Finiteness results for subgroups of finite extensions

Vítor Araújo

Universidade Federal da Bahia, Instituto de Matemática,
Av. Adhemar de Barros, S/N, Ondina,
40170-110 Salvador-BA, Brazil
email: vitor.d.araujo@ufba.br

Pedro V. Silva

Universidade Federal da Bahia, Instituto de Matemática, Brazil
and Centro de Matemática, Faculdade de Ciências, Universidade do Porto,
R. Campo Alegre 687, 4169-007 Porto, Portugal
email: pvsilva@fc.up.pt

Mihalis Sykiotis

Department of Mathematics, National and Kapodistrian University of Athens,
Panepistimioupolis, GR-157 84, Athens, Greece
e-mail: msykiot@math.uoa.gr

February 1, 2014

2010 Mathematics Subject Classification: 20E06, 20E07, 20E22

Keywords: finite extensions, Howson's Theorem, Hanna Neumann Conjecture, Takahasi's Theorem, periodic subgroups

ABSTRACT

We discuss in the context of finite extensions two classical theorems of Takahasi and Howson on subgroups of free groups. We provide bounds for the rank of the intersection of subgroups within classes of groups such as virtually free groups, virtually nilpotent groups or fundamental groups of finite graphs of groups with virtually polycyclic vertex groups and finite edge groups. As an application of our generalization of Takahasi's Theorem, we provide an uniform bound for the rank of the periodic subgroup of any endomorphism of the fundamental group of a given finite graph of groups with finitely generated virtually nilpotent vertex groups and finite edge groups.

1 Introduction

Some famous theorems on subgroups of free groups involve finiteness conditions. Part of them admit generalizations to further classes of groups, and constructions such as free products, finite extensions or graphs of groups have been involved in most of them.

For instance, Howson's Theorem states that the intersection of two finitely generated subgroups H, K of a free group is also finitely generated. In his seminal paper [11], Howson also provided an upper bound on the rank of $H \cap K$ with respect to the ranks of H and K, namely (for H and K nontrivial):

$$\text{rk}(H \cap K) \leq 2\text{rk}(H)\text{rk}(K) - \text{rk}(H) - \text{rk}(K) + 1.$$

Later on, Hanna Neumann improved this upper bound to

$$\text{rk}(H \cap K) \leq 2(\text{rk}(H) - 1)(\text{rk}(K) - 1) + 1$$

and conjectured that the factor 2 could be removed, the famous *Hanna Neumann Conjecture*. The Conjecture was finally proved in 2011 by Friedman and Mineyev (independently):

Theorem 1.1 [7, 20] *Let F be a free group and let $K_1, K_2 \leq F$ be finitely generated and nontrivial. Then*

$$\text{rk}(K_1 \cap K_2) \leq (\text{rk}(K_1) - 1)(\text{rk}(K_2) - 1) + 1.$$

Howson's Theorem led to the concept of Howson group: a group G is a *Howson group* if the intersection of finitely generated subgroups of G is still finitely generated. Kapovich has shown that many hyperbolic groups fail this property [13], but it is easy to show that Howson groups are closed under finite extension and so in particular virtually free groups are Howson groups. More generally, the class of Howson groups is closed under graphs of groups, where the edge groups are finite (see [30, Theorem 2.13 (1)] for a proof). But can we get some rank formula as in the case of free groups? A recent paper of Zakharov [33] provides an upper bound for the rank of the intersection of two *free* finitely generated subgroups of a virtually free group. In the case of free products, upper bounds for the Kurosh rank of the intersection of subgroups have been obtained by various authors. See for instance [1], and the references therein, where Theorem 1.1 is extended to free products of right-orderable groups.

We introduce in Section 3 the concept of *strongly Howson group*, when an uniform bound for the rank of $H \cap K$ can be obtained from bounds on the ranks of H and K. We show that the class of strongly Howson groups is closed under finite extensions and compute bounds using an improved version of Schreier's Lemma, which can be obtained with the help of Stallings automata. These bounds are then applied to several particular cases such as virtually free, virtually polycyclic, virtually nilpotent, and more generally fundamental groups of finite graphs of groups with virtually polycyclic vertex groups and finite edge groups.

