

Spectral Lanczos' tau method for systems of nonlinear integro-differential equations*

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Abstract

In this paper an extension of the spectral Lanczos' tau method to systems of nonlinear integro-differential equations is proposed. This extension includes (i) linearization coefficients of orthogonal polynomials products issued from nonlinear terms and (ii) recursive relations to implement matrix inversion whenever a polynomial change of basis is required and (iii) orthogonal polynomial evaluations directly on the orthogonal basis. All these improvements ensure numerical stability and accuracy in the approximate solution. Exposed in detail, this novel approach is able to significantly outperform numerical approximations with other methods as well as different tau implementations. Numerical results on a set of problems illustrate the impact of the mathematical techniques introduced.

Keywords: spectral (tau) method, nonlinear systems of differential equations

Mathematics Subject Classification: 65L60, 65H10, 68N01

1 Introduction

The tau method is a spectral method, originally developed by Lanczos in the 30's [5] that delivers polynomial approximations to the solution of differential problems. The method tackles both initial and boundary value problems with ease. It is a spectral method thus ensuring excellent error properties, whenever the solution is smooth.

Initially developed for linear differential problems with polynomial coefficients, it has been used to solve broader mathematical formulations: functional coefficients, nonlinear differential and integro-differential equations. Several studies applying the tau method have been performed to approximate the solution of differential linear and non-linear equations [2, 7], partial differential equations [8, 9] and integro-differential equations [1, 11], among others. Nevertheless, in all these works the tau method is tuned for the approximation of specific problems and not offered as a general purpose numerical tool.

A barrier to use the method as a general purpose technique has been the lack of automatic mechanisms to translate the integro-differential problem by an algebraic one. Furthermore and most importantly, problems often require high-order polynomial approximations, which brings numerical instability issues. The tau method inherits numerical instabilities from the large condition number associated with large matrices representing

*This work was partially supported by CMUP (UID/MAT/00144/2013), which is funded by FCT (Portugal) with national (MEC) and European structural funds (FEDER) under the partnership agreement PT2020.

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algebraically the actions of the integral, differential or integro-differential operator on the coefficients of the series solution.

In this work numerical instabilities related with high-order polynomial approximations, in the tau method, are tackled allowing for the deployment of a general framework to solve integro-differential problems. We aim at contributing to broadcast the tau method for the scientific community and industry as it provides polynomial solutions with good error properties. The `Tau Toolbox` [13] is a MATLAB tool to solve integro-differential problems by the tau method. It aggregates all contributions available, enhances the use of the method by developing more stable algorithms and offers efficient implementations.

2 Preliminaries

We begin by introducing the notation for the algebraic formulation of the tau method.

Assume throughout that $\mathcal{P} = [P_0, P_1, \dots] \subseteq \mathbb{P}$ is an orthogonal basis for the polynomials space \mathbb{P} of any non-negative integer degree, $\mathcal{X} = [1, x, x^2, \dots] \in \mathbb{P}$ the power basis for \mathbb{P} . Furthermore, consider that $y(x) = \sum_{i \geq 0} a_i P_i = \mathcal{P} \mathbf{a}$ is a formal series with coefficients $\mathbf{a} = [a_0, a_1, \dots]^T$. For the power basis, $y(x) = \sum_{i \geq 0} a_i x^i = \mathcal{X} \mathbf{a}_{\mathcal{X}}$.

Lemma 1 illustrates matrices \mathbf{M} , \mathbf{N} and \mathbf{O} that set, respectively, polynomial multiplication, differentiation and integration into algebraic operations.

Lemma 1. *Let \mathbf{V} be the triangular matrix such that $\mathcal{P} = \mathcal{X} \mathbf{V}$ and $\mathbf{a} = \mathbf{V}^{-1} \mathbf{a}_{\mathcal{X}}$. Then $xy = \mathcal{P} \mathbf{M} \mathbf{a}$, $\frac{d}{dx} y = \mathcal{P} \mathbf{N} \mathbf{a}$ and $\int y dx = \mathcal{P} \mathbf{O} \mathbf{a}$ where*

$$\mathbf{M} = \mathbf{V}^{-1} \mathbf{M}_{\mathcal{X}} \mathbf{V}, \quad \mathbf{M}_{\mathcal{X}} = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \ddots \end{bmatrix}, \quad (1)$$

$$\mathbf{N} = \mathbf{V}^{-1} \mathbf{N}_{\mathcal{X}} \mathbf{V}, \quad \mathbf{N}_{\mathcal{X}} = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 2 & & \\ & & 0 & 3 & \\ & & & \ddots & \ddots \\ & & & & \ddots & \ddots \end{bmatrix} \quad (2)$$

and

$$\mathbf{O} = \mathbf{V}^{-1} \mathbf{O}_{\mathcal{X}} \mathbf{V}, \quad \mathbf{O}_{\mathcal{X}} = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & \frac{1}{2} & 0 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \ddots \end{bmatrix}. \quad (3)$$

