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A general method for deriving some semi-classical properties of perturbed second degree forms: The case of the Chebyshev form of second kind

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ABSTRACT

We present a new general method and the corresponding symbolic algorithm *PSDF* for deriving some semi-classical properties of perturbed second degree forms namely: the Stieltjes function, the Stieltjes equation, the functional equation, the class, a structure relation and the second order linear differential equation. We give new explicit results for some perturbed of order 3 of the second kind Chebyshev polynomials.

1. Introduction

The general method and the algorithm *PSDF* presented in this work are based on the algebraic approach of orthogonal polynomials introduced by Pascal Maroni mainly in Refs. [1–6], in particular on an implementation of a set of operations defined in the topological dual space of the vectorial space of polynomials. The basic idea consists in dealing directly with the linear forms, determined by their moments or equivalently by the corresponding Stieltjes series, and their interrelationships, and not with the integral representations of them [5].

Perturbation corresponds to a modification on the first coefficients of the recurrence relation of order two satisfied by orthogonal polynomial sequences. This transformation can promote a deep change of properties; nevertheless there is a large set of forms that are preserved by perturbation: the second degree forms. In other words, the perturbed of a second degree form still is a second degree form. Moreover, a second degree form is also a semi-classical one [5,7]. The general

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method presented herein is based on this crucial fact. We remark that, in general, the perturbed of a semi-classical form is not semi-classical, but a Laguerre–Hahn [5] that satisfies a fourth-order differential equation [8–11]. Laguerre–Hahn forms [12,13] generalize semi-classical and second degree forms. It is worthy to mention here that among the classical forms only certain Jacobi forms are of second degree [14]; from them other second degree forms can be generated by applying several transformations [7,15,14,16]. Furthermore, all self-associated forms are also of second degree [17].

We notice that perturbed orthogonal polynomials have some possible applications [18–20,9,21], which motivate further their study. In fact, during the last years, several authors have worked on this subject considering perturbations of several orders with more or less free parameters with respect to classical, semi-classical, Laguerre–Hahn and others families, studying several properties like generating functions, Stieltjes functions, structure relations and differential equations, separation and the distribution function of zeros and integral representations among others. With respect to the co-recursive case, we would like to cite [22,23,18,24,10], for the co-dilated situation refer to [16,21], for the co-modified [25,26,8], for the generalized co-polynomials see [19,27]. Also, we call the attention to the general Refs. [28,29]. Furthermore, there are some specific works about perturbed Chebyshev families namely [16] on the co-dilated case of the second kind form and [15,30] concerning all the four forms.

It is well known that the four Chebyshev forms [31–33] are the most important cases of second degree forms [15] due to their remarkable properties and utility in applied mathematics, physics and other sciences [34]. In particular, for the purposes of perturbation, the form of second kind is the most simple among them, because it is self-associated, therefore it is often taken as study case in the mentioned literature. So, it seemed important to us to clarify and explicit some semi-classical properties of perturbed Chebyshev polynomials of second kind.

In this work we present a general method, and the corresponding symbolic algorithm, intended to explicit some semi-classical properties of perturbed second degree forms, namely: the Stieltjes function, the Stieltjes equation, the functional equation, the class, a structure relation and the second order linear differential equation. Moreover, we provide the first moments of the perturbed forms. The advantage of this method is its generality: it is intended to work for any perturbation and any second degree form and can be implemented in an algebraic manipulator.

The Chebyshev form of second kind is taken as study example and we give new explicit results for the generalized co-recursive and co-dilated cases of order three. In the same way, other perturbations can be treated and the same procedure can be applied to the other three forms of Chebyshev. This will be the subject of a forthcoming article [35]. Moreover, the characteristic elements presented in this work can be useful in order to obtain other ones like integral representations or make the study of zeros that are crucial in quadrature formulas of numerical integration.

Let us summarize the content of this article. In Section 2, we establish the theoretical framework, we recall the mathematical background necessary to understand the subject of perturbed second degree forms. In particular, we have collected the most important formulas and procedures that compose the general method closely following Refs. [5–7,12–14]. In Section 3, we introduce the method and the algorithm *PSDF—Perturbed Second Degree Forms*. In last section we apply the method step by step to the Chebyshev form of second kind and we give the corresponding new results concerning the above mentioned perturbations. Also, we derive a closed formula for the generating functions of any perturbed Chebyshev family and we compute them in the two treated cases. Notice that often in applications one is interested on numerical concrete values of parameters so that the given formulas will be quite simplified.

2. Theoretical framework

2.1. General definitions and features

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its topological dual space. The effect of the ***form*** or ***functional*** $u \in \mathcal{P}'$ on $f \in \mathcal{P}$ will be denoted by $\langle u, f\rangle$. In particular $(u)_n := \langle u, x^n\rangle$, $n \geq 0$, are called the moments of u. Give u, is equivalent to give the sequence of moments $(u)_n$, $n \geq 0$, or the formal series $F\big(u\big)(z) := \sum_{n\geq 0}(u)_n z^n$, or the so-called ***formal Stieltjes function*** [5]

$$S\big(u\big)(z) := -z^{-1}F\big(u\big)(z^{-1}) = -\sum_{n\geq 0}\frac{(u)_n}{z^{n+1}}.$$

Let $f,\ p \in \mathcal{P},\ u,\ v \in \mathcal{P}'$. By transposition of the operations in $\mathcal{P}$, we have the following operations in $\mathcal{P}'$ [4]. Left-multiplication of a form by a polynomial fu, $\langle fu, p\rangle := \langle u, fp\rangle$. Derivative of a form $u' = Du$, $\langle u', p\rangle := -\langle u, p'\rangle$. Division of a form by a first degree polynomial $(x-c)^{-1}u$, $\langle (x-c)^{-1}u, p\rangle := \langle u, \theta_c p\rangle$, where θ_c is the divided difference operator

$$\big(\theta_c p\big)(x) := \frac{p(x)-p(c)}{x-c},\quad c\in\mathbb{C},\ x\neq c;\qquad (\theta_c p)(c) = p'(c).$$

Cauchy product of two forms uv, $\langle uv, p\rangle := \langle u, vp\rangle$, where $\big(vp\big)(x) := \langle v, \frac{xp(x)-\xi p(\xi)}{x-\xi}\rangle$ is the right-multiplication of a form by a polynomial.

Let us consider a polynomial sequence $\{P_n\}_{n\geq 0}$ such that $\deg P_n = n$, $n \geq 0$, then there exists a unique sequence $\{u_n\}_{n\geq 0}$, $u_n \in \mathcal{P}'$, $n \geq 0$, called the ***dual sequence*** of $\{P_n\}_{n\geq 0}$ such that $\langle u_n, P_m\rangle = \delta_{n,m}$, $n,\ m \geq 0$.

Let $\{P_n^{(1)}\}_{n\geq 0}$ be the ***associated sequence*** of $\{P_n\}_{n\geq 0}$, with respect to u_0, then [23,5,17]

$$P_n^{(1)}(x) := \big(u_0\theta_0 P_{n+1}\big)(x) = \left\langle u_0, \frac{P_{n+1}(x) - P_{n+1}(\xi)}{x-\xi} \right\rangle. \tag{1}$$

More generally, the ***successive associated sequences*** [23,5] are defined by iteration

$$P_n^{(r+1)} = \big(P_n^{(r)}\big)^{(1)}, \qquad u_n^{(r+1)} = (u_n^{(r)})^{(1)}, \quad n,\ r \geq 0. \tag{2}$$

A polynomial sequence $\{P_n\}_{n\geq 0}$ is ***orthogonal*** with respect to $u \in \mathcal{P}'$ if and only if

$$\langle u, P_n P_m \rangle = k_n \delta_{n,m}, \quad n, m \geq 0;\ k_n \neq 0,\ n \geq 0. \tag{3}$$

Consequently $\deg P_n = n$, $n \geq 0$, and any P_n can be taken monic, i.e., with unit leading coefficient ($P_n(x) = x^n + \cdots$), then the sequence $\{P_n\}_{n\geq 0}$ is called a ***monic orthogonal polynomial sequence*** (MOPS). Necessarily $u = (u)_0 u_0$, $(u)_0 \neq 0$; the form u is normalized if $(u)_0 = 1$, in this case $u = u_0$. In this work, we will always consider monic polynomial sequences and ***normalized forms***. A form u is ***regular*** [5] if it is possible to associate with it an orthogonal sequence fulfilling (3). A sequence $\{P_n\}_{n\geq 0}$ is orthogonal with respect to a normalized form u (thus $u = u_0$) if and only if there are two sequences of coefficients $\{\beta_n\}_{n\geqslant 0}$ and $\{\gamma_{n+1}\}_{n\geqslant 0}$, with $\gamma_{n+1} \neq 0$, $n \geqslant 0$ such that $\{P_n\}_{n\geq 0}$ verifies the following initial conditions and ***recurrence relation of order two*** [23]

$$\begin{cases} P_0(x) = 1, \qquad P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad \gamma_{n+1} \neq 0,\ n \geq 0. \end{cases} \tag{4}$$

