INDEX INTEGRAL REPRESENTATIONS FOR CONNECTION BETWEEN CARTESIAN, CYLINDRICAL, AND SPHEROIDAL SYSTEMS

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ABSTRACT. In this paper, we present two new index integral representations for connection between Cartesian, cylindrical, and spheroidal coordinate systems in terms of Bessel, MacDonald, and conical functions. Our result is mainly motivated by solution of the boundary value problems in domains composed of both Cartesian and hyperboloidal boundaries, and the need for new integral representations that facilitate the transformation between these coordinates. As a byproduct, the special cases of our results will produce new proofs to known index integrals and provide some new integral identities.

1. INTRODUCTION

Hyperboloidal coordinates are of particular importance in modeling of a new class of experimental arrangements for the measurements of microscopic features of various material samples. In scanning probe microscopy (SPM), the frequently occurring interaction of a probe with the sample can be modeled with a geometric hybridity, where a one-sheeted hyperboloid of revolution describes the probe and the $z = 0$ plane or another planar boundary describes the sample. In particular when solving the Laplace equation, the solutions in the hyperboloidal domain can be expressed as integrals involving the conical functions. Therefore, the ability to express the Cartesian coordinates in terms of an integral involving the conical functions is of great importance (see[7], [9], and [8] for a detailed discussion). Here, among other results, we provide a proof of such an integral representation for the coordinate z,

$$z = -\pi z_0 \int_1^\infty \eta' \, d\eta' \int_0^\infty \frac{q \tanh \pi q}{\cosh \pi q} P^0_{-\frac{1}{2}+iq}(0) \left[P^0_{-\frac{1}{2}+iq}(\mu) - P^0_{-\frac{1}{2}+iq}(0) \right] P^0_{-\frac{1}{2}+iq}(\eta') P^0_{-\frac{1}{2}+iq}(\eta) \, dq, \tag{1.1}$$

where z_0 is a scale factor that defines the focal distance of the hyperboloid in the spheroidal (μ, η, ϕ) coordinate system, and $P^0_{-\frac{1}{2}+iq}$ denotes the conical functions. This integral expansion comprises the key element in the study of the Coulomb interaction of the SPM's probe with a sample surface.

In Section 2, we provide some background formulas and uniform asymptotic expansions for the conical and MacDonald's functions in relevant regimes of the parameters. There is an extensive classical literature for this which we quote, e.g., Prudnikov, Brychkov, and Marichev [10], [11], [12] among others. Section 3 contains statement and the proof of a new particular inverse Kontorovich–Lebedev transform which is central to the proof of (1.1) and all other applications mentioned in this paper. Our approach is based on application of Mellin and inverse Mellin transforms. In Section 4 is the proof of integral representation

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(1.1). In fact, we prove a more general result from which (1.1) follows as a special case. In section 5, we provide some application of the obtained inverse Kontorovich–Lebedev transform to give new proofs of some already known index integral transforms and also provide some new index integral transforms emphasizing the broader impact of our result.

Finally, it should be mentioned that for the sake of clarity of our presentation, proofs of some elementary facts, results, and observations are omitted. In such situations, we have provided sufficient references.

2. MacDonald's and conical functions

Recall that the *MacDonald's functions* are defined by

$$K_{iq}(\alpha) = \int_0^\infty e^{-\alpha\cosh x}\cos(qx)\,dx, \tag{2.1}$$

where $\alpha > 0$ and $q \geq 0$. For detailed facts regarding MacDonald's functions and their properties, we refer the reader to any classical reference in this regard (see, e.g., [1], [4], Vol II, [5]). Here we mention those results which are used in this work. First of all note that (2.1) implies

$$|K_{iq}(\alpha)| \leq \int_0^\infty e^{-\alpha\cosh x}\,dx \leq \int_0^\infty e^{-\alpha(1+\frac{1}{2}x^2)}\,dx = \sqrt{\frac{\pi}{2\alpha}}e^{-\alpha}. \tag{2.2}$$

Another useful estimate (see [15], p.15) is given by

$$|K_{iq}(\alpha)| \leq \sqrt{\frac{\pi}{2\alpha\cos\delta}}e^{-\delta q}e^{-\alpha\cos\delta} \quad \text{for all} \quad \delta \in [0,\pi/2). \tag{2.3}$$

Using the well known equivalent expression of $K_{iq}(\alpha)$ in terms of modified Bessel functions (see [2], p.458), one can show that for each $A > 0$

$$K_{iq}(\alpha) = -q^{-1}\mathrm{Im}[e^{iq\ln(\frac{\alpha}{2})}\Gamma(1-iq)] + E_q(\alpha) \tag{2.4}$$

for all $q > 0$ and $\alpha \in (0, A]$, where $|E_q(\alpha)| \leq C_A\alpha^2/\sqrt{q\sinh(\pi q)}$ and the constant C_A depends only on A. For the asymptotic expansion of $K_{iq}(\alpha)$ (see, e.g., [4] p.88 or [15] p.20) we have

$$K_{iq}(\alpha) = \sqrt{\frac{2\pi}{q}}e^{-\frac{\pi q}{2}}\left[\sin\left(q\ln q - q - q\ln(\frac{\alpha}{2}) + \frac{\pi}{4}\right) + O(q^{-1})\right], \quad \text{as} \quad q \to \infty \tag{2.5}$$

uniformly for $\alpha \in (0, A]$ with $A > 0$. From (2.1) we see that $K_{iq}(\alpha)$ is continuous at each $(\alpha, q) \in (0,\infty)\times[0,\infty)$; by (2.2) it is bounded for (α, q) in any set $[\epsilon,\infty)\times[0,\infty)$, where $\epsilon > 0$; and by (2.4) it is bounded for (α, q) in any set $(0, A]\times[\epsilon,\infty)$, where $\epsilon, A > 0$.