Another famous result, known as Takahasi's Theorem, states the following:

Theorem 1.2 [31] *Let F be a free group and let $K_1 \leq K_2 \leq \ldots$ be an ascending chain of finitely generated subgroups of F. If the rank of the subgroups in the chain is bounded, then the chain is stationary.*

Bogopolski and Bux proved recently an analogue of Takahasi's Theorem for fundamental groups of closed compact surfaces [5, Proposition 2.2]. We say that a group G is a *Takahasi group* if every ascending chain $H_1 \leq H_2 \leq \cdots$ of subgroups each of rank $\leq M$ in G, is stationary. We prove, in

Section 4, that the class of Takahasi groups is closed under finite extensions and finite graphs of groups with virtually polycyclic vertex groups and finite edge groups.

We provide an application of the generalized Takahasi's Theorem in Section 5. Finally, using previous work of the third author [29], we show that the periodic subgroup is finitely generated for every endomorphism of the fundamental group of a finite graph of groups with finitely generated virtually nilpotent vertex groups and finite edge groups. As a consequence, we can bound the periods for each particular endomorphism of such a group.

2 Preliminaries

We collect in this section some standard group-theoretic concepts and results. The reader is referred to [9, 14, 18] for details.

Given a group G and $X \subseteq G$, we denote by $\langle X \rangle$ the subgroup of G generated by X. If G is finitely generated, the *rank* of G is defined as

$$\mathrm{rk}(G) = \min\{|X| : G = \langle X \rangle\}.$$

We denote by F_A the free group on an alphabet A. A free group of rank n is generically denoted by F_n. The standard way of describing finitely generated subgroups of a free group is by means of Stallings automata, a construction designed by Stallings under a different formalism [28].

To simplify things, we define an *automaton* to be a structure of the form $\mathcal{A} = (A, Q, q_0, T, E)$ where:

- A is a finite alphabet;
- Q is a set (vertices);
- $q_0 \in Q$ (initial vertex);
- $T \subseteq Q$ (terminal vertices);
- $E \subseteq Q \times A \times Q$ (edges).

The automaton is finite if Q is finite.

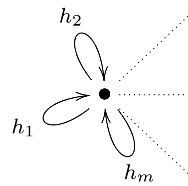
A *finite nontrivial path* in $\mathcal{A}$ is a sequence

$$p_0 \xrightarrow{a_1} p_1 \xrightarrow{a_2} \ldots \xrightarrow{a_n} p_n$$

with $(p_{i-1}, a_i, p_i) \in E$ for $i = 1, \ldots, n$. Its *label* is the word $a_1 \ldots a_n \in A^*$. It is said to be a *successful* path if $p_0 = q_0$ and $p_n \in T$. We consider also the *trivial path* $p \xrightarrow{1} p$ for $p \in Q$. It is successful if $p = q_0 \in T$.

The *language* $L(\mathcal{A})$ *recognized by* $\mathcal{A}$ is the set of all labels of successful paths in $\mathcal{A}$. For details on automata, the reader is referred to [4, 24].

Let $H \leq F_A$ be finitely generated. Taking a finite set of generators $h_1, \ldots, h_n$ of H in reduced form, we start with the so-called flower automaton $\mathcal{F}(H)$ (on the alphabet $A \cup A^{-1}$), where *petals* (of variable length) labelled by the words h_i are glued to a basepoint q_0 (both initial and terminal):



We include also an edge of the form $q \xrightarrow{a^{-1}} p$ for every edge of the form $p \xrightarrow{a} q$. Then we proceed by successively folding pairs of edges of the form $q \xleftarrow{a} p \xrightarrow{a} r$ ($a \in A \cup A^{-1}$). The final automaton $\mathcal{S}(H)$ does not depend on the folding sequence nor even on the original finite generating set, and is known as the *Stallings automaton* of H. For details and applications of Stallings automata, see [2, 14, 19].

One of the classical applications of Stallings automata provides a solution for the generalized word problem of F_A (see [2, Proposition 2.5]): given $u \in F_A$ in reduced form, we have

$$u \in H \quad \Leftrightarrow \quad u \in L(\mathcal{S}(H)). \tag{1}$$

Another famous application (see [2, Proposition 2.6]) is the rank formula

$$\mathrm{rk}(H) = e - v + 1,$$

where v denotes the number of vertices of $\mathcal{S}(H)$ and e denotes the number of positive edges of $\mathcal{S}(H)$ (i.e. edges labelled by letters of A). In the particular case where H is of finite index in F_A, we get $v = [F_A : H]$ and $e = [F_A : H]|A|$, hence (see [2, Proposition 2.8])

$$\mathrm{rk}(H) = [F_A : H](|A| - 1) + 1. \tag{2}$$

Given a class $\mathcal{C}$ of groups, we say that a group G is:

- *virtually* $\mathcal{C}$ if G has a finite index subgroup in $\mathcal{C}$;
- $\mathcal{C}$*-by-finite* if G has a finite index normal subgroup in $\mathcal{C}$.

