for the corresponding tail cells.

Definition 2.1 (i) Given a network, the *input set of a cell* of the network is the set of arrows directed to that cell.
(ii) Two cells of a network are said to be *(input) isomorphic* if there is an arrow-type preserving bijection between the corresponding input sets.
(iii) A *homogeneous* coupled cell network is a network in which all cells are (input) isomorphic.
(iv) A *regular* coupled cell network is a homogenous network with only one arrow type. For a regular network, the *valency* is the number of arrows of the input set of any cell and the *adjacency matrix* is the matrix where the (i, j) entry is the number of arrows from cell j to cell i, assuming the set of cells is $\{1, \ldots, n\}$. If v is the valency of a regular network then the corresponding adjacency matrix has v constant row sum.
(v) For a general coupled cell network with set of cells $\{1, \ldots, n\}$ and k arrow equivalence classes, we define k adjacency matrices, one for each arrow type, say $A_1, \ldots, A_k \in M_{n\times n}\left(\mathbf{Z}_0^+\right)$, in the following way: the (i, j) entry of the matrix A_p is the number of arrows of type p from

cell j to cell i.
(vi) A coupled cell network is said *injective* if its adjacency matrices are injective. $\diamond$

In Figure 1, we have from left to the right, a two-cell regular network with valency two, a three-cell regular network with valency two, and a six-cell homogeneous network with two arrow types. Note that for each of the three networks, all cells are input isomorphic. For example, for the six-cell network, every cell receives two solid arrows and two dashed arrows.

Coupled cell systems

Following [22, 14, 13], the connection between coupled cell systems and coupled cell networks is made in the following way: to each coupled cell c is associated a choice of *cell phase space* P_c which is assumed to be a finite-dimensional real vector space, say $\mathbf{R}^k$ for some $k > 0$. If cells c and d are cell equivalent then it is required that $P_c = P_d$ and the two spaces are identified canonically. If $\mathcal{C} = \{1, \ldots, n\}$ denotes the set of cells of the network, then the *total phase space* P of the coupled cell system is the direct product of the cell phase spaces, $\prod_{c \in \mathcal{C}} P_c$, and we employ the coordinate system $x = (x_c)_{c \in \mathcal{C}}$ on P. Given a network $\mathcal{G}$ and a fixed choice of the total phase space P, we describe now the coupled cell systems that correspond to the class of the systems of ordinary differential equations, $\dot{X} = F(X),\ X \in P$, compatible with the structure of the network. The system associated with cell j has the form

$$\dot{x}_j = f_j\left(x_j; x_{i_1}, \ldots, x_{i_m}\right)$$

where the first argument x_j in f_j represents the internal dynamics of the cell and each of the remaining variables x_{i_p} represents a coupling between cell i_p and cell j. Thus $x_j \in P_j, x_{i_p} \in P_{i_p},\ p = 1, \ldots, m$ and we assume $f_j :\ P_j \times P_{i_1} \times \cdots \times P_{i_m} \to P_j$ is smooth. Moreover, identical couplings directed to cell j correspond to the invariance of f_j under permutation of the corresponding variables. Systems associated with (input) isomorphic cells are identical up to permutation of the variables accordingly to the input sets of the cells. The vector fields F are called *$\mathcal{G}$-admissible*.

If for example we consider the six-cell homogeneous network on the right of Figure 1, as all cells are input isomorphic, we have that coupled cell systems having structure consistent with this network must be of the following form: the system associated with cell ij has the form

$$\dot{x}_{ij} = f\left(x_{ij}; x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}\right)$$

where the first argument x_{ij} in f represents the internal dynamics of the cell and each of the remaining variables x_{i_p} represents a coupling between cell i_p and cell ij. All cells must have the same phase space, say $\mathbf{R}^k$ for some $k > 0$, and so the total phase space is then $\left(\mathbf{R}^k\right)^6$. Moreover, $f :\ \mathbf{R}^k \times \left(\mathbf{R}^k\right)^4 \to \mathbf{R}^k$ is smooth, and invariant say, under the second and third coordinates (representing the couplings associated with solid arrows), and under the fourth and fifth coordinates (representing the couplings associated with dashed arrows). Note that the cell systems are given by the same function f, as all cells are input isomorphic.

3 Product networks

We define now the cartesian and the Kronecker (tensor) products of identical-edge networks. Following Golubistky *et al.* [12], we consider the cartesian product of networks as a variant of the usual definition of cartesian product of digraphs where the arrows types from each component network involved in the product remain distinct in the product. We adopt the usual definition of Kronecker product of digraphs for the Kronecker product of networks.

3.1 Cartesian product network

Definition 3.1 For $i = 1, 2$, consider the network $\mathcal{N}_i$ with set of cells (nodes) $\mathcal{C}_i$, set of arrows $\mathcal{E}_i$, and adjacency matrix A_i. Following [12], we define the *cartesian product* of $\mathcal{N}_1$ and $\mathcal{N}_2$, denoted by $\mathcal{N}_1 \boxtimes \mathcal{N}_2$, as the network with set of cells (nodes) the cartesian product $\mathcal{C}_1 \times \mathcal{C}_2$ and with adjacency matrices $A_1 \otimes \mathrm{Id}_{\#\mathcal{C}_2}$, $\mathrm{Id}_{\#\mathcal{C}_1} \otimes A_2$. $\diamond$

Assume that $\mathcal{N}_i$ has r_i cells, say $\mathcal{C}_i = \{1, \ldots, r_i\}$, for $i = 1, 2$. Denote by ij the cell of $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ corresponding to $(i, j) \in \mathcal{C}_1 \times \mathcal{C}_2$. It follows then that there is an arrow from cell ij to cell kl in $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ if and only if:

$$i = k \text{ and } (j, l) \in \mathcal{E}_2, \text{ or } j = l \text{ and } (i, k) \in \mathcal{E}_1 \,. \tag{3.1}$$

Observe that the product network $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ has two adjacency matrices $A_1 \otimes \mathrm{Id}_{\#\mathcal{C}_2}$, $\mathrm{Id}_{\#\mathcal{C}_1} \otimes A_2$ and so two arrow types: the arrows of the type corresponding to $A_1 \otimes \mathrm{Id}_{\#\mathcal{C}_2}$ connect cell ij to cell kj when $(i, k) \in \mathcal{E}_1$; the arrows of the type corresponding to $\mathrm{Id}_{\#\mathcal{C}_1} \otimes A_2$ connect cell ij to cell il when $(j, l) \in \mathcal{E}_2$.

Remark 3.2 Definition 3.1 generalizes the definition of product network of [12] where it is defined the product of regular networks. $\diamond$

Example 3.3 Consider the 2-cell regular network $\mathcal{N}_1$, the 3-cell regular network $\mathcal{N}_2$ and the homogeneous product network $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ in Figure 1. $\diamond$

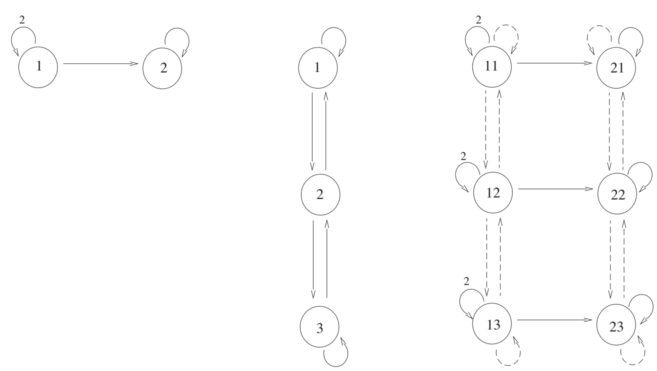

Figure 1: (From left to right) A 2-cell network $\mathcal{N}_1$. A 3-cell network $\mathcal{N}_2$. The cartesian product network $\mathcal{N}_1 \boxtimes \mathcal{N}_2$.