Next, we recall some facts regarding the *conical functions* $P^0_{-1/2+iq}(x)$ used throughout this paper. The most general form of the conical functions is given in terms of the hypergeometric function ${}_2F_1$ by

$$P^0_{-1/2+iq}(x) = {}_2F_1\left(\tfrac{1}{2} - iq, \tfrac{1}{2} + iq; 1; \tfrac{1-x}{2}\right), \tag{2.6}$$

where $q \geq 0$ and $x > -1$ (cf. [4], Vol. I, pp.122 or [13] 7-2-5). The conical functions have the alternative forms

$$P^0_{-1/2+iq}(\mu) = \frac{\sqrt{2}}{\pi}\cosh(\pi q)\int_0^\infty \frac{\cos(qt)}{\sqrt{\mu+\cosh t}}dt, \quad \text{where} \quad -1 < \mu < 1,\ q \geq 0, \tag{2.7}$$

and

$$P^0_{-1/2+iq}(\mu) = \frac{\sqrt{2}}{\pi^{3/2}}\cosh(\pi q)\int_0^\infty e^{-\kappa\mu}\frac{K_{iq}(\kappa)}{\sqrt{\kappa}}d\kappa, \quad \text{where} \ -1 < \mu < 1,\ q > 0. \tag{2.8}$$

Using an alternative representation of $P^0_{-1/2+iq}(\eta)$ (cf. [7] eq. 2.36), one gets

$$|P^0_{-1/2+iq}(\eta)| \leq 1, \quad \text{for all} \quad \eta \geq 1 \text{ and } q \geq 0. \tag{2.9}$$

Moreover, we have that $P^0_{-1/2+iq}(1) = 1$ for all $q \geq 0$; $P^0_{-1/2+iq}(\eta)$ is continuous and bounded on the set $[1,\infty) \times [0,\infty)$; $P_q(\mu)$ is continuous on the set $S = \{(\mu, q) \in (-1,1] \times [0,\infty)\}$ and bounded on compact subsets of S. It follows also that $P^0_{-1/2+iq}(\eta)$ and $P^0_{-1/2+iq}(\mu)$ are analytic in $q > 0$ for fixed $\eta \in [1,\infty)$ and $\mu \in (-1,1]$, respectively. For the asymptotic expansions of $P^0_{-1/2+iq}(\eta)$ and $P^0_{-1/2+iq}(\mu)$, we mention two useful equalities for our purposes; namely,

$$P^0_{-1/2+iq}(\eta) = \sqrt{\frac{2}{\pi \sinh \zeta}}\, q^{-\frac{1}{2}} \left[\cos(q\zeta - \frac{\pi}{4}) + O(q^{-1})\right] \quad \text{as} \quad q \to \infty, \tag{2.10}$$

uniformly for $\zeta \in [\epsilon, \infty), \epsilon > 0$, where $\eta = \cosh \zeta$; and

$$P^0_{-1/2+iq}(\mu) = \frac{1}{\sqrt{2\pi \sin \theta}}\, q^{-\frac{1}{2}} e^{\theta q} \left[1 + O(q^{-1})\right] \quad \text{as} \quad q \to \infty, \tag{2.11}$$

uniformly for $\theta \in [\epsilon, \frac{\pi}{2}], \epsilon > 0$, where $\mu = \cos \theta$. To see these facts and a detailed account on conical functions and their properties, we refer the reader to any of the classical references [3], [4], [5], [13]. We also use the following estimates for $J_0(u)$ and $J_1(u)$.

$$|J_0(u)| \leq C\, u^{-\frac{1}{2}} \quad \text{and} \quad |J_1(u)| \leq C\, u^{-\frac{1}{2}} \quad \text{for all } u > 0, \tag{2.12}$$

where $C > 0$ is a constant. The estimates given in (2.12) follows from the asymptotic expansions for $J_0(u)$ and $J_1(u)$ as $u \to \infty$ (see, e.g., [2] p.518).

Throughout this paper we employ the hyperboloidal coordinates (μ, η, ϕ) in $\mathbb{R}^3$. Fixing $z_0 > 0$, they are defined by (see, e.g., [9]).

$$x = R \cos \phi, \qquad y = R \sin \phi, \qquad \text{and} \qquad z = z_0\, \mu\eta, \tag{2.13}$$

where

$$R = z_0 \sqrt{(\eta^2 - 1)(1 - \mu^2)}, \qquad -1 \leq \mu \leq 1, \qquad \eta \geq 1, \qquad \text{and} \qquad 0 \leq \phi \leq 2\pi. \tag{2.14}$$

The $\eta =$ constant and $\mu =$ constant level surfaces are confocal hyperboloids and ellipsoids of revolution about the z–axis, respectively.

3. An Index Integral Representation

For brevity, we will write P_q for $P^0_{-1/2+iq}$ throughout the rest of this paper.

In this section we investigate the validity of a new integral expansion for the Bessel function in terms of MacDonald and Conical functions of complex lower index $-\frac{1}{2} + iq$. More precisely, we give a rigorous proof regarding the type of convergence and divergence of the integral

$$e^{-kz} J_0(kR) = \sqrt{\frac{2}{\pi k}} \int_0^\infty q \tanh(\pi q) K_{iq}(k) P_q(\mu) P_q(\eta) dq \tag{3.1}$$

where $k > 0$, $z = \mu\eta \geq 0$ and $R = \sqrt{(\eta^2 - 1)(1 - \mu^2)}$. In fact, if one considers μ and η as the spheroidal coordinates with natural restrictions on their domains, then (3.1) gives a new relation between the cylindrical and spheroidal coordinates. This relation will be exploited extensively in the later sections. In particular, we obtain some new index integrals and also give a proof of our integral expansion of the

Cartesian coordinate z in terms of conical functions discussed in the introduction. Before presenting our main result, recall that the Mellin transform of the function $f(k)$ is defined by

$$\mathcal{M}\left[f(k)\right](s) = F(s) = \int_0^\infty f(k)\, k^{s-1}\, dk, \tag{3.2}$$

For basic properties of this transform such as existence, uniqueness, and convolution, we refer the reader to [12], [14], or any classical text on this topic.

