Bifurcations from Regular Quotient Networks: A First Insight

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Abstract

We consider regular (identical-edge identical-node) networks whose cells can be grouped into classes by an equivalence relation. The identification of cells in the same class determines a new network - the quotient network. In terms of the dynamics this corresponds to restricting the coupled cell systems associated with a network to flow-invariant subspaces given by equality of certain cell coordinates. Assuming a bifurcation occurs for a coupled cell system restricted to the quotient network, we ask how that bifurcation lifts to the overall space. Surprisingly, for certain networks, new branches of solutions occur besides the ones that occur in the quotient network. To investigate this phenomenon we develop a systematic method that enumerates all networks with a given quotient. We also prove necessary conditions for the existence of solutions branches not predicted by the quotient. We then apply our method to two particular quotient networks; namely, two- and three-cell bidirectional rings. We show there are no additional bifurcating solution branches when the quotient network is a two-cell bidirectional ring. However, two of the 12 five-cell networks that have the three-cell bidirectional ring as a quotient network

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exhibit bifurcating solutions that do not occur in the quotient itself. Thus, network architecture sometimes forces the existence of bifurcating branches in addition to the ones determined by the quotient.

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1 Introduction

A theory for coupled cell networks has been developed in [12, 8, 7]. In this theory a *cell* is a system of ordinary differential equations and a *coupled cell network* is a collection of interacting cells. The *network architecture* is a directed graph whose nodes represent cells and whose arrows represent couplings between cells. Such networks form an interesting class of dynamical systems that have been used as models in different areas such as biology, economics, physics and ecology. See for example Strogatz [13], Wang [14], Stewart [11], Lieberman *et al.* [10], Boccaletti *et al.* [3] and references therein.

In this paper we only consider *regular networks*: networks where each node has the same differential equation (up to reordering of coordinates) and one kind of coupling. The general theory associates a class of *admissible* vector fields to each network. In a regular network let $x_j \in \mathbf{R}^k$ be the coordinates of the jth cell, where k is the dimension of the internal dynamics in each cell. The jth coordinate of an admissible vector field of an n-cell regular network has the form

$$\dot{x}_j = f(x_j, \overline{x_{\sigma_j(1)}, \ldots, x_{\sigma_j(v)}}) \qquad j = 1, \ldots, n \tag{1.1}$$

where v is the *valency* of the network, $\sigma_j(i)$ is the index of the ith cell that couples to cell j. The overbar indicates the coupling coordinates are invariant under permutations of the coupling cells; this invariance is assumed since there is just one kind of coupling. Since there is only one kind of node, we assume that the function $f : \mathbf{R}^k \times (\mathbf{R}^k)^v \to \mathbf{R}^k$ is independent of j. In general, the theory permits *self-coupling* ($\sigma_j(i) = j$ for some i and j) and *multiple arrows* ($\sigma_j(i_1) = \sigma_j(i_2)$ for some $i_1 \neq i_2$ and j).

The three-cell bidirectional ring pictured in Figure 1 is an example of a regular network architecture and the admissible vector fields have the form

$$\begin{array}{rcl} \dot{x}_1 &=& f(x_1, \overline{x_2, x_3}) \\ \dot{x}_2 &=& f(x_2, \overline{x_1, x_3}) \\ \dot{x}_3 &=& f(x_3, \overline{x_1, x_2}) \end{array} \tag{1.2}$$

where the overline indicates that

$$f(a, b, c) = f(a, c, b). \tag{1.3}$$

Note that the bidirectional ring has $\mathbf{S}_3$-symmetry.

In this paper we discuss a surprising feature of synchrony-breaking bifurcations in certain regular networks. To describe this feature we need to define synchrony subspaces. A *polydiagonal* is a subspace of the network phase space $(\mathbf{R}^k)^n$ that is defined by equalities between cell coordinates; that is, $x_i = x_j$ for certain pairs of cells i, j. A *synchrony*

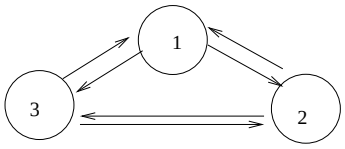

Figure 1: Three-cell bidirectional ring.

subspace is a polydiagonal that is flow-invariant for every admissible vector field. It follows that solutions whose initial conditions are in Δ stay inside Δ for all time. Solutions in Δ are (partially) synchronous since $x_i(t) = x_j(t)$ for all time t. It is straightforward to note that setting all coordinates equal in a regular network yields a synchrony subspace. Let

$$\Delta_0 = \{(x, \ldots, x) \in (\mathbf{R}^k)^n\}.$$

We assume that an admissible vector field F has a (fully) synchronous equilibrium $X_0 \in \Delta_0$. Let E^c be the center subspace of $(dF)_{X_0}$. The equilibium X_0 has a synchrony-breaking bifurcation if $E^c \setminus \Delta_0$ is nonempty; that is, there is a vector in E^c that is not in Δ_0.

Suppose that $\Delta \supset \Delta_0$ is another synchrony subspace and that $F|\Delta$ has a synchrony-breaking bifurcation. One might expect that all bifurcating solutions remain inside Δ. In this paper we show that there are five-cell examples where this supposition is false, and in those cases we analyse the actual bifurcations.

To understand this observation more fully we need to describe more of the general theory. We can associate to each polydiagonal Δ a coloring of the network nodes. In this coloring two cells i, j have the same color precisely when $x_i = x_j$ is part of the definition of Δ. Theorem 4.3 of [8] states that Δ is a synchrony-subspace if and only if the coloring associated to Δ is balanced. For regular networks, a coloring is *balanced* if any pair of cells with color r have the same number of inputs from cells of color b for each b. An example of a balanced coloring of the bidirectional ring is given in Figure 2 (left). Note that each blue cell has one white cell and one blue cell as inputs. Hence $x_2 = x_3$ is a synchrony subspace.

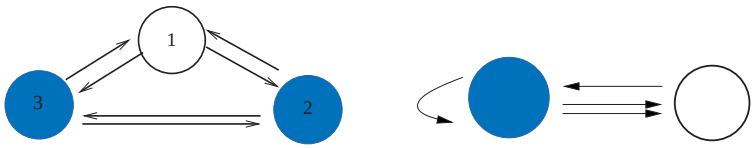

Figure 2: Two-cell quotient of the three-cell bidirectional ring.

The restriction of an admissible vector field (1.2) to the synchrony subspace $x_2 = x_3$ has the form

$$\begin{aligned} \dot{x}_1 &= f(x_1, \overline{x_2, x_2}) \\ \dot{x}_2 &= f(x_2, \overline{x_1, x_2}). \end{aligned} \tag{1.4}$$

These vector fields are admissible with respect to the network (with multiple arrows and self-coupling) in Figure 2 (right). Theorem 5.2 of [8] shows that for any synchrony subspace Δ there is always a network, called the *quotient network*, such that the restrictions

of admissible vector fields to Δ are the admissible vector fields of the quotient network. Observe that a given network can be the quotient network of many different networks. However, a quotient network is a regular network if and only if the original network is a regular network. Part of this paper will address the 'inverse' problem: Given a regular network Q, enumerate the networks G that have Q as a quotient network. We develop techniques to enumerate such networks and point out that these techniques can also be useful when trying to find networks with a pre-defined dynamics. See Aguiar *et al.* [2].

As noted, dynamics on a quotient network describe (partially) synchronous dynamics on the whole network. Specifically, admissible vector fields associated to the two-cell network in Figure 2 (right) describes the dynamics of the three-cell network in Figure 2 (left) in which cells 2 and 3 form a subset of cells or a population of cells that move synchronously. More generally, a balanced coloring enables us to partition the cells in a network into subsets of cells or *subpopulations* of cells, where the cells within each subpopulation move synchronously. From this perspective, it seems surprising that bifurcations within the quotient network may force, because of the network architecture of the larger network, solutions in which these subpopulations do not move synchronously. However, we will present examples of five-cell networks with three-cell bidirectional rings as quotients where bifurcations within the ring dynamics leads to solutions that break synchrony in the five-cell network.

The adjacency matrix A of a regular network G is the matrix whose i, j entry is the number of arrows that connect cell j to cell i. Results in [9, 5] relate the eigenvalues of the Jacobian J_G of a coupled cell system at X_0 with the eigenvalues of A. In order for bifurcations within the quotient network Q to lead to nonsynchronous solutions in the larger network G the center subspace of J_G must be larger than the center subspace of J_Q. We present results that relate the eigenvalues of the adjacency matrix of the network G with those of the adjacency matrix of the quotient Q. These results provide an easy way to identify networks G for which the dimension of the center subspaces of J_G and J_Q are the same.

We organize the paper as follows. In Section 2 we develop a general method to enumerate all networks that admit a given quotient. Moreover, we relate the eigenvalue structure of the adjacency matrices of such networks with the adjacency matrix corresponding to the quotient.

In Section 3.1, we enumerate connected networks that admit the two-cell bidirectional ring as a quotient network. We show that codimension-one steady-state synchrony-breaking bifurcations for coupled cell systems associated with that quotient do not lead to new branches besides the ones guaranteed by the quotient.

In Section 3.2 we enumerate the four and five-cell networks admitting the three-cell bidirectional ring quotient network. Up to isomorphism, there are two four-cell and twelve five-cell networks. See Figures 4 and 5. In Theorem 3.2 we show that only two such networks can exhibit branches of steady-state solutions not predicted by bifurcation in the three-cell bidirectional ring. In Theorems 3.4 and 3.5 we show that, generically, the coupled cell systems associated with networks 4 and 6 in Figure 5 have additional branches. The proofs of these theorems involve long computations which are given in the Appendix.

2 Networks with a Quotient network

In this paper we consider *regular networks* – networks with one kind of node and one kind of coupling, and where the number of edges directed to each cell is equal for all cells.

For a valency l regular network G and a balanced equivalence relation $\bowtie$ on its cells, the *quotient network* $G_{\bowtie}$ is also a valency l regular network defined naturally as follows: the set of cells of $G_{\bowtie}$ is formed by one cell of each color (each $\bowtie$-equivalence class); the edges in the quotient network are the projection of edges in the original network. Specifically, given a cell in the quotient, representing the cells with a color i, the number of edges directed from a cell representing the cells with a color j to that cell is equal to the number of edges that any cell with color i receives from cells with color j in the network G.

In this section we derive general results concerning the enumeration of all networks that admit a given quotient and the eigenvalue structure of the corresponding adjacency matrices in order to address the main question of the paper.

2.1 Enumeration

Given a regular network Q with p cells we present a general method to enumerate the n-cell (regular) networks, where $n > p$, that admit Q as a quotient network. This is equivalent to determining the n-cell networks G that admit a balanced equivalence relation $\bowtie$ such that Q is the quotient network of G by $\bowtie$.

In what follows we denote by A_G the $n \times n$ adjacency matrix of an n-cell regular network G with cells $\mathcal{C} = \{1, \ldots, n\}$. The ij-entry of A_G is the number of directed edges from cell j to cell i.