Moreover,

$$\beta_n = \frac{\langle u, xP_n^2(x)\rangle}{k_n}, \qquad \gamma_{n+1} = \frac{k_{n+1}}{k_n}, \quad n \geq 0.$$

If $\{P_n\}_{n\geq 0}$ is orthogonal, then $\{P_n^{(r)}\}_{n\geq 0}$ is also orthogonal and fulfils [5]

$$\begin{cases} P_0^{(r)}(x) = 1, \qquad P_1^{(r)}(x) = x - \beta_0^{(r)}, \\ P_{n+2}^{(r)}(x) = (x - \beta_{n+1}^{(r)})P_{n+1}^{(r)}(x) - \gamma_{n+1}^{(r)}P_n^{(r)}(x), \quad n, r \geq 0, \end{cases} \tag{5}$$

where

$$\beta_n^{(r)} = \beta_{n+r}, \qquad \gamma_{n+1}^{(r)} = \gamma_{n+1+r}, \quad n,\ r \geq 0. \tag{6}$$

The ***co-recursive*** [22,23] of a MOPS $\{P_n\}_{n\geq 0}$ is a sequence $\{P_n(\mu;\cdot)\}_{n\geq 0}$ such that

$$\begin{cases} P_0(\mu; x) = 1, \qquad P_1(\mu; x) = x - \beta_0 - \mu, \\ P_{n+2}(\mu; x) = (x - \beta_{n+1})P_{n+1}(\mu; x) - \gamma_{n+1}P_n(\mu; x), \quad n \geq 0, \end{cases} \tag{7}$$

where $\mu \in \mathbb{C} - \{0\}$. The sequence $\{P_n(\mu;\cdot)\}_{n\geq 0}$ is orthogonal with respect to the form $u(\mu)$ given by [5]

$$S\big(u(\mu)\big)(z) = \frac{S\big(u\big)(z)}{1 + \mu S\big(u\big)(z)}. \tag{8}$$

More generally, let $\{\widetilde{P}_n\}_{n\geq 0}$ be the r-***perturbed*** sequence of a MOPS $\{P_n\}_{n\geq 0}$ defined by [5]

$$\begin{cases} \widetilde{P}_0(x) = 1, \qquad \widetilde{P}_1(x) = x - \tilde{\beta}_0, \\ \widetilde{P}_{n+2}(x) = (x - \tilde{\beta}_{n+1})\widetilde{P}_{n+1}(x) - \tilde{\gamma}_{n+1}\widetilde{P}_n(x), \quad n \geq 0, \end{cases} \tag{9}$$

where

$$\tilde{\beta}_0 = \beta_0 + \mu_0, \tag{10}$$

$$\tilde{\beta}_n = \beta_n + \mu_n, \qquad \mu_n \in \mathbb{C}; \qquad \tilde{\gamma}_n = \lambda_n\gamma_n, \quad \lambda_n \in \mathbb{C} - \{0\},\ 1 \leq n \leq r, \tag{11}$$

$$\tilde{\beta}_n = \beta_n; \qquad \tilde{\gamma}_n = \gamma_n, \quad n \geq r + 1. \tag{12}$$

Either $\mu_r \neq 0$ or $\lambda_r \neq 1$. Thus, the co-recursive case corresponds to the perturbed case of order 0. With the notation $\mu := (\mu_1, \ldots, \mu_r)$, $\lambda := (\lambda_1, \ldots, \lambda_r)$, $r \geq 1$, we put

$$\widetilde{P}_n(x) = P_n\left(\mu_0; \begin{matrix}\mu \\ \lambda\end{matrix}; r; x\right), \quad n \geq 0,$$

and we say that the sequence $\{\widetilde{P}_n\}_{n\geq 0}$ is orthogonal with respect to the ***perturbed form*** $\tilde{u} := u\left(\mu_0;\ {}^{\mu}_{\lambda}\ ; r\right)$ given by [5]

$$S\big(\tilde{u}\big)(z) = -\frac{U_r(z) + V_r(z)S\big(u\big)(z)}{X_r(z) + Y_r(z)S\big(u\big)(z)}, \tag{13}$$

with the ***transfer polynomials***

$$U_r(z) = \gamma_r\left\{\widetilde{P}^{(1)}_{r-1}(z)P^{(1)}_{r-2}(z) - \lambda_r P^{(1)}_{r-1}(z)\widetilde{P}^{(1)}_{r-2}(z)\right\} - \mu_r P^{(1)}_{r-1}(z)\widetilde{P}^{(1)}_{r-1}(z), \tag{14}$$

$$V_r(z) = \gamma_r\left\{\widetilde{P}^{(1)}_{r-1}(z)P_{r-1}(z) - \lambda_r P_r(z)\widetilde{P}^{(1)}_{r-2}(z)\right\} - \mu_r P_r(z)\widetilde{P}^{(1)}_{r-1}(z), \tag{15}$$

$$X_r(z) = \gamma_r\left\{\widetilde{P}_r(z)P^{(1)}_{r-2}(z) - \lambda_r P^{(1)}_{r-1}(z)\widetilde{P}_{r-1}(z)\right\} - \mu_r P^{(1)}_{r-1}(z)\widetilde{P}_r(z), \tag{16}$$

$$Y_r(z) = \gamma_r\left\{\widetilde{P}_r(z)P_{r-1}(z) - \lambda_r P_r(z)\widetilde{P}_{r-1}(z)\right\} - \mu_r P_r(z)\widetilde{P}_r(z), \tag{17}$$

where $\widetilde{P}^{(1)}_n$ are the associated of the perturbed polynomials $\widetilde{P}_n$. As usual, we suppose that $P_n(x) = 0$, $P^{(r)}_n(x) = 0$ and $\widetilde{P}_n(x) = 0$ for $n < 0$.

We can consider two elementary particular cases of a perturbation of order r for $r > 0$, when there is only one parameter of perturbation. That is, when $\mu_k = 0$, $k = 0(1)r - 1$, $\mu_r \neq 0$, $\lambda_k = 1$, $k = 1(1)r$ and when $\mu_k = 0$, $k = 0(1)r$, $\lambda_k = 1$, $k = 1(1)r-1$, $\lambda_r \neq 1$. In the first case, we do a ***perturbation by translation*** of the recurrence coefficient β_r and, in the second case, we do a ***perturbation by dilatation*** of the recurrence coefficient γ_r. In both situations, the operator corresponding to the recurrence relation of order 2 is perturbed, whereas for $r = 0$ the operator does not change, but the initial condition $P_1(x) = x - \beta_0$ becomes $P_1(x) = x - (\beta_0 + \mu_0)$ leading to a different solution of the same recurrence relation. In literature, these cases are often designated as ***r-generalized co-recursive*** and ***r-generalized co-dilated*** cases, respectively. Explicit results given in this work in Section 4 concern these kind of perturbations.

2.2. Shohat–Favard method

From the Shohat–Favard method [1], one can establish a general procedure [36][1] to obtain the first moments $(u)_{n=0,\ldots,nmax}$ of a normalized regular form u from the first recurrence coefficients $\{\beta_n\}_{n=0,\ldots,nmax-1}$, $\{\gamma_{n+1}\}_{n=0,\ldots,nmax-2}$ of the corresponding MOPS $\{P_n(x)\}_{n\geq 0}$. Let us write P_n in the canonical basis

$$P_n(x) = \sum_{\nu=0}^{n} p_{n,\nu}x^{\nu}, \quad n \geq 0. \tag{18}$$

Obviously that

$$p_{n,\nu} = 0,\ \nu > n,\ \nu < 0,\ n < 0; \qquad p_{n,n} = 1, \quad n \geq 0. \tag{19}$$

Replacing in the recurrence relation (4) the polynomials by their expressions given by (18), we have

$$p_{1,0} = -\beta_0, \tag{20}$$

$$p_{n+2,\nu} = p_{n+1,\nu-1} - \beta_{n+1}p_{n+1,\nu} - \gamma_{n+1}p_{n,\nu}, \quad 0 \leq \nu \leq n, \tag{21}$$

$$p_{n+2,n+1} = p_{n+1,n} - \beta_{n+1}, \quad n \geq 0. \tag{22}$$

Then, from the orthogonality condition $\langle u, B_n\rangle = 0$, $n \geq 1$, we obtain the moments

$$(u)_0 = 1, \qquad (u)_n = -\sum_{\nu=0}^{n-1} p_{n,\nu}(u)_{\nu}, \quad n \geq 1. \tag{23}$$

2.3. Laguerre–Hahn forms

A normalized ***regular*** form u is a ***Laguerre–Hahn form*** [12,13] if its formal Stieltjes function $S\big(u\big)(z)$ satisfies the *Riccati equation*

$$A(z)S'\big(u\big)(z) = B(z)S^2\big(u\big)(z) + C(z)S\big(u\big)(z) + D(z), \tag{24}$$

[1] Ref. [36] is accompanied by a symbolic implementation written in *Mathematica*® available in *netlib*.

where A, B, C and D are polynomials. The sequence $\{P_n\}_{n\geq 0}$ orthogonal with respect to u is also called a ***Laguerre–Hahn sequence***. With $A = \Phi$, Eq. (24) is equivalent to [12,13]

$$(\Phi u)' + \psi u + B(x^{-1}u^2) = 0,$$

with

$$C(z) = -\Phi'(z) - \psi(z),$$
$$D(z) = -\big(u\theta_0\Phi\big)'(z) - \big(u\theta_0\psi\big)(z) - \big(u^2\theta_0^2 B\big)(z).$$

Perturbed and associated of Laguerre–Hahn forms still are Laguerre–Hahn forms [5], because the corresponding formal Stieltjes functions satisfy the Riccati equations (33) and (43) (see next section). The same is true for the structure relation (49), which is also valid for Laguerre–Hahn sequences [5].