Remark 3.4 If the eigenvalues of A_1 are $\lambda_i, i = 1, \dots, s$, with $s \le r_1$, with algebraic multiplicities $m_a(\lambda_i)$, then the eigenvalues of $A_1 \otimes \mathrm{Id}_{r_2}$ are $\lambda_i, i = 1, \dots, s$ with algebraic multiplicities $r_2 \times m_a(\lambda_i)$. Moreover, for $i = 1, \dots, s$, the generalized eigenspace $G_{\lambda_i}^{A_1 \otimes \mathrm{Id}_{r_2}}$ of $A_1 \otimes \mathrm{Id}_{r_2}$ is given by

$$G_{\lambda_i}^{A_1 \otimes \mathrm{Id}_{r_2}} = G_{\lambda_i}^{A_1} \otimes \mathbf{R}^{r_2},$$

with $G_{\lambda_i}^{A_1}$ the generalized eigenspace of λ_i for the matrix A_1. Analogously, if the eigenvalues of A_2 are $\beta_j, j = 1, \dots, t$, with $t \le r_2$, with algebraic multiplicities $m_a(\beta_j)$, then the eigenvalues of $\mathrm{Id}_{r_1} \otimes A_2$ are $\beta_j, j = 1, \dots, t$ with algebraic multiplicities $r_1 \times m_a(\beta_j)$. Moreover, for $j = 1, \dots, t$, the generalized eigenspace $G_{\beta_j}^{\mathrm{Id}_{r_1} \otimes A_2}$ of $\mathrm{Id}_{r_1} \otimes A_2$ is given by

$$G_{\beta_j}^{\mathrm{Id}_{r_1} \otimes A_2} = \mathbf{R}^{r_1} \otimes G_{\beta_j}^{A_2},$$

with $G_{\beta_i}^{A_2}$ the generalized eigenspace of β_j for the matrix A_2. ◇

Remark 3.5 Let $\mathcal{N}_1, \mathcal{N}_2$ be two networks.
(i) The cartesian product networks $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ and $\mathcal{N}_2 \boxtimes \mathcal{N}_1$ are isomorphic.
(ii) The cartesian product network $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ is injective if and only if $\mathcal{N}_1$ and $\mathcal{N}_2$ are injective networks.
(iii) The cartesian product network $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ is an homogeneous network if and only if both networks $\mathcal{N}_1$ and $\mathcal{N}_2$ are regular. More specifically, the number of input classes of $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ is the product of the numbers of input classes of the two networks $\mathcal{N}_1$ and $\mathcal{N}_2$. ◇

3.2 Kronecker (tensor) product network

Definition 3.6 For $i = 1, 2$, consider the network $\mathcal{N}_i$ with set of cells (nodes) $\mathcal{C}_i$, set of edges $\mathcal{E}_i$, and adjacency matrix A_i. Following [23], we define the *Kronecker product* of $\mathcal{N}_1$ and $\mathcal{N}_2$, denoted by $\mathcal{N}_1 \otimes \mathcal{N}_2$, as the network with set of cells (nodes) the cartesian product $\mathcal{C}_1 \times \mathcal{C}_2$ and with adjacency matrix $A_1 \otimes A_2$. ◇

As before, assume that $\mathcal{N}_i$ has r_i cells, say $\mathcal{C}_i = \{1, \dots, r_i\}$, for $i = 1, 2$ and denote by ij the cell of $\mathcal{N}_1 \otimes \mathcal{N}_2$ corresponding to $(i, j) \in \mathcal{C}_1 \times \mathcal{C}_2$. It follows then that there is an arrow from cell ij to cell kl in $\mathcal{N}_1 \otimes \mathcal{N}_2$ if and only if:

$$(i, k) \in \mathcal{E}_1 \text{ and } (j, l) \in \mathcal{E}_2 \,. \tag{3.2}$$

Example 3.7 In Figure 2, we show the Kronecker product network $\mathcal{N}_1 \otimes \mathcal{N}_2$ of the 2-cell and 3-cell networks of Example 3.3. ◇

Remark 3.8 If u and w are eigenvectors of A_1 and A_2 associated to eigenvalues λ_1 and λ_2, respectively, then $v = u \otimes w$ is an eigenvector of $A_1 \otimes A_2$ associated to the eigenvalue $\lambda_1\lambda_2$. More generally, every eigenvector v of $A_1 \otimes A_2$ associated to an eigenvalue λ is given by a linear combination $v = \sum_i \alpha_i u_i \otimes w_i$, with u_i and w_i eigenvectors of A_1 and A_2 associated with eigenvalues λ_i^1 and λ_i^2, respectively, such that $\lambda_i^1 \lambda_i^2 = \lambda$. ◇

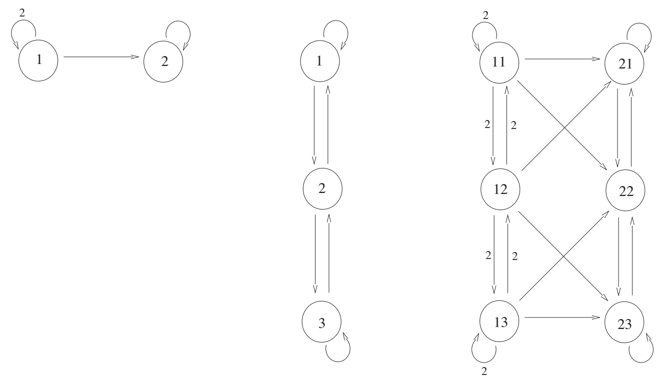

Figure 2: (From left to right) A 2-cell network $\mathcal{N}_1$. A 3-cell network $\mathcal{N}_2$. The Kronecker product network $\mathcal{N}_1 \otimes \mathcal{N}_2$.

Proposition 3.9 *Every generalized eigenspace $G_\mu^{A_1 \otimes A_2}$ of $A_1 \otimes A_2$ is given by*

$$G_\mu^{A_1 \otimes A_2} = \oplus_{i,j} \left(G_{\lambda_i}^{A_1} \otimes G_{\beta_j}^{A_2} \right),$$

with $G_{\lambda_i}^{A_1}$ and $G_{\beta_j}^{A_2}$ generalized eigenspaces of A_1 and A_2 associated to eigenvalues λ_i and β_j, respectively, such that $\lambda_i \beta_j = \mu$.

Proof The result follows from Remark 3.8. $\square$

Remark 3.10 Let $\mathcal{N}_1, \mathcal{N}_2$ be two networks.
(i) The Kronecker product networks $\mathcal{N}_1 \otimes \mathcal{N}_2$ and $\mathcal{N}_2 \otimes \mathcal{N}_1$ are isomorphic.
(ii) The Kronecker product network $\mathcal{N}_1 \otimes \mathcal{N}_2$ is injective if and only if $\mathcal{N}_1$ and $\mathcal{N}_2$ are injective networks.
(iii) The Kronecker product network $\mathcal{N}_1 \otimes \mathcal{N}_2$ is regular if and only if $\mathcal{N}_1$ and $\mathcal{N}_2$ are regular networks. $\diamond$

4 Lattices of synchrony subspaces

In this section we start by recalling the definition of synchrony subspace for a network and some results concerning the set of all synchrony subspaces for a network, which forms a lattice under the subspace inclusion operation. We then establish the relation between the lattices of synchrony subspaces for the cartesian and Kronecker product networks.