Theorem 3.3. *For $k > 0$, the following statements hold.*

(a) *If $\mu, \eta > 0$ and $\mu^2 + \eta^2 > 1$, then the integral (3.1) converges absolutely.*
(b) *If $\mu = 0$, $\eta \geq 1$ or $\eta = 0$, $\mu \geq 1$, then the integral (3.1) converges conditionally.*
(c) *If $(\mu, \eta) \in (0,1) \times (0,1)$ and $\mu^2 + \eta^2 = 1$, then the integral (3.1) converges conditionally.*
(d) *If $(\mu, \eta) \in [0,1) \times [0,1)$ and $\mu^2 + \eta^2 < 1$, then the integral (3.1) diverges.*

Proof. To prove part (a), we first assume that $\mu, \eta \in [1, \infty)$ and denote the right-hand side of (3.1) by

$$I_{\mu,\eta}(k) = \sqrt{\frac{2}{\pi k}} \int_0^\infty q \tanh(\pi q) K_{iq}(k) P_q(\mu) P_q(\eta)\, dq. \tag{3.4}$$

Using asymptotic behavior of the modified Bessel function and conical functions (see formulas (2.5), (2.10)), it follows that the modulus of the integrand in (3.4) is $O(e^{-\frac{\pi}{2}q})$; therefore, the integral (3.4) converges absolutely in this case. Let $I^*_{\mu,\eta}$ denote the Mellin transform of $I_{\mu,\eta}(k) e^{-k}\sqrt{k}$; that is,

$$\begin{aligned} \mathcal{M}\left[I_{\mu,\eta}(k) e^{-k}\sqrt{k}\right](s) = I^*_{\mu,\eta}(s) &= \int_0^\infty I_{\mu,\eta}(k) e^{-k} k^{s-1/2} dk \\ &= \sqrt{\frac{2}{\pi}} \int_0^\infty \int_0^\infty q \tanh(\pi q) K_{iq}(k) P_q(\mu) P_q(\eta) e^{-k} k^{s-1}\, dq\, dk. \end{aligned} \tag{3.5}$$

In view of (2.3) and the fact that $|P_q(\mu)P_q(\eta)| \leq 1$ (see Section 2), it follows that the integrand in last equality of (3.5) belongs to $L^1\big(\mathbb{R}^+ \times \mathbb{R}^+, dq \times dk\big)$. Therefore $I^*_{\mu,\eta}$ is well defined, the first integral in (3.5) converges absolutely, and one can interchange the order of integration in the last double integral via Fubini's theorem. Now, using the relation (see [12], identity 8.4.23.3)

$$\int_0^\infty e^{-k} K_{iq}(k) k^{s-1} dk = 2^{-s}\sqrt{\pi}\, \frac{\Gamma(s+iq)\Gamma(s-iq)}{\Gamma(s+1/2)} \qquad \operatorname{Re} s > 0, \tag{3.6}$$

one can rewrite (3.5) as

$$I^*_{\mu,\eta}(s) = \frac{2^{1/2-s}}{\Gamma(s+1/2)} \int_0^\infty q \tanh(\pi q)\, \Gamma(s+iq)\, \Gamma(s-iq)\, P_q(\mu)\, P_q(\eta)\, dq. \tag{3.7}$$

The following integral representation can be found in [14];

$$P_q(x) = \frac{2}{\pi} \cosh(\pi q) \int_0^\infty J_0(cy) K_{2iq}(y)\, dy, \quad \text{where} \quad c = \sqrt{\frac{x-1}{2}}\,. \tag{3.8}$$

Substituting (3.8) into (3.7) with $a = \sqrt{\frac{\mu-1}{2}}$ and $b = \sqrt{\frac{\eta-1}{2}}$ yields

$$I^*_{\mu,\eta}(s) = \tfrac{2^{3/2-s}}{\pi^2\Gamma(s+1/2)} \int_0^\infty \int_0^\infty \int_0^\infty q \sinh(2\pi q) \Gamma(s+iq)\Gamma(s-iq) J_0(ay) J_0(bu) K_{2iq}(y) K_{2iq}(u) du dy dq. \tag{3.9}$$

By virtue of the Stirling asymptotic formula for gamma–functions (see [1], [14]) we have

$$\big|\Gamma(s+iq)\big| = O\left(e^{-\pi q/2} q^{\mathrm{Re} s-1/2}\right) \quad \text{as} \quad q\to\infty. \tag{3.10}$$

Therefore taking into account the asymptotic properties of Bessel functions (2.3), together with the inequality (2.12), one can easily verify the absolute convergence of integral (3.9). Consequently, we can apply Fubini's theorem to interchange the order of integration in (3.9). Now the inner integral with respect to q can be calculated with the aid of relation (2.16.53.1) in [11]. This implies

$$I^{*}_{\mu,\eta}(s) = \frac{2^{1/2-3s}}{\Gamma(s+1/2)} \int_0^\infty \int_0^\infty J_0(ay) J_0(bu) \left(\frac{y^2u^2}{y^2+u^2}\right)^s K_{2s}\left(\sqrt{u^2+y^2}\right) du\, dy. \tag{3.11}$$

On the other hand, relation (2.3.16.1) in [10] gives

$$\left(\frac{y^2u^2}{y^2+u^2}\right)^s K_{2s}\left(\sqrt{u^2+y^2}\right) = \frac{1}{2}\int_0^\infty t^{2s-1}\, e^{-\left(t\,\frac{y^2+u^2}{2uy} - \frac{uy}{2t}\right)}\, dt. \tag{3.12}$$

The change of variable $\frac{1}{8}t^2 \mapsto t$ in (3.12) and substitution of the result into (3.11) brings us to the equality

$$\begin{aligned} I^{*}_{\mu,\eta}(s)\Gamma\left(\tfrac{1}{2}+s\right) &= 2^{-3/2}\int_0^\infty\int_0^\infty\int_0^\infty J_0(ay)\, J_0(bu)\, e^{-\left(\sqrt{8t}\,\frac{y^2+u^2}{2uy}+\frac{uy}{2\sqrt{8t}}\right)} t^{s-1}\, du\, dy\, dt \\ &= \mathcal{M}\left[2^{-3/2}\int_0^\infty\int_0^\infty J_0(ay)\, J_0(bu)\, e^{-\left(\sqrt{8t}\,\frac{y^2+u^2}{2uy}+\frac{uy}{2\sqrt{8t}}\right)}\, du\, dy\right]. \end{aligned} \tag{3.13}$$

Note also that in the first equality of (3.13) we have interchanged the order of integration since the modulus of the integrand is dominated by $\exp\left\{-\left(\sqrt{8t}\,\frac{y^2+u^2}{2uy}+\frac{uy}{2\sqrt{8t}}\right)\right\}$ for all $u,y\geq 1$, $t\geq 0$, $\mu,\eta\geq 1$ and it is bounded in the neighborhood of zero.