2.4. Semi-classical forms

Let (Φ, ψ) be two polynomials, Φ monic, $\deg\psi \geq 1$. The ***pair*** (Φ, ψ) is called **admissible** if the ***functional equation*** that it generates,

$$(\Phi u)' + \psi u = 0, \tag{25}$$

possesses at least one normalized regular solution u [5]. Let $t = \deg\Phi$ and $p = \deg\psi$. With an admissible pair (Φ, ψ) it is possible to associate an integer

$$s := \max(p-1, t-2). \tag{26}$$

A normalized solution u fulfilling (25) also satisfies equations $\chi(\Phi u)' + \chi\psi u = 0$, for any polynomial χ, or $(\chi\Phi u)' + (\chi\psi - \chi'\Phi)u = 0$. Then $s_1 = \max(p_1 - 1, t_1 - 2) = s + \deg\chi$, with $t_1 = \deg(\chi\Phi)$, $p_1 = \deg(\chi\psi - \chi'\Phi)$. Thus, let us consider the function $u \longrightarrow s(u) \subseteq \mathbb{N}$. The minimum element of $s(u)$ will be called the ***class*** of the form u [5].

A regular form u is ***semi-classical*** (SC) [5,6] if it satisfies (25), where the pair (Φ, ψ) is admissible. The sequence $\{P_n\}_{n\geq 0}$ orthogonal with respect to u is also called a ***semi-classical sequence***, it is said to be of class s, if u is of class s.

For any semi-classical form u the pair $(\widehat{\Phi}, \hat{\psi})$ which realizes the minimum of $s(u)$ is unique. A ***classical sequence*** (Hermite, Laguerre, Bessel and Jacobi) [6,37] appears as a semi-classical sequence of class zero.

Given a semi-classical form u it is necessary to know whether the integer s associated with (Φ, ψ) is the minimum of $s(u)$. A normalized semi-classical form u satisfying (25) is of class $s = \max(\deg\psi - 1, \deg\Phi - 2)$ if and only if [6]

$$\prod_c \Big(|\psi(c) + \Phi'(c)| + |\langle u, \theta_c\psi + \theta_c^2\Phi\rangle|\Big) \neq 0, \tag{27}$$

where c goes over the set of zeros of Φ. When it is possible to simplify (25) by the factor $x - c$, we obtain the new equation $\big((\theta_c\Phi)u\big)' + \big(\theta_c\psi + \theta_c^2\Phi\big)u = 0$, then u is of class less than or equal to $s - 1$.

For any regular normalized form u the following two assertions are equivalent [5,6].

1. There are two polynomials Φ and ψ, Φ monic, such that

$$(\Phi u)' + \psi u = 0. \tag{28}$$

2. There are two polynomials A and C, A monic, such that

$$A(z)S'\big(u\big)(z) = C(z)S\big(u\big)(z) + D(z), \tag{29}$$

where

$$D(z) = -\big(u\theta_0\Phi\big)'(z) - \big(u\theta_0\psi\big)(z).$$

If s is given by (26), then $\deg C \leq s + 1$ and $\deg D \leq s$. The form u is of class

$$s = \max(\deg C - 1, \deg D) \tag{30}$$

if and only if A, C and D have no common factor [5].

The link between (28) and (29) is [5]

$$A(z) = \Phi(z), \qquad C(z) = -\Phi'(z) - \psi(z). \tag{31}$$

From (29), it is obvious that semi-classical sequences are the particular case of Laguerre–Hahn sequences corresponding to $B = 0$ in Eq. (24).

2.4.1. Perturbed sequences of semi-classical sequences

If u is a normalized semi-classical form fulfilling (28), or equivalently, if $S(u)$ fulfils (29) written as

$$A_0(z)S'(u)(z) = B_0(z)S^2(u)(z) + C_0(z)S(u)(z) + D_0(z), \tag{32}$$

with $A_0(z) = \Phi(z)$ and $B_0(z) = 0$, then the formal Stieltjes function $S(\tilde{u})$ of $\tilde{u} = u\left(\mu_0; \frac{\mu}{\lambda}; r\right)$, satisfies

$$\widetilde{A}(z)S'(\tilde{u})(z) = \widetilde{B}(z)S^2(\tilde{u})(z) + \widetilde{C}(z)S(\tilde{u})(z) + \widetilde{D}(z). \tag{33}$$

If $r = 0$ (*co-recursive* sequences), from (8), we obtain

$$\widetilde{A}(z) = A_0(z), \qquad \widetilde{B}(z) = B_0(z) - \mu C_0(z) + \mu^2 D_0(z), \tag{34}$$

$$\widetilde{C}(z) = C_0(z) - 2\mu D_0(z), \qquad \widetilde{D}(z) = D_0(z). \tag{35}$$

If $r \geq 1$, from (13), we obtain [5]

$$\widetilde{A}(z) = A_0(z)\Big(U_r(z)Y_r(z) - V_r(z)X_r(z)\Big), \tag{36}$$

$$\widetilde{B}(z) = B_0(z)X_r^2(z) - C_0(z)X_r(z)Y_r(z) + D_0(z)Y_r^2(z) + A_0(z)\Big(X_r'(z)Y_r(z) - X_r(z)Y_r'(z)\Big), \tag{37}$$

$$\begin{aligned}\widetilde{C}(z) \;=\; & 2\Big(B_0(z)U_r(z)X_r(z) + D_0(z)V_r(z)Y_r(z)\Big) - C_0(z)\Big(U_r(z)Y_r(z) + V_r(z)X_r(z)\Big) \\ & + A_0(z)\Big(U_r'(z)Y_r(z) - U_r(z)Y_r'(z) + V_r(z)X_r'(z) - V_r'(z)X_r(z)\Big),\end{aligned} \tag{38}$$

$$\widetilde{D}(z) = B_0(z)U_r^2(z) - C_0(z)U_r(z)V_r(z) + D_0(z)V_r^2(z) + A_0(z)\Big(U_r'(z)V_r(z) - U_r(z)V_r'(z)\Big) \tag{39}$$

where $U_r(z), V_r(z), X_r(z)$ and $Y_r(z)$ are the *transfer polynomials* given by (14)–(17). Thus, in general, the perturbed of a semi-classical form is not semi-classical, but is a Laguerre–Hahn form.

2.4.2. Associated sequences of semi-classical sequences and a structure relation

If u is a normalized semi-classical form fulfilling (28), then the formal Stieltjes function $S(u^{(1)})(z)$ of its associated form $u^{(1)}$ satisfies [5]

$$A_1(z)S'(u^{(1)})(z) = B_1(z)S^2(u^{(1)})(z) + C_1(z)S(u^{(1)})(z) + D_1(z), \tag{40}$$

where

$$A_1(z) = \Phi(z), \qquad B_1(z) = \gamma_1 D(z), \qquad C_1(z) = -C(z) + 2(z-\beta_0)D(z), \tag{41}$$

$$D_1(z) = \gamma_1^{-1}\big\{-\Phi(z) - (z-\beta_0)C(z) + (z-\beta_0)^2 D(z)\big\}. \tag{42}$$

For the associated form of order r, we get [5]

$$A_r(z)S'(u^{(r)})(z) = B_r(z)S^2(u^{(r)})(z) + C_r(z)S(u^{(r)})(z) + D_r(z), \quad r \geq 0, \tag{43}$$

where

$$A_0(z) = \Phi(z), \qquad B_0(z) = 0, \qquad C_0(z) = C(z), \qquad D_0(z) = D(z), \tag{44}$$

$$A_{r+1}(z) = \Phi(z), \quad r > 0, \tag{45}$$

$$B_{r+1}(z) = \gamma_{r+1}D_r(z), \quad r > 0, \tag{46}$$

$$C_{r+1}(z) = -C_r(z) + 2(z-\beta_r)D_r(z), \quad r > 0, \tag{47}$$

$$\gamma_{r+1}D_{r+1}(z) = -\Phi(z) + B_r(z) - (z-\beta_r)C_r(z) + (z-\beta_r)^2 D_r(z), \quad r > 0. \tag{48}$$

We have $\deg C_r \leq s+1, \deg D_r \leq s, r \geq 0$, where s is the class of u. Thus, in general the associated of a semi-classical form is a Laguerre–Hahn form.