Definition 2.1 Let G be a regular coupled cell network with n cells $\mathcal{C} = \{1, \ldots, n\}$ and let A_G be the corresponding adjacency matrix whose columns we denote by $A_{G_1}, \ldots, A_{G_n}$. Let $\bowtie$ be an equivalence relation on $\mathcal{C}$ with classes $I_1, \ldots, I_p$. Denote by $\overline{A}_G$ the $n \times p$-matrix whose columns $C_1, \ldots, C_p$ are defined by

$$C_j = \sum_{i \in I_j} A_{G_i}.$$

We say that the matrix A_G is $\bowtie$-*balanced* if for each $j = 1, \ldots, p$, the rows for $i \in I_j$ of $\overline{A}_G$ are identical. $\diamond$

Proposition 2.2 *Let G be an n-cell regular network and A_G the corresponding adjacency matrix. An equivalence relation $\bowtie$ on the set of cells of G is balanced if and only if the matrix A_G is $\bowtie$-balanced.*

Proof The definition of balanced equivalence relation in terms of the adjacency matrix associated with the network gives the result. To see that, let G be a regular coupled cell network with n cells, say $\mathcal{C} = \{1, \ldots, n\}$, A_G the corresponding $n \times n$ adjacency matrix with columns $A_{G_1}, \ldots, A_{G_n}$, and $\bowtie$ a balanced equivalence relation on $\mathcal{C}$ with classes $I_1, \ldots, I_p$. Let $j \in \{1, \ldots, p\}$. Now observe that each entry i in the jth column of the matrix $\overline{A}_G$ as defined in Definition 2.1 represents the number of cells in the class I_j that

are tail cells of edges with head cell i. Moreover, because $\bowtie$ is balanced it follows that if i, i' belong to the same class, then the corresponding entries in column j of $\overline{A}_G$ are equal. □

Proposition 2.3 *Let G be a regular network, A_G the corresponding adjacency matrix and $\bowtie$ a balanced equivalence relation on the set of cells of G with classes $I_1, \ldots, I_p$. For each $j = 1, \ldots, p$, choose any $j_i \in I_j$. Then the adjacency matrix of the quotient network $G_{\bowtie}$ is the $p \times p$ submatrix of $\overline{A}_G$ whose jth row is the row j_i of $\overline{A}_G$.*

Proof By Proposition 2.2, since $\bowtie$ is balanced, the matrix A_G is balanced. Now the definition of quotient network leads to the above result. □

Definition 2.4 Given $t \in \mathbf{Z}_0^+$ and $r, s \in \mathbf{N}$ define

$$M_{rs}(t) = \left\{ M = [m_{ij}]_{\substack{1 \le i \le r, \\ 1 \le j \le s}} \in M_{r \times s}(\mathbf{Z}_0^+) : \sum_{j=1}^{s} m_{ij} = t, \ i = 1, \ldots, r \right\}.$$

◇

Theorem 2.5 *Let Q be a regular network with p cells and adjacency matrix $A_Q = [q_{ij}]_{1 \le i,j \le p}$. A regular network G with n cells, say $\mathcal{C} = \{1, \ldots, n\}$, admits the quotient network Q if and only if there is a partition of $\mathcal{C}$ into p-parts, say $I_1, \ldots, I_p$, such that, after relabeling the cells if necessary, the adjacency matrix A_G of G has the following block structure:*

$$\begin{bmatrix} Q_{11} & \cdots & Q_{1p} \\ \vdots & \cdots & \vdots \\ Q_{p1} & \cdots & Q_{pp} \end{bmatrix}$$

where

$$Q_{ij} \in M_{\#I_i \, \#I_j}(q_{ij}), \quad \text{for } i, j = 1, \ldots, p.$$

Here $\#I_l$ denotes the cardinality of the set I_l.

Proof By definition, an n-cell regular network G admits a quotient network Q if and only if there is some balanced equivalence relation $\bowtie$ on the set of cells of G such that Q is the quotient of G by $\bowtie$. Moreover, by Proposition 2.2, the corresponding adjacency matrix is $\bowtie$-balanced and by Proposition 2.3, up to an isomorphism of the cells, the adjacency matrix of G has the above form. □

Using Theorem 2.5, we describe below an algorithm that enumerates all the regular networks G with n cells admitting a given regular quotient network Q with p cells. It starts by partitioning the set $\{1, \ldots, n\}$ into all possible parts with sizes $d_1, \ldots, d_p$ such that $0 < d_1 \le \cdots \le d_p$. For each of these partitions, it enumerates the adjacency matrices of all non-isomorphic connected networks that admit a balanced equivalence coloring with equivalence classes of sizes $d_1, \ldots, d_p$ such that the corresponding quotient is Q. In this

algorithm, we use two subroutines. One to check if a given network is isomorphic to any of the networks of a given set of networks. The other to check if a network is connected.

Algorithm 2.6 Given a regular network Q with p cells and adjacency matrix $A_Q = [q_{ij}]_{1\leq i,j\leq p}$ and a positive integer $n > p$, this algorithm finds the set S_G of the adjacency matrices of all non-isomorphic connected networks with n cells admitting the quotient Q.

1 [Compute the set of partitions] Compute the set $\mathcal{P}_n$ of vectors $(d_1, \ldots, d_p)$ such that $0 < d_1 \leq \cdots \leq d_p$ and $d_1 + \cdots + d_p = n$.

2 [Initialize] Set $S_G = \emptyset$.

3 [Compute sub-blocks for each partition] Given $(d_1, \ldots, d_p) \in \mathcal{P}_n$ do the following: set $\mathcal{P}_n = \mathcal{P}_n \setminus \{(d_1, \ldots, d_p)\}$; for $i, j = 1, \ldots, p$ compute $M_{d_i\, d_j}(q_{ij})$.

4 [Actualize A_G] For each $(Q_{11}, \ldots, Q_{pp}) \in M_{d_1\, d_1}(q_{11}) \times \ldots \times M_{d_p\, d_p}(q_{pp})$ do the following: let

$$M = \begin{bmatrix} Q_{11} & \cdots & Q_{1p} \\ \vdots & \cdots & \vdots \\ Q_{p1} & \cdots & Q_{pp} \end{bmatrix};$$

if Connected(M)=1 then: if CheckIsomorphic(S_G,M) = 0 then $S_G = S_G \cup \{M\}$.

5 [Finish?] If $\mathcal{P}_n = \emptyset$ then output S_G and terminate the algorithm. Otherwise go to step 3.

$\diamond$

The next algorithm determines if a $n \times n$ square matrix M is the adjacency matrix of a connected n-cell network. Note that the graph can have multiple arrows. Moreover, a network is connected if and only if given any two distinct cells, there is a path formed by undirected edges connecting them. Thus we start by forming the connection matrix $C = [c_{ij}]_{1\leq i,j\leq n}$ defined in the following way: (i) for all i, j we have $c_{ij} = c_{ji}$; $c_{ij} = 1$ if cell j has at least one directed edge to i or cell j has at least one directed edge to cell i, and zero otherwise; (ii) $c_{ii} = 1$ for all i. Then the graph is connected if and only if all the entries of C^n are nonzero.

Algorithm 2.7 (Connected) Connected(M) verifies if an $n \times n$ square matrix M is the adjacency matrix of a connected regular network. Let $A = [a_{ij}]_{1\leq i,j\leq n} = M + M^t + \mathrm{Id}_n$ and $C = [c_{ij}]_{1\leq i,j\leq n}$ where each c_{ij} is 0 if $a_{ij} = 0$ and 1 otherwise. If C^n has zero entries returns 0, otherwise returns 1. $\diamond$

Algorithm 2.8 (CheckIsomorphic) CheckIsomorphic(S_G,M) verifies if an $n \times n$ square matrix M is the adjacency matrix of a regular network isomorphic to one of the regular networks that have adjacency matrix in S_G. If for some matrix $A \in S_G$ there is a permutation matrix $P_\sigma \in \mathbf{S}_n$ such that

$$M = P_\sigma A P_\sigma^{-1}$$

then returns 1. Otherwise returns 0. $\diamond$

2.2 Eigenvalue Structure of Adjacency Matrices

We present now a few properties related with the structure of the adjacency matrices of regular networks admitting balanced colorings. In the cases where the quotients associated with the balanced colorings have no self-coupling, we can conclude some important remarks.

Theorem 2.9 *Let G be an n-cell regular network. Let $\bowtie$ be a balanced equivalence relation on the set of cells of G with p classes. Let A_Q be the $p \times p$ adjacency matrix of the quotient network Q of G by $\bowtie$.*

Then the adjacency matrix of G is similar to a matrix with the following block structure:

$$\begin{bmatrix} A_Q & R \\ 0_{(n-p)\times p} & B \end{bmatrix} \tag{2.5}$$

where R is a $p \times (n-p)$ matrix and B is a $(n-p) \times (n-p)$ matrix.

Proof Let A_G be the adjacency matrix associated to network G. We can interpret A_G as the matrix of a linear G-admissible vector field with respect to the canonical basis of $\mathbf{R}^n$, say $(e_1, \ldots, e_n)$. If

$$\Delta_{\bowtie} = \{x \in \mathbf{R}^n : \ i \bowtie j \Rightarrow x_i = x_j\}$$

then $A_G(\Delta_{\bowtie}) \subseteq \Delta_{\bowtie}$. Moreover $\left(\sum_{i\in I_1} e_i, \ldots, \sum_{i\in I_p} e_i\right)$ is a basis of $\Delta_{\bowtie}$ and $A_G|_{\Delta_{\bowtie}}$ with respect to this basis is the adjacency matrix A_Q of the quotient network Q of G by $\bowtie$. Denote by $I_1, \ldots, I_p$ the $\bowtie$-equivalence classes. Choose $s_j \in I_j$, $j = 1, \ldots, p$ and let $S = \mathcal{C} \setminus \{s_1, \ldots, s_p\}$. We can complete the above basis of $\Delta_{\bowtie}$ with the elements of $\{e_i : \ i \in S\}$ obtaining a basis of $\mathbf{R}^n$. The matrix A_G with respect to that basis has the structure (2.5). □

Remark 2.10 We make a few observations related to the above theorem when $\bowtie$ determines a quotient network with no self-coupling. That is, when there are no connections between cells in the same $\bowtie$-class.

(a) Following the proof of the theorem we can complete the basis of $\Delta_{\bowtie}$ with the elements of $\{e_i : \ i \in S\}$ choosing any ordering in S such that the cells in the same $\bowtie$-equivalence class are contiguous. In this case, B is an $(n-p)\times(n-p)$ matrix with diagonal blocks that are null square matrices of order $\#I_j - 1$, for all $j \in \{1, \ldots, p\}$ such that $\#I_j > 1$.

(b) Consider the special case where $\bowtie$ is a balanced equivalence relation on the set of cells of G with classes $I_1, \ldots, I_{n-1}$ such that $\#I_j = 2$ for some $j = 1, \ldots, n-1$ and $\#I_i = 1$ for $i \neq j$. Then the eigenvalues of the adjacency matrix associated to G are the eigenvalues of A_Q plus the zero eigenvalue.

(c) More generally, suppose $\bowtie$ is a balanced equivalence relation on the set of cells of G with classes $I_1, \ldots, I_{n-s}$ such that $\#I_j = s + 1$ for some $j = 1, \ldots, n-s$ and

$\#I_i = 1$ for $i \neq j$. Then the eigenvalues of the adjacency matrix associated to G are the eigenvalues of the matrix A_Q, with the same algebraic multiplicity, plus the zero eigenvalue, with algebraic multiplicity s.

$\diamond$

Algorithm 2.11 Given an n-cell regular network G with adjacency matrix A_G and a balanced equivalence relation $\bowtie$ on the set of cells of G with classes $I_1, \ldots, I_p$ such that $i_j = \#I_j,\ j = 1, \ldots, p$. This algorithm computes a matrix similar to A_G and with the block structure (2.5).

1 For each equivalence class I_l, $l = 1, \ldots, p$ substitute the first column of the matrix A_G indexed by a cell in I_l by the sum of all the columns of A_G indexed by cells in I_l.

2 Permute the columns of A_G such that the first column is the sum of the columns for class I_1, the second column is the sum of the columns for class I_2, and so on until column p. The next columns are the remaining columns of A_G.

3 Permute the rows of A_G such that the first row is the first row of the matrix A_G indexed by a cell in I_1, the second row is the first row of the matrix A_G indexed by a cell in I_2, and so on until row p. The next rows are the remaining rows of A_G indexed by cells in I_1, then the remaining rows of A_G indexed by cells in I_2, and so on until the remaining rows of A_G indexed by cells in I_p.

4 According to Proposition 2.3 the $p \times p$ submatrix of A_G with the first p rows and the first p columns of A_G is the matrix A_Q associated with the quotient network Q and the first p columns of the rows indexed by cells in the same equivalence class are identical.

5 For each equivalence class I_l, $l = 1, \ldots, p$ subtract the first row of the matrix A_G indexed by a cell in I_l to all the other rows of A_G indexed by cells in I_l. This way the first p columns of the last $n - p$ rows of A_G have null entries.