If the class $\mathcal{C}$ is closed under isomorphism and taking subgroups, then the two concepts coincide. That is the case for free, nilpotent, polycyclic and strongly polycyclic groups.

If F is a finite index subgroup of G, we also say that G is a *finite extension* of F. If $[G : F] = m$, we may decompose G as a disjoint union of right cosets

$$G = Fb_1 \cup \ldots \cup Fb_m \tag{3}$$

with $b_1 = 1$. We shall refer to (3) as a *standard decomposition* of G with respect to F.

The next simple result is essential to handle subgroups of finite extensions:

Proposition 2.1 *Let G be a finite extension of a group F with standard decomposition (3). Let $H \leq G$ and write $K = H \cap F$. Then there exist $I \subseteq \{2, \ldots, m\}$ and $x_i \in F$ $(i \in I)$ such that*

$$H = K \cup (\bigcup_{i \in I} Kx_ib_i). \tag{4}$$

Proof. Let

$$I = \{i \in \{2, \ldots, m\} \mid H \cap Fb_i \neq \emptyset\}.$$

Since $b_1 = 1$, we may write

$$H = K \cup (\bigcup_{i \in I} K_i b_i)$$

for some nonempty $K_i \subseteq F$ ($i \in I$). For each $i \in I$, fix $x_i \in K_i$. It remains to be proved that $K_i = Kx_i$.

Clearly, $Kx_i b_i \subseteq HK_i b_i \subseteq H^2 = H$, hence $Kx_i \subseteq K_i$. Conversely, let $y \in K_i$. Then $yx_i^{-1} = (yb_i)(x_i b)^{-1} \in HH^{-1} = H$. Since also $yx_i^{-1} \in K_i K_i^{-1} \subseteq FF^{-1} = F$, we get $yx_i^{-1} \in K$ and so $y \in Kx_i$. Thus $K_i = Kx_i$ as required. □

Now we recall the definitions of several other classes of groups which play a part in this paper.

A group G is *residually finite* if the intersection of all normal subgroups of finite index is equal to 1. Since any subgroup of finite index in a group G contains a normal subgroup of finite index in G, it follows that a finite extension of a residually finite group is residually finite.

Let G be a group. Given $H, K \leq G$, write

$$[H, K] = \langle hkh^{-1}k^{-1} : h \in H \; k \in K \rangle \leq G.$$

The *lower central series* of G is the sequence

$$G = G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \ldots,$$

where $G_n = [G, G_{n-1}]$ for every $n \geq 1$. The group G is *nilpotent* if $G_n = \{1\}$ for some $n \geq 1$. The minimum such n is the *nilpotency class* of G. Clearly, an abelian group is nilpotent of nilpotency class ≤ 1. A subgroup of a nilpotent group of class n is nilpotent of class $\leq n$.

A group G is called *polycyclic* if it admits a subnormal series

$$G = G_0 \triangleright G_1 \triangleright \ldots \triangleright G_n = \{1\} \tag{5}$$

such that G_{i-1}/G_i is cyclic for $i = 1, \ldots, n$. The minimum such n is the *polycyclic rank* of G and is denoted by $\mathrm{prk}(G)$. The *Hirsch number* $h(G)$ of G is defined as the number of infinite factors G_{i-1}/G_i, which is independent from the subnormal series. In particular, $h(G) \leq \mathrm{prk}(G)$. If G_{i-1}/G_i is infinite cyclic for every i, we say that G is *strongly polycyclic*. Every polycyclic group has a normal strongly polycyclic group of finite index. By [10], polycyclic groups are residually finite.

The class of (strongly) polycyclic groups is closed under taking subgroups. Moreover, a simple induction on $\mathrm{prk}(G)$ shows that if G is polycyclic then

$$\mathrm{rk}(H) \leq \mathrm{prk}(G) \quad \text{for every } H \leq G. \tag{6}$$

A nilpotent group is polycyclic if and only if it is finitely generated. For details on polycyclic groups, see [32].

