L_2-interpretation of the Kontorovich-Lebedev integrals

Semyon B. YAKUBOVICH

Abstract

We give an interpretation of the familiar Kontorovich-Lebedev transformation as an isometric isomorphism between weighted L_2-spaces. The convergence of the corresponding integrals is in the mean square sense with respect to the related norm of the space. Mapping properties of the Kontorvich-Lebedev operators are investigated.

Keywords: *Kontorovich-Lebedev transform, modified Bessel functions, Parseval equality, Fourier integrals*

AMS subject classification: 44A15

1 Introduction and auxiliary results

In this note we deal with the following reciprocal formulas of the Kontorovich-Lebedev transformation (cf. [2], [4], [8])

$$g(x) = \int_0^\infty K_{2i\tau}(2\sqrt{x}) f(\tau) d\tau, \tag{1.1}$$

$$f(\tau) = \frac{2}{\pi |\Gamma(2i\tau)|^2} \int_0^\infty K_{2i\tau}(2\sqrt{x}) g(x) \frac{dx}{x}, \tag{1.2}$$

where $\Gamma(z)$ is Euler's gamma-function and $K_\mu(z)$ is the modified Bessel function [1], which satisfies the differential equation

$$z^2 \frac{d^2u}{dz^2} + z\frac{du}{dz} - (z^2 + \mu^2)u = 0,$$

and has the asymptotic behaviour

$$K_\mu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}[1 + O(1/z)], \qquad z \to \infty, \tag{1.3}$$

and near the origin

$$z^{|\mathrm{Re}\mu|} K_\mu(z) = 2^{\mu-1}\Gamma(\mu) + o(1),\ z \to 0,\ \mu \neq 0, \tag{1.4}$$

$$K_0(z) = -\log z + O(1),\ \ z \to 0. \tag{1.5}$$

We will call f and g as originals and images under the Kontorovich-Lebedev transform (1.1), respectively. These functions will be from the weighted L_2-spaces $L_2(\mathbb{R}_+;\omega(x)dx)$ with respect to the measure $\omega(x)dx$ equipped with the norm

$$||f||_{L_2(\mathbb{R}_+;\omega(x)dx)} = \left(\int_0^\infty |f(x)|^2\omega(x)dx\right)^{1/2}. \tag{1.6}$$

Namely, the Kontorovich-Lebedev transform (1.1) will be investigated as an isometric isomorphism

$$g : L_2(\mathbb{R}_+;|\Gamma(2i\tau)|^2d\tau) \leftrightarrow L_2(\mathbb{R}_+;x^{-1}dx). \tag{1.7}$$

Moreover, our goal is to show that integrals (1.1), (1.2) can be generally interpreted as

$$g(x) = \text{l.i.m.}_{N\to\infty}\int_0^N K_{2i\tau}(2\sqrt{x})f(\tau)d\tau, \tag{1.8}$$

$$f(\tau) = \text{l.i.m.}_{N\to\infty}\frac{2}{\pi|\Gamma(2i\tau)|^2}\int_{1/N}^N K_{2i\tau}(2\sqrt{x})g(x)\frac{dx}{x}, \tag{1.9}$$

where the convergence is in mean square with respect to the norms of the spaces $L_2(\mathbb{R}_+;x^{-1}dx)$ and $L_2(\mathbb{R}_+;|\Gamma(2i\tau)|^2d\tau)$, respectively. Finally we establish the isometry by the Parseval equality

$$\int_0^\infty |g(x)|^2\frac{dx}{x} = \frac{\pi}{2}\int_0^\infty |f(\tau)|^2|\Gamma(2i\tau)|^2d\tau, \tag{1.10}$$

which is proved, for instance, in [11].