$\diamond$

Remark 2.12 If $\bowtie$ determines a quotient network with no self-coupling the matrix B in (2.5) will have diagonal blocks corresponding to null square matrices if in step 2 of the above algorithm the last $n - p$ columns are ordered such that the columns of A_G indexed by cells in I_1 appear first then followed by the columns of A_G indexed by cells in I_2, and so on. $\diamond$

Theorem 2.13 *Let Q be a p-cell regular network with no self-coupling and G an n-cell network, with $n > p$, that admits Q as a quotient network.*

Assume there are networks $Q_0 = Q$, $Q_1, \ldots, Q_{n-p} = G$ such that for $j = 1, \ldots, n - p$, Q_j has $p + j$ cells and Q_j admits Q_{j-1} as a quotient network. Then the eigenvalues of the adjacency matrix associated to G are the eigenvalues of the adjacency matrix associated to the quotient network Q, with the same algebraic multiplicity, plus the eigenvalue zero, with algebraic multiplicity $n - p$.

Proof The result follows from Theorem 2.9 and Remark 2.10. □

2.3 Codimension-one Bifurcations

In Proposition 2.14 below, which is a generalization of [9, Proposition 3.1], we use the eigenvalue structure of adjacency matrices of networks G admitting a quotient Q to describe the linearization J of the admissible vector fields for the networks at the bifurcation point.

Consider an n-cell regular network G and denote the total phase space by $P = (\mathbf{R}^k)^n$ where $\mathbf{R}^k$ is the phase space of each cell. Recall that the coupled cell systems associated to such networks have the form $\dot{X} = F(X, \lambda)$, where $X = (x_1, \ldots, x_n) \in P$, $\lambda \in \mathbf{R}$ and the n coordinate functions of F are defined by the same function f. As mentioned above we assume that there exists a synchronous equilibrium in the synchronous subspace $\triangle_0 = \{(x, x, \ldots, x) : x \in \mathbf{R}^k\}$, which we may assume, after a change of coordinates, is at the origin. Let $J = (dF)_{(0,0)}$ and $J^c = J|E^c$, where E^c denotes the center subspace.

Let $\alpha = (d_{x_i} f)_0$ be the linearized internal dynamics and let $\beta = (d_{x_j} f)_0$ be the linearized coupling. Note that α and β are $k \times k$ matrices. Denote the eigenvalues of A by $\mu_1, \ldots, \mu_n$ where μ_1 corresponds to the synchrony eigenvector $(1, \ldots, 1) \in \triangle_0$ and is equal to the valency of the network.

The proof of the following proposition is similar to the one given for Proposition 3.1 in [9].

Proposition 2.14 *The eigenvalues of J are the union of the eigenvalues of the $k \times k$ matrices $\alpha + \mu_j \beta$, $j = 1, \ldots, n$, including algebraic multiplicity.*

The eigenvalues of each of the n matrices $\alpha + \mu_j \beta$ of order $k \times k$ are generically simple. So the possible steady-state bifurcation types do not depend on k, and we assume $k = 1$. In this case the $n \times n$ matrix J has n eigenvalues $\gamma_j = \alpha + \mu_j \beta$, where α and β are 1×1 matrices. Say, γ_1 corresponds to the synchrony eigenvector $(1, \ldots, 1) \in \triangle_0$.

Codimension-one bifurcations divide into *steady-state* (J^c has a zero eigenvalue) and *Hopf* bifurcation (J^c has purely imaginary eigenvalues). Each of these bifurcation types divide into *synchrony-preserving* ($E^c \subset \triangle_0$) and *synchrony-breaking* ($E^c \not\subset \triangle_0$). We focus in this paper on synchrony-breaking steady-state bifurcations from a synchronous equilibrium.

Suppose $\Delta \supset \Delta_0$ is a synchrony subspace of the total phase space. Then $J^c(\Delta \cap E^c) \subseteq \Delta \cap E^c$. Denote the quotient network associated to Δ by Q. Assume a codimension-one steady-state bifurcation occurs in a coupled cell system associated to Q (and hence for G). In this paper we investigate whether this bifurcation also gives rise to branches of steady-state solutions outside of Δ.

3 Two Sample Cases

In this section we apply our results to regular quotient networks with two and three cells.

3.1 Networks with Two-Cell Bidirectional Ring Quotient

Following the methods in Section 2.1, we describe now all connected networks G that admit the two-cell bidirectional ring (see Figure 3) as a quotient network. We prove then that for any such G, the synchrony-breaking bifurcations that occur in G lead only to branches of solutions guaranteed by the quotient two-cell network.

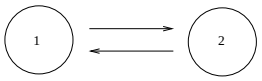

Figure 3: Two-cell bidirectional ring.

The adjacency matrix of the two-cell bidirectional ring quotient network Q is

$$A_Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

An n-cell network admitting Q as quotient has a balanced coloring of two colors such that one group I_1, say of n_1 cells, receives one color, and the other group I_2, say with n_2 cells, receives the other color. Thus $n_1 + n_2 = n$. Moreover, in order for the coloring to be balanced and leading to the given quotient Q, cells inside each group do not connect between them and each cell of one group receives an edge from a cell of the other group. The graphs corresponding to such networks are called *bipartite* graphs. The adjacency matrix of any such network has the form

$$\begin{bmatrix} 0 & Q_{12} \\ Q_{21} & 0 \end{bmatrix}$$

where each Q_{ij} is an $n_i \times n_j$-matrix having rows with one nonzero entry equal to 1. Here, we are enumerating so that the cells from group I_1 appear first. Coupled cell systems consistent with the quotient network have the form

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2, \lambda) \\ \dot{x}_2 &= f(x_2, x_1, \lambda) \end{aligned} \tag{3.6}$$

where $x_1, x_2 \in \mathbf{R}$ and $\lambda \in \mathbf{R}$ is the bifurcation parameter. As mentioned in Section 2.3, there is no loss of generality in assuming that each cell-phase is one-dimensional. The eigenvalues of the adjacency matrix of the two-cell ring are $1, -1$ and the eigenvalues of the Jacobian of the associated systems at the origin are $f_u(0) + f_v(0), f_u(0) - f_v(0)$ where $f_u(0), f_v(0)$ denote the derivative of f with respect to the first and second variable, respectively, evaluated at the origin. (Recall Proposition 2.14.)

Assuming $f(0, 0, \lambda) = 0$ and $f_u(0) - f_v(0) = 0$, generically we have a codimension-one steady-state synchrony-breaking bifurcation from the trivial equilibrium to a branch of steady-state solutions satisfying $x_1 \neq x_2$. Observe that the critical value $f_u(0) - f_v(0)$ is associated with the eigenvalue -1 of the adjacency matrix of the two-cell directed ring. Now, any coupled cell system admitting this quotient network with the restriction to the synchronous subspace given by (3.6) admits this branch of solutions.

The first step to analyze if there are additional branches outside the synchronous subspace is to determine the multiplicity of the critical eigenvalue $f_u(0) - f_v(0)$. This multiplicity is greater than one if and only if the adjacency matrix A_G of G has the eigenvalue -1 with multiplicity greater than one. We show that this cannot happen.

Observe that, as any G has a bipartite structure, we have necessarily an l-cycle where $l \geq 2$: Just fix one cell of one group. Say cell a_1 of group I_1. Then there is a cell of group I_2, say b_2, that has a directed edge to a_1. Again, there is a cell of group I_1 that has a directed edge to cell b_2. If it is a_1, we have a 2-cycle. If not, suppose it is a_2. We repeat this process. There will be a stage where necessarily a cell of one group that has already appeared will connect to a cell of the other group and so close an l-cycle where $l \geq 2$.

Now, if G contains two or more cycles, again because any cell receives only one edge from another cell, these cycles cannot receive inputs from other cells and they can only send inputs. It follows then that the graph is disconnected, which we exclude from our discussion. Therefore, if G is connected, we can enumerate the cells in the graph G such that the adjacency matrix A_G has the following block structure:

$$\begin{bmatrix} C_l & 0 \\ B_1 & B_2 \end{bmatrix}$$

where C_l corresponds to the adjacency matrix of an l-cycle, B_1 is an $(n-l) \times l$-matrix and B_2 is an $(n-l) \times (n-l)$ lower triangular matrix with zero entries at the diagonal.

To see the structure of the matrix B_2 we should consider the subnetwork consisting of the cells not belonging to the l-cycle and the connections between them. There is at least one cell in the subnetwork that does not receive any connection from the other cells in the subnetwork and so the matrix B_2 has at least one row with all the entries equal to zero. The cells of the subnetwork can be rearranged into groups such that the cells in each group are 'sequentially connected'. In the intersection of any two groups of cells there is at most one cell. That common cell receives at most one connection from only one of the cell groups. Thus, there is an ordering of the cells in the subnetwork such that the adjacency matrix B_2 is a lower triangular matrix with zero entries at the diagonal. So, the matrix B_2 has the eigenvalue zero with $(n-l)$ algebraic multiplicity.

Since A_Q has eigenvalues ± 1, it follows then that C_l has eigenvalues ± 1. Moreover, as C_l corresponds to an l-cycle permutation matrix, the eigenvalues are the lth roots of unity. Thus l is even and the real parts of the other eigenvalues of C_l are not equal to -1. Therefore, A_G has the eigenvalue -1 with algebraic multiplicity one and it has no other eigenvalues with real part -1. Thus, the algebraic multiplicity of the critical eigenvalue $f_u(0) - f_v(0)$ of the Jacobian at the origin for the coupled cell systems associated with G is one, and there are no other eigenvalues with real part equal to $f_u(0) - f_v(0)$.

3.2 Networks with Three-cell Bidirectional Ring Quotient

We now consider the three-cell bidirectional ring Q of Figure 1. We observe that Q is the only $\mathbf{S}_3$-symmetric three-cell network which has neither self-coupling nor multiple arrows. (All the other three-cell $\mathbf{S}_3$-symmetric networks are *ODE-equivalent* to Q in the sense that they all generate the same space of admissible vector fields, see Dias and Stewart [4]. Moreover, Q has minimal number of edges among all such networks. Following Aguiar and Dias [1], Q is the *minimal network* of the ODE-class.)

Five-cell Networks: Enumeration

Using the method described in Section 2.1 we find, up to isomorphism, the four-cell and five-cell networks admitting the quotient Q. In Theorem 3.1 we show that there are twelve such five-cell networks, see Figure 5. Analogous computations prove the existence, up to isomorphism, of two four-cell networks admitting the quotient Q, see Figure 4.

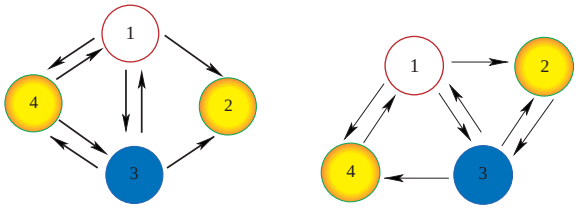

Figure 4: Four-cell networks with the three-cell bidirectional ring as a quotient network.

Theorem 3.1 *Let G be a five-cell network. The network G admits the three-cell bidirectional ring quotient network Q if only if it is isomorphic to one of the twelve coupled cell networks in Figure 5.*

Proof We start by observing that by definition, a network G has the three-cell bidirectional ring quotient network Q if and only if Q is the quotient of G by a balanced equivalence relation $\bowtie$ on the set of the five cells of G having three equivalence classes. Say I_1, I_2, I_3.

Let $C = \{1, 2, 3, 4, 5\}$ be the set of cells of G and $A = [a_{ij}]_{1\leq i,j\leq 5}$ the corresponding adjacency matrix. Let $A_Q = [q_{ij}]_{1\leq i,j\leq 3}$ be the adjacency matrix of the quotient network Q.

By Theorem 2.5, relabeling the cells if necessary, the adjacency matrix A of G satisfies:

$$A = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix}$$

where

$$Q_{ij} \in M_{\#I_i \#I_j}(q_{ij}), \qquad \text{for } i, j = 1, \ldots, 3.$$

As the quotient network Q has no self-coupling, that is, $q_{ii} = 0$ for $i = 1, 2, 3$, we have that Q_{11}, Q_{22} and Q_{33} are zero matrices.

Up to a renumbering of the cells, there are only two possible partitions of the set of five cells into three equivalence classes, I_1, I_2, I_3, as follows:

(a) $\#I_1 = \#I_2 = 1$ and $\#I_3 = 3$,

(b) $\#I_1 = 1$ and $\#I_2 = \#I_3 = 2$.

