Finiteness of Walrasian equilibria

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Abstract

The paper establishes that the Thom-Boardman stratification result of singularity theory applies generically to the world of aggregate excess demand functions. This, in turn, guarantees that all Walrasian equilibria — including non-regular ones — are finite.

Keywords: Arrow-Debreu, transversality, exchange economy, singularity, critical equilibria

JEL classification: D50, C62, C02

1 Introduction

Intuitively, failure of local isolation at a critical economy corresponds to aggregate excess demand being 'flat' at the critical equilibrium, meaning that all higher order derivatives are zero. Indeed, because the perturbation must affect the curvature of the equilibrium manifold, a linear perturbation such as the one at the heart of Debreu's [3] proof no longer suffices. An example of a higher-order perturbation is found in Mas-Colell [7, Proposition 8.8.3], who shows that for a *one-dimensional* parameterization in the space of agents' characteristics, a 'flat' aggregate excess demand is not generic in the parameterization of the first agent's utility. It is instructive to note that Mas-Colell must resort to a *quadratic* perturbation of utility.

In this paper, the objective is to extend the result to higher dimensional parameterizations. The standard transversality argument would require that the economy be perturbed so as to transform its associated aggregate excess demand (henceforth AED) function into a function with isolated zeros. Clearly, the challenge of this type of exercise is that (1) the perturbations of the underlying economy (in particular, preferences) be sufficiently rich to affect all higher-order derivatives of the AED and (2) that such "deeply" perturbed preferences still satisfy the standard choice axioms. We circumvent this challenge by, essentially, proceeding in reverse: using a fact recently established by Castro *et al.* [2], we can perturb AED *directly* and rest assured that the *primitives* of the perturbed AED are close to those of the original one.

We use the notion of *Boardman* map and show that it is a sufficient condition for local isolation of all price equilibria. Finiteness then follows from the standard boundary condition and compactness of the endowment set. Furthermore, we show that, generically (i.e., for the complement of a meagre subset of the set of underlying economies), AED is indeed a Boardman map.

In addition to providing an alternative and more direct proof of local isolation of all equilibria (i.e., both regular and critical ones), establishing Boardman stratification provides us with additional information about the generic structure of the aggregate excess demand functions and, in turn, about equilibrium manifolds.

In the next section, we introduce notation and some results for later use. Section 3 establishes the main results. We conclude with a short discussion.

2 Notation and preliminary results

Consider an economy with L commodities ($\ell = 1, \dots, L$) and I agents ($i = 1, \dots, I$). Let Ω be the non-negative orthant of $\mathbb{R}^L$ and let each agent i be defined by her endowment $\omega^i \in \Omega$ and her preferences $\succsim_i$, a complete order on Ω with the following "rationality" properties:

(P1) completeness, reflexivity, and transitivity.

If $x \succsim_i y$ and $y \succsim_i x$, then x is indifferent to y and we write $x \sim_i y$. If $x \succsim_i y$ but not $x \sim_i y$, then x is strictly preferred to y and we write $x \succ_i y$. We call the partial preference order $\succ_i$ continuous if it satisfies:

(P2) continuity ($\{x : y \succ_i x\}$ and $\{y : y \succ_i x\}$ are open).

In addition, we shall assume non-satiation and strict convexity:

(P3) non-satiation ($x \geq y$ ($x_\ell \geq y_\ell$, $\forall \ell = 1, \dots, L$) and $x \neq y \Rightarrow x \succ_i y$);

(P4) strict convexity ($x \sim_i y$ and $x \neq y \Rightarrow \forall \alpha \in (0, 1)$, $\alpha x + (1 - \alpha) y \succ_i x$).

As in Castro et al. [2], let Ξ denote the space of all such preferences. Furthermore, denote $\mathcal{E} \equiv (\Xi \times \Omega)^I$ with typical element $e = (\succ^1, \omega^1, \dots \succ^I, \omega^I)$ the space of all L by I economies described by preferences and endowments. If the preferences $\succ^i$ of at least one agent (to be taken as the first agent, without loss of generality) can be represented by a twice continuously differentiable utility, we refer to these as C^2 preferences and to $e \in \mathcal{E}$ as a C^2 economy.