In order to proceed our goals we give some additional properties of the kernel function $K_{2i\tau}(2\sqrt{x})$. Indeed, the modified Bessel function $K_{2i\tau}(2\sqrt{x})$ [1] can be defined by the relation

$$K_{2i\tau}(2\sqrt{x}) = \frac{\pi}{2i\sinh 2\pi\tau}\left[I_{-2i\tau}(2\sqrt{x}) - I_{2i\tau}(2\sqrt{x})\right], \tag{1.11}$$

where $I_{2i\tau}(2\sqrt{x})$ is in turn, the modified Bessel function, which can be written in terms of the series

$$I_{2i\tau}(2\sqrt{x}) = \sum_{m=0}^\infty \frac{x^{m+i\tau}}{m!\Gamma(m+2i\tau+1)} = \frac{e^{i\tau\log x}}{\Gamma(2i\tau+1)} + \sum_{m=1}^\infty \frac{x^{m+i\tau}}{m!\Gamma(m+2i\tau+1)}. \tag{1.12}$$

When $x,\tau \in \mathbb{R}_+$, then $K_{2i\tau}(2\sqrt{x})$ is real-valued and represents the kernel of the Kontorovich-Lebedev transform (1.1), (1.8). At the same time it can be given for instance, by the Mellin-Barnes integral (see [8], relation (1.113))

$$K_{2i\tau}(2\sqrt{x}) = \frac{1}{4\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(s+i\tau)\Gamma(s-i\tau)x^{-s}ds, \tag{1.13}$$

where $x > 0, \gamma > 0, \tau \in \mathbb{R}$ or by the Fourier integral [1]

$$K_{2i\tau}(2\sqrt{x}) = \int_0^\infty e^{-2\sqrt{x}\cosh u} \cos 2\tau u du. \tag{1.14}$$

Finally we note in this section that various L_2-theorems of the composition type for the Kontorovich-Lebedev transform were proved in [7], [9]. Its distributional analog was studied in [6]. These results were generalized on the so-called Kontorovich-Lebedev type integral transformations (see [10]).

2 Interpretation of the Kontorovich-Lebedev integrals

In this section we will prove the reciprocity and clarify L_2 - interpretation of the Kontorovih-Lebedev integrals (1.8), (1.9) for any $f \in L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$. However, let us first illustrate by examples the possible convergence problems for Lebesgue integrals (1.1), (1.2). For instance, if

$$g(x) = \begin{cases} \frac{1}{\log x}, & \text{if } 0 < x \le \frac{1}{2}, \\ 0, & \text{if } x > \frac{1}{2}, \end{cases}$$

then it evidently belongs to $L_2(\mathbb{R}_+; x^{-1}dx)$ but due to the asymptotic formula (1.4) integral (1.2) is divergent for this function. Analogously, taking into account the asymptotic behaviour with respect to the index of the modified Bessel function [8] we have that for each $x > 0$ $K_{2i\tau}(2\sqrt{x}) = O\left(\frac{e^{-\pi\tau}}{\sqrt{\tau}}\right)$, $\tau \to +\infty$. Hence, if similarly

$$f(\tau) = \begin{cases} 0, & \text{if } 0 < \tau \le 2, \\ \frac{e^{\pi\tau}}{\log \tau}, & \text{if } \tau > 2, \end{cases}$$

then invoking Strling's asymptotic formula for gamma-functions [7] we see that $f \in L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$ but integral (1.1) is divergent for all $x > 0$.

Let us take first $f \in L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$ vanishing outside of some interval $(0, a)$, $a > 0$. Then (1.1) exists as a Lebesgue integral. Invoking (1.11), (1.12) we find

$$\left(\int_0^\infty |g(x)|^2 \frac{dx}{x}\right)^{1/2} = \left(\int_0^\infty \left|\int_0^a K_{2i\tau}(2\sqrt{x}) f(\tau) d\tau\right|^2 \frac{dx}{x}\right)^{1/2}$$

$$\le \left(\int_0^1 \left|\int_0^a K_{2i\tau}(2\sqrt{x}) f(\tau) d\tau\right|^2 \frac{dx}{x}\right)^{1/2} + \left(\int_1^\infty \left|\int_0^a K_{2i\tau}(2\sqrt{x}) f(\tau) d\tau\right|^2 \frac{dx}{x}\right)^{1/2}. \tag{2.1}$$

Hence

$$\left(\int_0^1 \left|\int_0^a K_{2i\tau}(2\sqrt{x}) f(\tau) d\tau\right|^2 \frac{dx}{x}\right)^{1/2} \le \frac{\pi}{2} \left(\int_0^1 \left|\int_0^a \frac{f(\tau)}{\sinh 2\pi\tau} I_{-2i\tau}(2\sqrt{x}) d\tau\right|^2 \frac{dx}{x}\right)^{1/2}$$

$$+\frac{\pi}{2}\left(\int_0^1 \left|\int_0^a \frac{f(\tau)}{\sinh 2\pi\tau} I_{2i\tau}(2\sqrt{x})d\tau\right|^2 \frac{dx}{x}\right)^{1/2} = I_1 + I_2. \tag{2.2}$$