We consider now the two cases separately.
Case (a)
Since $q_{12} = 1$ and $q_{21} = 1$ we have $Q_{12} = [1]$ and $Q_{21} = [1]$. Moreover, the matrices Q_{31}

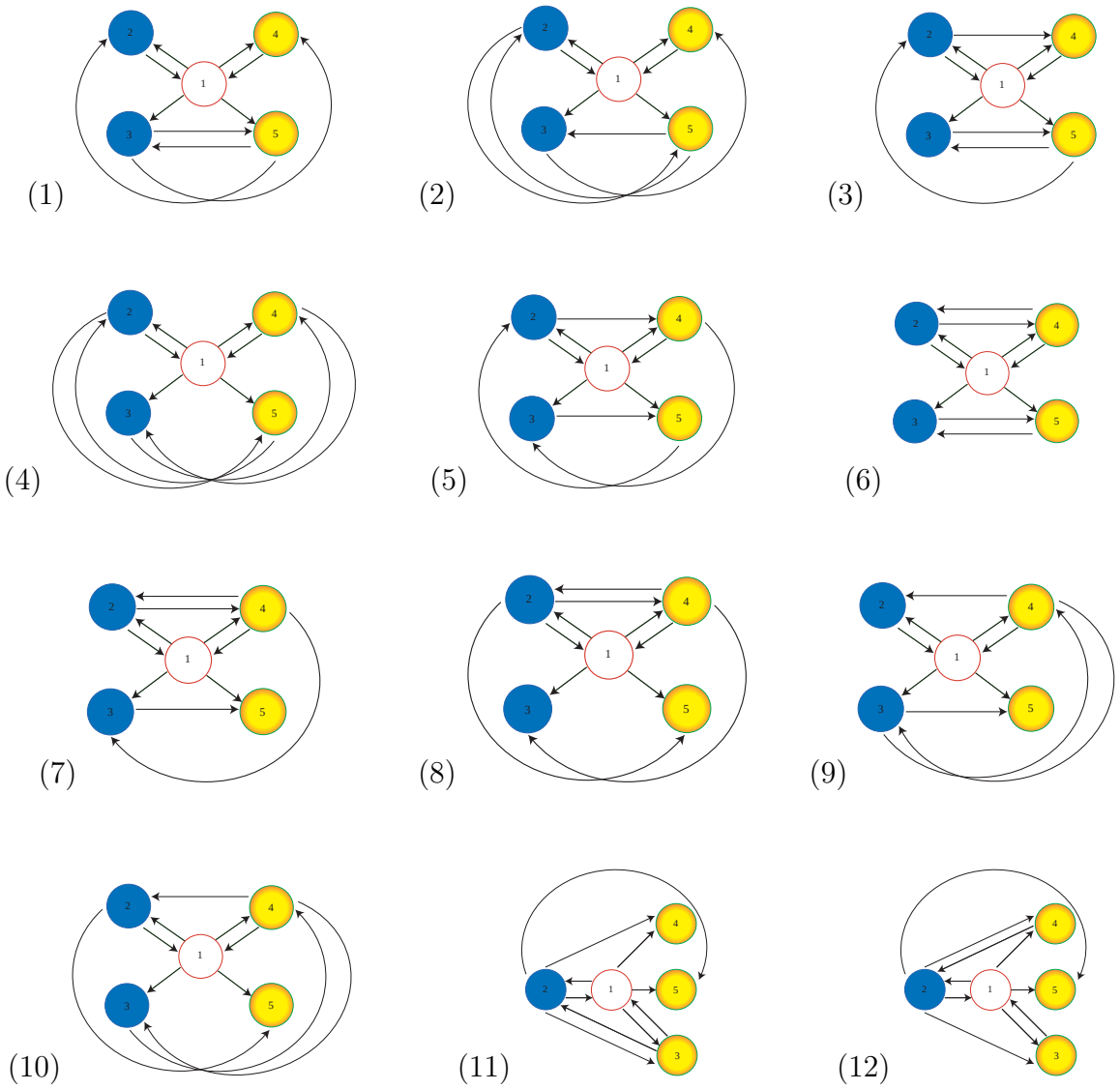

Figure 5: Five-cell networks with the three-cell bidirectional ring as a quotient network.

and Q_{32} are column vectors of order 3×1 with all entries equal to 1 since $q_{31} = 1$ and $q_{32} = 1$. Hence, the structure of A is

$$A = \left[\begin{array}{cc|ccc} 0 & 1 & \multicolumn{3}{c}{Q_{13}} \\ \hline 1 & 0 & \multicolumn{3}{c}{Q_{23}} \\ \hline 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{array}\right].$$

As the quotient network Q has valency 2, this implies that one of the entries in Q_{13} is 1 and the others are zero, and one of the entries in Q_{23} is 1 and the others are zero. This

corresponds to nine hypotheses for the matrix A which, up to isomorphism, correspond to the last two matrices in Table 1, the adjacency matrices of the last two networks in Figure 5.
Case (b)
Since $q_{12} = 1$ and $q_{13} = 1$ we have that the matrices Q_{12} and Q_{13} of order 1×2 have one entry equal to 1 and the other entry is zero. Without loss of generality, we can assume that $Q_{12} = [1\,0]$ and $Q_{13} = [1\,0]$. Moreover, the matrices Q_{21} and Q_{31} are column vectors of order 2×1 with all entries equal to 1 since $q_{21} = 1$ and $q_{31} = 1$. This implies the following structure for A:

$$A = \left[\begin{array}{c|cc|cc} 0 & 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Q_{23}}} \\ 1 & 0 & 0 & & \\ \hline 1 & \multicolumn{2}{c|}{\multirow{2}{*}{Q_{32}}} & 0 & 0 \\ 1 & & & 0 & 0 \end{array}\right].$$

The valency 2 of the quotient network Q implies that

$$Q_{23}, Q_{32} \in \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

which consists in sixteen hypotheses for matrix A. It is easy to see that, up to isomorphism there are only ten hypotheses for A, the first ten matrices in Table 1, which correspond to the adjacency matrices of the first ten networks in Figure 5. □

Five-cell Networks: Codimension-one Steady-state Bifurcation

The coupled cell systems associated to the bidirectional ring Q of Figure 1 satisfy

$$\begin{array}{rcl} \dot{x}_1 & = & f(x_1, \overline{x_2, x_3}, \lambda) \\ \dot{x}_2 & = & f(x_2, \overline{x_1, x_3}, \lambda) \\ \dot{x}_3 & = & f(x_3, \overline{x_1, x_2}, \lambda) \end{array} \tag{3.7}$$

where $x_1, x_2, x_3 \in \mathbf{R}$, the bifurcation parameter is $\lambda \in \mathbf{R}$ and $f(u, v, w, \lambda)$ is a smooth function. The adjacency matrix of Q is

$$A_Q = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

with eigenvalues $2, -1, -1$ and associated eigenspaces

$$E_2 =< (1,1,1) >,\ E_{-1} =< (1,-1,0), (1,0,-1) > .$$

By Proposition 2.14 the eigenvalues of the Jacobian J at the origin, say for $\lambda = 0$, are $f_u(0) - 2f_v(0)$, $f_u(0) - f_v(0)$ and $f_u(0) - f_v(0)$. Assuming $f(0,0,0,\lambda) = 0$ and $f_u(0) = f_v(0)$, it is known that generically there are three branches of steady-state solutions bifurcating from the trivial equilibrium. See for example [6, Ch 1].

(1) $\begin{bmatrix} 0&1&0&1&0\\ 1&0&0&0&1\\ 1&0&0&0&1\\ 1&0&1&0&0\\ 1&0&1&0&0 \end{bmatrix}$	(2) $\begin{bmatrix} 0&1&0&1&0\\ 1&0&0&0&1\\ 1&0&0&0&1\\ 1&0&1&0&0\\ 1&1&0&0&0 \end{bmatrix}$	(3) $\begin{bmatrix} 0&1&0&1&0\\ 1&0&0&0&1\\ 1&0&0&0&1\\ 1&1&0&0&0\\ 1&0&1&0&0 \end{bmatrix}$
(4) $\begin{bmatrix} 0&1&0&1&0\\ 1&0&0&0&1\\ 1&0&0&1&0\\ 1&0&1&0&0\\ 1&1&0&0&0 \end{bmatrix}$	(5) $\begin{bmatrix} 0&1&0&1&0\\ 1&0&0&0&1\\ 1&0&0&1&0\\ 1&1&0&0&0\\ 1&0&1&0&0 \end{bmatrix}$	(6) $\begin{bmatrix} 0&1&0&1&0\\ 1&0&0&1&0\\ 1&0&0&0&1\\ 1&1&0&0&0\\ 1&0&1&0&0 \end{bmatrix}$
(7) $\begin{bmatrix} 0&1&0&1&0\\ 1&0&0&1&0\\ 1&0&0&1&0\\ 1&1&0&0&0\\ 1&0&1&0&0 \end{bmatrix}$	(8) $\begin{bmatrix} 0&1&0&1&0\\ 1&0&0&1&0\\ 1&0&0&1&0\\ 1&1&0&0&0\\ 1&1&0&0&0 \end{bmatrix}$	(9) $\begin{bmatrix} 0&1&0&1&0\\ 1&0&0&1&0\\ 1&0&0&1&0\\ 1&0&1&0&0\\ 1&0&1&0&0 \end{bmatrix}$
(10) $\begin{bmatrix} 0&1&0&1&0\\ 1&0&0&1&0\\ 1&0&0&1&0\\ 1&0&1&0&0\\ 1&1&0&0&0 \end{bmatrix}$	(11) $\begin{bmatrix} 0&1&1&0&0\\ 1&0&1&0&0\\ 1&1&0&0&0\\ 1&1&0&0&0\\ 1&1&0&0&0 \end{bmatrix}$	(12) $\begin{bmatrix} 0&1&1&0&0\\ 1&0&0&1&0\\ 1&1&0&0&0\\ 1&1&0&0&0\\ 1&1&0&0&0 \end{bmatrix}$

Table 1: Adjacency matrices associated to the networks in Figure 5.

We analyze the codimension-one steady-state synchrony-breaking bifurcation imposed by the same degeneracy condition for systems associated to networks with the quotient Q. This bifurcation leads to the three bifurcating solution branches lying in the synchrony subspace associated with Q. Additional branches can only exist for the networks with adjacency matrix admitting the eigenvalue -1 with algebraic multiplicity greater than 2. That is, the center subspace at bifurcation for the corresponding coupled cell systems is not contained in the synchronous subspace associated with the quotient.

By Remark 2.10 (b) it follows immediately that for the four-cell networks with quotient Q the algebraic multiplicity of the eigenvalue -1 does not change and thus no new branches appear. As we show in the next theorem, this is not the case for only two of the five-cell networks with quotient Q.

Theorem 3.2 *Consider the coupled cell systems associated with the twelve five-cell networks (in Figure 5) that admit the three-cell bidirectional ring quotient network Q. Assume that a codimension-one steady-state synchrony-breaking bifurcation associated with the eigenvalue $f_u(0) - f_v(0)$ occurs for the coupled cell systems associated with Q. Then, generically, only the coupled cell systems associated with the two networks 4 and 6 of Figure 5 admit additional branches of steady-state solutions besides the three branches lying*

in the synchrony subspace associated with Q.

Proof As we have remarked before, the critical space of the Jacobian of the coupled cell systems at the bifurcation point is determined by the eigenvalue structure of the adjacency matrix of the network. Recall Proposition 2.14. Specifically, the degeneracy condition $f_u(0) - f_v(0) = 0$ corresponds to the -1 eigenvalue of the adjacency matrix of Q. Additional branches of steady-state solutions for the five-cell coupled cell systems, besides the three branches lying in the synchrony subspace associated with Q, may arise only when the algebraic multiplicity of the eigenvalue -1 of the adjacency matrix increases, i.e., it is greater than 2. All the five-cell networks, except networks $4, 5, 6$ of Figure 5, admit a four-cell quotient network in Figure 4. Hence, by Theorem 2.13, there are no new branches of steady-state solutions for those networks. It remains to analyze the networks $4, 5, 6$ in Figure 5.