Next, using the translation property of the Mellin transform and the fact that $\Gamma(s)=\mathcal{M}\left[e^{-t}\right](s)$, it follows that $\Gamma\left(\frac{1}{2}+s\right)=\mathcal{M}\left[\sqrt{t}\,e^{-t}\right](s)$. This observation together with the convolution property of the Mellin transform (e.g. [12], [14]) and (3.5) imply

$$I^{*}_{\mu,\eta}(s)\Gamma\left(\tfrac{1}{2}+s\right) = \mathcal{M}\left[\sqrt{t}\,\int_0^\infty I_{\mu,\eta}(k) e^{-k-\frac{t}{k}}\,\frac{dk}{k}\right]. \tag{3.14}$$

Our last application of the Mellin transform is its uniqueness property (see [3], [4]), which in view of (3.13) and (3.14) gives

$$\int_0^\infty I_{\mu,\eta}(k) e^{-k-\frac{t}{k}}\,\frac{dk}{k} = \frac{2^{-3/2}}{\sqrt{t}}\int_0^\infty\int_0^\infty J_0(ay)\, J_0(bu)\, e^{-\left(\sqrt{8t}\,\frac{y^2+u^2}{2uy}+\frac{uy}{2\sqrt{8t}}\right)}\, du\, dy, \quad (t>0). \tag{3.15}$$

Inspired by the fact that the left–hand side of (3.15) represents a modified Laplace transform of the function $e^{-k}I_{\mu,\eta}(k)$ (see [3], [14]); we show that one can also rewrite the right–hand side of (3.15) in a similar fashion. We start with a polar coordinates substitution in the right–hand side of (3.15); that is,

$$\int_0^\infty I_{\mu,\eta}(k) e^{-k-\frac{t}{k}}\,\frac{dk}{k} = \frac{2^{-3/2}}{\sqrt{t}}\int_0^{\pi/2}\int_0^\infty J_0(ar\sin\varphi)\, J_0(br\cos\varphi)\, e^{-\left(\frac{\sqrt{8t}}{\sin 2\varphi}+\frac{r^2\sin 2\varphi}{4\sqrt{8t}}\right)}\, r dr d\varphi. \tag{3.16}$$

The latter integral in (3.16) can be calculated with respect to the variable r via relation (2.12.39.3) in [11]. As a result, we have

$$\int_0^\infty I_{\mu,\eta}(k)e^{-k-\frac{t}{k}}\frac{dk}{k} = 2\,I_0(ab\sqrt{8t})\int_0^{\pi/2}\exp\left(-\sqrt{8t}\;\frac{1+a^2\sin^2\varphi+b^2\cos^2\varphi}{\sin 2\varphi}\right)d\varphi, \tag{3.17}$$

where $I_0(z)$ denotes the modified Bessel function (see [6]). Letting $\sqrt{8t}\mapsto t$ and substituting $u=\tan\phi$ in (3.17), it follows from the relation (2.3.16.1) in [10] that

$$\int_0^\infty I_{\mu,\eta}(k)e^{-k-\frac{t^2}{8k}}\frac{dk}{k} = 2I_0(abt)K_0\left(t\sqrt{(1+a^2)(1+b^2)}\right). \tag{3.18}$$

In view of the property of Bessel functions $J_0(iz)=I_0(z)$ (see [6]) and relation (2.12.10.1) in [11], we write the right–hand side of (3.18) as

$$\int_0^\infty I_{\mu,\eta}(k)\,e^{-k-\frac{t^2}{8k}}\,\frac{dk}{k} = \int_0^\infty e^{-kz}\,J_0(kR)\,e^{-k-\frac{t^2}{8k}}\,\frac{dk}{k}, \quad \text{where} \quad t>0. \tag{3.19}$$

As a result of the uniqueness theorem for the modified Laplace transform of integrable functions (see [3], [14]) it follows from (3.19) that

$$I_{\mu,\eta}(k) = e^{-kz}J_0(kR), \quad \text{for} \quad \mu,\nu\in[1,\infty). \tag{3.20}$$

This proves the assertion of part (a), for $\mu,\nu\in[1,\infty)$. Since $P_q(z)$ is analytic in the half-plane Re z >-1, one can easily see that (3.1) also holds for the cases $\mu\in(0,1)$, $\eta\ge 1$ or $\mu\in(0,1)$, $\eta\ge 1$. Moreover, in these cases, the uniform estimates (2.10) and (2.11) imply that (3.1) converges absolutely and uniformly for $\arccos\mu\in\left[\varepsilon,\ \frac{\pi}{2}-\varepsilon\right]$, $\eta\ge 1$ or $\arccos\eta\in\left[\varepsilon,\ \frac{\pi}{2}-\varepsilon\right]$, $\mu\ge 1$, for all $k>0$. Finally, we turn our attention to the last remaining case of part (a); that is, $(\mu,\eta)\in(0,1)\times(0,1)$. Employing the estimates (2.5), (2.11), and the trivial identity (see [10])

$$\arccos\mu+\arccos\eta = \arccos\left(\mu\eta-\sqrt{(1-\mu^2)(1-\eta^2)}\right),$$

one observes that for sufficiently large $A>0$

$$\begin{aligned}\int_A^\infty q\tanh(\pi q)\big|K_{iq}(k)P_q(\mu)P_q(\eta)\big|\,dq &\le C\int_A^\infty \frac{\tanh(\pi q)}{\sqrt{q}}e^{-q\left(\frac{\pi}{2}-\arccos\mu-\arccos\eta\right)}dq \\ &= C\int_A^\infty \frac{\tanh(\pi q)}{\sqrt{q}}e^{-q\left(\frac{\pi}{2}-\arccos\left(\mu\eta-\sqrt{(1-\mu^2)(1-\eta^2)}\right)\right)}\,dq,\end{aligned} \tag{3.21}$$

where $C>0$ denotes an absolute constant. Clearly, the last integral in (3.21) converges uniformly if $\arccos\left(\mu\eta-\sqrt{(1-\mu^2)(1-\eta^2)}\right)\le\frac{\pi}{2}-\varepsilon$ $(\varepsilon>0)$, which is equivalent to the condition $\mu^2+\eta^2>1$. This completes the proof of part (a).