Proof. See [10] for \mathbf{M} and \mathbf{N} . It is then easy to extend to \mathbf{O} (see [4]). \square

The next proposition shows how to translate a linear ordinary differential and integral operators, with polynomial coefficients, into an algebraic representation.

Proposition 2. *The v th order, $v \in \mathbb{N}$, ordinary linear differential operator $\mathcal{D}y = \sum_{k=0}^v p_k \frac{d^k y}{dx^k}$ and the γ th order, $\gamma \in \mathbb{N}$, ordinary linear integral operator $\mathcal{S}y = \sum_{\ell=0}^{\gamma} p_{\ell} (\int y dx)^{\ell}$ acting on \mathbb{P} , are casted on \mathcal{P} by, respectively,*

$$\mathcal{D}y = \mathcal{P} \mathbf{D} \mathbf{a}, \quad \mathbf{D} = \sum_{k=0}^v p_k (\mathbf{M}) \mathbf{N}^k \quad (4)$$

and

$$\mathcal{S}y = \mathcal{P} \mathbf{S} \mathbf{a}, \quad \mathbf{S} = \sum_{\ell=0}^{\gamma} p_{\ell} (\mathbf{M}) \mathbf{O}^{\ell}, \quad (5)$$

with $p_r(\mathbf{M}) = \sum_{i=0}^{n_r} p_{r,i} \mathbf{M}^i$, $r = k, \ell$ and $n_r \in \mathbb{N}_0$.

Proof. Note that

$$\mathcal{X}M_{\mathcal{X}}\mathbf{a}_{\mathcal{X}} = xy, \quad \mathcal{X}N_{\mathcal{X}}\mathbf{a}_{\mathcal{X}} = \frac{d}{dx}y \quad \text{and} \quad \mathcal{X}O_{\mathcal{X}}\mathbf{a}_{\mathcal{X}} = \int ydx$$

for $M_{\mathcal{X}}$, $N_{\mathcal{X}}$ and $O_{\mathcal{X}}$ as in Lemma 1. Then,

- (i) $xy = \mathcal{X}M_{\mathcal{X}}\mathbf{a}_{\mathcal{X}} = \mathcal{X}VMV^{-1}\mathbf{a}_{\mathcal{X}} = \mathcal{P}M\mathbf{a}$;
- (ii) $\frac{d}{dx}y = \mathcal{X}N_{\mathcal{X}}\mathbf{a}_{\mathcal{X}} = \mathcal{X}VNV^{-1}\mathbf{a}_{\mathcal{X}} = \mathcal{P}N\mathbf{a}$ and thus $\mathcal{D}y = \sum_{k=0}^V p_k \frac{d^k y}{dx^k} = \mathcal{P}D\mathbf{a}$;
- (iii) $\int ydx = \mathcal{X}O_{\mathcal{X}}\mathbf{a}_{\mathcal{X}} = \mathcal{X}VOV^{-1}\mathbf{a}_{\mathcal{X}} = \mathcal{P}O\mathbf{a}$ and thus $\mathcal{S}y = \sum_{\ell=0}^Y p_{\ell} (\int ydx)^{\ell} = \mathcal{P}S\mathbf{a}$.