By Theorem 2.9 and Remark 2.10, if $A = [a_{ij}]_{1 \le i,j \le 5}$ is the adjacency matrix of a five-cell network with quotient Q, then it is similar to a matrix of the form

$$\begin{bmatrix} A_Q & R \\ 0_{2\times 2} & B \end{bmatrix}, \tag{3.8}$$

where A_Q is the 3×3 adjacency matrix of the bidirectional ring Q of Figure 1 and

$$B = \begin{bmatrix} 0 & a_{35} - a_{25} \\ a_{53} - a_{43} & 0 \end{bmatrix}. \tag{3.9}$$

The two eigenvalues of B depend on whether cell 3 connects or not to cells 4 or 5 and whether cell 5 connects or not to cells 2 or 3. Moreover, they are both zero or symmetric with values ± 1 or $\pm i$, since the trace of B is zero.

For network 5 the matrix B has eigenvalues $\pm i$. For the networks 4 and 6 it has eigenvalues ± 1. So the algebraic multiplicity of the eigenvalue -1 increases only for networks 4 and 6. We prove the generic existence of additional branches of steady-state solutions for the coupled cell systems associated with these two networks in Theorems 3.4 and 3.5. □

Remark 3.3 Observe that network 4 is $\mathbf{Z}_2 =< (24)(35) >$-symmetric and network 6 is $\mathbf{Z}_2 \times \mathbf{Z}_2 =< (24), (35) >$-symmetric. In both cases the symmetry implies matrix B in (3.9) to be symmetric and so to have symmetric real eigenvalues.

The eigenvalue structure for the adjacency matrices for the networks 4 and 6 is summarized in Table 2.

Network 6: Additional Branches of Solutions

Following the discussion given in Section 1, more precisely, using (1.1) it is straightforward to give the form of the admissible vector fields for network 6 of Figure 5

$$\begin{aligned} \dot{x}_1 &= f(x_1, \overline{x_2, x_4}, \lambda) \\ \dot{x}_2 &= f(x_2, \overline{x_1, x_4}, \lambda) \\ \dot{x}_3 &= f(x_3, \overline{x_1, x_5}, \lambda) \\ \dot{x}_4 &= f(x_4, \overline{x_1, x_2}, \lambda) \\ \dot{x}_5 &= f(x_5, \overline{x_1, x_3}, \lambda) \end{aligned}$$

Net	Eigenvalues	Eigenspaces
(4)	$2,-1,-1,-1,1$	$E_2 =< (1,1,1,1,1) >$ $E_{-1} =< (1,-1,-1,0,0),(1,0,0,-1,-1) >$ $E_1 =< (0,-1,1,1,-1) >$
(6)	$2,-1,-1,-1,1$	$E_2 =< (1,1,1,1,1) >$ $E_{-1} =< (1,-1,-1,0,0),(1,0,0,-1,-1),(1,0,-1,-1,0) >$ $E_1 =< (0,0,1,0,1) >$

Table 2: Eigenvalues and eigenspaces of the adjacency matrix of networks 4 and 6 of Figure 5.

where $x_i \in \mathbf{R}$, the bifurcation parameter is $\lambda \in \mathbf{R}$ and $f:\ \mathbf{R}^4 \to \mathbf{R}$ is smooth.

We show that bifurcations for network 6 associated to the critical eigenvalue $f_u(0) - f_v(0) = 0$ lead to nine nontrivial bifurcating branches that are either transcritical or pitchfork.

In order to state the following results we present a list of expressions involving the first and second derivatives of f with respect to the first and the second variable at the origin, that we denote by $f_u(0), f_v(0), f_{uu}(0)$, etc.

Let

$$C = f_{uu}(0) - 2f_{uv}(0) - f_{vv}(0) + 2f_{vw}(0) \tag{3.10}$$

$$D = (f_{uuu}(0) - 3f_{uuv}(0) + 3f_{uvv}(0) - f_{vvv}(0)) \tag{3.11}$$

$$E = f_{uu}(0) + f_{vv}(0) - 2f_{uv}(0) \tag{3.12}$$

$$F = E - \frac{2}{3}\frac{f_u(0)D}{f_{uu}(0) - f_{vv}(0)} \tag{3.13}$$

$$G = 2D - 3\frac{f_{uu}(0) - f_{vv}(0)}{f_u(0)}E \tag{3.14}$$

$$L = \frac{f_{vv}(0) - f_{uu}(0) + 6f_{uv}(0) - 6f_{vw}(0)}{f_{uu}(0) - f_{vv}(0)}\ . \tag{3.15}$$

Theorem 3.4 *Consider a coupled cell system associated to network* 6 *satisfying the following nondegeneracy conditions:*

$$f_u(0) = f_v(0) \neq 0, \qquad f_{u\lambda}(0) - f_{v\lambda}(0) \neq 0, \qquad f_{uu}(0) - f_{vv}(0) \neq 0,$$
$$C \neq 0 \neq F, \qquad G \neq 0 \neq L.$$

Then, there are eight transcritical branches of solutions and one pitchfork branch of solutions bifurcating from the trivial solution. See Table 3 for the form of the solution branches. All these solutions are unstable.

The proof of Theorem 3.4, see Appendix A, consists basically in listing the polydiagonal subspaces of $\mathbf{R}^5$ that are flow-invariant by all admissible vector fields associated with the network 6 and then by showing the existence of eight bifurcating branches of steady-state solutions contained in those flow-invariant subspaces. Finally, we prove that besides those branches there is only one more branch.

Zero Eigenvalue	Eigenvectors	Synchrony	Growth Rates
$f_u(0) - f_v(0)$	$(0,0,1,0,-1)$	$x_1 = x_2 = x_3,\ x_4 = x_5$	$\lambda \quad \lambda \quad \lambda \quad \lambda \quad \lambda$
	$(-2,1,1,1,1)$	$x_1 = x_4 = x_5,\ x_2 = x_3$	$\lambda \quad \lambda \quad \lambda \quad \lambda \quad \lambda$
	$(1,-1,-1,0,0)$	$x_2 = x_3 = x_4 = x_5$	$\lambda \quad \lambda \quad \lambda \quad \lambda \quad \lambda$
		$x_1 = x_2 = x_5,\ x_3 = x_4$	$\lambda \quad \lambda \quad \lambda \quad \lambda \quad \lambda$
		$x_1 = x_3 = x_4,\ x_2 = x_5$	$\lambda \quad \lambda \quad \lambda \quad \lambda \quad \lambda$
		$x_1 = x_4,\ x_3 = x_5$	$\lambda \quad \lambda \quad \lambda \quad \lambda \quad \lambda$
		$x_1 = x_2,\ x_3 = x_5$	$\lambda \quad \lambda \quad \lambda \quad \lambda \quad \lambda$
		$x_1 = x_2 = x_4 = 0$	$0 \quad 0 \quad \lambda^{1/2} \quad 0 \quad \lambda^{1/2}$
		$x_2 = x_4$	$\lambda \quad \lambda \quad \lambda^{1/2} \quad \lambda \quad \lambda^{1/2}$

Table 3: Form of asynchronous branches of equilibria for network 6. The nonzero terms in $x_1, \ldots, x_5$ indicated in the last column are the approximation at lowest order in λ. The three first solution branches are inside the synchrony subspace associated to the quotient bidirectional ring.

Network 4: Additional Branches of Solutions

The form of the coupled systems associated to network 4 is:

$$\begin{aligned}
\dot{x}_1 &= f(x_1, \overline{x_2, x_4}, \lambda) \\
\dot{x}_2 &= f(x_2, \overline{x_1, x_5}, \lambda) \\
\dot{x}_3 &= f(x_3, \overline{x_1, x_4}, \lambda) \\
\dot{x}_4 &= f(x_4, \overline{x_1, x_3}, \lambda) \\
\dot{x}_5 &= f(x_5, \overline{x_1, x_2}, \lambda)
\end{aligned}$$

where we are assuming $x_i \in \mathbf{R}$, the bifurcation parameter is $\lambda \in \mathbf{R}$ and $f : \mathbf{R}^4 \to \mathbf{R}$ is smooth. We show that bifurcations occurring for these vector fields and the critical eigenvalue $f_u(0) - f_v(0)$ (which has algebraic and geometric multiplicity respectively three and two) lead to six nontrivial bifurcating solution branches that are transcritical.

Theorem 3.5 *Consider a coupled cell system associated to network* 4 *satisfying the following nondegeneracy conditions:*

$$f_u(0) = f_v(0) \neq 0, \qquad f_{uu}(0) - f_{vv}(0) \neq 0, \qquad f_{vw}(0) - f_{uv}(0) \neq 0, \tag{3.16}$$

$$f_{u\lambda}(0) - f_{v\lambda}(0) \neq 0, \qquad A \neq 0, \qquad D \neq 0, \tag{3.17}$$

where

$$\begin{aligned}
A &= f_{uu}(0) - 2f_{uv}(0) - f_{vv}(0) + 2f_{vw}(0), \\
D &= \frac{1}{2}\left(f_{uu}(0) + f_{vv}(0) - 2f_{uv}(0)\right) + f_u(0)E, \\
E &= \frac{f_{vvv}(0) - f_{uuu}(0) + 3f_{uuv}(0) - 3f_{uvv}(0)}{3(f_{uu}(0) - f_{vv}(0))}.
\end{aligned} \tag{3.18}$$

Then, there are six transcritical branches of solutions bifurcating from the trivial solution. See Table 4 for the form of the solution branches.

Zero E'value	Generalized E'vectors	Synchrony
$f_u(0) - f_v(0)$	$(1,1,1,-2,-2)$ $(2,-1,-1,-1,-1)$ $(0,-1,2,-1,2)$	$x_1 = x_2 = x_3$, $x_4 = x_5$ $x_1 = x_4 = x_5$, $x_2 = x_3$ $x_2 = x_3 = x_4 = x_5$
		$x_2 = x_4$, $x_3 = x_5$ $x_2 = x_5$ $x_3 = x_4$

Table 4: Form of asynchronous branches of equilibria for network 4. In all solution branches the growth rate in each coordinate $x_1, \ldots, x_5$, at lowest order in λ, is of order λ. The three first branches of solutions are inside the synchronous subspace associated to the quotient bidirectional ring.

Appendix B contains the proof of Theorem 3.5. In the proof we start by considering the polydiagonal subspaces of $\mathbf{R}^5$ that are flow-invariant by all admissible vector fields associated with the network 4 and by computing the bifurcating branches of steady-state solutions contained in those subspaces. We show that there are four such branches. Then we prove the existence of two more nontrivial branches.

4 Conclusions

In the first part of this paper we obtained two general results. First, we developed an algorithm that enumerates all networks that admit a given quotient network. Second, we found necessary conditions that identify those networks with a given quotient, that could exhibit branches not predicted by the quotient. These conditions are given in terms of the eigenvalue structure of the adjacency matrix.

In the second part of the paper we apply our results to two quotient networks. The intuitive answer to the question on how steady-state bifurcations lift from the quotient network seemed to be that in general there would be no additional branches of solutions for the full network besides the ones in the quotient. The answer turned out to be the opposite for one of the two quotient network examples we discuss here. We show that among all the four-cell and five-cell networks that quotient to the three-cell bidirectional ring there are two exhibiting bifurcating solution branches not occurring in the quotient itself. This result is interesting since it shows that sometimes the network architecture forces additional bifurcating branches of solutions other than the ones determined by the quotient network. We also show that there are no new bifurcating solution branches for the systems associated with networks having the two-cell bidirectional ring as a quotient network.

As we increase the number of cells, the number of networks that have a specific quotient network increases exponentially. However, we note that most of the five cell networks (nine out of twelve) admit a four cell quotient network that quotients to the three-cell bidirectional ring. Using the results of Section 2.2 we conclude immediately that no new branches can arise for those five cell networks. This property generalizes to networks with any number of cells. Specifically, it can easily be argued that many of the n-cell networks that quotient to the three-cell bidirectional ring also quotient to an $(n-1)$-cell network

that quotients to the three-cell bidirectional ring. Recursively, we obtain networks with n cells that admit a chain of quotient networks Q_k with k cells for $3 \leq k \leq n-1$ such that Q_3 is the three-cell bidirectional ring. Using the results of Section 2.2 again we conclude that no new branches arise for that chain of networks. Those networks certainly form a big subset of all the networks that quotient to the three-cell bidirectional ring.