To prove part (b), we first assume that $\eta>1$. It follows from estimates (2.5), (2.10), and (2.11) that for sufficiently large $A>0$

$$\begin{aligned}&\int_A^\infty q\tanh(\pi q)K_{iq}(k)P_q(\mu)P_q(\eta)\,dq \\ &= O\left(\int_A^\infty e^{q\left(\arccos\mu-\frac{\pi}{2}\right)}\frac{\tanh(\pi q)}{\sqrt{q}}\sin\left(q\log\left(\frac{2q}{k}\right)-q+\frac{\pi}{4}\right)\cos\Big(q\ \mathrm{arccosh}\left(\eta-\frac{\pi}{4}\right)\Big)\,dq\right),\end{aligned} \tag{3.22}$$

for all $\mu \in [0, \varepsilon]$, where $\varepsilon > 0$ is sufficiently small, and $k > 0$. Now the Abel's test implies the uniform convergence of the integral (3.22). Therefore, one can let $\mu = 0$ in (3.1) with the aid of an obvious limiting process, part (a), and continuity properties of the conical functions mentioned in Section 2. For $\eta = 1$, we need some extra argument. In this case $P_q(1) = 1$; therefore, the estimate (2.10), and thus (3.22), does not hold. In order to overcome this difficulty, we use (see [6]) the special case identity $P_q(0) = \left|\Gamma\left(\frac{3}{4} + \frac{iq}{2}\right)\right|^{-2}$ for the conical functions in (3.1). This implies, for $\eta > 1$,

$$J_0\left(k\sqrt{\eta^2 - 1}\right) = \sqrt{\frac{2}{k}} \int_0^\infty q \tanh(\pi q) \left|\Gamma\left(\frac{3}{4} + \frac{iq}{2}\right)\right|^{-2} K_{iq}(k)\, P_q(\eta)\, dq. \tag{3.23}$$

In fact, (3.23) coincides with a particular case of the relation (2.17.27.21) in [12] . A similar argument as the one given in (3.22) with the asymptotic estimates (2.5), (2.10), and (3.10) imply

$$\int_A^\infty \frac{q \tanh(\pi q)}{\left|\Gamma\left(\frac{3}{4} + \frac{iq}{2}\right)\right|^2} K_{iq}(k) P_q(\eta)\, dq = O\left(\int_A^\infty \sin\left(q \log\left(\frac{2q}{k}\right) - q + \frac{\pi}{4}\right) \cos\Big(q \operatorname{arccosh}\left(\eta - \tfrac{\pi}{4}\right)\Big)\, dq\right), \tag{3.24}$$

where $A > 0$ is chosen sufficiently large. An application of integration by parts shows that the right–hand side of (3.24) is of order

$$O\left(\operatorname{arccosh} \eta \int_A^\infty \cos\left(q \log\left(\frac{2q}{k}\right) - q(1 - \operatorname{arccosh} \eta) + \frac{\pi}{4}\right) \frac{dq}{\log(2q/k)}\right),$$

for all $\eta \in [1, 1+\varepsilon]$, where $\varepsilon > 0$ is sufficiently small. Now the uniform convergence of the integral in (3.23) follows from Dirichlet test. As a result, we can let $\eta = 1$ in (3.23) using the same argument outlined for the case $\mu = 0$. Finally, noting (3.1) is symmetric in μ and η and $R(\mu, 0) = \sqrt{\mu^2 - 1}$, the case $\eta = 0$ and $\mu \geq 1$ can be treated in an exact same way as the one given above. This completes the proof of part (b).

Next suppose $\mu, \eta \in (0, 1)$. If $\mu^2 + \eta^2 = 1$, then trivially $\arccos \mu + \arccos \eta = \frac{\pi}{2}$. Thus from a similar argument as the one given in (3.21), we have for sufficiently large $A > 0$

$$\int_A^\infty q \tanh(\pi q) K_{iq}(k) P_q(\mu) P_q(\eta)\, dq = O\left(\int_A^\infty \frac{\tanh(\pi q)}{\sqrt{q}} \sin\left(q \log\left(\frac{2q}{k}\right) - q + \frac{\pi}{4}\right) dq\right) < \infty$$

due to the Dirichlet test, which proves the assertion of part (c). If $\mu^2 + \eta^2 < 1$, then again a similar estimate as the one given in (3.21) implies for large $A > 0$

$$\int_A^\infty q \tanh(\pi q) K_{iq}(k) P_q(\mu) P_q(\eta)\, dq = O\left(\int_A^\infty e^{q\left(\arccos\left(\mu\eta - \sqrt{(1-\mu^2)(1-\eta^2)}\right) - \frac{\pi}{2}\right)}\, dq\right) \to \infty$$

as $A \to \infty$, due to the fact that $\arccos\left(\mu\eta - \sqrt{(1-\mu^2)(1-\eta^2)}\right) > \frac{\pi}{2}$. This proves part (d) and completes the proof of the theorem. □

We close this section by pointing out certain limitation of the formula (3.1) with respect to the range of variables μ and η. To see this note that the uniform asymptotic formula (2.11) remains valid for $\theta = \arccos \mu \in \left(\frac{\pi}{2}, \pi - \varepsilon\right]$, where $\varepsilon > 0$ (see [6]). As a result, if for instance we assume $\mu \in (-1, 0)$ and $\eta > -1$, then in view of of (2.5) it follows that for $A > 0$ sufficiently large

$$\int_A^\infty q \tanh(\pi q) K_{iq}(k) P_q(\mu) P_q(\eta) dq = O\left(\int_A^\infty P_q(\eta) \exp\left(q\left[\theta - \frac{\pi}{2}\right]\right) dq\right),$$

where clearly the latter integral approaches infinity as $A \to \infty$. We summarize the above observation in the following remark.