4.1 The lattice of synchrony subspaces for a coupled cell network

A network structure imposes the existence of certain flow-invariant subspaces for any coupled cell system associated with that structure. These subspaces are called synchrony subspaces. Following [22, 14]:

Definition 4.1 Given a network $\mathcal{G}$, a *synchrony subspace* Δ of the total phase space P is a subspace of P characterized by a set of cell coordinates equalities which is flow-invariant for all $\mathcal{G}$-admissible vector fields on P. $\diamond$

From [22, 14] it follows that for any choice of the total phase space P, a polydiagonal subspace Δ is a synchrony subspace if and only if a combinatorial condition at the network is satisfied. Specifically, the relation on the network set of cells, defined by making two cells equivalent if and only if the corresponding cell coordinates are being identified on Δ, has to be *balanced*: two equivalent cells must have input sets isomorphic through an isomorphism that preserves the arrow types. For details, see [22, 14]. As a corollary of this result it follows that for any choice of the total phase space, a polydiagonal subspace is a synchrony subspace if and only it is left invariant for *all linear* network admissible vector fields choosing the cell phase spaces to be $\mathbf{R}$. Equivalently, a polydiagonal subspace is a synchrony subspace if and only it is left invariant by all the network adjacency matrices. See [3, Section 2.2].

Now recall that a *lattice* is a partially ordered set X such that every pair of elements $x, y \in X$ has a *unique least upper bound or join*, denoted by $x \vee y$, and a *unique greatest lower bound or meet*, denoted by $x \wedge y$. A *complete lattice* is a lattice where every subset $Y \subseteq X$ has a unique least upper bound or join, and a unique greatest lower bound or meet. See for example Davey and Priestley [10].

Let $V_{\mathcal{G}}$ be the set of synchrony subspaces for $\mathcal{G}$ and note that the intersection of two synchrony subspaces for a network is again a synchrony subspace. Taking the partial order on $V_{\mathcal{G}}$ given by inclusion $\subseteq$ of spaces, it follows that $V_{\mathcal{G}}$ is a complete lattice, where the meet operation is the intersection and the join can be defined in terms of the meet. See Stewart [21] (or [3, Section 3.3]). Moreover, the lattice $V_{\mathcal{G}}$, as a set, it is a subset of the lattice of the invariant subspaces under the network adjacency matrices.

Example 4.2 The lattice of synchrony subspaces for the 3-cell network $\mathcal{N}_2$ at Figure 2 is formed by three spaces: two are trivial, corresponding to the full total phase space P and the diagonal space $\{\mathbf{x} : x_1 = x_2 = x_3\}$, and the third one is $\{\mathbf{x} : x_1 = x_3\}$. $\diamond$

4.2 Relation between the lattices of synchrony subspaces for the cartesian and Kronecker products

Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be networks with only one edge type. Consider the product networks $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ and $\mathcal{N}_1 \otimes \mathcal{N}_2$, and the corresponding lattices of synchrony subspaces $V_{\mathcal{N}_1 \boxtimes \mathcal{N}_2}$ and $V_{\mathcal{N}_1 \otimes \mathcal{N}_2}$.

Assume the networks $\mathcal{N}_1$ and $\mathcal{N}_2$ have r_1 and r_2 cells, respectively. Let A_1 and A_2 be the adjacency matrices of $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively. It follows then that the adjacency matrices of $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ are the $r_1 r_2 \times r_1 r_2$-matrices $A_1 \otimes \mathrm{Id}_{r_2}$ and $\mathrm{Id}_{r_1} \otimes A_2$, and $A_1 \otimes A_2$ is the adjacency matrix of $\mathcal{N}_1 \otimes \mathcal{N}_2$. We have the following relation between the matrices $A_1 \otimes \mathrm{Id}_{r_2}$, $\mathrm{Id}_{r_1} \otimes A_2$ and $A_1 \otimes A_2$:

Remark 4.3 The adjacency matrices $A_1 \otimes \mathrm{Id}_{r_2}$ and $\mathrm{Id}_{r_1} \otimes A_2$ of $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ satisfy

$$(A_1 \otimes \mathrm{Id}_{r_2}) \circ (\mathrm{Id}_{r_1} \otimes A_2) = (\mathrm{Id}_{r_1} \otimes A_2) \circ (A_1 \otimes \mathrm{Id}_{r_2}) = A_1 \otimes A_2\,.$$

$\diamond$

Lemma 4.4 *A subspace S of $\mathbf{R}^{r_1} \otimes \mathbf{R}^{r_2}$ which is invariant under both $A_1 \otimes \mathrm{Id}_{r_2}$ and $\mathrm{Id}_{r_1} \otimes A_2$ it is also invariant under $A_1 \otimes A_2$.*

Proof If a subspace S is invariant under $A_1 \otimes \mathrm{Id}_{r_2}$ and $\mathrm{Id}_{r_1} \otimes A_2$ then, from Remark 4.3, we have

$$(A_1 \otimes A_2)(S) \;= (A_1 \otimes \mathrm{Id}_{r_2}) \circ (\mathrm{Id}_{r_1} \otimes A_2)(S) \subseteq (A_1 \otimes \mathrm{Id}_{r_2})(S) \subseteq S\,.$$

Thus, S is $A_1 \otimes A_2$-invariant. $\square$

Proposition 4.5 *Given two networks $\mathcal{N}_1$ and $\mathcal{N}_2$ with only one edge type, we have the following inclusion relation between the lattices of synchrony subspaces for their product networks:*

$$V_{\mathcal{N}_1 \boxtimes \mathcal{N}_2} \subseteq V_{\mathcal{N}_1 \otimes \mathcal{N}_2}\,.$$

Proof For $i = 1, 2$, denote by A_i the adjacency matrix of $\mathcal{N}_i$. Then the adjacency matrices of $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ are the matrices $A_1 \otimes \mathrm{Id}_{r_2}$ and $\mathrm{Id}_{r_1} \otimes A_2$, and $A_1 \otimes A_2$ is the adjacency matrix of $\mathcal{N}_1 \otimes \mathcal{N}_2$. Using Lemma 4.4 and the fact that a synchrony subspace for a coupled cell network is a polydiagonal subspace that is left invariant under the adjacency matrices of the network, we obtain $V_{\mathcal{N}_1 \boxtimes \mathcal{N}_2} \subseteq V_{\mathcal{N}_1 \otimes \mathcal{N}_2}$. $\square$

The following example illustrates that, in general, the inclusion in Proposition 4.5 is strict.