Next, consider the set of smooth (twice differentiable) pure exchange economies $\mathcal{Z}$ with L commodities ($\ell = 1, \dots, L$) and I ($i = 1, \dots, I$) agents, generated by each element $e \in \mathcal{E}$ and described by aggregate excess demand functions $z \in \mathcal{Z}$ depending smoothly on prices and endowments $\omega \in \Omega \subseteq \mathbb{R}^L_+$. Normalize prices to lie in the $(L-1)$-dimensional unit simplex $\Delta \equiv \{p \in \mathbb{R}^L_+ \mid \sum_{\ell=1}^L = 1\}$. Note that an *equilibrium price* is $p \in \Delta$ such that $z(p, \omega) = 0$. We write $z(p, \omega) = z(p)$, when ω is held fixed.

The relationship between twice differentiable (C^2) AED functions in $\mathcal{Z}$ and the underlying preferences and endowments in $\mathcal{E}$ was established in Castro et al. [2, Corollary 4.2] and is as follows (see also Theorem 4.1 therein):

Theorem 2.1 (Castro et al. [2]). *Let $z_0(p) \in \mathcal{Z}$ be the AED for a C^2 economy $e_0 \in \mathcal{E}$ with L goods and I agents characterized by C^2-preferences*

$\succ_0^i$ satisfying (P1)–(P4) and endowments ω_0^i, $i = 1, \ldots, I$. An AED $z(p)$ is a perturbation of $z_0(p)$ if and only if $z(p)$ is the AED for an economy $e \in \mathcal{E}$ with L goods and I agents such that the new preferences $\succ^1$ of the first agent are perturbations of $\succ_0^1$ and the new endowments ω^1 are perturbations of ω_0^1.

Note that we define a *perturbation* of a point x_0 in a topological space X as a point $x \in X$ which is contained in an arbitrarily small open neighborhood of the original point x_0.

Let X and Y be smooth manifolds. In what follows, we shall make use of the *jet-space* $J^k(X, Y)$, whose elements are k-jets. The k-jet of a map $f : X \to Y$ is written $j^k f$. We shall simply write J^k for $J^k(X, Y)$ when no confusion is possible. We want to think of $j^k f(p)$ as a description of the Taylor expansion of f at $p \in X$ up to order k. Note that $j^k f$ is a smooth map. See Golubitsky and Guillemin [5, chapter II, §2] for details. In the present context, we shall use $X = \Delta$ and $Y = \mathbb{R}^{L-1}$.

Definition 2.2. *The* first-order Thom singularity sets *of f are defined as*

$$S_r(f) = \{x \in X \mid D_x f \text{ has corank } r\},$$

where $D_x f$ denotes the Jacobian of f at x. Furthermore, define $\Sigma_r(X, Y)$ as the subset of the jet-space $J^1(X, Y)$ comprising all 1-jets of functions $g : X \to Y$ with corank r:

$$\Sigma_r(X, Y) = \{j^1 g \in J^1 \mid Dg \text{ has corank } r\}.$$

As with J^k, we shall henceforth simply write Σ_r for $\Sigma_r(X, Y)$.[1] It follows from the definitions that

$$S_r(f) = (j^1 f)^{-1}(\Sigma_r).$$

In the context of AED functions $z \in \mathcal{Z}$, we have

$$S_r(z) = \{p \in \Delta \mid D_p z \text{ has corank } r\}.$$

We note the following fact about Σ_r:

[1] Readers familiar with the notations used in Golubitsky and Guillemin [5] and in Gibson [4] should note that we use $S_r(z)$ to denote singular subsets of points, as in Golubitsky and Guillemin [5], and Σ_r to denote subsets of jet space, as in Gibson [4].

Lemma 2.3 (Levine [6, p. 10]). *The set Σ_r is a smooth submanifold of J^1 of codimension r^2.*

Definition 2.4 (See Golubitsky and Guillemin [5, Def. 1.5]). *Call the function z* 1-generic *if the 1-jet of z is transversal to every Σ_r, $r = 1, \ldots L-1$.*

Proposition 2.5. *If z is 1-generic, then $S_r(z)$ is a smooth submanifold of Δ of codimension r^2.*

Proof. We have $S_r(z) = (j^1 z)^{-1}(\Sigma_r)$, which, by Lemma 2.3 and by the Implicit Function Theorem, is a smooth manifold of the same codimension as Σ_r. (See also Golubitsky and Guillemin [5, p. 143].) $\square$

Let a smooth function $f : X \to Y$ be 1-generic, so that the first-order Thom singularity set $S_r(f) \equiv \{x \in X \mid D_x f \text{ has corank } r\}$ is a manifold. The *second-order Thom singularity sets* $S_{r,s}(f)$ are defined as the sets of points where the map $f : S_r(f) \to \mathbb{R}^{L-1}$ drops rank s.