It is sufficient to estimate integral I_1, since I_2 can be treated in the same manner. The above condition on f implies $f(\tau)\Gamma(2i\tau) \in L_1(\mathbb{R}_+; d\tau)$. Therefore the Fourier transform of this product exists as a Lebesgue integral. Hence appealing to (1.12), the Minkowski inequality, the supplement and reduction formulas for the Gamma-function we find

$$I_1 \le \frac{1}{2}\left(\int_0^1 \left|\int_0^a e^{-i\tau\log x} f(\tau)\Gamma(2i\tau)d\tau\right|^2 \frac{dx}{x}\right)^{1/2}$$

$$+\frac{1}{2}\left(\int_0^1 \left|\int_0^a e^{-i\tau\log x} f(\tau)\Gamma(2i\tau)\sum_{m=1}^{\infty}\frac{x^m}{m!(-2i\tau+1)_m}d\tau\right|^2 \frac{dx}{x}\right)^{1/2}.$$

By simple substitution $y = -\log x$ and the Parseval formula for the Fourier transform [5] we come out with

$$\frac{1}{2}\left(\int_0^1 \left|\int_0^a e^{-i\tau\log x} f(\tau)\Gamma(2i\tau)d\tau\right|^2 \frac{dx}{x}\right)^{1/2}$$

$$= \sqrt{\frac{\pi}{2}}\left(\int_0^\infty \left|\frac{1}{\sqrt{2\pi}}\int_0^a e^{i\tau y} f(\tau)\Gamma(2i\tau)d\tau\right|^2 dy\right)^{1/2}$$

$$\le \sqrt{\frac{\pi}{2}}\left(\int_{-\infty}^\infty \left|\frac{1}{\sqrt{2\pi}}\int_0^a e^{i\tau y} f(\tau)\Gamma(2i\tau)d\tau\right|^2 dy\right)^{1/2}$$

$$= \sqrt{\frac{\pi}{2}}\left(\int_0^a |f(\tau)\Gamma(2i\tau)|^2 d\tau\right)^{1/2} \le \sqrt{\frac{\pi}{2}}||f||_{L_2(\mathbb{R}_+;|\Gamma(2i\tau)|^2d\tau)} < \infty.$$

Meanwhile, by virtue of the generalized Minkowski and the Schwarz inequalities we find

$$\frac{1}{2}\left(\int_0^1 \left|\int_0^a e^{-i\tau\log x} f(\tau)\Gamma(2i\tau)\sum_{m=1}^{\infty}\frac{x^m}{m!(-2i\tau+1)_m}d\tau\right|^2 \frac{dx}{x}\right)^{1/2}$$

$$\le \frac{1}{2}\int_0^a |f(\tau)\Gamma(2i\tau)| \left(\int_0^1 \left|\sum_{m=1}^{\infty}\frac{x^m}{m!(-2i\tau+1)_m}\right|^2 \frac{dx}{x}\right)^{1/2} d\tau$$

$$\le \frac{1}{2}\left(\int_0^a |f(\tau)\Gamma(2i\tau)|^2 d\tau\right)^{1/2} \left(\int_0^\infty \int_0^1 \left|\sum_{m=1}^{\infty}\frac{x^m}{m!(-2i\tau+1)_m}\right|^2 \frac{dx}{x}d\tau\right)^{1/2}$$

$$\leq \frac{1}{2}||f||_{L_2(\mathbb{R}_+;|\Gamma(2i\tau)|^2d\tau)} \left(\int_0^\infty \frac{d\tau}{1+4\tau^2}\right)^{1/2} \sum_{m=1}^\infty \frac{1}{m!}$$

$$= \frac{(e-1)\sqrt{\pi}}{4}||f||_{L_2(\mathbb{R}_+;|\Gamma(2i\tau)|^2d\tau)} < \infty.$$

Treating I_2 in the same manner we combine with (2.2) to conclude that its left-hand side is bounded for any $f \in L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$, which is zero outside of a finite interval $(0,a)$.