□

Let $K(x, t)$ be a two-variable polynomial, or a two-variable polynomial approximation of a two-variable function. Then, for $\mathcal{P}|_{x=t} = [P_0(t), P_1(t), \dots]$ and $K \in \mathbb{R}^{n_x \times n_t}$,

$$K(x, t) = \sum_{i=0}^{n_x} \sum_{j=0}^{n_t} k_{i,j} P_i(x) P_j(t) = \mathcal{P}K\mathcal{P}^T|_{x=t}. \quad (6)$$

Lemma 3. For the integral operator $\int^x K(x, t)y(t)dt$, where \int^x stands for the calculation of the integral at x , $K(x, t) = \sum_{i=0}^{n_x} \sum_{j=0}^{n_t} k_{i,j} P_i(x) P_j(t)$ and $y(x) = \mathcal{P}\mathbf{a}$, one has $\int^x K(x, t)y(t)dt = \mathcal{P}S\mathbf{a}$, where $S = \sum_{i=0}^{n_x} \sum_{j=0}^{n_t} k_{i,j} P_i(M) O P_j(M)$.

Proof. Using Lemma 1 one can deduce that

$$\begin{aligned} \int^x x^i t^j y(t) dt &= \mathcal{X}(VM^i V^{-1})(VOV^{-1})(VM^j V^{-1})V\mathbf{a} \\ &= \mathcal{P}M^i O M^j \mathbf{a}, \end{aligned}$$

and therefore

$$\int^x P_i(x) P_j(t) y(t) dt = \mathcal{P}P_i(M) O P_j(M) \mathbf{a}.$$

□

Lemma 4. $\int_{x_0}^x P_i(x) P_j(t) y(t) dt = \mathcal{P}(P_i(M) - \mathbf{e}_{i+1} \mathcal{P}|_{x=x_0}) O P_j(M) \mathbf{a}$, where \mathbf{e}_{i+1} is the $(i+1)$ th column of the identity matrix.

Proof. From Lemma 3

$$\begin{aligned} \int_{x_0}^x P_i(x) P_j(t) y(t) dt &= \mathcal{P}P_i(M) O P_j(M) \mathbf{a} - \mathcal{P}|_{x=x_0} P_i(M) O P_j(M) \mathbf{a} \\ &= \mathcal{P}(P_i(M) - \mathbf{e}_{i+1} \mathcal{P}|_{x=x_0}) O P_j(M) \mathbf{a}. \end{aligned}$$

Note that it is easy to understand that $\mathcal{X}|_{x=x_0} M^i \mathcal{X} = \mathbf{e}_{i+1} \mathcal{X}|_{x=x_0}$.

□

Lemma 5. $\int_a^b P_i(x) P_j(t) y(t) dt = \mathcal{P} \mathbf{e}_{i+1} (\mathcal{P}|_{x=b} - \mathcal{P}|_{x=a}) O P_j(M) \mathbf{a}$.

Proof. Immediate since it is a particular case of Lemma 4.

□

Proposition 6. The linear Volterra integral operator $\mathcal{S}_V y = \int_{x_0}^x K(x, t)y(t)dt$ and the Fredholm integral operator $\mathcal{S}_F y = \int_a^b K(x, t)y(t)dt$, with degenerate kernel $K(x, t) \approx \sum_{i=0}^{n_x} \sum_{j=0}^{n_t} k_{i,j} P_i(x) P_j(t)$, smooth and continuous, acting on y have, respectively, the following algebraic representation

$$S_V y = \sum_{i=0}^{n_x} \sum_{j=0}^{n_t} k_{i,j} \left(P_i(M) - \mathbf{e}_{i+1} \mathcal{P}|_{x=x_0} \right) O P_j(M) \mathbf{a} \quad (7)$$

and

$$S_F y = \sum_{i=0}^{n_x} \sum_{j=0}^{n_t} k_{ij} e_{i+1} \left(\mathcal{P}|_{x=b} - \mathcal{P}|_{x=a} \right) O P_j(M) a \quad (8)$$

for e_i the i th column of the identity matrix.

Proof. Equation (7) can be immediately obtained from Lemma 4 (see e.g. [12]) and equation (8) from Lemma 5 (see e.g. [4]). \square

3 The tau method for integro-differential problems

An approximate polynomial solution y_n for the linear integro-differential problem

$$\begin{cases} \mathcal{D}y + \mathcal{S}y + \mathcal{S}_V y + \mathcal{S}_F y = f \\ c_i(y) = s_i, i = 1, \dots, v \end{cases}, \quad (9)$$

is obtained in the tau sense by solving a perturbed system

$$\begin{cases} \mathcal{D}y_n + \mathcal{S}y_n + \mathcal{S}_V y_n + \mathcal{S}_F y_n = f + \tau_n \\ c_i(y_n) = s_i, i = 1, \dots, v \end{cases}, \quad (10)$$

where f is a λ th degree polynomial (or a polynomial approximation of a function), τ_n is the residual and $c_i(y) = s_i, i = 1, \dots, v$ the initial and/or boundary conditions.