Any semi-classical sequence $\{P_n\}_{n\geq 0}$ satisfies the following ***structure relation*** [5]

$$\Phi(x)P_{n+1}'(x) - B_0(x)P_n^{(1)}(x) = \frac{1}{2}\Big(C_{n+1}(x) - C_0(x)\Big)P_{n+1}(x) - \gamma_{n+1}D_{n+1}(x)P_n(x), \quad n \geq 0, \tag{49}$$

with $B_0(x) = 0$.

2.4.3. The second order linear differential equation

Any polynomial P_{n+1} from a semi-classical sequence $\{P_n\}_{n\geq 0}$ satisfies the following ***second-order linear differential equation*** [5]

$$J(x;n)P''_{n+1}(x) + K(x;n)P'_{n+1}(x) + L(x;n)P_{n+1}(x) = 0, \quad n \geq 0, \tag{50}$$

with $\deg J(\cdot;n) \leq 2s+2$, $\deg K(\cdot;n) \leq 2s+1$, $\deg L(\cdot;n) \leq 2s$, $n \geq 0$, and

$$J(x;n) = \Phi(x)D_{n+1}(x), \quad n \geq 0, \tag{51}$$

$$K(x;n) = C_0(x)D_{n+1}(x) - W\big(\Phi, D_{n+1}(x)\big)(x), \quad n \geq 0, \tag{52}$$

$$L(x;n) = W\left(\frac{1}{2}(C_{n+1} - C_0), D_{n+1}\right)(x) - D_{n+1}(x)\sum_{\nu=0}^{n} D_\nu(x), \quad n \geq 0, \tag{53}$$

where $W(f,g)$ denotes the wronskian of f and g. Reciprocally, if any polynomial P_{n+1} from an orthogonal sequence fulfils (50)–(53), then the sequence $\{P_n\}_{n\geq 0}$ is semi-classical [5].

2.5. Second degree forms

A normalized regular form u is a ***second degree form*** (SD) [5] if there are polynomials B and C, B monic, such that, $S\big(u\big)(z)$ satisfy the *quadratic Stieltjes equation*

$$B(z)S^2\big(u\big)(z) + C(z)S\big(u\big)(z) + D(z) = 0, \tag{54}$$

where

$$D(z) = \big(u\theta_0 C\big)(z) - \big(u^2\theta_0^2 B\big)(z).$$

The regularity of u means that we must have $B \neq 0$, $C^2 - 4BD \neq 0$ and $D \neq 0$ [5]. The sequence $\{P_n\}_{n\geq 0}$ orthogonal with respect to u is also called a ***second degree sequence***.

From (54), it is obvious that second degree forms are the particular case of Laguerre–Hahn forms corresponding to take $A = 0$ in Eq. (24).

2.5.1. Perturbed sequences of second degree sequences

If u is a second degree form, then the perturbed form of u, $\tilde{u} = u(\mu_0; {}^{\mu}_{\lambda}; r)$, $r \geq 0$, is also of second degree and satisfies

$$\widetilde{B}(z)S^2\big(\tilde{u}\big)(z) + \widetilde{C}(z)S\big(\tilde{u}\big)(z) + \widetilde{D}(z) = 0, \tag{55}$$

with

$$k_r\widetilde{B}(z) = B(z)X_r^2(z) - C(z)X_r(z)Y_r(z) + D(z)Y_r^2(z), \tag{56}$$

$$k_r\widetilde{C}(z) = 2\big\{B(z)U_r(z)X_r(z) + D(z)V_r(z)Y_r(z)\big\} - C(z)\big\{U_r(z)Y_r(z) + V_r(z)X_r(z)\big\}, \tag{57}$$

$$k_r\widetilde{D}(z) = B(z)U_r^2(z) - C(z)U_r(z)V_r(z) + D(z)V_r^2(z), \tag{58}$$

where k_r is a normalization constant chosen in order to make $\widetilde{B}$ monic and $U_r(z)$, $V_r(z)$, $X_r(z)$ and $Y_r(z)$ are the *transfer polynomials*. These identities can be obtained from (36)–(39) taking $A(z) = A_0(z) = 0$.

2.5.2. Second degree forms as semi-classical forms

If u is a second degree form, then u is a semi-classical form and satisfies Eq. (28) with [5,7]

$$k\Phi(x) := B(x)\big\{C^2(x) - 4B(x)D(x)\big\}, \tag{59}$$

$$k\psi(x) := -\frac{3}{2}B(x)\big\{C^2(x) - 4B(x)D(x)\big\}', \tag{60}$$

where k is a normalization constant chosen in order to make $\Phi(x)$ monic [7]. Equivalently, if u is a second degree form, then $S\big(u\big)(z)$ fulfils the *affine Stieltjes equation* [7]

$$\widehat{A}(z)S'\big(u\big)(z) = \widehat{C}(z)S\big(u\big)(z) + \widehat{D}(z), \tag{61}$$

where

$$\widehat{A}(z) = B(z)\big\{C^2(z) - 4B(z)D(z)\big\}, \tag{62}$$

$$\widehat{C}(z) = 2B(z)\big\{B'(z)D(z) - D'(z)B(z)\big\} + C(z)\big\{C'(z)B(z) - B'(z)C(z)\big\}, \tag{63}$$

$$\widehat{D}(z) = B(z)\big\{C'(z)D(z) - D'(z)C(z)\big\} + D(z)\big\{C'(z)B(z) - B'(z)C(z)\big\}. \tag{64}$$

Among the classical forms, only the Jacobi forms $\mathcal{J}(k - \frac{1}{2}, l - \frac{1}{2})$, for $k + l \geq 0$, $k \in \mathbb{Z}$, $l \in \mathbb{Z}$, are second degree forms [14, Theorem p.445].

3. A new general method for expliciting some semi-classical properties of perturbed second degree forms

In this section, we begin by giving a description of the method followed by the corresponding algorithm intended to be implemented in an algebraic manipulator.

3.1. Description of the general method

The method consists in four main steps. In the first one, we define the starting data of the original second degree form needed in the sequel of computations, they are the recurrence coefficients, the closed formula of the Stieltjes function and the coefficients of the quadratic Stieltjes equation. In step two we do the perturbation and we compute the closed formula of the Stieltjes function and the coefficients of the Stieltjes equation of the perturbed form as second degree form. In step three we determine some elements of the perturbed form as semi-classical form, namely the coefficients of the functional equation and of the Stieltjes equation and the class. In the last step, we begin by finding and demonstrating by induction the closed formulas of the coefficients of the structure relation and from them we obtain directly the closed formulas of the coefficients of the differential equation.

Let us detail better each step of the algorithm.

The starting point of computations is constituted by the recurrence coefficients β_n, γ_{n+1} of the recurrence relation (4), the closed formula of the Stieltjes function $S(u)(z)$ and the characteristic elements $B^{SD}(z)$, $C^{SD}(z)$ and $D^{SD}(z)$ of the Eq. (54) of the original second degree form u. Then we consider the perturbed form $\tilde{u} = u(\mu_0; {}^{\mu}_{\lambda}; r)$ and the corresponding modified recurrence coefficients $\widetilde{\beta}_n$ and $\widetilde{\gamma}_{n+1}$ given by (10)–(12) from which we compute the first perturbed polynomials $\{\widetilde{P}_n\}_{n=0,\dots,nmax}$ using (9) and from their canonical coefficients, we apply the Shohat–Favard algorithm to obtain some first moments $(\tilde{u})_{n=0,\dots,nmax}$ by (19)–(23). Next we want to calculate the coefficients of the Stieltjes equation (55) satisfied by the perturbed form $\tilde{u}$ as second degree form, i.e. $\widetilde{B}^{SD}(z)$, $\widetilde{C}^{SD}(z)$ and $\widetilde{D}^{SD}(z)$ by means of the equalities (56)–(58) in which intervenes the transfer polynomials $U_r(z)$, $V_r(z)$, $X_r(z)$ and $Y_r(z)$ stated by (14)–(17). We need to compute some first polynomials of the sequences $\{P_n\}_{n\geq 0}$, $\{\widetilde{P}_n\}_{n\geq 0}$ and $\{\widetilde{P}_n^{(1)}\}_{n\geq 0}$ to get the transfer polynomials, using the relations (4), (5), (7) and (9). The closed formula of the Stieltjes function $S(\tilde{u})(z)$ is obtained from the transfer polynomials by means of (8) or (13).

In the sequel, we pretend to obtain some semi-classical properties of $\tilde{u}$.

We begin by determining the coefficients $\widetilde{\Phi}^{SC}(x)$ and $\widetilde{\psi}^{SC}(x)$ of the functional equation (25) applying the identities (59) and (60) and also the coefficients $\widetilde{A}^{SC}(z)$, $\widetilde{C}^{SC}(z)$ and $\widetilde{D}^{SC}(z)$ of the Stieltjes equation (61) using (62)–(64). These two steps are accomplished from $\widetilde{B}^{SD}(z)$, $\widetilde{C}^{SD}(z)$ and $\widetilde{D}^{SD}(z)$. At this moment, we are able to determine the class of the semi-classical form $\tilde{u}$ from $\widetilde{\Phi}^{SC}(x)$ and $\widetilde{\psi}^{SC}(x)$ or from $\widetilde{A}^{SC}(z)$, $\widetilde{C}^{SC}(z)$ and $\widetilde{D}^{SC}(z)$. In both cases it is necessary to assure that we get the minimum value of s. In the case we use (26) from the coefficients of the functional equation, we should verify the criterion (27), and in the case we employ (30) from the coefficients of the Stieltjes equation, we should cancel any existent common factor between $\widetilde{A}^{SC}(z)$, $\widetilde{C}^{SC}(z)$ and $\widetilde{D}^{SC}(z)$.