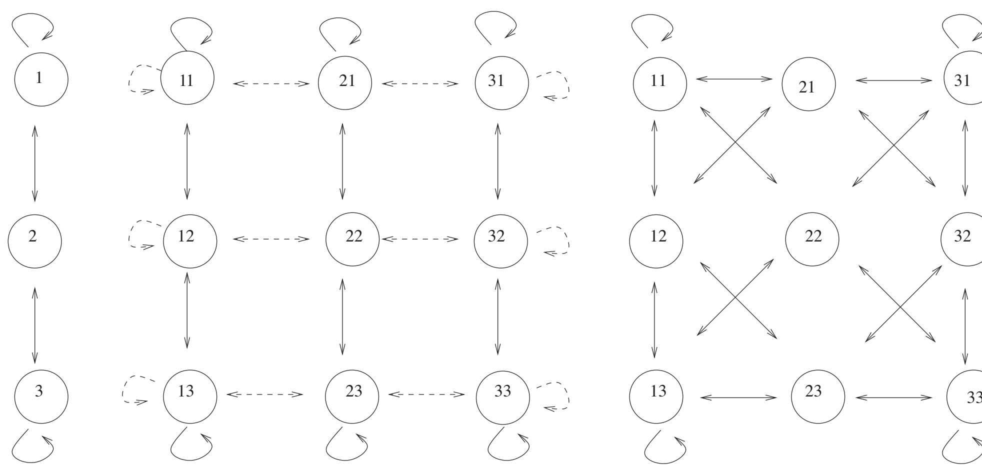

Figure 3: (From left to right) A 3-cell network $\mathcal{N}$. The cartesian product network $\mathcal{N} \boxtimes \mathcal{N}$. The Kronecker product network $\mathcal{N} \otimes \mathcal{N}$.

Example 4.6 Consider the network $\mathcal{N}$ and the product networks $\mathcal{N} \boxtimes \mathcal{N}$, $\mathcal{N} \otimes \mathcal{N}$, in Figure 3. Take the subspace $S = \{\mathbf{x} :\ x_{11} = x_{33},\ x_{12} = x_{23},\ x_{21} = x_{32}\}$. Note that S is a synchrony subspace for $\mathcal{N} \otimes \mathcal{N}$ but it is not a a synchrony subspace for $\mathcal{N} \boxtimes \mathcal{N}$. Thus the network $\mathcal{N}$ is an example where $V_{\mathcal{N} \boxtimes \mathcal{N}} \subsetneqq V_{\mathcal{N} \otimes \mathcal{N}}$. $\diamond$

5 The lattice of synchrony subspaces for the Kronecker product

In this section, we address the problem of relating the lattice of synchrony subspaces for a Kronecker product network with the lattices of synchrony subspaces for the component coupled cell networks of the product. We start by reviewing certain results concerning the lattice of invariant subspaces for the Kronecker product of matrices, which relate with the lattices of invariant subspaces for each component matrix in the product. Motivated by these results, we then prove similar results for the lattice of synchrony subspaces for the Kronecker product of networks.

5.1 The lattice of invariant subspaces for the Kronecker product of matrices

Let P_1, P_2 be two real vector spaces and consider the linear subspace $P_1 \otimes P_2$. As in Kubrusly [18], we define regular subspaces of $P_1 \otimes P_2$ in the following way.

Definition 5.1 A subspace of the tensor product $P_1 \otimes P_2$ is *regular* if it is of the form $S_1 \otimes S_2$, with S_i a subspace of P_i, $i = 1, 2$. Otherwise, it is *irregular*. $\diamond$

From the results in Lemma 1 of Kubrusly [18], we have:

Lemma 5.2 *For $i = 1, 2$, let A_i be a linear operator on a linear space P_i, and S_i a subspace of P_i. We have:*

1. *If S_i is an invariant subspace for A_i, $i = 1, 2$, then $S = S_1 \otimes S_2$ is an invariant subspace for $A_1 \otimes A_2$.*
2. *If $S = S_1 \otimes S_2$ is an invariant subspace for $A_1 \otimes A_2$, then, for $i = 1$ or $i = 2$, S_i is an invariant subspace for A_i.*
3. *If $S = S_1 \otimes S_2$ is an invariant subspace for $A_1 \otimes A_2$ and if $S_i \not\subseteq \ker(A_i)$, for $i = 1, 2$, then S_i is an invariant subspace for A_i. Particular case:*
   (a) *If $S = S_1 \otimes S_2$ is nonzero and invariant for $A_1 \otimes A_2$ and if A_i, for $i = 1, 2$, is injective, then S_i is an invariant subspace for A_i.*

According to Lemma 5.2, we present next an example to illustrate that if the matrices A_1 and A_2 are not both injective, then we can have $S = S_1 \otimes S_2$ invariant for $A_1 \otimes A_2$ with only S_1 or S_2 invariant for A_1 or A_2, respectively.

Example 5.3 Consider the linear matrices

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

The generalized eigenspaces of A_1 are $G_2^{A_1} =< (1,1) >$ and $G_0^{A_1} =< (1,-1) >$ and the generalized eigenspaces of A_2 are $G_2^{A_2} =< (1,1,1) >$ and $G_0^{A_2} =< (1,-1,0),(0,0,1) >$. Thus, the generalized eigenspaces of $A_1 \otimes A_2$ are $G_4^{A_1 \otimes A_2} =< (1,1) \otimes (1,1,1) >$ and

$$G_0^{A_1 \otimes A_2} =< (1,1) \otimes (1,-1,0), (1,1) \otimes (0,0,1), (1,-1) \otimes (1,1,1), (1,-1) \otimes (1,-1,0), (1,-1) \otimes (0,0,1) > .$$

Note that the linear operators A_1, A_2 (and $A_1 \otimes A_2$) are not injective. As $(1,-1) \otimes (1,1,1)$ and $(1,-1) \otimes (0,0,1)$ are eigenvectors of $A_1 \otimes A_2$ associated with the eigenvalue 0, it follows that $(1,-1) \otimes (1,1,1) + (1,-1) \otimes (0,0,1)$ is also an eigenvector of $A_1 \otimes A_2$. Thus the subspace $< (1,-1) > \otimes < (1,1,2) >$ is invariant for $A_1 \otimes A_2$, where the subspace $< (1,-1) >$ is invariant for A_1 but the subspace $< (1,1,2) >$ is not invariant for A_2. $\diamond$

As above, for $i = 1, 2$, let A_i be a linear operator on a linear space P_i, and S_i a subspace of P_i. Let $\mathrm{Lat}(A_1) \otimes \mathrm{Lat}(A_2)$ denote the collection of all nonzero regular subspaces $S_1 \otimes S_2$, where each S_i is an A_i-invariant subspace of P_i, and let $\mathrm{RLat}(A_1 \otimes A_2)$ denote the collection of all regular invariant subspaces of $P_1 \otimes P_2$ for $A_1 \otimes A_2$. From the results in Theorem 1 of Kubrusly [18], we have:

Theorem 5.4 *RLat$(A_1 \otimes A_2)$ is a lattice. If each A_i is injective, then*

$$\mathit{RLat}(A_1 \otimes A_2) \setminus \{0\} = \mathit{Lat}(A_1) \otimes \mathit{Lat}(A_2) \subseteq \mathit{Lat}(A_1 \otimes A_2) \setminus \{0\},$$

and the inclusion may be proper even though every A_i is injective.

Theorem 5.4 states that for injective matrices, A_1 and A_2, the lattice of regular invariant subspaces for $A_1 \otimes A_2$ is given by the product of the invariant subspaces for A_1 with the invariant subspaces for A_2. Nevertheless, as the theorem states and as the following example illustrates, there can be other invariant subspaces for $A_1 \otimes A_2$, the irregular ones.