Problem (9) has a matrix representation given by

$$\begin{cases} Ca = s \\ (D + S + S_V + S_F)a = f \end{cases}, \quad (11)$$

where $C = [c_{ij}]_{v \times \infty}$, $c_{ij} = c_i(P_{j-1}), i = 1, \dots, v, j = 1, 2, \dots$, $a = [a_0, a_1, \dots]^T$ the coefficients of y in \mathcal{P} , $s = [s_1, \dots, s_v]^T$, D, S, S_V and S_F as defined in, respectively, (4)-(8), and $f = [f_0, \dots, f_\lambda, 0, 0, \dots]^T$ the right hand side of the system.

Choosing an integer $n \geq v + \lambda$, an $(n-1)$ th degree polynomial approximate solution $y_n = \mathcal{P}_n a_n$ is obtained by truncating system (11) to its first n columns. Moreover, restricting this system to its first $n + v + h$ equations, a linear system of dimension $n \times n$ is obtained, which is equivalent to introduce a polynomial residual

$$\tau_n = (\mathcal{D}y - \mathcal{D}y_n) + (\mathcal{S}y - \mathcal{S}y_n) + (\mathcal{S}_V y - \mathcal{S}_V y_n) + (\mathcal{S}_F y - \mathcal{S}_F y_n). \quad (12)$$

4 Nonlinear approach for integro-differential problems

Nonlinear differential problems are tackled with linear approximations and solving a set of linear problems.

Let G be the nonlinear operator acting on an appropriate space of smooth functions

$$G(y^{(-\gamma)}, \dots, y^{(-1)}, y^{(0)}, y^{(1)}, \dots, y^{(v)}) = 0, \quad (13)$$

where $y^{(\ell)} = \frac{d^\ell y}{dx^\ell}$ for $\ell \in \mathbb{Z}^+$, $y^{(\ell)} = (\int y dt)^\ell$ for $\ell \in \mathbb{Z}^-$ and includes Volterra and Fredholm terms.

If G is \mathbb{C}^1 in a neighborhood Ω of $\omega = (y^{(-\gamma)}, \dots, y^{(-1)}, y^{(0)}, y^{(1)}, \dots, y^{(v)})$ and if $\omega_0 \in \Omega$ is an approximation of ω , then a linear operator T can be defined, represented by the order one Taylor polynomial centered at ω_0

$$T(\omega) = G(\omega_0) + \sum_{i=-\gamma}^v \frac{\partial G}{\partial y^{(i)}} \Big|_{\omega_0} (y^{(i)} - y_0^{(i)})$$

As in the Newton method for algebraic equations, we can replace G by T in (13) and solve the approximated equation

$$\sum_{i=-\gamma}^{\nu} \frac{\partial G}{\partial y^{(i)}}|_{\omega_0} y^{(i)} = -G(\omega_0) + \sum_{i=-\gamma}^{\nu} \frac{\partial G}{\partial y^{(i)}}|_{\omega_0} y_0^{(i)}. \quad (14)$$

Applying the Tau method to the linear differential equation (14) and taking $\omega_1 = (y_1^{(-\gamma)}, \dots, y_1^{(\nu)})$ as the solution, if $\omega_1 \in \Omega$ we can repeat the process, obtaining an iterative procedure, solving for ω_k the linear differential equation

$$\sum_{i=-\gamma}^{\nu} \frac{\partial G}{\partial y^{(i)}}|_{\omega_{k-1}} y_k^{(i)} = -G(\omega_{k-1}) + \sum_{i=-\gamma}^{\nu} \frac{\partial G}{\partial y^{(i)}}|_{\omega_{k-1}} y_{k-1}^{(i)}, \quad k = 1, 2, \dots$$

5 Contributions to stability

In this section we summarize some of the mathematical techniques developed for the tau method to provide stable algorithms for the Tau Toolbox library.

Let $\mathcal{P} = [P_0(x), P_1(x), \dots]$ be an orthogonal basis satisfying $xP_j = \alpha_j P_{j+1} + \beta_j P_j + \gamma_j P_{j-1}$, $j \geq 0$, $P_0 = 1$, $P_{-1} = 0$.