Our final goal is to obtain closed formulas for the polynomial coefficients $\widetilde{J}^{SC}(x; n)$, $\widetilde{K}^{SC}(x; n)$ and $\widetilde{L}^{SC}(x; n)$, $\forall n \geq 0$, of the second order linear differential equation (50) concerning the form $\tilde{u}$. These closed formulas can be obtained directly from the closed formulas of the coefficients $\widetilde{\Phi}^{SC}(z)$, $\widetilde{C}^{SC}_{n+1}(z)$ and $\widetilde{D}^{SC}_{n+1}(z)$, $\forall n \geq 0$ of the structure relation (49) for the form $\tilde{u}$ by means of the equalities (51)–(53) assuming that it is possible to find a closed formula, valid for any $n \geq 0$, for the finite summation that appears in (53). In fact, the major difficult of this work lies in the closed formulas of $\widetilde{C}^{SC}_{n+1}(z)$ and $\widetilde{D}^{SC}_{n+1}(z)$, $\forall n \geq 0$. The coefficients $\widetilde{B}^{SC}_{n+1}(z)$, $\widetilde{C}^{SC}_{n+1}(z)$ and $\widetilde{D}^{SC}_{n+1}(z)$ for $n = 0, 1, \dots$ until a maximal order $nmax$ can always be computed recursively from the relations (41)–(42) or (44)–(48), starting from $\widetilde{\Phi}^{SC}(z)$, $\widetilde{C}^{SC}(z)$ and $\widetilde{D}^{SC}(z)$. The recurrence coefficients of the perturbed form $\tilde{u}$ given by (10)–(12) are also required at this step. In general it is not possible to solve analytically the recurrence relations (41)–(42) or (44)–(48) in order to obtain closed formulas for the coefficients of the structure relation valid $\forall n \geq 0$. Otherwise, in the case we have a model for these closed formulas, we can use these same relations to do a demonstration by induction. Considering that the coefficients of $\widetilde{B}^{SC}_{n+1}(z)$, $\widetilde{C}^{SC}_{n+1}(z)$ and $\widetilde{D}^{SC}_{n+1}(z)$ in the canonical basis $\langle 1, z, z^2, \dots, z^k, \dots\rangle$ are polynomials in n of limited and fixed degree, often of low degree, it is possible to find those closed formulas from the first few elements for $n = 0, 1, \dots, nmax$ by an interpolation procedure.

Finally, we notice that it is important to factorize $J_n(x; n)$, $K_n(x; n)$ and $L_n(x; n)$, because often there are common factors between them that can be simplified in the differential equation. All these tasks can be accomplish by the implementation of the next algorithm in an automatic manipulator.

3.2. PSDF – perturbed second degree forms – algorithm

- **Step 1**

  For the **original second degree** form u give the **starting data**:

  1.1 Recurrence coefficients: β_n, γ_{n+1}, $n \geq 0$.

  1.2 Closed formula for the Stieltjes function: $S(u)(z)$.

  1.3 For the form u as **second degree** form give:

Coefficients of the Stieltjes equation (54): $B^{SD}(z), C^{SD}(z), D^{SD}(z)$.
1.4 Perturbation: $(\mu_0;\ {}^{\mu}_{\lambda};\ r), \mu = (\mu_1, \ldots, \mu_r), \lambda = (\lambda_1, \ldots, \lambda_r), r \geq 1$.

- **Step 2**
  For the **perturbed** form $\tilde{u} = u(\mu_0;\ {}^{\mu}_{\lambda};\ r)$ **compute**:
  2.1 Recurrence coefficients: $\widetilde{\beta}_n$ and $\widetilde{\gamma}_{n+1}$ given by (10)–(12).
  2.2 First MOPS $P_n, \widetilde{P}_n, P_n^{(1)}, \widetilde{P}_n^{(1)}$ using (4), (5), (7) and (9).
  2.3 First moments: $(\tilde{u})_n,\ n = 0, \ldots, nmax$ from (19)–(23).
  2.4 Transfer polynomials: $U_r(z), V_r(z), X_r(z), Y_r(z)$ stated by (14)–(17).
  2.5 Closed formula for the Stieltjes function $S(\tilde{u})(z)$ given by (8) or (13).
  2.6 For the **perturbed** form $\tilde{u}$ as **second degree** form **compute**:
  Coefficients of the Stieltjes equation (55): $\widetilde{B}^{SD}(z), \widetilde{C}^{SD}(z), \widetilde{D}^{SD}(z)$ given by (56)–(58).
- **Step 3**
  For the **perturbed** form $\tilde{u}$ as **semi-classical** form, from $\widetilde{B}^{SD}(z), \widetilde{C}^{SD}(z), \widetilde{D}^{SD}(z)$, **compute**:
  3.1 Coefficients of the functional equation (25): $\widetilde{\Phi}^{SC}(x), \widetilde{\psi}^{SC}(x)$ given by (59) and (60).
  3.2 Coefficients of the Stieltjes equation (61): $\widetilde{A}^{SC}(z), \widetilde{C}^{SC}(z), \widetilde{D}^{SC}(z)$ using (62)–(64).
  3.3 **Compute** the class s: from the polynomials of step 3.1 by (26) and (27), or from the polynomials of step 3.2 by (30).
- **Step 4**
  For the **perturbed** form $\tilde{u}$ as **semi-classical** form:
  4.1 From $\widetilde{\Phi}^{SC}(z), \widetilde{C}^{SC}(z), \widetilde{D}^{SC}(z)$ **find by an interpolation procedure** and **prove automatically by induction** closed formulas for the coefficients $\widetilde{B}^{SC}_{n+1}(z), \widetilde{C}^{SC}_{n+1}(z), \widetilde{D}^{SC}_{n+1}(z), n \geq 0$, given by (41)–(42) or (44)–(48) of the Stieltjes equations (40) or (43) of $\tilde{u}^{(r)}$ and of the **structure relation** (49) of $\tilde{u}$.[2]
  4.2 From closed formulas of $\widetilde{\Phi}^{SC}(z), \widetilde{C}^{SC}_{n+1}(z), \widetilde{D}^{SC}_{n+1}(z), n \geq 0$ **compute** closed formulas for the coefficients $\widetilde{J}^{SC}(x; n), \widetilde{K}^{SC}(x; n), \widetilde{L}^{SC}(x; n), n \geq 0$, given by (51)–(53) of the **second order linear differential equation** (50) of $\tilde{u}$.

4. New results for some perturbed of the Chebyshev form of second kind

In this section, we present new explicit results obtained applying the algorithm *PSDF* step by step in the cases of the following two perturbations of order three of the second kind Chebyshev form $\mathcal{U}$

$$\mathcal{U}\left(0;\ \begin{matrix}0, 0, \mu_3\\ 1, 1, 1\end{matrix};\ 3\right), \quad \mu_3 \neq 0; \qquad \mathcal{U}\left(0;\ \begin{matrix}0, 0, 0\\ 1, 1, \lambda_3\end{matrix};\ 3\right), \quad \lambda_3 \neq 0,\ \lambda_3 \neq 1.$$

The computations of *PSDF* start from the recurrence coefficients, the Stieltjes function and the coefficients of the Stieltjes equation of $\mathcal{U}$ as second degree form [7,15] given respectively by

$$\beta_n = 0, \qquad \gamma_{n+1} = \frac{1}{4}, \quad n \geq 0, \tag{65}$$
$$S(\mathcal{U})(z) = -\frac{2}{z + \sqrt{z^2 - 1}},$$
$$A^{SD}(x) = 0, \qquad B^{SD}(x) = 1, \qquad C^{SD}(x) = 4x, \qquad D^{SD}(x) = 4.$$

For the sake of completeness, we present next some other characteristic elements of $\mathcal{U}$ and $\{P_n(x)\}_{n\geq 0}$ [37,7,15]. Also, recall that $\mathcal{U}$ is a Jacobi classical form with parameters $\alpha = \beta = \frac{1}{2}$.