Example 5.5 Consider the injective linear operators on $\mathbf{R}^2$ and $\mathbf{R}^3$, respectively, given by:

$$A_1 = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The generalized eigenspaces for A_1 are $G_2^{A_1} =< u_1 >=< (1,1) >$ and $G_1^{A_1} =< u_2 >=< (0,1) >$. The generalized eigenspaces for A_2 are $G_2^{A_2} =< w_1 >=< (1,1,1) >$, $G_{-1}^{A_2} =< w_2 >=< (1,-2,1) >$ and $G_1^{A_2} =< w_3 >=< (1,0,-1) >$. Consider now the Kronecker product matrix

$$A_1 \otimes A_2 = \begin{bmatrix} 2A_2 & 0A_2 \\ 1A_2 & 1A_2 \end{bmatrix} \cong \begin{bmatrix} 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

According to Proposition 3.9, we have the following generalized eigenspaces for $A \otimes B$,

$$\begin{aligned}
G_4^{A_1 \otimes A_2} &= G_2^{A_1} \otimes G_2^{A_2} =< u_1 \otimes w_1 >, \\
G_2^{A_1 \otimes A_2} &= \left(G_2^{A_1} \otimes G_1^{A_2}\right) \oplus \left(G_1^{A_1} \otimes G_2^{A_2}\right) =< u_1 \otimes w_3, u_2 \otimes w_1 >, \\
G_{-2}^{A_1 \otimes A_2} &= G_2^{A_1} \otimes G_{-1}^{A_2} =< u_1 \otimes w_2 >, \\
G_{-1}^{A_1 \otimes A_2} &= G_1^{A_1} \otimes G_{-1}^{A_2} =< u_2 \otimes w_2 >, \\
G_1^{A_1 \otimes A_2} &= G_1^{A_1} \otimes G_1^{A_2} =< u_2 \otimes w_3 > .
\end{aligned}$$

We have $\mathrm{RLat}(A_1 \otimes A_2) \setminus \{0\} \subsetneq \mathrm{Lat}(A_1 \otimes A_2) \setminus \{0\}$ since, for example, the $A_1 \otimes A_2$-invariant subspace $< u_1 \otimes w_1, u_2 \otimes w_1, u_2 \otimes w_2, u_2 \otimes w_3 >$ is an irregular subspace.

$\diamond$

5.2 The lattice of synchrony subspaces for the Kronecker product of networks

Motivated by the results in Section 5.1, we prove in this section how to get the lattice of regular synchrony subspaces for the Kronecker product of networks from the the lattices of synchrony subspaces for the component networks of the product. We also show that, in general, the same is not possible for the all lattice of synchrony subspaces for the Kronecker product network.

As before, let P_1, P_2 be two real vector spaces, say $\mathbf{R}^n$ and $\mathbf{R}^m$, and consider the linear space $P_1 \otimes P_2 \cong \mathbf{R}^{nm}$. We start by recalling that a synchrony subspace for an identical-edge n-cell network is a polydiagonal subspace that is left invariant by the $n \times n$ network adjacency matrix. (See Definition 4.1 and the discussion that follows that definition.)

Lemma 5.6 *A regular subspace $S = S_1 \otimes S_2$ of $P_1 \otimes P_2$ is polydiagonal if and only if each S_i is a polydiagonal subspace of P_i, $i = 1, 2$.*

Proof Let $S = S_1 \otimes S_2$ be a regular subspace of $P_1 \otimes P_2$ where both S_1 and S_2 are polydiagonal subspaces of P_1 and P_2, respectively. Trivially, we have that $S_1 \otimes P_1$ and $P_1 \otimes S_2$ are polydiagonal subspaces of $P_1 \otimes P_2$. As $S = S_1 \otimes S_2 = S_1 \otimes P_2 \cap P_1 \otimes S_2$, it follows that S is a polydiagonal subspace of $P_1 \otimes P_2$.

Assume now that $S = S_1 \otimes S_2$ is a regular subspace of $P_1 \otimes P_2$ where for example S_1 is not a polydiagonal subspace of P_1. Recall that $P_1 = \mathbf{R}^n$ and $P_2 = \mathbf{R}^m$. We prove that then S cannot be a polydiagonal subspace of $P_1 \otimes P_2$. Assume that S_1 has dimension r and S_2 has dimension s. Take the polydiagonal subspace P_{S_1} of P_1 given by the intersection of all polydiagonal subspaces of P_1 containing S_1. If P_{S_1} has dimension $n - i$ we have that all the vectors of P_{S_1} and of S_1 satisfy i independent equalities. Moreover, as S_1 is not a polydiagonal, that is, S_1 is strictly contained in P_{S_1}, we have that $r < n - i$. Take also the polydiagonal subspace P_{S_2} of P_2 given by the intersection of all polydiagonal subspaces of P_2 containing S_2. If P_{S_2} has dimension $n-j$ we have that all the vectors of P_{S_2} and of S_2 satisfy j independent equalities, and $s \leq m - j$. The polydiagonal $P_{S_1} \otimes P_{S_2}$ has dimension $(n-i)(m-j)$ and all the vectores in $P_{S_1} \otimes P_{S_2}$ and in $S_1 \otimes S_2$ satisfy $jn + i(m - j)$ independent equalities. In fact, as $S = S_1 \otimes S_2$, and we know that the maximum number of independent equalities satisfied by all the vectors of S_1 is i and of S_2 is j, then the maximum number of independent equalities satisfied by all the vectors of $S_1 \otimes S_2$ is $jn + i(m - j)$. Since $S_1 \otimes S_2$ has dimension $rs < (n - i)(m - j)$, we conclude that $S = S_1 \otimes S_2$ cannot be a polydiagonal subspace. □

Lemma 5.7 *For $i = 1, 2$, let $\mathcal{N}_i$ be an identical-edge coupled cell network. Assume $\mathcal{N}_1$ has n cells and $\mathcal{N}_2$ has m cells. Take P_1, P_2 to be the real vector spaces $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively. For $i = 1, 2$, let S_i a subspace of P_i. We have:*

1. *If S_i is a synchrony subspace for the network $\mathcal{N}_i$, for $i = 1, 2$, then the subspace $S = S_1 \otimes S_2$ of $P_1 \otimes P_2$ is a regular synchrony subspace for $\mathcal{N}_1 \otimes \mathcal{N}_2$.*

2. *If $S = S_1 \otimes S_2$ is a regular synchrony subspace for $\mathcal{N}_1 \otimes \mathcal{N}_2$, then at least one of the S_i is a synchrony subspace for $\mathcal{N}_i$.*

3. *If $\mathcal{N}_1$ and $\mathcal{N}_2$ are injective networks and $S = S_1 \otimes S_2$ is a regular synchrony subspace for $\mathcal{N}_1 \otimes \mathcal{N}_2$, then S_1 and S_2 are synchrony subspaces for $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively.*

Proof If S_i, $i = 1, 2$, is a synchrony subspace for $\mathcal{N}_i$ with adjacency matrix A_i, by Lemma 5.2, the regular subspace $S = S_1 \otimes S_2$ is invariant under $A_1 \otimes A_2$, the adjacency matrix of $\mathcal{N}_1 \otimes \mathcal{N}_2$; by Lemma 5.6, as S_1, S_2 are polydiagonal subspaces, $S_1 \otimes S_2$ is also a polydiagonal subspace. Thus $S_1 \otimes S_2$ is a regular synchrony subspace for $\mathcal{N}_1 \otimes \mathcal{N}_2$, proving statement 1.

Assume now that $S = S_1 \otimes S_2$ is a regular synchrony subspace for $\mathcal{N}_1 \otimes \mathcal{N}_2$. Thus S is a polydiagonal subspace of $P_1 \otimes P_2$ that is left invariant under $A_1 \otimes A_2$, the adjacency matrix of $\mathcal{N}_1 \otimes \mathcal{N}_2$. By Lemma 5.6, we have that both S_1 and S_2 are polydiagonal subspaces. By Lemma 5.2 we have that at least one of the S_i is invariant under A_i, and so it is a synchrony subspace for $\mathcal{N}_i$, proving statement 2.