Remark 3.25. If either $\mu \in (-1,0)$ or $\eta \in (-1,0)$, then the integral (3.1) diverges.

4. An Integral Expansion for z

In this section, we give a proof of Theorem 4.9 from which the integral expansion (1.1) follows as a consequence. The main key is provided by Proposition 4.2 below. In fact, Proposition 4.2 is an important application of Theorem 3.3 and contains the new index integral formula (4.4). Here, we assume some basics regarding the definition and properties of conical functions $Q_\nu(z)$ of the second kind (see for example [6] for a detailed discussion). For our purpose, we mention the facts that $Q_\nu(z)$ is analytic in the half–plane Re $z > 1$ and has the following uniform asymptotic behavior at infinity (see [1], [6])

$$Q_\nu(z) = O\left(\frac{\sqrt{\pi}}{2^{\nu+1}}\frac{\Gamma(1+\nu)}{\Gamma(\nu+3/2)}z^{-\nu-1}\right) \quad \text{as} \quad z\to\infty, \tag{4.1}$$

which can be easily obtained from Q_ν's representation in terms of the Gauss hypergeometric function.

Proposition 4.2. *Let $k > 0$ and $\mu, \eta \geq 0$. Then*

$$\frac{\sqrt{2}}{\pi^{3/2}}\int_0^\infty e^{-kz}J_0(kR)K_{iq}(k)\,\frac{dk}{\sqrt{k}} = sech(\pi q)P_q(\mu)P_q(\eta), \tag{4.3}$$

where the integral converges absolutely. Moreover if μ_j and η_j $(j = 1,2)$ satisfy either of the conditions

(1) $(\mu_j,\eta_j) \in [0,1)\times(1,\infty)$ or $(\mu_j,\eta_j)\in(1,\infty)\times[0,1)$,
(2) $(\mu_j,\eta_j)\in(1,\infty)\times(1,\infty)$,
(3) $(\mu_j,\eta_j)\in(0,1)\times(0,1)$ such that $\eta_j^2+\mu_j^2>1$,

then

$$\int_0^\infty q\frac{\tanh(\pi q)}{\cosh(\pi q)}P_q(\mu_1)P_q(\eta_1)P_q(\mu_2)P_q(\eta_2)\,dq = \frac{1}{\pi^2\sqrt{R_1R_2}}\,Q_{-1/2}\left(\frac{(z_1+z_2)^2+R_1^2+R_2^2}{2R_1R_2}\right), \tag{4.4}$$

where $z_j = \mu_j\,\eta_j$ and $R_j = \sqrt{(\eta_j^2-1)(1-\mu_j^2)}$.

Proof. First, recall that the Kontorovich -Lebedev(KL) transform of a fucntion $f(k)$ is defined by

$$F(q) = \int_0^\infty K_{iq}(k)f(k)\,\frac{dk}{\sqrt{k}}, \qquad q\in\mathbb{R}, \tag{4.5}$$

whenever the latter integral exists. Our proof is based on the Plancherel theorem and Parseval's identity for the KL–transform. In brief, the Plancherel theorem states that F defines a bounded (linear) operator from $L^2(\mathbb{R}^+,dk)$ onto $L^2\big(\mathbb{R}^+, q\sinh(\pi q)dq\big)$ with its bounded inverse given by

$$f(k) = \frac{2}{\pi^2}\int_0^\infty q\sinh(\pi q)\frac{K_{iq}(k)}{\sqrt{k}}F(q)\,dq. \tag{4.6}$$

Moreover, the following Parseval type identity holds

$$\frac{2}{\pi^2}\int_0^\infty q\sinh(\pi q)F_1(q)F_2(q)dq = \int_0^\infty f_1(k)f_2(k)\,dk, \tag{4.7}$$

where F_1 and F_2 denote KL–transforms of f_1 and f_2; respectively. For the mentioned facts and further properties of KL-transform, we refer the reader to [14] and/or [15].

Now, suppose $0 < \mu \leq 1$ and $\eta \geq 1$. The asymptotic behavior of Bessel functions (2.12) implies that $e^{-kz}J_0(kR)$ belongs to $L_2(\mathbb{R}_+;dk)$. Consequently, from the integral representation (3.1) and (4.6), it follows that (4.3) holds and the integral converges absolutely in this case. Furthermore, one can easily observe that the absolute and uniform convergence of integral (4.3) remains true for $\mu \geq 0$ and $\eta \geq 0$. Thus, the validity of (4.3) carries over to $\mu, \nu \geq 0$ with the aid of properties of conical functions P_q and the uniform convergence of the integral in (4.3).

Next, for $j = 1, 2$, let $F_j(q) = \mathrm{sech}(\pi q) P_q(\mu_j) P_q(\eta_j)$. Then under either of the conditions (1), (2), or (3), the asymptotic behavior of p_q (see (2.10), (2.11)) implies that $F_j \in L^2\big(\mathbb{R}^+, q \sinh(\pi q) dq\big)$. Thus, in view of the Parseval's identity (4.7) and the index integral (4.3), we have that

$$\int_0^\infty q \frac{\tanh(\pi q)}{\cosh(\pi q)} P_q(\mu_1) P_q(\eta_1) P_q(\mu_2) P_q(\eta_2)\, dq = \int_0^\infty e^{-k(z_1+z_2)}\, J_0(kR_1)\, J_0(kR_2)\, dk. \tag{4.8}$$

Finally, the relations (2.12.38.1) and (2.12.8.2) in [11] imply the equality of the right–hand sides of (4.8) and (4.4). Also note that in view of the asymptotic behavior of P_q, either of the conditions (1), (2), or (3) guarantees the absolute and uniform convergence of the integral (4.4). This proves the proposition. □

Now we are in the position to state the main result of this section.