$$f(x,t) = \sum_{n\geq 0} P_n(x)t^n = \frac{1}{1 + t\left(\frac{1}{4}t - x\right)};$$
$$(\mathcal{U})_n = \frac{2}{\Gamma\left(\frac{3}{2}\right)} \sum_{\nu=0}^{n} \binom{n}{\nu} (-1)^{n-\nu} 2^{\nu} \frac{\Gamma\left(\nu + \frac{3}{2}\right)}{\Gamma(\nu + 3)}, \quad n \geq 0;$$
$$\{(\mathcal{U})_n\}_{n=0,\ldots,10} = \left\{1, 0, \frac{1}{4}, 0, \frac{1}{8}, 0, \frac{5}{64}, 0, \frac{7}{128}, 0, \frac{21}{512}, 0, \frac{33}{1024}, 0, \frac{429}{16384}, 0\right\};$$
$$\Phi^{SC}(x) = x^2 - 1, \qquad \psi^{SC}(x) = -3x;$$
$$A^{SC}(x) = x^2 - 1, \qquad B^{SC}(x) = 0, \qquad C^{SC}(x) = x, \qquad D^{SC}(x) = 2;$$
$$A_n^{SC}(x) = x^2 - 1; \qquad B_0^{SC}(x) = 0, \qquad B_n^{SC}(x) = \frac{n}{2}, \quad n \geq 1,$$

[2] In fact, only the coefficients $\widetilde{C}^{SC}_{n+1}(z), \widetilde{D}^{SC}_{n+1}(z), n \geq 0$ are necessary for the structure relation and for the differential equation as we can see from identities (49) and (51)–(53).

$$C_n^{SC}(x) = (2n+1)x, \qquad D_n^{SC}(x) = 2(n+1), \quad n \geq 0;$$
$$k_n = \frac{1}{4^n}, \quad n \geq 1; \qquad A_n^{SD}(x) = 0, \qquad B_n^{SD}(x) = 1, \qquad C_n^{SD}(x) = 4x, \qquad D_n^{SD}(x) = 4, n \geq 1.$$

Generalizing what is done in [16, Sec.5], let us deduce a closed formula for the generating function of any perturbed Chebyshev sequence $\{\widetilde{P}_n(x)\}_{n\geq 0}$ of order r, $r \geq 0$

$$\tilde{f}(x,t) = \sum_{n\geq 0} \widetilde{P}_n(x) t^n. \tag{66}$$

From (9), (12) and (65), we have

$$\widetilde{P}_{n+r+2}(x) = x\widetilde{P}_{n+r+1}(x) - \frac{1}{4}\widetilde{P}_{n+r}(x), \quad n \geq 0, \ r \geq 0.$$

Multiplying both sides by t^{n+r+2} and summing for $n \geq 0$, we get

$$\sum_{n\geq 0} \widetilde{P}_{n+r+2}(x) t^{n+r+2} = xt \sum_{n\geq 0} \widetilde{P}_{n+r+1}(x) t^{n+r+1} - \frac{1}{4} t^2 \sum_{n\geq 0} \widetilde{P}_{n+r}(x) t^{n+r}.$$

Taking into account (66), we obtain the desired formula as follows:

$$\tilde{f}(x,t) - \sum_{n=0}^{r+1} \widetilde{P}_k(x) t^k = xt\Big(\tilde{f}(x,t) - \sum_{n=0}^{r} \widetilde{P}_k(x) t^k\Big) - \frac{1}{4} t^2 \Big(\tilde{f}(x,t) - \sum_{n=0}^{r-1} \widetilde{P}_k(x) t^k\Big),$$
$$\tilde{f}(x,t)\left(1 + t\left(\frac{1}{4}t - x\right)\right) = \sum_{n=0}^{r+1} \widetilde{P}_k(x) t^k - xt \sum_{n=0}^{r} \widetilde{P}_k(x) t^k + \frac{1}{4} t^2 \sum_{n=0}^{r-1} \widetilde{P}_k(x) t^k,$$
$$\tilde{f}(x,t) = \frac{\sum_{n=0}^{r+1} \widetilde{P}_k(x) t^k - xt \sum_{n=0}^{r} \widetilde{P}_k(x) t^k + \frac{1}{4} t^2 \sum_{n=0}^{r-1} \widetilde{P}_k(x) t^k}{1 + t\left(\frac{1}{4}t - x\right)}, \quad r \geq 0.$$

4.1. Elementary perturbation of order three by translation

The perturbed form	**Recurrence coefficients of $\tilde{u}$**	**The generating function**
$\tilde{u} = \mathcal{U}\left(0; \begin{smallmatrix} 0,\ 0,\ \mu_3 \\ 1,\ 1,\ 1 \end{smallmatrix}; 3\right), \ \mu_3 \neq 0.$	$\tilde{\beta}_n = \mu_3 \delta_{n,3}, \ n \geq 0; \ \tilde{\gamma}_{n+1} = \frac{1}{4}, \ n \geq 0.$	$\tilde{f}(x,t) = \frac{1+t^4\mu_3\left(\frac{x}{2}-x^3\right)}{1+t\left(\frac{t}{4}-x\right)}.$

First moments: $(\tilde{u})_{n=0(1)14} = \Big\{1, 0, \frac{1}{4}, 0, \frac{1}{8}, 0, \frac{5}{64}, \frac{\mu_3}{64}, \frac{1}{128}\left(7 + 2\mu_3^2\right), \frac{1}{64}\mu_3\left(2 + \mu_3^2\right), \frac{1}{512}\left(21 + 20\mu_3^2 + 8\mu_3^4\right),$
$\frac{1}{256}\mu_3\left(11 + 12\mu_3^2 + 4\mu_3^4\right), \frac{33+66\mu_3^2+56\mu_3^4+16\mu_3^6}{1024}, \frac{1}{256}\mu_3\left(13 + 23\mu_3^2 + 16\mu_3^4 + 4\mu_3^6\right), \frac{429+1456\mu_3^2+1952\mu_3^4+1152\mu_3^6+256\mu_3^8}{16384}\Big\}.$

Transfer polynomials: $U_r(x) = -\frac{1}{16}(-1+2x)^2(1+2x)^2\mu_3, \ V_r(x) = \frac{1}{64}\left(1 - 8x\mu_3 + 48x^3\mu_3 - 64x^5\mu_3\right),$
$X_r(x) = \frac{1}{64}\left(-1 - 8x\mu_3 + 48x^3\mu_3 - 64x^5\mu_3\right), \ Y_r(x) = -\frac{1}{4}x^2\left(-1+2x^2\right)^2\mu_3.$

The Stieltjes function of $\tilde{u}$: $S(\tilde{u})(z) = \frac{-2\left(-1+6z\mu_3-32z^3\mu_3+32z^5\mu_3\right)+4(-1+2z)^2(1+2z)^2\mu_3\sqrt{-1+z^2}}{z\left(-1+24z\mu_3-80z^3\mu_3+64z^5\mu_3\right)+\sqrt{-1+z^2}\left(-1-8z\mu_3+48z^3\mu_3-64z^5\mu_3\right)}.$

Coefficients of the Stieltjes equation of $\tilde{u}$ as second degree form

$\widetilde{B}^{SD}(x) = x^7 - x^6\mu_3 - \frac{3x^5}{2} + x^4\mu_3 + \frac{5x^3}{8} - \frac{x^2\mu_3}{4} - \frac{x}{16} - \frac{1}{256\mu_3},$
$\widetilde{C}^{SD}(x) = 2x^6 - 2x^5\mu_3 - \frac{5x^4}{2} + \frac{3x^3\mu_3}{2} + \frac{3x^2}{4} - \frac{x\left(1+16\mu_3^2\right)}{64\mu_3} - \frac{1}{32}, \ \widetilde{D}^{SD}(x) = x^5 - x^4\mu_3 - x^3 + \frac{x^2\mu_3}{2} + \frac{3x}{16} - \frac{1+4\mu_3^2}{64\mu_3}.$

Coefficients of the functional equation of $\tilde{u}$ **$\tilde{u}$ is a semi-classical form of class 7.**

$\widetilde{\Phi}^{SC}(x) = (x^2-1)\widetilde{B}^{SD}(x), \ \widetilde{\psi}^{SC}(x) = -3x\widetilde{B}^{SD}(x).$

Coefficients of the Stieltjes equation of $\tilde{u}$ as semi-classical form

$\widetilde{A}^{SC}(x) = \widetilde{\Phi}^{SC}(x), \ \widetilde{C}^{SC}(x) = -6x^8 + 5x^7\mu_3 + 13x^6 - 9x^5\mu_3 - \frac{35x^4}{4} + \frac{17x^3\mu_3}{4} + \frac{15x^2}{8} - \frac{x\left(1+128\mu_3^2\right)}{256\mu_3} - \frac{1}{16},$
$\widetilde{D}^{SC}(x) = -5x^7 + 4x^6\mu_3 + \frac{39x^5}{4} - \frac{13x^4\mu_3}{2} - \frac{43x^3}{8} + \frac{9x^2\mu_3}{4} + \frac{47x}{64} - \frac{1+16\mu_3^2}{128\mu_3}.$

Coefficients of the Stieltjes equation of $\tilde{u}^{(n)}$ as Laguerre–Hahn form

$\widetilde{A}_0^{SC}(x) = \widetilde{\Phi}^{SC}(x); \ \widetilde{B}_0^{SC}(x) = 0, \ \widetilde{B}_n^{SC}(x) = \frac{1}{4}\widetilde{D}_{n-1}^{SC}(x), \ n \geq 1,$