We remark that the two networks with five cells that quotient to the three-cell ring and have additional branches (networks 4 and 6) are symmetric. Further more only the symmetry of the network 4 leaves invariant the synchrony subspace associated to the three-cell ring. As we have proved, some of the additional solution branches of network 6 are forced by the symmetry. But the other seem not to be explained by the symmetry. It would be interesting to clarify in general the relation between the existence of symmetry and the occurrence of branches of solutions besides the ones in the quotient.

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A Proof of Theorem 3.4

In Table 5 we list the polydiagonal subspaces of $\mathbf{R}^5$ that are flow-invariant by all admissible vector fields associated with the network 6.

Symmetry	$\mathbf{Z}_2 \times \mathbf{Z}_2 =< (24), (35) >$
Synchrony subspaces	$\Delta_{00} = \{x : \ x_2 = x_4\}$ $\Delta_{01} = \{x : \ x_3 = x_5\}$ $\Delta_{02} = \{x : \ x_1 = x_2\}$ $\Delta_{03} = \{x : \ x_1 = x_4\}$ $\Delta_1 = \{x : \ x_2 = x_3, \ x_4 = x_5\}$ $\Delta_2 = \{x : \ x_1 = x_2 = x_4\} = \mathrm{Fix}(< (124), \ (12) >)$ $\Delta_3 = \{x : \ x_2 = x_4, \ x_3 = x_5\} = \mathrm{Fix}(< (24), \ (35) >)$ $\Delta_4 = \{x : \ x_2 = x_5, \ x_3 = x_4\}$ $\Delta_5 = \{x : \ x_3 = x_5, \ x_1 = x_4\}$ $\Delta_6 = \{x : \ x_1 = x_2, \ x_3 = x_5\}$ $\Delta_{11} = \{x : \ x_1 = x_2 = x_3, \ x_4 = x_5\} \subseteq \Delta_1$ $\Delta_{12} = \{x : \ x_2 = x_3, \ x_1 = x_4 = x_5\} \subseteq \Delta_1$ $\Delta_{13} = \{x : \ x_2 = x_3 = x_4 = x_5\} \subseteq \Delta_1, \Delta_4, \Delta_3$ $\Delta_{21} = \{x : \ x_1 = x_2 = x_4, \ x_3 = x_5\} \subseteq \Delta_2, \Delta_3, \Delta_5, \Delta_6$ $\Delta_{41} = \{x : \ x_1 = x_2 = x_5, \ x_3 = x_4\} \subseteq \Delta_4$ $\Delta_{43} = \{x : \ x_1 = x_3 = x_4, \ x_2 = x_5\} \subseteq \Delta_4$

Table 5: Symmetry and synchrony subspaces associated to the five-cell network 6 of Figure 5.

The critical space associated with the Jacobian at the origin is given by

$$E_{f_u(0)-f_v(0)} = \{x \in \mathbf{R}^5 : \ x_4 = -x_1 - x_2, \ x_5 = -x_1 - x_3\}.$$

We start by proving the existence of eight bifurcating branches of steady-state solutions contained in the flow-invariant subspaces Δ_1, Δ_4, Δ_6, Δ_5 and Δ_2. We end by proving that besides those branches there is only one more branch.

Consider the coupled cell systems associated to network 6. Those systems restricted to Δ_1 have the form

$$\begin{aligned} \dot{x}_1 &= f(x_1, \overline{x_2, x_4}, \lambda) \\ \dot{x}_2 &= f(x_2, \overline{x_1, x_4}, \lambda) \\ \dot{x}_4 &= f(x_4, \overline{x_1, x_2}, \lambda) \end{aligned}$$

and are $\mathbf{S}_3$-symmetric. They correspond to the coupled cell systems associated to the bidirectional ring in Figure 1. Observe that $E_{f_u(0)-f_v(0)} \cap \Delta_1$ is two-dimensional. The bifurcations for these systems have been studied in [6, Ch 1]. Provided the nondegeneracy conditions $f_{u\lambda}(0) - f_{v\lambda}(0) \neq 0$ and $C \neq 0$ are satisfied, codimension-one bifurcations lead to three nontrivial transcritical symmetry related branches whose form is given in Table 3. We obtain branches of solutions in the following flow-invariant subspaces of Δ_1: Δ_{11}, Δ_{12} and Δ_{13}. Moreover, using the $\mathbf{Z}_2 \times \mathbf{Z}_2$-symmetry of the network 6 we obtain two more transcritical branches in the flow-invariant subspaces Δ_{41} and Δ_{43}. Note that the coupled cell systems associated with the network 6 restricted to the flow-invariant space Δ_4 also correspond to the admissible vector fields for the bidirectional ring of Figure 1.

The coupled cell systems associated with the network 6 restricted to Δ_6 have the form

$$\begin{aligned} \dot{x}_1 &= f(x_1, \overline{x_1, x_4}, \lambda) \\ \dot{x}_3 &= f(x_3, \overline{x_1, x_3}, \lambda) \\ \dot{x}_4 &= f(x_4, \overline{x_1, x_1}, \lambda) \end{aligned} \ .$$

These correspond to the coupled cell systems associated with the three-cell network 26 that appears in [9], see Figure 6. We can conclude that there is one transcritical bifurcating branch in Δ_6 provided $f_{u\lambda}(0) - f_{v\lambda}(0) \neq 0$ and $C \neq 0$ (C in (3.10)). Using the $\mathbf{Z}_2 \times \mathbf{Z}_2$-symmetry of the network 6 we obtain one more transcritical branch in the flow-invariant space Δ_5.

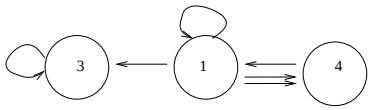

Figure 6: Three-cell network 26 in [9].

Now, taking the coupled cell systems associated with the network 6 restricted to the flow-invariant subspace $\{x : \ x_1 = x_2 = x_4 = 0\} \subseteq \Delta_2$ we obtain

$$\begin{aligned} \dot{x}_3 &= f(x_3, 0, x_5, \lambda) \\ \dot{x}_5 &= f(x_5, 0, x_3, \lambda) \end{aligned} \tag{A.19}$$

which are $\mathbf{Z}_2$-symmetric. Observe that

$$E_{f_u(0)-f_v(0)} \cap \{x : \ x_1 = x_2 = x_4 = 0\} =< (0, 0, 1, 0, -1) >$$

where the $\mathbf{Z}_2\times\mathbf{Z}_2$-symmetry of the network 6 acts as $-\mathrm{Id}$. Generically, we have a pitchfork branch of solutions bifurcating from the trivial solution provided $f_{u\lambda}(0) - f_{v\lambda}(0) \neq 0$ and $G \neq 0$, where G is given in (3.14).

We end the proof by showing that generically, there exists only one more transcritical branch of solutions bifurcating from the trivial solution in the flow-invariant subspace $x_2 = x_4$ denoted by Δ_{00}.

For the systems associated with network 6 the equations for cells $1, 2, 4$ decouple from the equations for cells $3, 5$ and are $\mathbf{S}_3$-symmetric:

$$\begin{array}{l} f(x_1, \overline{x_2, x_4}, \lambda) = 0 \\ f(x_2, \overline{x_1, x_4}, \lambda) = 0 \\ f(x_4, \overline{x_1, x_2}, \lambda) = 0 \end{array} \cdot$$

Thus, the codimension-one synchrony-breaking bifurcations in cells $1, 2, 4$ lead to three symmetry-breaking transcritical branches where two of the cells are synchronized and the trivial branch. Denote the first three branches by:

- $(x_1, x_2, x_4, \lambda) = (A(\lambda), A(\lambda), B(\lambda), \lambda)$;
- $(x_1, x_2, x_4, \lambda) = (A(\lambda), B(\lambda), A(\lambda), \lambda)$;
- $(x_1, x_2, x_4, \lambda) = (B(\lambda), A(\lambda), A(\lambda), \lambda)$.

Easy computations show that

$$A'(0) = \frac{2}{C}\left(f_{u\lambda}(0) - f_{v\lambda}(0)\right), \tag{A.20}$$

$$B'(0) = -2A'(0) \tag{A.21}$$

where C is given in (3.10).

Each of these branches and the trivial branch can be entered into the equations for cells $3, 5$ obtaining a system in the variables x_3, x_5:

$$\begin{array}{l} f(x_3, \overline{x_1(\lambda), x_5}, \lambda) = 0 \\ f(x_5, \overline{x_1(\lambda), x_3}, \lambda) = 0 \end{array} . \tag{A.22}$$

Observe that since $f(u, \overline{v, w}, \lambda)$ is invariant in the v, w variables, $f_u(0) = f_v(0)$ and $f(0, 0, 0, \lambda) \equiv 0$, the Taylor expansion of f at the origin is

$$\begin{aligned} f(u, \overline{v, w}, \lambda) &= f_u(0)(u + v + w) + \tfrac{1}{2}f_{uu}(0)u^2 + \tfrac{1}{2}f_{vv}(0)(v^2 + w^2) + f_{vw}(0)vw \\ &\quad + f_{uv}(0)u(v + w) + f_{u\lambda}(0)u\lambda + f_{v\lambda}(0)(v + w)\lambda + O(3). \end{aligned}$$

Moreover, $f(u, \overline{v, w}, \lambda) - f(w, \overline{v, u}, \lambda)$ vanishes when $u = w$. Hence,

$$f(u, \overline{v, w}, \lambda) - f(w, \overline{v, u}, \lambda) = (u - w)h(u, v, w, \lambda) \tag{A.23}$$

where

$$h(u, v, w, \lambda) = \frac{1}{2}(f_{uu}(0) - f_{vv}(0))(u + w) + (f_{uv}(0) - f_{vw}(0))v + (f_{u\lambda}(0) - f_{v\lambda}(0))\lambda + O(2)\,. \tag{A.24}$$

Thus, (A.22) is equivalent to

$$\begin{array}{l} f(x_3, \overline{x_1(\lambda), x_5}, \lambda) = 0 \\ f(x_3, \overline{x_1(\lambda), x_5}, \lambda) - f(x_5, \overline{x_1(\lambda), x_3}, \lambda) = 0 \end{array} \tag{A.25}$$

and so to

$$\begin{array}{l} f(x_3, \overline{x_1(\lambda), x_5}, \lambda) = 0 \\ (x_3 - x_5) h(x_3, x_1(\lambda), x_5, \lambda) = 0 \end{array} \tag{A.26}$$

where

$$\begin{aligned} h(x_3, x_1(\lambda), x_5, \lambda) &= \tfrac{1}{2}\left(f_{uu}(0) - f_{vv}(0)\right)(x_3 + x_5) + \left(f_{uv}(0) - f_{vw}(0)\right) x_1(\lambda) \\ &\quad + \left(f_{u\lambda}(0) - f_{v\lambda}(0)\right)\lambda + O(2). \end{aligned}$$

We distinguish the following three cases:
(a) If $x_1(\lambda) \equiv 0$ in (A.26) we obtain (A.19) deriving the pitchfork branch and the trivial branch of solutions in the flow-invariant space Δ_2.

