ON THE SQUARE OF STIELTJES'S TRANSFORM AND ITS CONVOLUTION WITH APPLICATIONS TO SINGULAR INTEGRAL EQUATIONS

S. YAKUBOVICH AND M.MARTINS

ABSTRACT. We consider mapping properties of the square of Stieltjes's transform, establishing its new relations with the square of the Hilbert transform (a singular integral) on the half-axis and proving the corresponding convolution and Titchmarsh's type theorems. Moreover, the obtained convolution method is applied to solve a new class of singular integral equations.

1. INTRODUCTION AND AUXILIARY RESULTS

Let $x \in \mathbb{R}_+,\ f \in L_p(\mathbb{R}_+),\ 1 \le p < \infty$ be a complex-valued function. It is known that the classical Laplace transform

$$(\mathscr{L} f)(x) = \int_0^\infty e^{-xt} f(t)dt, \quad x > 0, \tag{1}$$

is well defined and one can compute its square, simply changing the order of integration and calculating an elementary integral. This drives us to the operator of Stieltjes's transform. Namely, we obtain

$$(\mathscr{S} f)(x) = (\mathscr{L}^2 f)(x) = \int_0^\infty e^{-xs} \int_0^\infty e^{-st} f(t)dtds = \int_0^\infty \frac{f(t)}{x+t}dt, \quad x > 0, \tag{2}$$

where the change of the order of integration is allowed due to Fubini's theorem via the estimate, which is based on the Hölder inequality

$$\int_0^\infty e^{-xs} \int_0^\infty e^{-st} |f(t)|dtds \le \int_0^\infty e^{-xs} \left(\int_0^\infty e^{-qst}dt\right)^{1/q} ds \left(\int_0^\infty |f(t)|^p dt\right)^{1/p}$$

$$= q^{-1/q}||f||_p \Gamma\left(1-q^{-1}\right) x^{q^{-1}-1},\ \frac{1}{q}+\frac{1}{p} = 1.$$

Let us compute, in turn, the square of the Stieltjes transform (2) of an arbitrary $f \in L_p(\mathbb{R}_+),\ 1 \le p < \infty$. Similar motivations with the estimate

$$\int_0^\infty \frac{1}{x+s} \int_0^\infty \frac{|f(t)|}{s+t}dtds \le ||f||_p \int_0^\infty \frac{1}{x+s} \left(\int_0^\infty \frac{1}{(s+t)^q}dt\right)^{1/q} ds$$

$$= \left[\frac{\Gamma(q-1)}{\Gamma(q)}\right]^{1/q} \Gamma(q^{-1})\Gamma(1-q^{-1})||f||_p\, x^{q^{-1}-1},\ \frac{1}{q}+\frac{1}{p} = 1$$

Date: November 13, 2013.

2000 *Mathematics Subject Classification.* 44A15, 44A35, 45E05, 45E10 .

Key words and phrases. Stieltjes transform, convolution method, Mellin transform, Laplace transform, Titchmarsh theorem, singular integral equations .

and relations (2.2.6.24), (7.3.2.148) in [1], Vol. 1 and [1], Vol.3, respectively, lead us to the following transformation

$$(\mathscr{S}^2 f)(x) \equiv G(x) = \int_0^\infty \frac{\log(x)-\log(t)}{x-t} f(t)dt,\ x>0, \tag{3}$$

whose kernel has a removable singularity at the point $t=x$ and the integral exists in the Lebesgue sense.

In 1990 [2] the first author proposed a new method of convolution constructions for integral transforms, which is based on the double Mellin-Barnes integrals (see in [3], [5]). Following this direction, he established for the first time as an interesting particular case the convolution operator for the Stieltjes transform (2) (see [3], formula (24.38))

$$(f*g)_{\mathscr{S}}(x) = f(x)(Hg)(x) + g(x)(Hf)(x),\ x>0, \tag{4}$$

where

$$(Hf)(x) = \int_0^\infty \frac{f(t)}{t-x}dt \tag{5}$$

is the operator of the Hilbert transform. Moreover, it was proved the corresponding convolution theorem

$$(\mathscr{S}\,(f*g)_{\mathscr{S}})(x) = (\mathscr{S}f)(x)(\mathscr{S}g)(x) \tag{6}$$

in the class of functions, which is associated with the Mellin transform. Later [6] these results were extended on L_p-spaces and applied to a class of singular integral equations of convolution type (4).

Our main goal in this paper is to employ the convolution method to the transformation (3) in order to derive the related convolution operator, to prove the convolution and Titchmarsh's type theorems (the latter one is about the absence of divisors of zero in the convolution product) and to apply these results, finding solutions and solvability conditions for a new class of singular integral equations. However, we begin with investigation of mapping properties of the square of Stieltjes's transform (3), proving the inversion theorem for this transformation of the Paley-Wiener type, as it was done by M. Dzrbasjan for the classical Stieltjes transform (2) (see [7], Theorem 2.11).

Acknowledgments

The present investigation was supported, in part, by the "Centro de Matemática" of the University of Porto.

References

[1] A.P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, *Integrals and Series: Vol. 1: Elementary Functions*, Gordon and Breach, New York (1986); *Vol. 3: More Special Functions*, Gordon and Breach, New York (1990).

[2] S.Yakubovich, A constructive method for constructing integral convolutions, *Dokl. Akad. Nauk BSSR*, **34** (1990), N 7, 588-591 (in Russian).

[3] Nguyen Thanh Hai and S. Yakubovich, *The double Mellin-Barnes type integrals and their applications to convolution theory*. Series on Soviet and East European Mathematics, 6. World Scientific Publishing Co., Inc., River Edge, NJ (1992).

[5] S. Yakubovich and Yu. Luchko, *The Hypergeometric Approach to Integral Transforms and Convolutions*. Mathematics and its Applications, 287. Kluwer Academic Publishers Group, Dordrecht (1994).

[6] H.M. Srivastava and Vu Kim Tuan, A new convolution theorem for the Stieltjes transform and its application to a class of singular integral equations, *Arch. Math. (Basel)*, **64** (1995), N 2, 144-149.

[7] M. Dzrbasjan, *Integral transforms and representations of functions in the complex domain*, Izdat. "Nauka", Moscow (1966) (in Russian).

[8] E.C. Titchmarsh, *An Introduction to the Theory of Fourier Integrals*, Clarendon Press, Oxford (1937).

[9] D.V. Widder, *The Laplace Transform*, New Jersey (1946).

[10] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional integrals and derivatives. Theory and applications*, Gordon and Breach Science Publishers, Yverdon (1993).

Department of Mathematics, Fac. Sciences of University of Porto,Rua do Campo Alegre, 687; 4169-007 Porto (Portugal)

E-mail address: syakubov@fc.up.pt