If $S_{r,s}(f)$ turns out to be a manifold, we can define $S_{r,s,t}(f)$ similarly. This would be the case if f's 2-jet were transversal to all second-order Thom singularity sets, i.e., if $j^2 f \pitchfork \Sigma_{r,s}$. In such a case, we call f *2-generic*. This process may be continued indefinitely:

Theorem 2.6 (Boardman, see Golubitsky and Guillemin [5, Th. 5.1]). *For every sequence of integers $r_1 \geq r_2 \geq \cdots \geq r_k \geq 0$, one can define a fiber subbundle, $\Sigma_{r_1,\ldots,r_k}$ of $J^k(L-1, L-1)$, such that if $j^\ell f$ is transversal to all the manifolds $\Sigma_{r_1,\ldots,r_k}$ where $\ell < k$, then $S_{r_1,\ldots,r_k}(f)$ is well-defined and*

$$p \in S_{r_1,\ldots,r_k}(f) \iff j^k f(x) \in \Sigma_{r_1,\ldots,r_k}.$$

The following definition provides the transition from jet space to smooth functions. See Golubitsky and Guillemin [5, p. 157].

Definition 2.7. *A map f is called a* Boardman *map, if for all k, and for every sequence $r_1 \geq r_2 \geq \cdots \geq r_k \geq 0$, $j^k(f) \pitchfork \Sigma_{r_1,\ldots,r_k}$.*

We shall say that a set is *generic* if it is residual (a countable intersection of open dense sets). In a Baire space residual implies dense. A generic set can also be described as the complement of a meagre set (a countable union of nowhere dense closed sets). Hence, the generic economy is such that its properties are not only satisfied by almost all economies, but any economy can be slightly perturbed to satisfy those properties.

3 Finiteness of equilibria

Let $z : \Delta \to \mathbb{R}^{L-1}$ be an AED function. We obtain the following result:

Proposition 3.1. *If z is a Boardman map, then all its zeros are locally isolated.*

Proof. We proceed by contradiction. Suppose that for some economy some equilibria are not locally isolated. Then there exists a (non-degenerate) path $\mathcal{T} \subset \Delta$ such that $\forall p \in \mathcal{T}$, $z(p) = 0$. Assume the worst-case scenario in which $D_p z : \mathcal{T} \to \mathbb{R}^{L-1}$ has rank 0.[2] According to Lemma 3.2 below, since z is Boardman, the set of points in Δ where $D_p z$ loses all rank is a subset of $cl(S_{1,\ldots,1}(z))$, the closure of $S_{1,\ldots,1}(z)$; therefore we must have $\mathcal{T} \subset cl(S_{1,\ldots,1}(z))$. However, $\dim cl(S_{1,\ldots,1}(z)) = 0$ (see Gibson, example 8, p.186), whereas $\dim \mathcal{T} = 1$. Hence, the zeros of z are locally isolated. □

Lemma 3.2. *Let $f : X \to Y$, where $\dim X = \dim Y = n$. The set of points in X where Df drops all rank is exactly $S_{r_1,\ldots,r_n}(f)$ with $r_i = 1$, $i = 1, \ldots, n$.*

Proof. Following Gibson [4, p. 187], codim $\Sigma_{r_1,\ldots,r_k}$ = codim $S_{r_1,\ldots,r_k}(f) \geq r_1^2 + \cdots + r_k^2$. In addition, note that the set of points where Df loses all rank must satisfy $\sum_{i=1}^k r_i = n$. Suppose then that there is an $r_i > 1$, thus $r_i^2 > r_i$. Then codim $S_{r_1,\ldots,r_k}(f) > \sum_{i=1}^k r_i = n = \dim X$, which is impossible. □

Having established local isolation of zeros for Boardman maps, we proceed to show that generically, i.e., almost always, an AED function is a Boardman map.

We use transversality arguments, which require the deformation of z. A simple constant deformation would not work, for instance, since it does not alter the derivatives of z. Instead we need a polynomial deformation. This deformation should be obtained by a sufficiently rich perturbation of the economy's underlying primitives, that is, preferences and endowments. However, Castro *et al.* [2, Theorem 4.1] show that such a perturbation exists and that it is equivalent to a *direct* perturbation of z. As such, Thom's proof of jet transversality directly transfers to the set of AED functions.