(b) If $x_1(\lambda) \equiv A(\lambda)$ in (A.26) then:
(b.i) If $x_3 = x_5$ and $(x_1, x_2, x_4) = (A(\lambda), A(\lambda), B(\lambda))$ then we have solutions that satisfy $x_3 = x_5$ and $x_1 = x_2$. Thus, we obtain the transcritical branch of solutions in the flow-invariant space Δ_6. If $x_3 = x_5$ and $(x_1, x_2, x_4) = (A(\lambda), B(\lambda), A(\lambda))$ then we have solutions that satisfy $x_3 = x_5$ and $x_1 = x_4$. Thus, we obtain the transcritical branch of solutions in the flow-invariant space Δ_5.
(b.ii) If $x_3 \neq x_5$ then (A.26) is equivalent to

$$\begin{array}{l} f(x_3, \overline{A(\lambda), x_5}, \lambda) = 0 \\ h(x_3, A(\lambda), x_5, \lambda) = 0 \end{array} \tag{A.27}$$

Assuming $f_{uu}(0) - f_{vv}(0) \neq 0$, we can solve the second equation for example for x_5 as a function of x_3 and λ obtaining $x_5 = X_5(x_3, \lambda)$ where $X_5(0, 0) = 0$,

$$X_5(x_3, \lambda) = -x_3 - A'(0)\lambda + O(2) \tag{A.28}$$

and $A'(0)$ is given in (A.20). Substitution of (A.28) into the first equation in (A.27) leads to an equation in the two variables x_3, λ:

$$g(x_3, \lambda) \equiv f(x_3, \overline{A(\lambda), X_5(x_3, \lambda)}, \lambda).$$

Implicit differentiation with respect to x_3, λ and evaluation at the origin show that

$$g(0) = 0,\ g_{x_3}(0) = 0,\ g_\lambda(0) = 0,\ g_{x_3 x_3}(0) \neq 0,\ g_{\lambda\lambda}(0) \neq 0$$

with

$$\begin{aligned} g_{x_3 x_3}(0) &= F \\ g_{\lambda\lambda}(0) &= f_u(0)\left(A''(0) + \frac{\partial^2 X_5(x_3, \lambda)}{\partial \lambda^2}|_0\right) + 2A'(0)^2\left(f_{vv}(0) - f_{vw}(0)\right) \end{aligned} \tag{A.29}$$

where A' and G are given respectively in (A.20) and (3.13), and

$$\begin{aligned}
\left.\frac{\partial^2 X_5(x_3,\lambda)}{\partial x_3^2}\right|_0 &= -\tfrac{2}{3}\frac{D}{f_{uu}-f_{vv}} \\
\left.\frac{\partial^2 X_5(x_3,\lambda)}{\partial \lambda^2}\right|_0 &= \frac{2}{f_{uu}-f_{vv}}\left[f_{v\lambda\lambda}-f_{u\lambda\lambda}+(f_{vw}-f_{uv})A''-\frac{1}{3}D(A')^2+MA'\right] \\
M &= f_{uu\lambda}-f_{vv\lambda}+2f_{vw\lambda}-2f_{uv\lambda} \\
A'' &= \frac{2}{C}(f_{u\lambda\lambda}-f_{v\lambda\lambda})+\frac{4}{3f_uC}(f_{v\lambda}-f_{u\lambda})^2\,.
\end{aligned}$$

The derivatives of f and $A(\lambda)$ are all evaluated at the origin and C, D are given in (3.10), (3.11).

We have two transcritical branches if $g_{x_3x_3}(0)g_{\lambda\lambda}(0)<0$. As we know that

$$\begin{aligned}
f(A(\lambda),\overline{A(\lambda),B(\lambda)},\lambda)=0 \\
f(B(\lambda),\overline{A(\lambda),A(\lambda)},\lambda)=0,
\end{aligned} \tag{A.30}$$

it follows this is the case. That is, the system (A.27) in the variables x_3, x_5, λ has the two transcritical branches of solutions $(x_3,x_5,\lambda)=(A(\lambda),B(\lambda),\lambda)$ and $(x_3,x_5,\lambda)=(B(\lambda),A(\lambda),\lambda)$. Thus we obtain the transcritical branches in the flow-invariant spaces Δ_{11} and Δ_{41} if $(x_1,x_2,x_4)=(A(\lambda),A(\lambda),B(\lambda))$ and the transcritical branches in the flow-invariant spaces Δ_{43} and Δ_{12} if $(x_1,x_2,x_4)=(A(\lambda),B(\lambda),A(\lambda))$.

(c) If $x_1(\lambda)\equiv B(\lambda)$ in (A.26) then:
(c.i) If $x_3=x_5$ as $(x_1,x_2,x_4)=(B(\lambda),A(\lambda),A(\lambda))$, we obtain solutions satisfying $x_3=x_5$ and $x_2=x_4$. Thus we obtain the transcritical branch of solutions in the flow-invariant space Δ_{13}.
(c.ii) If $x_3\neq x_5$ then (A.26) is equivalent to

$$\begin{aligned}
f(x_3,B(\lambda),x_5,\lambda)=0 \\
h(x_3,B(\lambda),x_5,\lambda)=0
\end{aligned} \tag{A.31}$$

Again, since we are assuming $f_{uu}(0)-f_{vv}(0)\neq 0$, we can solve the second equation for example for x_5 as a function of x_3 and λ obtaining $x_5=X_5(x_3,\lambda)$ where $X_5(0,0)=0$,

$$X_5(x_3,\lambda)=-x_3+LA'\lambda+O(2) \tag{A.32}$$

with L given in (3.15). Substitution of (A.32) into the first equation in (A.31) leads to an equation in the two variables x_3,λ:

$$g(x_3,\lambda) \equiv f(x_3,\overline{B(\lambda),X_5(x_3,\lambda)},\lambda).$$

Direct calculations show that

$$g(0)=0,\ g_{x_3}(0)=0,$$

$$g_\lambda(0)=-6f_u(0)\frac{f_{u\lambda}(0)-f_{v\lambda}(0)}{f_{uu}(0)-f_{vv}(0)}\neq 0$$

and $g_{x_3x_3}(0)$ is given by (A.29), which is nonzero by assumption. Thus, we obtain a transcritical branch of solutions in the flow-invariant subspace $x_2 = x_4$ where $(x_1, x_2, x_4) = (B(\lambda), A(\lambda), A(\lambda))$ that we call of *parabolic* type since the branch is transcritical in the variables x_1, x_2, x_4 and cells x_3, x_5 have a rate of growth of order $\lambda^{1/2}$.

We end the proof with a final remark on the instability of the solutions. The instability of the solutions of the branches obtained in cases (b) and (c) follows immediately from the following two facts: equations for cells 1, 2 and 4 decouple from the ones for cells 3 and 5 implying a block structure for the Jacobian matrix at any point; equations for cells 1, 2 and 4 are $\mathbf{S}_3$-symmetric and it is known that in this case the non-trivial solutions are generically unstable. The solutions of the pitchfork branch obtained in case (a) correspond to the trivial branch of solutions of the system with equations for cells 1, 2 and 4. In this case, if the branch is supercritical the instability follows from the instability of the trivial branch of solutions for λ positive near zero. □

B Proof of Theorem 3.5

In Table 6 we list the synchronous polydiagonal subspaces of $\mathbf{R}^5$ that are flow-invariant by all admissible vector fields associated to the network 4.

The critical space at the origin is given by

$$E_{f_u(0)-f_v(0)} = \{x \in \mathbf{R}^5 : \ x_1 = -x_3 - \frac{4}{3}x_4 + \frac{1}{3}x_5, x_2 = x_3 + x_4 - x_5\}.$$

Consider the coupled cell systems associated to network 4. We have that these restricted to Δ_1 have the form

$$\begin{aligned} \dot{x}_1 &= f(x_1, \overline{x_2, x_4}, \lambda) \\ \dot{x}_2 &= f(x_2, \overline{x_1, x_4}, \lambda) \\ \dot{x}_4 &= f(x_4, \overline{x_1, x_2}, \lambda) \end{aligned}$$

and are $\mathbf{S}_3$-symmetric. These systems correspond to the coupled cell systems associated to the network in Figure 1. As mentioned before, under the assumptions $f_{u\lambda}(0) - f_{v\lambda}(0) \neq 0$

Symmetry	$\mathbf{Z}_2 =< (24)(35) >$
Synchrony subspaces	$\Delta_{00} = \{x : \ x_3 = x_4\}$ $\Delta_{01} = \{x : \ x_2 = x_5\}$ $\Delta_1 = \{x : \ x_2 = x_3, \ x_4 = x_5\}$ $\Delta_2 = \{x : \ x_2 = x_4, \ x_3 = x_5\} = \mathrm{Fix}(< (24)(35) >)$ $\Delta_3 = \{x : \ x_2 = x_5, \ x_3 = x_4\}$ $\Delta_{11} = \{x : \ x_1 = x_2 = x_3, \ x_4 = x_5\} \subseteq \Delta_1$ $\Delta_{12} = \{x : \ x_2 = x_3, \ x_1 = x_4 = x_5\} \subseteq \Delta_1$ $\Delta_{13} = \{x : \ x_2 = x_3 = x_4 = x_5\} \subseteq \Delta_1$

Table 6: Symmetry and synchrony subspaces associated to the five-cell network 4 of Figure 5.

and $A \neq 0$, codimension-one bifurcations lead to three nontrivial transcritical symmetry related branches whose form is given in Table 4. That is, we obtain branches of solutions in the following flow-invariant subspaces of Δ_1: Δ_{11}, Δ_{12} and Δ_{13}. The solutions are unstable in the directions in the Δ_1 subspace.

Next we consider the flow-invariant subspace Δ_2. The coupled cell systems restricted to Δ_2 have the form

$$\begin{aligned}\dot{x}_1 &= f(x_1, \overline{x_2, x_2}, \lambda)\\ \dot{x}_2 &= f(x_2, \overline{x_1, x_3}, \lambda)\\ \dot{x}_3 &= f(x_3, \overline{x_1, x_2}, \lambda)\end{aligned}$$

with synchronous space $x_2 = x_3$. These systems correspond to the coupled cell vector fields associated with the three-cell network 11 in [9]. See Figure 7. Hence, by [9, Theorem 4.14] we conclude that there are two transcritical bifurcating branches of unstable solutions, provided $f_{u\lambda}(0) - f_{v\lambda}(0) \neq 0$ and $A \neq 0$. One of these branches occurs in the intersection of the plane $x_2 = x_3$ with Δ_2, that is, it lies in the space Δ_{13}. Observe that $E_{f_u(0)-f_v(0)} \cap \Delta_{13}$ is one-dimensional. Hence, this transcritical branch is unique and is the one identified previously.

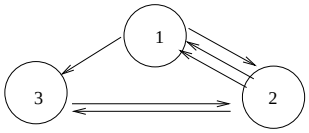

Figure 7: Three-cell network 11 in [9].

We show now that there are two more nontrivial branches in addition to these four transcritical branches. The proof consists in studying all the possible solutions of the following system:

$$\begin{aligned}&f(x_1, \overline{x_2, x_4}, \lambda) = 0\\ &f(x_2, \overline{x_1, x_5}, \lambda) = 0\\ &f(x_3, \overline{x_1, x_4}, \lambda) = 0\\ &f(x_4, \overline{x_1, x_3}, \lambda) = 0\\ &f(x_5, \overline{x_1, x_2}, \lambda) = 0.\end{aligned} \tag{B.33}$$

To find the solutions of (B.33) we apply (A.23) to cells 2,5 equations and cells 3,4 equations and we obtain

$$f(x_2, \overline{x_1, x_5}, \lambda) - f(x_5, \overline{x_1, x_2}, \lambda) = (x_2 - x_5)h(x_2, x_1, x_5, \lambda) = 0 \tag{B.34}$$

$$f(x_3, \overline{x_1, x_4}, \lambda) - f(x_4, \overline{x_1, x_3}, \lambda) = (x_3 - x_4)h(x_3, x_1, x_4, \lambda) = 0 \tag{B.35}$$

where h is defined by (A.24).

Thus, there are the following four possibilities for solutions of (B.33):

1. $x_2 = x_5, x_3 = x_4$;
2. $x_2 = x_5, h(x_3, x_1, x_4, \lambda) = 0$;
3. $x_3 = x_4, h(x_2, x_1, x_5, \lambda) = 0$;
4. $h(x_2, x_1, x_5, \lambda) = 0 = h(x_3, x_1, x_4, \lambda)$.

Case 1: $x_2 = x_5, x_3 = x_4$**.** This possibility corresponds to the restriction of network 4 equations to Δ_3. The restricted coupled cell systems have the form

$$\begin{aligned}\dot{x}_1 &= f(x_1, \overline{x_2, x_3}, \lambda)\\ \dot{x}_2 &= f(x_2, \overline{x_1, x_2}, \lambda)\\ \dot{x}_3 &= f(x_3, \overline{x_1, x_3}, \lambda)\end{aligned}$$

which correspond to the three-cell network 10 in [9]. See Figure 8. Hence, by [9, Table 3]

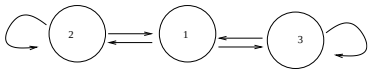

Figure 8: Three-cell network 10 in [9].

we conclude that there exists a unique transcritical bifurcating branch satisfying $x_2 = x_3 = x_4 = x_5$. This branch lies in Δ_{13} and it was identified previously.