If, additionally, A_1 and A_2 are injective then, by Lemma 5.2, both S_1 and S_2 are invariant under A_1 and A_2, respectively, and so are synchrony subspaces for $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively, proving statement 3. □

The following lemma will be used in the proof of Proposition 5.9.

Lemma 5.8 *Let A be the adjacency matrix of an n-cell network. There is no polydiagonal subspace of $\mathbf{R}^n$ contained in* $\ker(A)$.

Proof If a polydiagonal subspace P was contained in $\ker(A)$ then, in particular, the $(1, 1, \ldots, 1)$ vector would belong to $\ker(A)$. Since A is a nonnegative matrix, that would only be possible if A was the zero matrix, which is not the case. □

The following proposition generalizes the result of statement 3 of Lemma 5.7 for any identical-edge network, injective or not.

Proposition 5.9 *If S is a regular synchrony subspace for $\mathcal{N}_1 \otimes \mathcal{N}_2$ with $S = S_1 \otimes S_2$ then S_i is a synchrony subspace for $\mathcal{N}_i$, $i = 1, 2$.*

Proof Let A_i denote the adjacency matrix of $\mathcal{N}_i$, for $i = 1, 2$. Since $S = S_1 \otimes S_2$ is a synchrony subspace for $\mathcal{N}_1 \otimes \mathcal{N}_2$, it is a polydiagonal subspace that is invariant for $A_1 \otimes A_2$. As $S = S_1 \otimes S_2$ is a polydiagonal subspace, from Lemma 5.6, we have that both S_1 and S_2 are polydiagonal subspaces. To prove that S_1 and S_2 are synchrony subspaces it remains to prove that they are invariant subspaces for A_1 and A_2, respectively. If $\mathcal{N}_1 \otimes \mathcal{N}_2$ is an injective network that follows from statement 3 of Lemma 5.7. If $\mathcal{N}_1 \otimes \mathcal{N}_2$ is not an injective network, we have from Lemma 5.2, that at least one of the S_i, for $i = 1$ or $i = 2$, is an invariant subspace for N_i. But, in the situation where S_1 and S_2 are polydiagonal subspaces we cannot have that one of the S_i, for $i = 1$ or $i = 2$, is not an invariant subspace for A_i. In fact, if one of the S_is was not invariant for A_i, then it would have to be contained at $\ker(A_i)$ by Lemma 5.2. But, by Lemma 5.8, that is impossible since S_i is a polydiagonal subspace. □

As above, for $i = 1, 2$, let $\mathcal{N}_i$ be a network with adjacency matrix A_i. Thus A_i is a linear operator on the linear space P_i, say $P_1 = \mathbf{R}^n$ and $P_2 = \mathbf{R}^m$, if $\mathcal{N}_1$ has n cells and $\mathcal{N}_2$ has m cells. We generalize now the notation given at Section 5.1 for $\mathrm{Lat}(A_1) \otimes \mathrm{Lat}(A_2)$ and $\mathrm{RLat}(A_1 \otimes A_2)$. Let $V_{\mathcal{N}_1} \otimes V_{\mathcal{N}_2}$ denote the collection of all regular subspaces $S_1 \otimes S_2$, where each S_i is a synchrony subspace for $\mathcal{N}_i$. Let $\mathrm{RV}_{\mathcal{N}_1 \otimes \mathcal{N}_2}$ denote the collection of all regular synchrony subspaces of $P_1 \otimes P_2$ for $\mathcal{N}_1 \otimes \mathcal{N}_2$ and recall that $V_{\mathcal{N}_1 \otimes \mathcal{N}_2}$ denotes the lattice of synchrony subspaces for the Kronecker product network $\mathcal{N}_1 \otimes \mathcal{N}_2$. From the above results, we have:

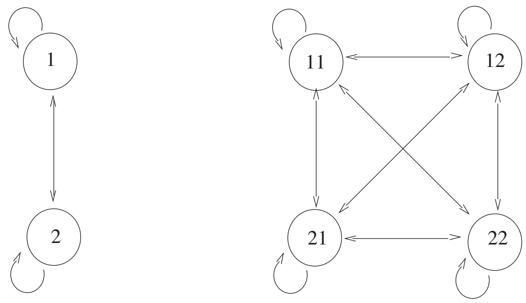

Figure 4: (Left) The network $\mathcal{N}$. (Right) The Kronecker product network $\mathcal{N} \otimes \mathcal{N}$.

Theorem 5.10 *$RV_{\mathcal{N}_1 \otimes \mathcal{N}_2}$ is a lattice and*

$$RV_{\mathcal{N}_1 \otimes \mathcal{N}_2} = V_{\mathcal{N}_1} \otimes V_{\mathcal{N}_2} \subseteq V_{\mathcal{N}_1 \otimes \mathcal{N}_2}$$

and the inclusion may be proper even though every $\mathcal{N}_i$ is injective.

Proof By the statement 2 of Lemma 5.7 and Proposition 5.9, a regular subspace $S_1 \otimes S_2$ of $P_1 \otimes P_2$ is a synchrony subspace for the Kronecker product network $\mathcal{N}_1 \otimes \mathcal{N}_2$ if and only if both S_1 and S_2 are synchrony subspaces for the newtorks $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively. We have then that $\mathrm{RV}_{\mathcal{N}_1 \otimes \mathcal{N}_2} = V_{\mathcal{N}_1} \otimes V_{\mathcal{N}_2}$. Moreover, $\mathrm{RV}_{\mathcal{N}_1 \otimes \mathcal{N}_2}$ is a lattice since it is the tensor of the two lattices $V_{\mathcal{N}_1}$ and $V_{\mathcal{N}_2}$. (See Corollary 2.7 of Grätzer *et al.* [15].) $\square$

We finish this section by showing examples where the inclusion $V_{\mathcal{N}_1} \otimes V_{\mathcal{N}_2} \subseteq V_{\mathcal{N}_1 \otimes \mathcal{N}_2}$ is proper, for both cases where the networks are injective or not. Moreover, our examples also show the existence of irregular and sum-irreducible synchrony subspaces for the Kronecker product – that is, there are synchrony subspaces that are not a sum of regular synchrony subspaces for the Kronecker product. The conclusion is that, although, taking in advance the knowledge about the synchrony lattices for the component networks involved in a product, the synchrony lattice for the Kronecker product network is not, in general, completely determined by those lattices. That goes in the same lines as it is for the lattices of invariant subspaces under tensor product of linear operators, as seen in Section 5.1.

Example 5.11 Consider the network $\mathcal{N}$ and the Kronecker product network $\mathcal{N} \otimes \mathcal{N}$ in Figure 4. The adjacency matrix of $\mathcal{N}$ is

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Note that the networks are not injective. The generalized eigenspaces of A are $G_2^A =< (1,1) >$ and $G_0^A =< (1,-1) >$. Thus, the generalized eigenspaces of $A \otimes A$ are $G_4^{A\otimes A} =< (1,1) \otimes (1,1) > \cong < (1,1,1,1) >$ and $G_0^{A\otimes A} =< (1,1) \otimes (1,-1), (1,-1) \otimes (1,1), (1,-1) \otimes (1,-1) >$. Note that $G_0^{A\otimes A} \cong < (1,-1,1,-1), (1,1,-1,-1), (1,-1,-1,1) >$. We have that $S = \{\mathbf{x} : \; x_{11} = x_{12} = x_{21}\} =< (1,1,1,1), (1,1,1,-3) >$ is an irregular synchrony subspace that cannot be given as a sum of regular synchrony subspaces. $\diamond$

Example 5.12 Consider the injective network $\mathcal{N}$ in Figure 5 with adjacency matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

This is network 26 of Leite [19]. The generalized eigenspaces of A are $G_2 =< (1,1,1) >$, $G_1 =< (0,1,0) >$ and $G_{-1} =< (-2,1,4) >$. The only nontrivial synchrony subspace for this network is $\{\mathbf{x} : \ x_1 = x_3\}$.