Orthogonal evaluation: If \mathcal{P}^* are the corresponding orthogonal polynomials shifted to $[a, b]$ and x is a vector, then the evaluation of $y_n(x) = \sum_{i=0}^n a_i P_i^*$ is directly computed in \mathcal{P}^* by the recursive relation

$$\begin{cases} P_0^* = [1, \dots, 1]^T \\ P_1^* = \frac{c_1 x + (c_2 - \beta_0) P_0^*}{\alpha_0} \\ P_i^* = \frac{(c_1 x + c_2 - \beta_{i-1} P_0^*) \odot P_{i-1}^* - \gamma_{i-1} P_{i-2}^*}{\alpha_{i-1}}, \quad i = 2, 3, \dots, n \end{cases}$$

where \odot is the element-wise product of two vectors, $c_1 = \frac{2}{b-a}$ and $c_2 = \frac{a+b}{a-b}$.

The Tau Toolbox proposes a `orthoval` function, instead of the `polyval` MATLAB one, to implement this functionality.

Change of basis by recurrence: Let V satisfy $a_{\mathcal{P}} = Va$. The coefficients of $W = V^{-1}$ can be computed without inverting V by the recurrence relation

$$\begin{cases} w_1 = e_1 \\ w_{j+1} = Mw_j, \quad j = 1, 2, \dots \end{cases}, \quad (15)$$

where M is such that $\mathcal{P}x = \mathcal{P}M$, w_j is the j th column of M and e_1 the first column of the identity matrix.

Avoiding similarity transformations: Matrix inversion presented at all similarity transformations must be avoided to ensure numerically stable computations. Recurrence relations to compute the elements of matrices M , N and O directly on \mathcal{P} can be computed, respectively, by

$$\begin{aligned} M &= [\mu_{i,j}]_{i,j=1}^n = \begin{cases} \mu_{j+1,j} = \alpha_{j-1}, & \mu_{j,j} = \beta_{j-1}, & \mu_{j-1,j} = \gamma_{j-1} \\ \mu_{i,j} = 0, & |i-j| > 1 \end{cases}, \\ N &= [\eta_{i,j}]_{i,j=1}^n = \begin{cases} \eta_{i,j+1} = \frac{\alpha_{i-1} \eta_{j,i-1} + (\beta_i - \beta_j) \eta_{j,i} + \gamma_{i+1} \eta_{j,i+1} - \gamma_j \eta_{j-1,i}}{\alpha_j} \\ \eta_{j,j+1} = \frac{\alpha_{j-1} \eta_{j,j-1} + 1}{\alpha_j} \end{cases}, \text{ and} \\ O &= [\theta_{i,j}]_{i,j=1}^n = \begin{cases} \theta_{j+1,j} = \frac{\alpha_j}{j+1} \\ \theta_{i+1,j} = -\frac{\alpha_i}{i+1} \sum_{k=i+2}^{j+1} \eta_{i,k} \theta_{k,j}, \quad i = j-1, \dots, 1, 0 \end{cases}. \end{aligned}$$

Linearization coefficients: Product of polynomials p and q in \mathcal{P} occurs in nonlinear problems. Usually both polynomials are translated first from \mathcal{P} to \mathcal{X} , then the convolution is applied and finally the product is translated back, $pq = V^{-1}\text{conv}(Vp, Vq)$. Alternatively, ensuring robustness, this product can be directly computed on \mathcal{P} using the linearization coefficients: $P_i P_j = \sum_{k=0}^{i+j} l_{i,j,k} P_k$, where $l_{i,j,k}$ are computed by recurrence relations [6].

For $p = \mathcal{P}a$ and $q = \mathcal{P}b$, of degree n , then $pq = \mathcal{P}c$ where

$$\begin{cases} c_k = \sum_{j=0}^k \sum_{i=\lfloor \frac{k+1-j}{2} \rfloor}^{n-j} \left(\frac{1}{2}\right)^{\delta_{i,j}} (a_i b_j + a_j b_i) l_{i,i+j,k}, & k = 0, \dots, n \\ c_{n+k} = \sum_{j=0}^{n-k} \sum_{i=\lfloor \frac{n-k-j}{2} \rfloor}^{n-j} \left(\frac{1}{2}\right)^{\delta_{i,j}} (a_i b_j + a_j b_i) l_{i,i+j,n+k}, & k = 1, \dots, n \end{cases}$$