Theorem 4.9. *Fix $z_0 > 0$. Let $z_1 = z_0\, \mu_1 \eta_1$ and $z_2 = z_0\, \mu_2 \eta_2$ denote the spheroidal coordinates representation of the z–coordinates of two points in $\mathbb{R}^3$, where $1 < \eta_j < \infty$ and $0 \leq \mu_j < 1$ ($j = 1, 2$).* Then

$$z_2 - z_1 = \pi z_0 \int_1^\infty \int_0^\infty q \frac{\tanh(\pi q)}{\cosh(\pi q)} P_q(0) \Big[P_q(\mu_1) P_q(\eta_1) - P_q(\mu_2) P_q(\eta_2) \Big] P_q(\eta)\, dq\, \eta d\eta. \tag{4.10}$$

Proof. Recall from (2.14) that $R_j = z_0 \sqrt{(\eta_j^2 - 1)(1 - \mu_j^2)}$, where $j = 1, 2$. By the identity (4.4) of Proposition 4.2

$$\int_0^\infty q \frac{\tanh(\pi q)}{\cosh(\pi q)} P_q(0) P_q(\eta) P_q(\mu_j) P_q(\eta_j)\, dq = \frac{\sqrt{z_0}}{\pi^2 \sqrt{(\eta^2 - 1) R_j}}\, Q_{-1/2} \left(\frac{z_j^2 + z_0^2(\eta^2 - 1) + R_j^2}{2 z_0 \sqrt{(\eta^2 - 1)} R_j} \right), \tag{4.11}$$

where $j = 1, 2$. Therefore, the identity (4.10) is equivalent to

$$z_2 - z_1 = \frac{z_0^{3/2}}{\pi} \int_1^\infty \frac{\eta}{(\eta^2 - 1)^{1/4}} \left[\frac{1}{\sqrt{R_1}}\, Q_{-1/2} \left(\frac{z_1^2 + R_1^2 + z_0^2(\eta^2 - 1)}{2 z_0 \sqrt{\eta^2 - 1}\, R_1} \right) - \frac{1}{\sqrt{R_2}}\, Q_{-1/2} \left(\frac{z_2^2 + R_2^2 + z_0^2(\eta^2 - 1)}{2 z_0 \sqrt{\eta^2 - 1}\, R_2} \right) \right] d\eta. \tag{4.12}$$

To prove (4.12), we reduce the problem via the change of variable $u = z_0 \sqrt{\eta^2 - 1}$ to the equivalent identity

$$z_2 - z_1 = \frac{1}{\pi} \int_1^\infty \sqrt{u} \left[\frac{1}{\sqrt{R_1}}\, Q_{-1/2} \left(\frac{z_1^2 + R_1^2 + u^2}{2 R_1 u} \right) - \frac{1}{\sqrt{R_2}}\, Q_{-1/2} \left(\frac{z_2^2 + R_2^2 + u^2}{2 R_2 u} \right) \right] du. \tag{4.13}$$

If $R_1 = R_2$ and $z_1 = z_2$, there is nothing to prove. So we may assume that either $R_1 \neq R_2$ or $z_1 \neq z_2$. Recall relation (2.18.3.9) in [12] ,

$$\int_0^\infty x^{\alpha - 1} Q_{-1/2} \left(\tfrac{a^2 + b^2 + x^2}{2 a x} \right) dx = \tfrac{\sqrt{\pi a}}{2}\, \Gamma\left(\tfrac{1}{4} + \tfrac{\alpha}{2}\right) \Gamma\left(\tfrac{1}{4} - \tfrac{\alpha}{2}\right) (a^2 + b^2)^{(2\alpha - 1)/4}\, P^0_{\alpha - 1/2} \left(\tfrac{b}{\sqrt{a^2 + b^2}} \right), \tag{4.14}$$

which is valid for $a, b > 0$ and $\mathrm{Re}\ \alpha < \frac{1}{2}$. Clearly, in our case $\alpha = 3/2$ and a direct application of (4.14) is not possible. However, one can still use this result if we look more carefully into the right–hand side of

(4.13). So let us denote the integrand in (4.13) by $I(\mu)$. Then the the asymptotic relation (4.1) imply

$$I(\mu) = \begin{cases} O\left(u\left(\frac{1}{(z_1^2+R_1^2+u^2)^{1/2}} - \frac{1}{(z_2^2+R_2^2+u^2)^{1/2}}\right)\right) = O\left(u\right), & \text{as} \quad u \to 0, \\ \\ O\left(u\left(\frac{1}{(z_1^2+R_1^2+u^2)^{1/2}} - \frac{1}{(z_2^2+R_2^2+u^2)^{1/2}}\right)\right) = O\left(u^{-2}\right), & \text{as} \quad u \to \infty. \end{cases}$$

Consequently, integral (4.13) converges absolutely. This means that one can extend (4.14) in the case of (4.13) from $\mathrm{Re}\,\alpha < \frac{1}{2}$ to $\alpha = \frac{3}{2}$. Therefore, we can set $\alpha = \frac{3}{2}$ in the right–hand side of (4.14). Finally, taking into account the fact $\Gamma(-1/2) = -2\sqrt{\pi}$ together with expression for the Legendre polynomial $P_1(z) = z$, we obtain

$$\begin{aligned} \frac{1}{\pi}\int_1^\infty \sqrt{u}\left[\tfrac{1}{\sqrt{R_1}}\, Q_{-1/2}\left(\tfrac{z_1^2+R_1^2+u^2}{2R_1 u}\right) - \tfrac{1}{\sqrt{R_2}}\, Q_{-1/2}\left(\tfrac{z_2^2+R_2^2+u^2}{2R_2 u}\right)\right] du \\ = (R_2^2+z_2^2)^{1/2} P_1\left(\frac{z_2}{\sqrt{R_2^2+z_2^2}}\right) - (R_1^2+z_1^2)^{1/2} P_1\left(\frac{z_1}{\sqrt{R_1^2+z_1^2}}\right) \\ = z_2 - z_1. \end{aligned}$$

This proves (4.13) and hence the identity (4.10). □

Corollary 4.15. *The integral representation* (1.1) *follows from Theorem 4.9 by letting* $\eta_1 = \eta_2$ *and* $\mu_2 = 0$ *in* (4.10).