Take the Kronecker product $\mathcal{N} \otimes \mathcal{N}$, see Figure 5, and the two-dimensional synchrony subspace for $\mathcal{N} \otimes \mathcal{N}$ given by $S = \{\mathbf{x} : \ x_{11} = x_{12} = x_{13} = x_{21} = x_{23} = x_{31} = x_{32} = x_{33}\}$. Note that

$$S =< (1,1,1,1,1,1,1,1,1), (0,0,0,0,1,0,0,0,0) > \cong < (1,1,1) \otimes (1,1,1), (0,1,0) \otimes (0,1,0) >$$

where $(1,1,1) \otimes (1,1,1)$ and $(0,1,0) \otimes (0,1,0)$ are eigenvectors of $A \otimes A$. Now S is a synchrony subspace for the Kronecker product which is sum-irreducible, as it is not the sum of proper synchrony subspaces for $\mathcal{N} \otimes \mathcal{N}$. Moreover, S is not a regular synchrony subspace. Thus, this is another example, where the Kronecker product has synchrony subspaces that cannot be obtained using the synchrony subspaces for the component networks of the product. $\diamond$

6 The lattice of synchrony subspaces for the cartesian product network

In this section, we show that the results presented in Section 5.2 for the Kronecker product of networks are also valid for the cartesian product of networks. More specifically, we prove that the regular synchrony lattice for the cartesian product of two networks coincides with that of the Kronecker product of the two networks and thus is given by the tensor product of the lattices of synchrony subspaces for the component networks of the product. Moreover, as for the Kronecker product, we show that, in general, we cannot get the synchrony lattice for the cartesian product of two networks from the lattices of synchrony subspaces for the component coupled cell networks of the product.

As before, let $\mathcal{N}_1$ and $\mathcal{N}_2$ be two identical-edge networks, say with n and m cells, respectively, and take $P_1 = \mathbf{R}^n$ and $P_2 = \mathbf{R}^m$. In what follows, we adopt the following notation: for $i = 1, 2$, denote by $\mathcal{N}_i^d$ the disconnected subnetwork of $\mathcal{N}_i$ with no edges. Thus the network $\mathcal{N}_1 \boxtimes \mathcal{N}_2^d$ is the subnetwork of $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ ignoring the edges corresponding to the network $\mathcal{N}_2$ – it is formed by r_2 copies of the network $\mathcal{N}_1$. Similarly, ignoring the edges corresponding to the network $\mathcal{N}_1$ we obtain r_1 copies of the network $\mathcal{N}_2$, that correspond to the network $\mathcal{N}_1^d \boxtimes \mathcal{N}_2$.

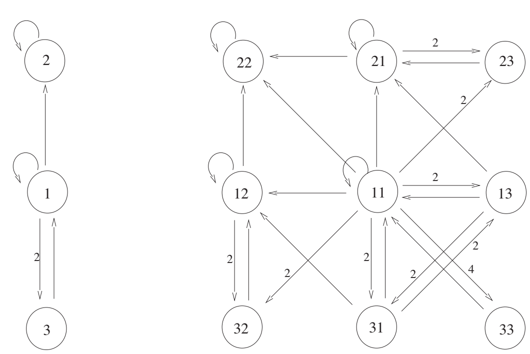

Figure 5: (Left) The network $\mathcal{N}$. (Right) The Kronecker product network $\mathcal{N} \otimes \mathcal{N}$.

Proposition 6.1 *A polydiagonal S is a synchrony subspace for the product network $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ if and only if S is a synchrony subspace for the subnetworks $\mathcal{N}_1 \boxtimes \mathcal{N}_2^d$ and $\mathcal{N}_1^d \boxtimes \mathcal{N}_2$.*

Proof Observe that the set of edges of the product network $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ is the disjoint union of the two sets of edges of the subnetworks $\mathcal{N}_1 \boxtimes \mathcal{N}_2^d$ and $\mathcal{N}_1^d \boxtimes \mathcal{N}_2$. Moreover, the types of edges of each of these subnetworks are distinct. Equivalently, the adjacency matrices of the product network $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ are the adjacency matrices of the two subnetworks $\mathcal{N}_1 \boxtimes \mathcal{N}_2^d$ and $\mathcal{N}_1^d \boxtimes \mathcal{N}_2$. Now recall that a polydiagonal subspace S is a synchrony subspace for $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ if and only if it is left invariant under its adjacency matrices. $\square$

Proposition 6.2 *The lattice of synchrony subspaces for the product network $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ is the set-wise intersection of the lattices of synchrony subspaces for the subnetworks $\mathcal{N}_1 \boxtimes \mathcal{N}_2^d$ and $\mathcal{N}_1^d \boxtimes \mathcal{N}_2$.*

Proof Follows trivially from Proposition 6.1. $\square$

Lemma 6.3 *If S_i is a synchrony space for the network $\mathcal{N}_i$, for $i = 1, 2$, then the subspace $S = S_1 \otimes S_2$ of $P_1 \otimes P_2$ is a regular synchrony space for $\mathcal{N}_1 \boxtimes \mathcal{N}_2$.*

Proof Let S_1 be a synchrony subspace for $\mathcal{N}_1$. Then $S_1 \otimes \mathbf{R}^{r_2}$ can be seen as r_2 distinct copies of the synchrony subspace S_1 for $\mathcal{N}_1$ – each one adapted to the coordinates of each of the r_2 copies of the network $\mathcal{N}_1$ in $\mathcal{N}_1 \boxtimes \mathcal{N}_2^d$. Thus, trivially, $S_1 \otimes \mathbf{R}^{r_2}$ is a synchrony subspace for $\mathcal{N}_1 \boxtimes \mathcal{N}_2^d$. Moreover, the adjacency matrix of the network $\mathcal{N}_1^d \boxtimes \mathcal{N}_2$ leaves $S_1 \otimes \mathbf{R}^{r_2}$ invariant. Thus $S_1 \otimes \mathbf{R}^{r_2}$ is also a synchrony subspace for $\mathcal{N}_1^d \boxtimes \mathcal{N}_2$. Similarly, we prove that if S_2 is a synchrony subspace for $\mathcal{N}_2$, then $\mathbf{R}^{r_1} \otimes S_2$ is a synchrony subspace for both networks $\mathcal{N}_1^d \boxtimes \mathcal{N}_2$ and $\mathcal{N}_1 \boxtimes \mathcal{N}_2^d$. By Proposition 6.1, $S_1 \otimes \mathbf{R}^{r_2}$ and $\mathbf{R}^{r_1} \otimes S_2$ are both synchrony subspaces for $\mathcal{N}_1 \boxtimes \mathcal{N}_2$. Thus $S_1 \otimes \mathbf{R}^{r_2} \cap \mathbf{R}^{r_1} \otimes S_2 = S_1 \otimes S_2$ is a synchrony for $\mathcal{N}_1 \boxtimes \mathcal{N}_2$. $\square$