In order to estimate the integral in (2.1) over $(1,\infty)$ we choose a small $\alpha > 0$ and we write

$$\left(\int_1^\infty \left|\int_0^a K_{2i\tau}(2\sqrt{x})f(\tau)d\tau\right|^2 \frac{dx}{x}\right)^{1/2} \leq \left(\int_1^\infty \left|\int_0^a K_{2i\tau}(2\sqrt{x})f(\tau)d\tau\right|^2 x^{\alpha-1}dx\right)^{1/2}$$

$$\leq \left(\int_0^\infty \left|\int_0^a K_{2i\tau}(2\sqrt{x})f(\tau)d\tau\right|^2 x^{\alpha-1}dx\right)^{1/2}$$

$$= \left(\int_0^\infty x^{\alpha-1}dx \int_0^a K_{2i\tau}(2\sqrt{x})f(\tau)d\tau \int_0^a K_{2iu}(2\sqrt{x})\overline{f(u)}du\right)^{1/2}$$

$$= \left(\int_0^a \int_0^a f(\tau)\overline{f(u)}d\tau du \int_0^\infty x^{\alpha-1}K_{2i\tau}(2\sqrt{x})K_{2iu}(2\sqrt{x})dx\right)^{1/2}, \tag{2.3}$$

where the change of the order of integration is guaranteed via Fubini's theorem by the absolute and uniform convergence of the latter iterated integral. Meanwhile the integral with respect to x in (2.3) is calculated, for instance, in [3, Vol. 2], [8]

$$\int_0^\infty x^{\alpha-1}K_{2i\tau}(2\sqrt{x})K_{2iu}(2\sqrt{x})dx = \frac{1}{4\Gamma(2\alpha)}\left|\Gamma\left(\alpha+i(u+\tau)\right)\Gamma\left(\alpha+i(\tau-u)\right)\right|^2. \tag{2.4}$$

Hence combining with (2.3) we obtain the estimate

$$\left(\int_1^\infty \left|\int_0^a K_{2i\tau}(2\sqrt{x})f(\tau)d\tau\right|^2 \frac{dx}{x}\right)^{1/2} \leq \frac{1}{2\Gamma^{1/2}(2\alpha)}\left(\int_0^a \int_0^a f(\tau)\overline{f(u)}\right.$$

$$\left.\times \left|\Gamma\left(\alpha+i(u+\tau)\right)\Gamma\left(\alpha+i(\tau-u)\right)\right|^2 d\tau du\right)^{1/2},\ \alpha > 0. \tag{2.5}$$

Invoking Schwarz's inequality for double integrals we deduce

$$\frac{1}{2\Gamma^{1/2}(2\alpha)}\left(\int_0^a \int_0^a f(\tau)\overline{f(u)}\left|\Gamma\left(\alpha+i(u+\tau)\right)\Gamma\left(\alpha+i(\tau-u)\right)\right|^2 d\tau du\right)^{1/2}$$

$$\leq \frac{1}{2\Gamma^{1/2}(2\alpha)} \left(\int_0^a |f(\tau)|^2 \int_0^a \left| \Gamma\left(\alpha + i(u+\tau)\right) \Gamma\left(\alpha + i(\tau - u)\right) \right|^2 du d\tau \right)^{1/2}$$

$$\leq \frac{1}{2\Gamma^{1/2}(2\alpha)} \left(\int_0^a |f(\tau)|^2 \int_0^\infty \left| \Gamma\left(\alpha + i(u+\tau)\right) \Gamma\left(\alpha + i(\tau - u)\right) \right|^2 du d\tau \right)^{1/2}. \tag{2.6}$$

But since (see [3, Vol.2], [8])

$$\int_0^\infty \left| \Gamma\left(\alpha + i(u+\tau)\right) \Gamma\left(\alpha + i(\tau - u)\right) \right|^2 du = \frac{\pi^{3/2} 2^{1-4\alpha} \Gamma(2\alpha)}{\Gamma(2\alpha + 1/2)} |\Gamma(2(\alpha + i\tau))|^2, \ \alpha > 0$$

we get from (2.5), (2.6)

$$\left(\int_1^\infty \left| \int_0^a K_{2i\tau}(2\sqrt{x}) f(\tau) d\tau \right|^2 \frac{dx}{x} \right)^{1/2} \leq \frac{\pi^{3/4} 2^{-1/2-2\alpha}}{\Gamma^{1/2}(2\alpha + 1/2)} \left(\int_0^a |f(\tau)|^2 |\Gamma(2(\alpha + i\tau))|^2 d\tau \right)^{1/2}. \tag{2.7}$$