Finally, we recall the concept of graph of groups, central in Bass-Serre theory [27].

Following Serre, a *graph* is a structure of the form $\Gamma = (V, E, \alpha, ^{-})$, where:

- V is a nonempty set (vertices);
- E is a set (edges);

- $\alpha : E \to V$ is a mapping;
- $^- : E \to E$ is an involution without fixed points.

Concepts such as cycle, connectedness, tree or subgraph are defined the obvious way. If Γ is connected and $T \subseteq E$ defines a subtree of Γ connecting all the vertices, we say that T is a *spanning tree* of Γ.

A (finite) *graph of groups* over a (finite) connected graph Γ is a structure of the form

$$\mathcal{G} = ((G_v)_{v\in V}, (G_e)_{e\in E}, (\alpha_e)_{e\in E}), \tag{7}$$

where:

- the G_v are groups for all $v \in V$ (vertex groups);
- the G_e are groups for all $e \in E$ (edge groups) satisfying $G_{\bar{e}} = G_e$;
- the $\alpha_e : G_e \to G_{e\alpha}$ are monomorphisms for all $e \in E$ (boundary monomorphisms).

The *fundamental group* $\pi_1(\mathcal{G}, T)$ of the graph of groups (7) with respect to a spanning tree T of Γ is the quotient of the free product

$$(*_{v\in V} G_v) * F_E$$

by the normal subgroup generated by the following elements:

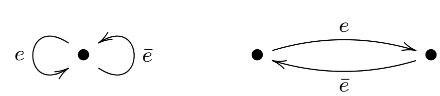
- $e\bar{e}$ $(e \in E)$;
- $t \in T$;
- $e^{-1}(g\alpha_e)e(g\alpha_{\bar{e}})^{-1}$ $(e \in E,\ g \in G_e)$.

The vertex groups are naturally embedded into $\pi_1(\mathcal{G}, T)$, which is independent of the chosen spanning tree T, up to isomorphism.

If the edge groups G_e are all trivial, then we get a free product

$$\pi_1(\mathcal{G}, T) = (*_{v\in V} G_v) * F_A, \tag{8}$$

where $E \setminus T = A \cup \bar{A}$ and $A \cap A^{-1} = \emptyset$. HNN extensions and amalgamated free products constitute important particular cases of this construction, by taking graphs with two edges, respectively of the form



Moreover, whenever Γ is finite, the fundamental group $\pi_1(\mathcal{G}, T)$ can be built from the vertex groups using a finite number of HNN extensions and amalgamated free products, where the associated/amalgamated subgroups are of the form $G_e\alpha_e$.

The nature of $\pi_1(\mathcal{G}, T)$ is conditioned by the nature of the vertex and edge groups. This is illustrated by the following well-known theorem of Karrass, Pietrowski and Solitar [16] (see also [26, Theorem 7.3]): a finitely generated group is virtually free if and only if it is the fundamental group of a finite graph of finite groups.

3 Howson's Theorem

We say that a group G is *strongly Howson* if

$$\sup\{\text{rk}(H_1 \cap H_2) \mid H_1, H_2 \leq G,\ \text{rk}(H_1) \leq n_1,\ \text{rk}(H_2) \leq n_2\} < \infty \tag{9}$$

for all $n_1, n_2 \in \mathbb{N}$. In this case, we can define a function $\xi_G : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by letting (9) be $(n_1, n_2)\xi_G$. Since every subgroup of a cyclic group is cyclic, we only care about the nontrivial cases $n_1, n_2 \geq 2$.

Clearly, if G is strongly Howson and $H \leq G$, then H is strongly Howson and $(n_1, n_2)\xi_H \leq (n_1, n_2)\xi_G$ for all $n_1, n_2 \in \mathbb{N}$.

Trivially, every strongly Howson group is a Howson group. We ignore if the converse is true.

Schreier's Lemma [25] (see also [9]) states that the inequality

$$\text{rk}(H) \leq [G : H]\text{rk}(G)$$

holds whenever H is a finite index subgroup of a finitely generated group G. The following improved version is well-known, but we give a proof for completeness, using Stallings automata:

Proposition 3.1 *Let H be a finite index subgroup of a finitely generated group G. Then* $\text{rk}(H) \leq [G : H](\text{rk}(G) - 1) + 1$.