Proposition 6.4 *If S is a regular synchrony subspace for $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ with $S = S_1 \otimes S_2$ then S_i is a synchrony subspace for $\mathcal{N}_i$, $i = 1, 2$.*

Proof If $S = S_1 \otimes S_2$ is a synchrony subspace for $\mathcal{N}_1 \boxtimes \mathcal{N}_2$, then it is a synchrony subspace for $\mathcal{N}_1 \otimes \mathcal{N}_2$, by the lattice inclusion $V_{\mathcal{N}_1 \boxtimes \mathcal{N}_2} \subseteq V_{\mathcal{N}_1 \otimes \mathcal{N}_2}$ proved in Proposition 4.5. The result then follows by Proposition 5.9. $\square$

Let $\mathrm{RV}_{\mathcal{N}_1 \boxtimes \mathcal{N}_2}$ denote the collection of all regular synchrony subspaces for $P_1 \otimes P_2$ for $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ and recall that $\mathrm{RV}_{\mathcal{N}_1 \otimes \mathcal{N}_2}$ denotes the lattice of regular synchrony subspaces for the Kronecker product network $\mathcal{N}_1 \otimes \mathcal{N}_2$ and $V_{\mathcal{N}_1 \boxtimes \mathcal{N}_2}$ denotes the lattice of synchrony subspaces for the cartesian product network $\mathcal{N}_1 \boxtimes \mathcal{N}_2$. From the above results, we have:

Theorem 6.5 *$RV_{\mathcal{N}_1 \boxtimes \mathcal{N}_2}$ is a lattice and*

$$RV_{\mathcal{N}_1 \boxtimes \mathcal{N}_2} = RV_{\mathcal{N}_1 \otimes \mathcal{N}_2} = V_{\mathcal{N}_1} \otimes V_{\mathcal{N}_2} \subseteq V_{\mathcal{N}_1 \boxtimes \mathcal{N}_2}$$

and the inclusion may be proper even though every $\mathcal{N}_i$ is injective.

Proof Note that, by Lemma 6.3 and Proposition 6.4, a regular subspace $S_1 \otimes S_2$ of $P_1 \otimes P_2$ is a synchrony subspace for the cartesian product network $\mathcal{N}_1 \boxtimes \mathcal{N}_2$ if and only if both S_1

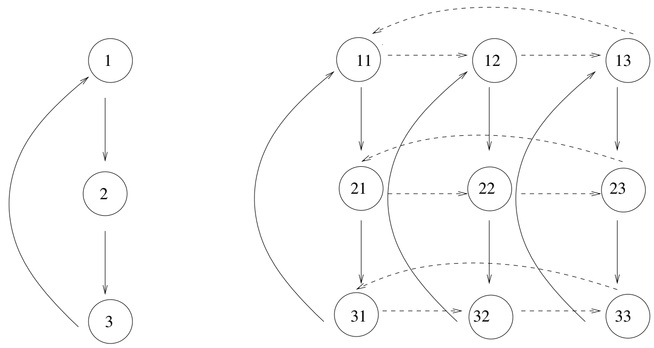

Figure 6: (Left) The 3-cell unidirectional ring network $\mathcal{N}$. (Right) The cartesian product network $\mathcal{N} \boxtimes \mathcal{N}$.

and S_2 are synchrony subspaces for the newtorks $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively. We have then that $\mathrm{RV}_{\mathcal{N}_1 \boxtimes \mathcal{N}_2} = V_{\mathcal{N}_1} \otimes V_{\mathcal{N}_2}$. By Theorem 5.10, it follows that $\mathrm{RV}_{\mathcal{N}_1 \boxtimes \mathcal{N}_2} = \mathrm{RV}_{\mathcal{N}_1 \otimes \mathcal{N}_2}$, and thus, it is a lattice. $\square$

We finish by showing an example where the inclusion $\mathrm{RV}_{\mathcal{N}_1 \boxtimes \mathcal{N}_2} \subseteq V_{\mathcal{N}_1 \boxtimes \mathcal{N}_2}$ is proper.

Example 6.6 Considering the cartesian product network $\mathcal{N} \boxtimes \mathcal{N}$ of Figure 6, we show that $\mathrm{RV}_{\mathcal{N} \boxtimes \mathcal{N}} \subsetneq V_{\mathcal{N} \boxtimes \mathcal{N}}$ by pointing out an irregular synchrony space in $V_{\mathcal{N} \boxtimes \mathcal{N}}$. For example, the polydiagonal $\{\mathbf{x} \in \mathbf{R}^9 : \ x_{11} = x_{22} = x_{33}, \ x_{12} = x_{23} = x_{31}, \ x_{13} = x_{21} = x_{32}\}$ is an irregular synchrony subspace for the product network $\mathcal{N} \boxtimes \mathcal{N}$ since it is not of the form $S_1 \otimes S_2$ with S_i a synchrony subspace for the network $\mathcal{N}$. $\diamond$

7 Conclusion

In this work we have considered coupled cell networks constructed as the product of smaller coupled cell networks, using two types of product: the cartesian and the Kronecker product. It is natural to realize that a network constructed as product of two networks, besides being a graph with more nodes and arrows, can as well be richer in some way. For example, from the symmetric point of view, it is possible that a product network has a bigger group of symmetries – bigger in the sense that, it includes symmetries that are not possible to describe mathematically from the symmetries of the graphs involved in the product. It follows then that, in particular, the dynamics for coupled cell systems associated with product networks will be in general, richer, depending on what kind of dynamics we are expecting to observe. One ultimate goal is to derive how far we can describe the dynamics of the coupled cell systems associated with a (large) product network based on the dynamics of the coupled cell systems consistent with the struture of the component networks of the product.

In this paper we have focused this dynamics question on the lattice of synchrony subspaces. In the analysis of how far it is possible to describe the set of synchrony subspaces for a product network from the sets of synchrony subspaces for the component networks, our answer is complete when we restrict to regular synchrony subspaces, that is, synchrony subspaces that are given by the tensor product of spaces. We show in Theorems 5.10 and 6.5 that, for both kinds of products considered here, the set of regular synchrony subspaces for the product is a lattice and it is the tensor product of the lattices of the synchrony subspaces for the component networks of the product. That is, a synchrony subspace for the product (cartesian or Kronecker) is regular if and only if it is the tensor of synchrony subspaces for the component networks of the product. Moreover, we also show that, in general, there are synchrony subspaces for the product that are not possible to describe using only the synchrony subspaces for the component networks.

In a future work, we intend to use our results relating the lattice of synchrony subspaces for a product coupled cell network with those of its component networks in the construction of heteroclinic networks in coupled cell systems associated with product networks whose component networks are known to support heteroclinic behaviour in the associated dynamics. We observe that it follows, in particular, from our results that if we take a coupled cell system consistent with a product network $\mathcal{N}_1 \otimes \mathcal{N}_2$ (or $\mathcal{N}_1 \boxtimes \mathcal{N}_2$), where it is known that at least one of the networks $\mathcal{N}_i$ can support heteroclinic cycles, then that behaviour will occur for the product coupled system (for appropriated choices of the product vector fields). This follows from the definition itself of synchrony subspace and the fact that the dynamics of coupled cell systems associated with $\mathcal{N}_i$ have to occur at the product coupled cell systems.

Acknowledgments

The authors were partially funded by the European Regional Development Fund through the program COMPETE and by the Portuguese Government through the FCT - Fundação para a Ciência e a Tecnologia under the projects PTDC/MAT/100055/2008 and PEst-C/MAT/UI0144/2013.

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