Hence passing to the limit through (2.7), when $\alpha \to 0+$ and invoking the uniform convergence by α of the integral in its right-hand side we come out with

$$\left(\int_1^\infty \left| \int_0^a K_{2i\tau}(2\sqrt{x}) f(\tau) d\tau \right|^2 \frac{dx}{x} \right)^{1/2} \leq \sqrt{\frac{\pi}{2}} \left(\int_0^a |f(\tau)|^2 |\Gamma(2i\tau)|^2 d\tau \right)^{1/2}$$

$$\leq \sqrt{\frac{\pi}{2}} ||f||_{L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)} < \infty.$$

Hence combining with (2.3), (2.1) and calling Fatou's lemma we get finally

$$\int_0^\infty |g(x)|^2 \frac{dx}{x} \leq \lim\inf_{\alpha \to 0+} \int_0^\infty |g(x)|^2 x^{\alpha - 1} dx \leq \frac{\pi}{2} ||f||^2_{L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)}. \tag{2.8}$$

Consequently, since the set of functions vanishing outside of some interval is dense in $L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$ we easily extend (2.8) for the whole L_2 - space defining the transformation (1.1) as a limit in mean with respect to the norm in $L_2(\mathbb{R}_+; x^{-1}dx)$ by formula (1.8), i.e. it satisfies

$$\int_0^\infty \left| g(x) - \int_0^N K_{2i\tau}(2\sqrt{x}) f(\tau) d\tau \right|^2 \frac{dx}{x} \to 0, \ N \to \infty.$$

Thus $g : L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau) \to L_2(\mathbb{R}_+; x^{-1}dx)$ is bounded and (see (2.8))

$$||g||_{L_2(\mathbb{R}_+; x^{-1}dx)} \leq \sqrt{\frac{\pi}{2}} ||f||_{L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)}. \tag{2.9}$$

Letting

$$g_N(x) = \begin{cases} g(x), & \text{if } \frac{1}{N} \le x \le N, \\ 0, & \text{if } x \in \mathbb{R}_+ \backslash [1/N, N], \end{cases}$$

and invoking an elementary identity $|\Gamma(2i\tau)|^2 = \pi[2\tau \sinh 2\pi\tau]^{-1}$ we appeal again to Fatou's lemma and arrive at the relations

$$\int_0^\infty \left| \int_{1/N}^N K_{2i\tau}(2\sqrt{x}) g(x) \frac{dx}{x} \right|^2 \frac{d\tau}{|\Gamma(2i\tau)|^2} \le \lim \inf_{\varepsilon \to 0+} \frac{2}{\pi} \int_0^\infty \tau \sinh(2\tau(\pi - \varepsilon))$$

$$\times \int_{1/N}^N K_{2i\tau}(2\sqrt{x}) g(x) \frac{dx}{x} \int_{1/N}^N K_{2i\tau}(2\sqrt{y}) \overline{g(y)} \frac{dy}{y} d\tau = \lim \inf_{\varepsilon \to 0+} \frac{2}{\pi} \int_{1/N}^N \int_{1/N}^N \frac{g(x)\overline{g(y)}}{xy}$$

$$\times \int_0^\infty \tau \sinh(2\tau(\pi - \varepsilon)) K_{2i\tau}(2\sqrt{x}) K_{2i\tau}(2\sqrt{y}) d\tau dx dy, \tag{2.10}$$

where the change of the order of integration is motivated by the Fubini theorem. But the latter integral with respect to τ is calculated for instance in [8, p. 43], namely

$$\frac{2}{\pi} \int_0^\infty \tau \sinh(2\tau(\pi - \varepsilon)) K_{2i\tau}(2\sqrt{x}) K_{2i\tau}(2\sqrt{y}) d\tau$$

$$= \frac{\sqrt{xy}}{2} \sin \varepsilon \frac{K_1(2(x + y - 2\sqrt{xy} \cos \varepsilon)^{1/2})}{(x + y - 2\sqrt{xy} \cos \varepsilon)^{1/2}},$$