Coefficients of the structure relation of $\tilde{u}$ as semi-classical form

$\widetilde{C}_0^{SC}(x) = \widetilde{C}^{SC}(x),\ \widetilde{C}_1^{SC}(x) = -4x^8 + 3x^7\mu_3 + \frac{13x^6}{2} - 4x^5\mu_3 - 2x^4 + \frac{x^3\mu_3}{4} - \frac{13x^2}{32} + \frac{x\left(-3+64\mu_3^2\right)}{256\mu_3} + \frac{1}{16}$,
$\widetilde{C}_2^{SC}(x) = -2x^8 + x^7\mu_3 + \frac{7x^6}{2} - 2x^5\mu_3 - \frac{13x^4}{8} + \frac{3x^3\mu_3}{4} + \frac{13x^2}{32} - \frac{x\left(5+64\mu_3^2\right)}{256\mu_3} - \frac{1}{16}$,
$\widetilde{C}_3^{SC}(x) = -x^7\mu_3 + \frac{x^6}{2} + x^5\mu_3 - \frac{3x^4}{8} - \frac{x^3\mu_3}{4} + \frac{x^2}{16} - \frac{7x}{256\mu_3} + \frac{1}{16}$,
$\widetilde{C}_n^{SC}(x) = (2n-6)x^8 - (2n-7)x^7\mu_3 - \frac{(6n-19)x^6}{2} + (2n-7)x^5\mu_3 + \frac{(10n-33)x^4}{8} - \frac{(2n-7)x^3\mu_3}{4} - \frac{(2n-7)x^2}{16} - \frac{(2n+1)x}{256\mu_3},\ n \geq 4$,
$\widetilde{D}_0^{SC}(x) = \widetilde{D}^{SC}(x),\ \widetilde{D}_1^{SC}(x) = -3x^7 + 2x^6\mu_3 + 5x^5 - 3x^4\mu_3 - \frac{29x^3}{16} + \frac{x^2\mu_3}{2} - \frac{1}{64\mu_3}$,
$\widetilde{D}_2^{SC}(x) = -x^7 + 2x^5 - \frac{x^4\mu_3}{2} - x^3 + \frac{x^2\mu_3}{4} + \frac{15x}{64} - \frac{3+16\mu_3^2}{128\mu_3}$,
$\widetilde{D}_n^{SC}(x) = (2n-5)x^7 - (2n-6)x^6\mu_3 - (3n-8)x^5 + (2n-6)x^4\mu_3 + \frac{(5n-14)x^3}{4} - \frac{(n-3)x^2\mu_3}{2} - \frac{(n-3)x}{8} - \frac{n+1}{128\mu_3},\ n \geq 3.$

Coefficients of the second order linear differential equation of $\tilde{u}$ as semi-classical form

$\widetilde{J}^{SC}(0;x) = -\frac{\widetilde{B}^{SD}(x)}{64\mu_3}(x^2-1)\Big(192x^7\mu_3 - 128x^6\mu_3^2 - 320x^5\mu_3 + 192x^4\mu_3^2 + 116x^3\mu_3 - 32x^2\mu_3^2 + 1\Big)$,
$\widetilde{J}^{SC}(1;x) = -\frac{\widetilde{B}^{SD}(x)}{128\mu_3}(x^2-1)\Big(128x^7\mu_3 - 256x^5\mu_3 + 64x^4\mu_3^2 + 128x^3\mu_3 - 32x^2\mu_3^2 - 30x\mu_3 + 16\mu_3^2 + 3\Big)$,
$\widetilde{J}^{SC}(n;x) = \frac{\widetilde{B}^{SD}(x)}{128\mu_3}(x^2-1)\Big\{128(-3+2n)x^7\mu_3 - 256(-2+n)x^6\mu_3^2 - 128(-5+3n)x^5\mu_3 + 256(-2+n)x^4\mu_3^2 +$
$32(-9+5n)x^3\mu_3 - 64(-2+n)x^2\mu_3^2 - 16(-2+n)x\mu_3 - 2 - n\Big\},\ n \geq 2.$
$\widetilde{K}^{SC}(0;x) = \frac{\widetilde{B}^{SD}(x)}{64\mu_3}x\Big(768x^7\mu_3 - 384x^6\mu_3^2 - 1984x^5\mu_3 + 960x^4\mu_3^2 + 1600x^3\mu_3 - 736x^2\mu_3^2 - 348x\mu_3 + 64\mu_3^2 - 3\Big)$,
$\widetilde{K}^{SC}(1;x) = \frac{\widetilde{B}^{SD}(x)}{128\mu_3}\Big\{512x^8\mu_3 - 1408x^6\mu_3 + 64x^5\mu_3^2 + 1280x^4\mu_3 - 224x^3\mu_3^2 - 324x^2\mu_3 + x(-3+4\mu_3)(3+4\mu_3) + 30\mu_3\Big\}$,
$\widetilde{K}^{SC}(n;x) = -\frac{\widetilde{B}^{SD}(x)}{128\mu_3}\Big\{512(-3+2n)x^8\mu_3 - 768(-2+n)x^7\mu_3^2 - 128(-31+20n)x^6\mu_3 + 1792(-2+n)x^5\mu_3^2 + 640(-5+$
$3n)x^4\mu_3 - 960(-2+n)x^3\mu_3^2 - 32(-25+14n)x^2\mu_3 + x\Big(-2\left(-3+128\mu_3^2\right) + n\left(3+128\mu_3^2\right)\Big) + 16(-2+n)\mu_3\Big\},\ n \geq 2.$
$\widetilde{L}^{SC}(0;x) = -\frac{\widetilde{B}^{SD}(x)}{64\mu_3}\Big(768x^7\mu_3 - 384x^6\mu_3^2 - 1984x^5\mu_3 + 960x^4\mu_3^2 + 1600x^3\mu_3 - 736x^2\mu_3^2 - 348x\mu_3 + 64\mu_3^2 - 3\Big)$,
$\widetilde{L}^{SC}(1;x) = -\frac{\widetilde{B}^{SD}(x)}{16\mu_3}\Big(96x^7\mu_3 - 232x^5\mu_3 + 166x^3\mu_3 - 32x^2\mu_3^2 - 16\mu_3^2 - 3\Big)$,
$\widetilde{L}^{SC}(n;x) = -\frac{\widetilde{B}^{SD}(x)}{128\mu_3}\Big\{128(-4+n)(1+n)(-3+2n)x^7\mu_3 - 256(-4+n)(-2+n)nx^6\mu_3^2$
$-64\left(62-9n-30n^2+6n^3\right)x^5\mu_3 + 256(-2+n)\left(-6-4n+n^2\right)x^4\mu_3^2 + 16\left(158-31n-54n^2+10n^3\right)x^3\mu_3$
$-64(-2+n)\left(-8-4n+n^2\right)x^2\mu_3^2 - 8(2+n)\left(21-16n+2n^2\right)x\mu_3 - (1+n)(2+n)(3+n)\Big\},\ n \geq 2.$

4.2. *Elementary perturbation of order three by dilatation*

The perturbed form

$\tilde{u} = \mathcal{U}\left(0;\ {}^{0,\,0,\,0}_{1,\,1,\,\lambda_3}\ ;3\right),\ \lambda_3 \neq 0,\ \lambda_3 \neq 1.$

Recurrence coefficients of $\tilde{u}$

$\tilde{\beta}_n = 0,\ n \geq 0;\ \tilde{\gamma}_{n+1} = \frac{1}{4}\big(1-(1-\lambda_3)\delta_{n+1,3}\big),\ n \geq 0.$

The generating function: $\tilde{f}(x,t) = \frac{1+t^4\left(\frac{1}{4}x^2(1-\lambda_3)+\frac{1}{16}(-1+\lambda_3)\right)}{1+t\left(\frac{t}{4}-x\right)}$.

First moments: $(\tilde{u})_{n=0(1)17} = \Big\{1, 0, \frac{1}{4}, 0, \frac{1}{8}, 0, \frac{1}{64}(4+\lambda_3), 0, \frac{1}{256}\left(8+5\lambda_3+\lambda_3^2\right), 0, \frac{(2+\lambda_3)\left(8+5\lambda_3+\lambda_3^2\right)}{1024}, 0,$
$\frac{(1+\lambda_3)\left(32+25\lambda_3+8\lambda_3^2+\lambda_3^3\right)}{4096}, 0, \frac{64+170\lambda_3+131\lambda_3^2+52\lambda_3^3+11\lambda_3^4+\lambda_3^5}{16384}, 0, \frac{128+494\lambda_3+474\lambda_3^2+245\lambda_3^3+75\lambda_3^4+13\lambda_3^5+\lambda_3^6}{65536}, 0\Big\}.$

Transfer polynomials: $U_r(x) = -\frac{1}{16}x\left(-1+4x^2\right)(-1+\lambda_3),\ V_r(x) = \frac{1}{64}\left(1+8x^2(-1+\lambda_3) - 16x^4(-1+\lambda_3)\right)$,
$X_r(x) = \frac{1}{4}x^4(1-\lambda_3) + \frac{1}{8}x^2(-1+\lambda_3) - \frac{\lambda_3}{64},\ Y_r(x) = -\frac{1}{32}x(-1+2x)(1+2x)\left(-1+2x^2\right)(-1+\lambda_3).$

The Stieltjes function of $\tilde{u}$: $S(\tilde{u})(z) = \frac{-2-12z^2(-1+\lambda_3)+16z^4(-1+\lambda_3)+\left(4z\left(1-4z^2\right)(-1+\lambda_3)\right)\sqrt{-1+z^2}}{z(4-3\lambda_3)+16z^3(-1+\lambda_3)-16z^5(-1+\lambda_3)+\sqrt{-1+z^2}\left(-8z^2(-1+\lambda_3)+16z^4(-1+\lambda_3)+\lambda_3\right)}$.