5. Further Applications and Remarks

In this section, we show some applications of the index integral (3.1). Furthermore, we discuss how special cases of (3.1) and its corollary; namely, Proposition 4.2, coincide with known integral formulas in literature.

The first application of (3.1) is a new index integral formula.

Corollary 5.1. *If* $k > 0$, *then the following identity holds.*

$$\int_0^\infty \frac{q\tanh(\pi q)}{\left|\Gamma\left(\frac{3}{4}+\frac{iq}{2}\right)\right|^2} K_{iq}(k)\, dq = \sqrt{\frac{k}{2}}. \tag{5.2}$$

Proof. Note $P_q(1) = 1$ and $J_0(0) = 1$. Now use part (b) of theorem 3.3 by letting $\eta = 1$ in (3.23). □

In view of part (a) of theorem 3.3 with $\eta = 1$, we obtain the following index integral

$$\int_0^\infty q\tanh(\pi q) K_{iq}(k) P_q(\mu)\, dq = \sqrt{\frac{\pi k}{2}} e^{-k\mu} \qquad (k, \mu > 0), \tag{5.3}$$

which coincides with relation (2.17.26.15) in [12]. Another application of theorem 3.3, part (a), with $\mu = 1$ gives the value

$$\int_0^\infty q\tanh(\pi q) K_{iq}(k)\, dq = \sqrt{\frac{\pi k}{2}} e^{-k} \qquad (k > 0), \tag{5.4}$$

which is the limit case of the relation (2.16.48.15) in [11].

Furthermore if $\mu = \eta \geq \frac{1}{\sqrt{2}}$, then parts (a) and (c) of theorem 3.3 imply

$$\int_0^\infty q\tanh(\pi q)K_{iq}(k)[P_q(\mu)]^2\,dq = \sqrt{\frac{\pi k}{2}}\,e^{-k\mu^2}I_0\Big(k(\mu^2-1)\Big), \tag{5.5}$$

which represents the corrected version of relation (2.17.29.4) in [12] .

Finally we conclude this section with the following two new index integrals.

Corollary 5.6. *Under the assumptions of proposition 4.2, the following holds.*

(1) *If we let either of the parameters* μ_j *or* η_j $(j = 1, 2\ )$ *equal* 1*, say* $\mu_1 = 1$*, then*

$$\int_0^\infty q\frac{\tanh(\pi q)}{\cosh(\pi q)}P_q(\eta_1)P_q(\mu_2)P_q(\eta_2)\,dq = \frac{1}{\pi}\frac{1}{\sqrt{(\eta_1+z_2)^2+R_2^2}}. \tag{5.7}$$

(2) *If* $\mu_1 = \mu_2 = \eta_1 = \eta_2 = a$*, where* $a \in \left(\frac{1}{\sqrt{2}}, \infty\right)\setminus\{1\}$*, then we have the index integral*

$$\int_0^\infty q\frac{\tanh(\pi q)}{\cosh(\pi q)}\left[P_q(a)\right]^4\,dq = \frac{1}{\pi^2|a^2-1|}Q_{-1/2}\left(\frac{a^4+2a^2-1}{(a^2-1)^2}\right). \tag{5.8}$$

Moreover, the limit case $a = 1$ *in* (5.8) *coincides with the known integral value* (see [13] and also [14])

$$\int_0^\infty q\frac{\tanh(\pi q)}{\cosh(\pi q)}\,dq = \frac{1}{2\pi}.$$

References

[1] M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Applied Mathematics Series 55, National Bureau of Standards, 1965.

[2] G. Arfken, *Mathematical Methods for Physicists*, Second edition, Academic Press Inc., 1970.

[3] V.A. Ditkin, A.P. Prudnikov, *Integral Transforms and Operational Calculus*, Pergamon Press, 1965.

[4] A. Erdelyi, W. Magnus, F. Oberhettinger, G. Tricomi, *Higher Transcendental Functions, Volumes I,II*, McGraw-Hill, New York, 1953.

[5] I. S. Gradshteyn, I. M. Ryzhik, *Tables of Integrals, Series and Products*, Academic Press Inc., 1980.

[6] N. N. Lebedev, *Special Functions and Their Applications*, Prentice-Hall Inc., 1965.st

[7] A. Passian, Collective Electronic Effects in Scanning Probe Microscopy, Ph. D. Dissertation, University of Tennessee, Knoxville, 2000.

[8] A. Passian, H. Simpson, S. Kouchekian, S. B. Yakubovich, On the orthogonality of the MacDonald's functions, *Journal of Mathematical Analysis and Application* **360** (2009), 380–390.

[9] A. Passian, S. Kouchekian, S. B. Yakubovich, T. Thundat, Properties of Index Transforms in Modeling of Nanostructures and Plasmonic Systems, *To appear in Journal of Mathematical Physics.*

[10] A. Prudnikov, Y. Brychkov, O. Marichev, *Integrals and Series, Volume I*, Gordon and Breach Science Publishers, 1986.

[11] A. Prudnikov, Y. Brychkov, O. Marichev, *Integrals and Series, Volume II*, Gordon and Breach Science Publishers, 1986.

[12] A. Prudnikov, Y. Brychkov, O. Marichev, *Integrals and Series, Volume III*, Gordon and Breach Science Publishers, 1989.

[13] I. N. Sneddon, *The Use of Integral Transforms*, McGraw-Hill, New York, 1972.

[14] S. B. Yakubovich and Yu. F. Luchko, *The Hypergeometric Approach to Integral Transforms and Convolutions*, (Kluwers Ser. Math. and Appl.: Vol. 287), Dordrecht, Boston, London (1994).

[15] S. B. Yakubovich, *Index Transforms*, World Scientific, 1996.

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