Computing with M: An efficient way to compute the powers of M is

$$M^k = [\mu_{i,j}^{(k)}], \quad \mu_{i,j}^{(k)} = \mu_{i-1,j}^{(k-1)} \alpha_{i-1} + \mu_{i,j}^{(k-1)} \beta_i + \mu_{i+1,j}^{(k-1)} \gamma_{i+1}.$$

Moreover, the evaluation of $y_n(M) = \sum_{i=0}^n a_{n,i} P_i(M)$ can be performed with

$$P_{j+1}(M) = \frac{(M - \beta_j I) P_j(M) - \gamma_j P_{j-1}(M)}{\alpha_j}, \quad j \geq 0,$$

$P_0(M) = I$ and $P_{-1}(M) = \emptyset$.

6 Numerical results

Example 7. Consider the integro-differential Fredholm nonlinear equation of the second kind [3], with exact solution $y(x) = \exp(-x)$,

$$\frac{d}{dx} y(x) + y(x) - \int_0^1 y(t)^2 dt = 0.5(e^{-2} - 1), \quad y(0) = 1. \quad (16)$$

Introducing new variables $y_1 = y$ and $y_2 = y_1^2$, we get $\frac{d}{dx} y_2 = 2y_1 \frac{d}{dx} y_1$. Linearizing it results $y_1 \frac{d}{dx} y_1 \approx y_1^{(k)} \frac{d}{dx} y_1 + \frac{d}{dx} y_1^{(k)} y_1 - y_1^{(k)} \frac{d}{dx} y_1^{(k)}$ and, therefore, problem (16) can be casted as

$$\begin{cases} \frac{d}{dx} y_1^{(k+1)} + y_1^{(k+1)} - \int_0^1 y_2^{(k+1)} dt = 0.5(e^{-2} - 1) \\ \frac{d}{dx} y_2^{(k+1)} - 2 \left(y_1^{(k)} \frac{d}{dx} y_1^{(k+1)} + y_1^{(k+1)} \frac{d}{dx} y_1^{(k)} \right) = -2y_1^{(k)} \frac{d}{dx} y_1^{(k)} \\ y_1^{(1)}(0) = 1, \quad y_2^{(1)}(0) = 1 \end{cases}, \quad (17)$$

and the coefficient matrix (ChebyshevT basis) of the linear system $\mathbb{T}_{\mathcal{F}} a_{\mathcal{F}} = b_{\mathcal{F}}$ is

$$\mathbb{T}_{\mathcal{F}} = \begin{bmatrix} T_0(0) & \dots & T_{n-1}(0) & \mathbf{0} \\ \mathbf{0} & & T_0(0) & \dots & T_{n-1}(0) \\ & \mathbf{M}_{\mathcal{F}} + \mathbf{I} & & & -\mathbf{S}_{F_{\mathcal{F}}} \\ -2 \left[y_1^{(k)}(\mathbf{M}_{\mathcal{F}}) \mathbf{N}_{\mathcal{F}} + y_1^{(k)'}(\mathbf{M}_{\mathcal{F}}) \mathbf{I} \right] & & & & \mathbf{N}_{\mathcal{F}} \end{bmatrix},$$

where $\mathbf{S}_{F_{\mathcal{F}}}$ by (8), making use of the Tau Toolbox `fred` function. The independent vector is $b_{\mathcal{F}} = \left[1, 1, 0.5(e^{-2} - 1), 0, \dots, 0, -2y_1^{(k)} \frac{d}{dx} y_1^{(k)} \right]^T$.

| n | [3] | | Tau Toolbox | |
|-----|----------------|----------|----------------|----------|
| | $\ e\ _\infty$ | CPU time | $\ e\ _\infty$ | CPU time |
| 5 | 9.63e-04 | 0.42 | 1.58e-04 | 0.03 |
| 9 | 1.28e-04 | 0.58 | 1.28e-09 | 0.03 |
| 17 | 2.87e-05 | 0.73 | 7.77e-16 | 0.04 |
| 33 | 5.61e-06 | 0.07 | 4.44e-16 | 0.07 |
| 65 | 2.39e-06 | 1.54 | 4.44e-16 | 0.37 |
| 129 | 1.28e-06 | 2.15 | 4.44e-16 | 2.35 |

Table 1: Comparison between the results at [3] and with Tau Toolbox .