where K_1 is the modified Bessel function of the order 1. Thus (2.10) yields

$$\int_0^\infty \left| \int_{1/N}^N K_{2i\tau}(2\sqrt{x}) g(x) \frac{dx}{x} \right|^2 \frac{d\tau}{|\Gamma(2i\tau)|^2}$$

$$\le \lim \inf_{\varepsilon \to 0+} \frac{\sin \varepsilon}{2} \int_{1/N}^N \int_{1/N}^N \frac{K_1(2(x + y - 2\sqrt{xy} \cos \varepsilon)^{1/2})}{(x + y - 2\sqrt{xy} \cos \varepsilon)^{1/2}} \frac{g(x)\overline{g(y)}}{\sqrt{xy}} dx dy. \tag{2.11}$$

Again with the Schwarz inequality for double integrals we find

$$\lim \inf_{\varepsilon \to 0+} \frac{\sin \varepsilon}{2} \int_{1/N}^N \int_{1/N}^N \frac{K_1(2(x + y - 2\sqrt{xy} \cos \varepsilon)^{1/2})}{(x + y - 2\sqrt{xy} \cos \varepsilon)^{1/2}} \frac{g(x)\overline{g(y)}}{\sqrt{xy}} dx dy$$

$$\le \lim \inf_{\varepsilon \to 0+} \frac{\sin \varepsilon}{2} \int_{1/N}^N \frac{|g(x)|^2}{x} dx \int_{1/N}^N \frac{K_1(2(x + y - 2\sqrt{xy} \cos \varepsilon)^{1/2})}{(x + y - 2\sqrt{xy} \cos \varepsilon)^{1/2}} dy$$

$$\le \lim \inf_{\varepsilon \to 0+} \frac{\sin \varepsilon}{2} \int_{1/N}^N \frac{|g(x)|^2}{x} dx \int_0^\infty \frac{K_1(2(x + y - 2\sqrt{xy} \cos \varepsilon)^{1/2})}{(x + y - 2\sqrt{xy} \cos \varepsilon)^{1/2}} dy$$

$$= \liminf_{\varepsilon\to 0+} \frac{\sin\varepsilon}{2}\int_{1/N}^{N}\frac{|g(x)|^2}{\sqrt{x}}dx\int_0^\infty \frac{K_1(2\sqrt{x}(1+t-2\sqrt{t}\cos\varepsilon)^{1/2})}{(1+t-2\sqrt{t}\cos\varepsilon)^{1/2}}dt$$

$$= \liminf_{\varepsilon\to 0+} \sin\varepsilon\int_{1/N}^{N}\frac{|g(x)|^2}{\sqrt{x}}dx\int_{-\cot\varepsilon}^\infty \frac{K_1(2\sin\varepsilon\sqrt{x}(u^2+1)^{1/2})}{(u^2+1)^{1/2}}(u\sin\varepsilon+\cos\varepsilon)du.$$

Taking into account an elementary inequality $xK_1(x)\le 1, x>0$ and the representation in terms of the modified Bessel function of zero index $K_1(x) = -\frac{d}{dx}K_0(x)$ we write the latter limit as follows

$$\liminf_{\varepsilon\to 0+} \sin\varepsilon\int_{1/N}^{N}\frac{|g(x)|^2}{\sqrt{x}}dx\int_{-\cot\varepsilon}^\infty \frac{K_1(2\sin\varepsilon\sqrt{x}(u^2+1)^{1/2})}{(u^2+1)^{1/2}}(u\sin\varepsilon+\cos\varepsilon)du$$

$$= \liminf_{\varepsilon\to 0+}\left[\frac{\sin 2\varepsilon}{2}\int_{1/N}^{N}\frac{|g(x)|^2}{\sqrt{x}}dx\int_{-\cot\varepsilon}^\infty \frac{K_1(2\sin\varepsilon\sqrt{x}(u^2+1)^{1/2})}{(u^2+1)^{1/2}}du\right.$$

$$\left.+\sin^2\varepsilon\int_{1/N}^{N}\frac{|g(x)|^2}{\sqrt{x}}dx\int_{-\cot\varepsilon}^\infty \frac{K_1(2\sin\varepsilon\sqrt{x}(u^2+1)^{1/2})}{(u^2+1)^{1/2}}udu\right]$$