Case 2: $x_2 = x_5, h(x_3, x_1, x_4, \lambda) = 0$**.** In this case to find the solutions of (B.33) is equivalent to solve the following system:

$$h(x_3, x_1, x_4, \lambda) = 0 \tag{B.36}$$
$$f(x_1, \overline{x_2, x_4}, \lambda) = 0 \tag{B.37}$$
$$f(x_2, \overline{x_1, x_2}, \lambda) = 0 \tag{B.38}$$
$$f(x_3, \overline{x_1, x_4}, \lambda) = 0. \tag{B.39}$$

By (A.24) it follows that

$$h(x_3, x_1, x_4, \lambda) = \frac{1}{2}(x_3+x_4)(f_{uu}(0)-f_{vv}(0))+(f_{uv}(0)-f_{vw}(0))x_1+(f_{u\lambda}(0)-f_{v\lambda}(0))\lambda+O(2).$$

Let

$$B = 2\frac{f_{vw}(0) - f_{uv}(0)}{f_{uu}(0) - f_{vv}(0)}, \qquad C = 2\frac{f_{v\lambda}(0) - f_{u\lambda}(0)}{f_{uu}(0) - f_{vv}(0)}.$$

Next we assume (3.16) and we solve (B.36) for x_4, which yields

$$x_4 = X_4(x_1, x_3, \lambda) = -x_3 + Bx_1 + C\lambda + O(2). \tag{B.40}$$

Substitution of (B.40) into (B.37)-(B.39) leads to

$$\begin{aligned}&f(x_1, \overline{x_2, X_4(x_1, x_3, \lambda)}, \lambda) = 0\\ &f(x_2, \overline{x_1, x_2}, \lambda) = 0\\ &f(x_3, \overline{x_1, X_4(x_1, x_3, \lambda)}, \lambda) = 0.\end{aligned} \tag{B.41}$$

The implicit function theorem guarantees that there exists a unique branch of solutions $X(\lambda) = (x_1(\lambda), x_2(\lambda), x_3(\lambda))$ of the system (B.41) satisfying $X(0) = 0$ provided $f_{uu}(0) - f_{vv}(0) \neq 0$ and $A \neq 0$.

Substitution of (B.40) into (B.39) leads to

$$g(x_3, x_1, \lambda) \equiv f(x_3, \overline{x_1, -x_3 + Bx_1 + C\lambda + O(2)}, \lambda) = 0.$$

A straightforward calculations shows that

$$\frac{\partial g}{\partial x_1}(0) = \frac{f_u(0)A}{f_{uu}(0) - f_{vv}(0)}$$

which is nonzero by (3.16) and (3.17). Hence, the implicit function theorem guarantees that there exists a unique solution $x_1 = X(x_3, \lambda)$ satisfying $X(0) = 0$ and

$$g(x_3, X(x_3, \lambda), \lambda) = f(x_3, \overline{X(x_3, \lambda), -x_3 + BX(x_3, \lambda) + C\lambda + O(2)}, \lambda) \equiv 0.$$

Implicit differentiation of g with respect to x_3 and λ, and evaluation at the origin leads to

$$X_{x_3}(0) = 0, \qquad X_\lambda(0) = -\frac{2}{A}(f_{v\lambda} - f_{u\lambda}).$$

Note that $X_\lambda(0) \neq 0$ since (3.17) holds. Hence,

$$x_1 = X_\lambda(0)\lambda + O(2) = -\frac{2}{A}(f_{v\lambda} - f_{u\lambda})\lambda + O(2). \tag{B.42}$$

Next we consider (B.38). Observe that

$$\frac{\partial f}{\partial x_2}(0) = f_u(0) + f_v(0)$$

which is nonzero since $f_u(0) = f_v(0) \neq 0$. Therefore, the implicit function theorem implies that there exists a unique solution $x_2 = Y(x_1, \lambda)$ such that $Y(0) = 0$ and

$$f(Y(x_1, \lambda), \overline{x_1, Y(x_1}, \lambda), \lambda) \equiv 0.$$

A straightforward calculation using implicit differentiation of f with respect to x_1 and λ shows that

$$Y_{x_1}(0) = -\frac{1}{2}, \qquad Y_\lambda(0) = 0.$$

Hence, $x_2 = -\frac{1}{2}x_1 + O(2)$. Substitution of (B.42) in this expression yields

$$x_2 = \frac{1}{A}(f_{v\lambda} - f_{u\lambda})\lambda + O(2). \tag{B.43}$$

To complete the analysis of case 2 we consider the equation (B.37). Substitution of (B.43), (B.40) and (B.42) into (B.37) leads to

$$l(x_3, \lambda) \equiv f(X(x_3, \lambda), \overline{Y(X(x_3, \lambda), \lambda), X_4(X(x_3, \lambda), x_3, \lambda)}, \lambda) = 0.$$

Implicit differentiation of l with respect to λ shows that the Taylor expansion of l at the origin is given by

$$l(x_3, \lambda) = f_v(0)\left(-x_3 + \frac{f_{v\lambda}(0) - f_{u\lambda}(0)}{A}\lambda\right) + O(2).$$

Hence, the solution of $l = 0$ is

$$x_3 = \frac{1}{A}(f_{v\lambda}(0) - f_{u\lambda}(0))\lambda + O(2). \tag{B.44}$$

Substitution of (B.44) into (B.40) yields

$$x_4 = \frac{1}{A}(f_{v\lambda} - f_{u\lambda})\lambda + O(2)$$

where A is given in (3.18). This branch of solutions is of transcritical type and is in the flow-invariant subspace satisfying $x_2 = x_5$. Moreover, it coincides at linear order (in λ) with the solution branch in the flow invariant subspace Δ_{13}. However, as we prove next, this is a new bifurcating solution branch and it does not lie in Δ_{13}.

Suppose that the transcritical solution branch we have found was in Δ_{13}. Under this assumption the solution branch must satisfy $x_3 = x_4$ and solve the system (B.36)-(B.39). That is, it is a solution of the following system

$$h(x_3, x_1, x_3, \lambda) = 0 \tag{B.45}$$
$$f(x_1, \overline{x_2, x_3}, \lambda) = 0 \tag{B.46}$$
$$f(x_2, \overline{x_1, x_2}, \lambda) = 0 \tag{B.47}$$
$$f(x_3, \overline{x_1, x_3}, \lambda) = 0. \tag{B.48}$$

Observe that the system consisting of equations (B.46)-(B.48) is precisely the one obtained in case 1. The only nontrivial solution of this system satisfies $x_2 = x_3 = x_4 = x_5$. Thus, to find the solutions of system (B.45)-(B.48) is equivalent to solve the system:

$$h(x_2, x_1, x_2, \lambda) = 0 \tag{B.49}$$
$$f(x_1, \overline{x_2, x_2}, \lambda) = 0 \tag{B.50}$$
$$f(x_2, \overline{x_1, x_2}, \lambda) = 0. \tag{B.51}$$

Note that (A.23) with $v = w$ can be applied to (B.50) and (B.51) reducing (B.49)-(B.51) to the following system

$$h(x_2, x_1, x_2, \lambda) = 0 \tag{B.52}$$
$$f(x_1, \overline{x_2, x_2}, \lambda) = 0 \tag{B.53}$$
$$(x_1 - x_2)h(x_1, x_2, x_2, \lambda) = 0. \tag{B.54}$$

There are the following two possibilities for the solutions of (B.52)-(B.54):
(a) $x_1 = x_2$ and $h(x_2, x_1, x_2, \lambda) = 0 = f(x_1, \overline{x_2, x_2}, \lambda)$;
(b) $h(x_2, x_1, x_2, \lambda) = 0 = h(x_1, x_2, x_2, \lambda) = 0 \quad$ and $\quad f(x_1, \overline{x_2, x_2}, \lambda) = 0$.

In case (a) the solutions are in the fully synchronous subspace. Hence, there is only the trivial solution branch which is not the case of the transcritical solution branch we have found.

In case (b) a straightforward calculation shows that the Jacobian of the system formed by the three equations, evaluated at the origin, has determinant given by

$$\frac{3}{2}\left(f_{u\lambda}(0) - f_{v\lambda}(0)\right) f_u(0) A$$

with A given in (3.18). Since we are assuming the nondegeneracy conditions $f_u(0) \neq 0$, $f_{u\lambda}(0) - f_{v\lambda}(0) \neq 0$, and $A \neq 0$ it follows that the determinant is nonzero.

Thus the only solution of system (B.45)-(B.48) is the trivial one. That is, the only solution of system (B.36)-(B.39) satisfying $x_3 = x_4$ is the trivial one. Since the transcritical solution branch we have found is nonzero at linear order, we conclude that it is outside Δ_{13} and it is a new bifurcating branch.

Case 3: $x_3 = x_4, \quad h(x_1, x_2, x_5, \lambda) = 0$**.** The analysis of this case follows precisely the steps described in Case 2, with the variables x_3, x_4 playing the role of the variables x_2, x_5 in Case 2. Similarly to case 2, we obtain a unique transcritical branch that is in the flow-invariant subspace $x_3 = x_4$. Equivalently, the $\mathbf{Z}_2 =< (24)(35) >$-symmetry of network 4 applied to the branch of solutions obtained in case 2 gives the same result.

Case 4: $h(x_2, x_1, x_5, \lambda) = 0 = h(x_3, x_1, x_4, \lambda)$**.** To find solutions of (B.33) satisfying conditions in case 4 we solve the system consisting of equations (B.36), (B.37), (B.39), and

$$h(x_2, x_1, x_5, \lambda) = 0 \tag{B.55}$$
$$f(x_2, \overline{x_1, x_5}, \lambda) = 0. \tag{B.56}$$

Observe that (B.36) was solved in Case 2 leading to (B.40). Using a similar procedure we solve (B.55) leading to

$$x_5 = X_5(x_1, x_2, \lambda) = -x_2 + Bx_1 + C\lambda + O(2). \tag{B.57}$$

Next we substitute (B.40) (considering the Taylor expansion of $X_4(x_1, x_3, \lambda)$ around the origin of degree 2) into (B.39) and (B.57)) (considering the Taylor expansion of $X_5(x_1, x_2, \lambda)$ around the origin of degree 2) into (B.56). Subtracting the resulting expressions we obtain

$$(x_2 - x_3)M(x_1, x_2, x_3, \lambda) = 0 \tag{B.58}$$

where

$$\begin{aligned} M(x_1, x_2, x_3, \lambda) \;=\; & (x_2 + x_3)\left(\frac{1}{2}(f_{uu} + f_{vv} - 2f_{uv}) + f_u(0)E\right) \\ & +[(1+B)f_{uv} - f_{vw} - Bf_{vv} - BEf_u(0)]x_1 \\ & +[C(f_{uv} - f_{vv}) + f_{u\lambda} - f_{v\lambda} - CEf_u(0)]\lambda + O(2). \end{aligned}$$

Hence, there are the following two possibilities for solutions of (B.58):
(a) $x_2 = x_3$;
(b) $M(x_1, x_2, x_3, \lambda) = 0$.

(a) $x_2 = x_3$. It is straightforward to see that by (B.57) and (B.40) the condition $x_2 = x_3$ implies $x_4 = x_5$. Recall that $\Delta_1 = \{x_2 = x_3, x_4 = x_5\}$ is a flow-invariant subspace and the solutions on this subspace were previously identified.

(b) $M(x_1, x_2, x_3, \lambda) = 0$. In this case we have to solve the following system:

$$\begin{aligned} &f(x_1, \overline{x_2, X_4(x_1, x_3, \lambda)}, \lambda) = 0 \\ &f(x_3, \overline{x_1, X_4(x_1, x_3, \lambda)}, \lambda) = 0 \\ &M(x_1, x_2, x_3, \lambda) = 0. \end{aligned} \tag{B.59}$$

By the implicit function theorem, provided $A \neq 0$, $f_{uu}(0) - f_{vv}(0) \neq 0$, $f_u(0) \neq 0$ and $D \neq 0$, we have a unique branch of solutions $(X_1(\lambda), X_2(\lambda), X_3(\lambda))$ such that $X_1(0) = X_2(0) = X_3(0) = 0$. This branch must correspond to the solution branch in Δ_2 not lying in Δ_{13}. □