Proposition 3.3. *Let $W_1, \ldots W_t$ be smooth submanifolds of J^k. The set of all smooth mappings $z : \mathbb{R}^{L-1} \to \mathbb{R}^{L-1}$ for which $j^k z : \mathbb{R}^{L-1} \to J^k$ is transverse to $W_1, \ldots W_t$ is dense in $C^\infty(\mathbb{R}^{L-1}, \mathbb{R}^{L-1})$.*

[2] If $\text{rank}(D_p(z)) > 0$, then we can confine our attention to the lower-dimensional price simplex where the loss of rank occurs.

Proof. See Gibson [4, Theorem 4.1]. □

Theorem 3.4. *The set of Boardman maps is a residual and dense subset of $C^\infty(X, Y)$.*

Proof. Density follows as a consequence of the Thom transversality theorem (Golubitsky and Guillemin [5, p. 157]) given that $C^\infty(X, Y)$ is a Baire space. □

As an immediate consequence, taking $X = \Delta$ and $Y = \mathbb{R}^{L-1}$, we have

Corollary 3.5. *The subset of AED functions which are Boardman is residual and dense in the set $\mathcal{Z}$ of all AED functions.*

We remark that if $L \leq 4$ then the set of Boardman maps is also open. See Wilson [10].

We have so far established that, generically, any AED function z has locally isolated zeros. Finiteness of such zeros requires the following standard assumption:

Definition 3.6. (Boundary Condition) *An AED $z \in \mathcal{Z}$ fulfills the* boundary condition *(BC), if for every $\omega \in \Omega$ and every sequence $(p_n)_{n\in\mathbb{N}} \in \Delta$ converging to the boundary $\partial\Delta$ as $n \to \infty$, $\|z(p_n, \omega)\|$ is unbounded.*

BC implies that for every $\omega \in \Omega$, the set of price equilibria is compact. We note that Mas-Colell and Neuefeind [8] show that BC is satisfied for a residual subset of economies.

Proposition 3.7. *If z is a Boardman map satisfying BC and Ω is compact, then the set of all price equilibria is finite.*

Proof. This follows directly from Proposition 3.1 and the fact that BC implies compactness of the set of price equilibria. □

We are now able to state and prove our main theorem.

Theorem 3.8. *Generically, that is, for all economies e in the complement of a meagre set, the number of equilibria is finite.*

Proof. Using Proposition 3.7 and Corollary 3.5, we know that there is a residual subset of $\mathcal{Z}$ for which the set of all price equilibria is finite. From Castro *et al.* [2, Theorem 4.1, Corollary 4.2] we know that $z \in \mathcal{Z}$ and $e \in \mathcal{E}$ are related by a continuous and open map. Therefore, to the residual and dense subset of $\mathcal{Z}$, there corresponds a residual and dense subset of economies $\mathcal{E}$. □

As noted earlier, the subset of generic economies is also open in the set of all economies $\mathcal{E}$, provided the number of commodities is four or less.

4 Discussion

We remark that if a map is Boardman, it is also 1-generic. However, 1-genericity is not a sufficiently strong property to ensure local isolation of zeros. For instance, in the case of three commodities (and thus two relative prices), the problem may be that the singularity of the singularity, written $S_{1,1}(z)$, is not a lower-dimensional submanifold of $S_1(z)$, thus producing a continuum of equilibria.

This paper adds to a long series of contributions on local isolation, determinacy, and other generic properties of the set of equilibrium prices, spawned by Debreu's [3] seminal 1970 paper. We single out two that are most closely related to ours. Allen [1] shows that the set of price equilibria is finite by working directly with AED functions. She does not, however, relate her result to the underlying primitives of preferences and endowments. Mas-Collel and Nachbar [9], on the other hand, work directly with deeper primitives, but only obtain countability of the equilibrium set. We bridge the remaining gap by showing that, for the generic economy, the set of all Walrasian equilibria is finite. Our approach also provides additional information about the structure of AED functions.

Acknowledgements: The authors wish to thank Beth Allen, Marcus Berliant, Jean-Marc Bonnisseau, Gaël Giraud, Peter Gothen, and John Nachbar for their very helpful conversations, and in particular, Marty Golubitsky for introducing the authors to each other.

Part of this work was done while the authors were visiting CERMSEM at Paris 1 whose hospitality is gratefully acknowledged. In particular, we wish to thank Cuong Le Van for the kind invitation and Bernard Cornet for providing us with most spacious working conditions.

Sofia Castro acknowledges support from Centro de Matemática da Universidade do Porto (CMUP) and Fundação para a Ciência e Tecnologia, through the programmes POCTI and POSI.

The paper originally circulated under the title *Thom Boardman stratification of aggregate excess demand.*

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