$$\le \liminf_{\varepsilon\to 0+}\left[\frac{\sin 2\varepsilon}{2}\int_{1/N}^{N}\frac{|g(x)|^2}{\sqrt{x}}dx\int_{-\infty}^\infty \frac{K_1(2\sin\varepsilon\sqrt{x}(u^2+1)^{1/2})}{(u^2+1)^{1/2}}du\right.$$

$$\left.+\sin^2\varepsilon\int_{1/N}^{N}\frac{|g(x)|^2}{\sqrt{x}}dx\int_{\cot\varepsilon}^\infty \frac{K_1(2\sin\varepsilon\sqrt{x}(u^2+1)^{1/2})}{(u^2+1)^{1/2}}udu\right]$$

$$\le \frac{\pi}{2}\int_{1/N}^{N}\frac{|g(x)|^2}{x}dx + \lim_{\varepsilon\to 0+}\frac{\sin^2\varepsilon}{2}\int_{1/N}^{N}\frac{|g(x)|^2}{\sqrt{x}}dx\int_{\cot^2\varepsilon}^\infty \frac{K_1(2\sin\varepsilon\sqrt{x}(u+1)^{1/2})}{(u+1)^{1/2}}du$$

$$= \int_{1/N}^{N}\frac{|g(x)|^2}{x}dx\left[\frac{\pi}{2} - \lim_{\varepsilon\to 0+}\frac{\sin\varepsilon}{2}\int_{\cot^2\varepsilon}^\infty \frac{d}{du}K_0(2\sin\varepsilon\sqrt{x}(u+1)^{1/2})du\right]$$

$$= \int_{1/N}^{N}\frac{|g(x)|^2}{x}dx\left[\frac{\pi}{2} + K_0(2\sqrt{x})\lim_{\varepsilon\to 0+}\frac{\sin\varepsilon}{2}\right] = \frac{\pi}{2}\int_{1/N}^{N}\frac{|g(x)|^2}{x}dx = \frac{\pi}{2}||g_N||^2_{L_2(\mathbb{R}_+;x^{-1}dx)} < \infty.$$

Combining with (2.10), (2.11) we obtain the exact inequality

$$\sqrt{\frac{\pi}{2}}||f_N||_{L_2(\mathbb{R}_+;|\Gamma(2i\tau)|^2d\tau)} \le ||g_N||_{L_2(\mathbb{R}_+;x^{-1}dx)}, \tag{2.12}$$

where we denote by

$$f_N(\tau) = \frac{2}{\pi|\Gamma(2i\tau)|^2}\int_{1/N}^{N}K_{2i\tau}(2\sqrt{x})g(x)\frac{dx}{x}$$

$$= \frac{2}{\pi|\Gamma(2i\tau)|^2} \int_0^\infty K_{2i\tau}(2\sqrt{x}) g_N(x) \frac{dx}{x}. \tag{2.13}$$

But evidently, $\{g_N\}_{N=1}^\infty$ is a Cauchy sequence and $||g - g_N||_{L_2(\mathbb{R}_+;x^{-1}dx)} \to 0, N \to \infty$. Therefore by virtue of (2.12) we get

$$\sqrt{\frac{\pi}{2}} ||f_N - f_M||_{L_2(\mathbb{R}_+;|\Gamma(2i\tau)|^2 d\tau)} \leq ||g_N - g_M||_{L_2(\mathbb{R}_+;x^{-1}dx)} \to 0, \ N, M \to \infty,$$

which means that $\{f_N\}_{N=1}^\infty$ is a Cauchy sequence as well. Then it has a limit in the space $L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$. We will prove that the limit is f, i.e.

$$\int_0^\infty \left| f(\tau) - \frac{2}{|\Gamma(2i\tau)|^2 \pi} \int_{1/N}^N K_{2i\tau}(2\sqrt{x}) g(x) \frac{dx}{x} \right|^2 |\Gamma(2i\tau)|^2 d\tau \to 0, N \to \infty,$$

where f is an original under the transformation g by formula (1.8). Therefore it will give the reciprocal form of the inverse operator (1.9).