The error presented in [3] is $\|e\|_\infty = \max_j |y_n(x_j) - y(x_j)|$, and therefore the same measure is applied to compare the results, see Table 1.

For all polynomial degree approximations the approximate solution provided by Tau Toolbox is clearly better than the one given by [3]. The error for $n = 129$ with [3] was reached with Tau Toolbox with a polynomial degree smaller than 9 and for degree 17 the Tau Toolbox was able to find the solution with machine precision order. Noteworthy is that for increasing polynomial degree the algorithm is stable, not showing any perturbations for higher degrees. The CPU time should not be compared between both approaches since results are reported from two distinct machines. It is however relevant to understand that the effort to solve the problem with Tau Toolbox is higher: if we take $n = 5$ as reference time then for $n = 128$ Tau Toolbox required $78.3 \times$ the reference computational cost whereas [3] necessitates $5.12 \times$. Robustness and stability comes at a price: more elaborate mathematics must be performed to reach such quality results. Nevertheless, one must point out that the CPU time to compute an accurate approximation was still low, only 2.35 seconds.

Example 8. Consider now the system of integro-differential equations with nonlinear Volterra term [1], with exact solution $y_1(x) = \sinh(x)$ and $y_2(x) = \cosh(x)$,

$$\begin{cases} \frac{d}{dx}y_1(x) + \frac{1}{2}\left(\frac{d}{dx}y_2(x)\right)^2 - \int_0^x (x-t)y_2(t) + y_2(t)y_1(t)dt = 1 \\ \frac{d}{dx}y_2(x) - \int_0^x (x-t)y_1(t) - y_2^2(t) + y_1^2(t)dt = 2x \\ y_1(0) = 0, \quad y_2(0) = 1 \end{cases} \quad (18)$$

As for the previous example, linearization is done first and the Volterra integral term S_V , following (7), is tackled using the Tau Toolbox `volt` function.

Fig. 1 shows the error after 5 iterations along the interval $[0, 1]$. For $n = 20$, Tau Toolbox was able to provide an approximate solution with machine precision all over the interval. For comparison purposes, results for the same problem in [1] with $n = 10$ are plotted together with Tau Toolbox library. The former, at the right part of the interval, can only reach single-precision accuracy, in contrast with the latter, which delivers double precision accuracy. For $n \geq 25$, Tau Toolbox reaches machine precision.

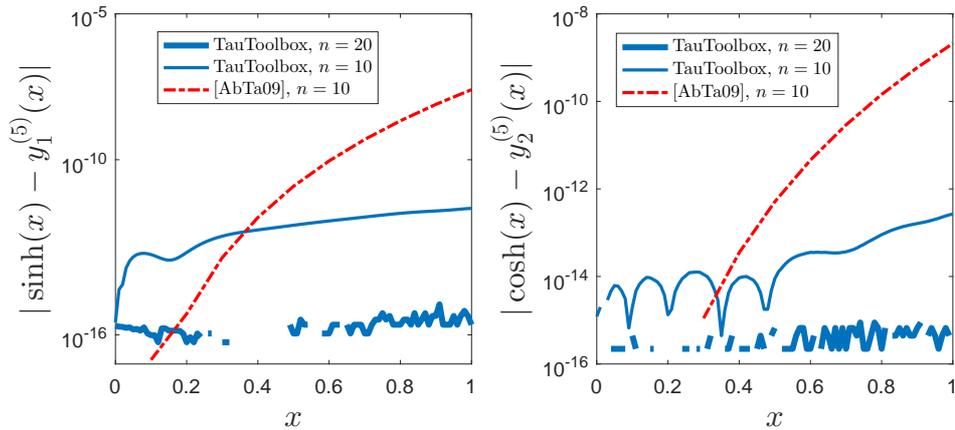


Figure 1: Comparison between [1] and Tau Toolbox for 10 (and 20) degree polynomial.

7 Conclusions

In this work we proposed the Lanczos' tau method for nonlinear integro-differential systems of equations. Contributions to improve the stability of the numerical implementation are presented. Numerical experiments illustrate the accuracy and efficiency of the new proposal, when compared with the results in the literature.

All these contributions are included at Tau Toolbox – a MATLAB library for the solution of integro-differential problems.

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