Coefficients of the Stieltjes equation of $\tilde{u}$ as second degree form

$\widetilde{B}^{SD}(x) = x^6 + \frac{1}{4}x^4(-4-\lambda_3) + \frac{1}{8}x^2(2+\lambda_3) - \frac{\lambda_3^2}{64(-1+\lambda_3)},\ \widetilde{C}^{SD}(x) = 2x^5 + \frac{1}{2}x^3(-3-\lambda_3) + \frac{x\left(-4+\lambda_3+2\lambda_3^2\right)}{16(-1+\lambda_3)}$,
$\widetilde{D}^{SD}(x) = x^4 + \frac{1}{4}x^2(-2-\lambda_3) - \frac{1}{16(-1+\lambda_3)}.$

Coefficients of the functional equation of $\tilde{u}$ $\tilde{u}$ **is a semi-classical form of class 6.**

$\widetilde{\Phi}^{SC}(x) = (x^2-1)\widetilde{B}^{SD}(x),\ \widetilde{\psi}^{SC}(x) = -3x\widetilde{B}^{SD}(x).$

Coefficients of the Stieltjes equation of $\tilde{u}$ as semi-classical form

$$\widetilde{A}^{SC}(x) = \widetilde{\Phi}^{SC}(x),\ \widetilde{C}^{SC}(x) = -5x^7 + \tfrac{3}{4}x^5(12+\lambda_3) + \tfrac{1}{8}x^3(-34-9\lambda_3) + \frac{x\left(-32+16\lambda_3+15\lambda_3^2\right)}{64(-1+\lambda_3)},$$
$$\widetilde{D}^{SC}(x) = -4x^6 + \tfrac{1}{2}x^4(13+\lambda_3) - \tfrac{3}{4}x^2(3+\lambda_3) + \frac{-4+\lambda_3+2\lambda_3^2}{32(-1+\lambda_3)}.$$

Coefficients of the Stieltjes equation of $\tilde{u}^{(n)}$ as Laguerre–Hahn form

$$\widetilde{A}_0^{SC}(x) = \widetilde{\Phi}^{SC}(x);\ \widetilde{B}_0^{SC}(x) = 0,\ \widetilde{B}_n^{SC}(x) = \tfrac{1}{4}\big(1-(1-\lambda_3)\delta_{n,3}\big)\widetilde{D}_{n-1}^{SC}(x),\ n \geq 1,$$

Coefficients of the structure relation of $\tilde{u}$ as semi-classical form

$$\widetilde{C}_0^{SC}(x) = \widetilde{C}^{SC}(x),\ \widetilde{C}_1^{SC}(x) = -3x^7 + \tfrac{1}{4}x^5(16+\lambda_3) + \tfrac{1}{8}x^3(-2-3\lambda_3) - \frac{x\left(-16+12\lambda_3+7\lambda_3^2\right)}{64(-1+\lambda_3)},$$
$$\widetilde{C}_2^{SC}(x) = -x^7 + \tfrac{1}{4}x^5(8-\lambda_3) + \tfrac{1}{8}x^3(-6+\lambda_3) - \frac{x\left(16-12\lambda_3+\lambda_3^2\right)}{64(-1+\lambda_3)},$$
$$\widetilde{C}_n^{SC}(x) = (2n-5)x^7 - \frac{(8n-20+(2n-7)\lambda_3)}{4}x^5 + \frac{(4n-10+(2n-7)\lambda_3)}{8}x^3 - \frac{\lambda_3(8+(2n-7)\lambda_3)}{64(-1+\lambda_3)}x,\ n \geq 3,$$
$$\widetilde{D}_0^{SC}(x) = \widetilde{D}^{SC}(x),\ \widetilde{D}_1^{SC}(x) = -2x^6 + 3x^4 + \tfrac{1}{8}x^2(-4-\lambda_3) - \frac{\lambda_3^2}{16(-1+\lambda_3)},\ \widetilde{D}_2^{SC}(x) = \frac{x^4}{2} - \frac{x^2}{4} + \frac{-4+\lambda_3}{32(-1+\lambda_3)},$$
$$\widetilde{D}_n^{SC}(x) = 2(n-2)x^6 - \frac{(4n-8+(n-3)\lambda_3)}{2}x^4 + \frac{(2n-4+(n-3)\lambda_3)}{4}x^2 - \frac{\lambda_3(4+(n-3)\lambda_3)}{32(-1+\lambda_3)},\ n \geq 3.$$

Coefficients of the second order linear differential equation of $\tilde{u}$ as semi-classical form

$$\widetilde{J}^{SC}(0;x) = -\frac{\widetilde{B}^{SD}(x)}{16(-1+\lambda_3)}(x^2-1)\Big\{32x^6(-1+\lambda_3) - 48x^4(-1+\lambda_3) + 2x^2(-1+\lambda_3)(4+\lambda_3) + \lambda_3^2\Big\},$$
$$\widetilde{J}^{SC}(1;x) = \frac{\widetilde{B}^{SD}(x)}{32(-1+\lambda_3)}(x^2-1)\Big\{16x^4(-1+\lambda_3) - 8x^2(-1+\lambda_3) + \lambda_3 - 4\Big\},$$
$$\widetilde{J}^{SC}(n;x) = \frac{\widetilde{B}^{SD}(x)}{32(-1+\lambda_3)}(x^2-1)\Big\{64(-1+n)x^6(-1+\lambda_3) - 16x^4(-1+\lambda_3)\Big(-2(2+\lambda_3) + n(4+\lambda_3)\Big) +$$
$$8x^2\Big(-2(1+\lambda_3) + n(2+\lambda_3)\Big) - \lambda_3(4-2\lambda_3+n\lambda_3)\Big\},\ n \geq 2.$$
$$\widetilde{K}^{SC}(0;x) = \frac{\widetilde{B}^{SD}(x)}{16(-1+\lambda_3)}x\Big\{96x^6(-1+\lambda_3) - 240x^4(-1+\lambda_3) - 2x^2(-92+\lambda_3)(-1+\lambda_3) + 16 - 12\lambda_3 - 7\lambda_3^2\Big\},$$
$$\widetilde{K}^{SC}(1;x) = -\frac{\widetilde{B}^{SD}(x)}{32(-1+\lambda_3)}x\Big\{16x^4(-1+\lambda_3) - 56x^2(-1+\lambda_3) + 13\lambda_3 - 4\Big\},$$
$$\widetilde{K}^{SC}(n;x) = -\frac{\widetilde{B}^{SD}(x)}{32(-1+\lambda_3)}x\Big\{192(-1+n)x^6(-1+\lambda_3) - 16x^4(-1+\lambda_3)\Big(-2(14+\lambda_3) + n(28+\lambda_3)\Big) +$$
$$8x^2(-1+\lambda_3)\Big(-2(15+7\lambda_3) + n(30+7\lambda_3)\Big) + n\left(32-16\lambda_3-13\lambda_3^2\right) + 2\left(-16+6\lambda_3+13\lambda_3^2\right)\Big\},\ n \geq 2.$$
$$\widetilde{L}^{SC}(0;x) = -\frac{\widetilde{B}^{SD}(x)}{16(-1+\lambda_3)}\Big\{96x^6(-1+\lambda_3) - 240x^4(-1+\lambda_3) - 2x^2(-92+\lambda_3)(-1+\lambda_3) + 16 - 12\lambda_3 - 7\lambda_3^2\Big\},$$
$$\widetilde{L}^{SC}(1;x) = -\frac{\widetilde{B}^{SD}(x)}{4(-1+\lambda_3)}\Big\{8x^2(-1+\lambda_3) - 4 + \lambda_3\Big\},$$
$$\widetilde{L}^{SC}(n;x) = -\frac{\widetilde{B}^{SD}(x)}{32(-1+\lambda_3)}\Big\{-64(-3+n)(-1+n)(1+n)x^6(-1+\lambda_3) + 16(-1+n)x^4(-1+\lambda_3)\Big(n(-8-5\lambda_3)$$
$$+6(-6+\lambda_3) + n^2(4+\lambda_3)\Big) - 8x^2(-1+\lambda_3)\Big(3n(-6+\lambda_3) - 6n^2(1+\lambda_3) + n^3(2+\lambda_3) + 2(11+5\lambda_3)\Big)$$
$$+(4-2\lambda_3+n\lambda_3)\Big(-8-4n(-2+\lambda_3) + 11\lambda_3 + n^2\lambda_3\Big)\Big\},\ n \geq 2.$$

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