To do this we appeal to (2.13) and integrating with respect to τ we come out with the equality

$$\int_0^\xi \tau f_N(\tau) |\Gamma(2i\tau)|^2 d\tau = \frac{2}{\pi} \int_{1/N}^N \int_0^\xi \tau K_{2i\tau}(2\sqrt{x}) g(x) \frac{dx d\tau}{x}, \tag{2.14}$$

where the change of the order of integration in the right-hand side of (2.14) is guaranteed by the absolute and uniform convergence of the integral by x. Hence observe with the Schwarz inequality and asymptotic behavior of the gamma-function that $\tau f_N(\tau)|\Gamma(2i\tau)|^2 \in L_1([0,\xi]; d\tau)$ and $g(x)x^{-1} \int_0^\xi \tau K_{2i\tau}(2\sqrt{x}) d\tau \in L_1(\mathbb{R}_+; dx)$. The latter fact is because (see [8]) $\int_0^\xi \tau K_{2i\tau}(2\sqrt{x}) d\tau \in L_2(\mathbb{R}_+; x^{-1}dx)$. Consequently, denoting by $\psi(\tau)$ the limit of the sequence $\{f_N\}_{N=1}^\infty$ in $L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$ we pass to the limit in (2.14) when $N \to \infty$ to have

$$\int_0^\xi \tau \psi(\tau) |\Gamma(2i\tau)|^2 d\tau = \frac{2}{\pi} \int_0^\infty \int_0^\xi \tau K_{2i\tau}(2\sqrt{x}) g(x) \frac{dx d\tau}{x}. \tag{2.15}$$

On the other hand, taking the Parseval equality (1.10), which is written for two functions $f_1, f_2 \in L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$ and the related transforms (1.8) g_1, g_2 we have

$$\int_0^\infty g_1(x) \overline{g_2(x)} \frac{dx}{x} = \frac{\pi}{2} \int_0^\infty f_1(\tau) \overline{f_2(\tau)} |\Gamma(2i\tau)|^2 d\tau.$$

Putting $f_1 = f$, $g_1 = g$ and

$$f_2(\tau) = \begin{cases} \tau, & \text{if } 0 \leq \tau \leq \xi, \\ 0, & \text{if } \tau > \xi, \end{cases}$$

we get

$$\int_0^{\xi} \tau f(\tau)|\Gamma(2i\tau)|^2 d\tau = \frac{2}{\pi}\int_0^{\infty}\int_0^{\xi} \tau K_{2i\tau}(2\sqrt{x})g(x)\frac{dxd\tau}{x}.$$

Hence comparing with (2.15) and differentiating with respect to ξ we prove that $\psi(\xi) = f(\xi)$ almost for all $\xi \in \mathbb{R}_+$.

Acknowledgments

The present investigation was supported, in part, by the "Centro de Matemática" of the University of Porto.

References

1. A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions*, Vol. II, McGraw-Hill, New York, London and Toronto (1953).

2. N.N. Lebedev, Sur une formule d'inversion, *C.R.Acad. Sci. URSS* **52**(1946), 655-658.

3. A.P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, *Integrals and Series: Vol. I: Elementary Functions; Vol. 2: Special Functions*, Gordon and Breach, New York (1986).

4. I.N. Sneddon, *The Use of Integral Transforms*, McGray Hill, New York (1972).

5. E.C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* , Clarendon Press, Oxford (1937).

6. S.B. Yakubovich and B. Fisher, On the Kontorovich-Lebedev transformation on distributions, *Proc. Amer. Math. Soc.* **122**, (1994), 773-777.

7. S.B. Yakubovich and Yu.F. Luchko, *The Hypergeometric Approach to Integral Transforms and Convolutions*, (Kluwers Ser. Math. and Appl.: Vol. 287), Dordrecht, Boston, London (1994).

8. S.B. Yakubovich, *Index Transforms*, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong (1996).

9. S.B. Yakubovich, *Index Transforms and Convolution Method*, Dr. Sc. Thesis. Minsk (1996).

10. S.B. Yakubovich and B. Fisher, A class of index transforms with general kernels, *Math. Nachr.* **200**, (1999), 165-182.

11. Yakubovich and J. de Graaf, On Parseval equalities and boundedness properties for Kontorovich-Lebedev type operators, *Novi Sad J. Math.*, **29** (1999), N 1, 185-205.

S.B.Yakubovich
Department of Pure Mathematics,
Faculty of Sciences,
University of Porto,
Campo Alegre st., 687
4169-007 Porto
Portugal
E-Mail: syakubov@fc.up.pt