Proof. Let $m = [G : H]$ and $n = \text{rk}(G)$. Then there exists an epimorphism $\varphi : F_n \to G$. It is straightforward that $[F_n : H\varphi^{-1}] = [G : H] = m$, hence it follows from (2) that

$$\text{rk}(H\varphi^{-1}) = [F_n : H\varphi^{-1}](n-1) + 1 = m(n-1) + 1.$$

Since $H = (H\varphi^{-1})\varphi$ yields $\text{rk}(H) \leq \text{rk}(H\varphi^{-1})$, we are done. □

Note that this bound is tight in view of (2).

Now we can prove the following result:

Theorem 3.2 *Let G be a finite extension of a strongly Howson group F and let $m = [G : F]$. Then G is strongly Howson and*

$$(n_1, n_2)\xi_G \leq (m(n_1 - 1) + 1, m(n_2 - 1) + 1)\xi_F + m - 1$$

for all $n_1, n_2 \geq 1$.

Proof. Let $H_1, H_2 \leq G$ with $\text{rk}(H_j) \leq n_j$ for $j = 1, 2$. We must show that

$$\text{rk}(H_1 \cap H_2) \leq (m(n_1 - 1) + 1, m(n_2 - 1) + 1)\xi_F + m - 1. \tag{10}$$

By Proposition 2.1, for $j = 1, 2$ we may write

$$H_j = K_j \cup (\bigcup_{i \in I_j} K_j x_i^{(j)} b_i)$$

with $K_j = H_j \cap F$, $I_j \subseteq \{2, \ldots, m\}$ and $x_i^{(j)} \in F$ $(i \in I_j)$.

For all $h, h' \in H_j$,

$$Fh = Fh' \Rightarrow h'h^{-1} \in F \Rightarrow h'h^{-1} \in H_j \cap F = K_j \Rightarrow K_j h = K_j h',$$

hence

$$[H_j : K_j] \leq [G : F] = m$$

and Proposition 3.1 yields

$$\mathrm{rk}(K_j) \leq m(\mathrm{rk}(H_j) - 1) + 1 \leq m(n_j - 1) + 1. \tag{11}$$

On the other hand, writing $K = K_1 \cap K_2 = H_1 \cap H_2 \cap F$, it follows from Proposition 2.1 that

$$H_1 \cap H_2 = K \cup (\bigcup_{i \in I} K y_i b_i)$$

for some $I \subseteq \{2, \dots, m\}$ and $y_i \in F$ $(i \in I)$. Since

$$H_1 \cap H_2 = \langle K \cup \{y_i b_i \mid i \in I\} \rangle,$$

we get

$$\mathrm{rk}(H_1 \cap H_2) \leq \mathrm{rk}(K) + |I| \leq \mathrm{rk}(K) + m - 1. \tag{12}$$

In view of (11), we get

$$\mathrm{rk}(K) \leq (m(n_1 - 1) + 1, m(n_2 - 1) + 1)\xi_F$$

and so (12) yields

$$\mathrm{rk}(H_1 \cap H_2) \leq (m(n_1 - 1) + 1, m(n_2 - 1) + 1)\xi_F + m - 1.$$

Therefore (10) holds as required. $\square$

We apply Theorem 3.2 to some classes of groups, starting with the straightforward virtually free case:

Corollary 3.3 *Let G be a virtually free group with a free subgroup F of index m. Then G is strongly Howson and*

$$(n_1, n_2)\xi_G \leq m^2(n_1 - 1)(n_2 - 1) + m$$

for all $n_1, n_2 \geq 1$.

Proof. By Theorem 1.1, we have

$$(k_1, k_2)\xi_F \leq (k_1 - 1)(k_2 - 1) + 1$$

for all $k_1, k_2 \geq 1$. By Theorem 3.2, we get

$$\begin{aligned}(n_1, n_2)\xi_G &\leq (m(n_1 - 1) + 1, m(n_2 - 1) + 1)\xi_F + m - 1 \leq (m(n_1 - 1))(m(n_2 - 1)) + m \\ &= m^2(n_1 - 1)(n_2 - 1) + m\end{aligned}$$

In particular, G is strongly Howson. $\square$

In a recent paper, Zakharov proved the following theorem:

Theorem 3.4 [33, Theorem 2] *Let G be a virtually free group and let $H_1, H_2 \leq G$ be finitely generated, free and nontrivial. Then*

$$\mathrm{rk}(H_1 \cap H_2) \leq 6n(\mathrm{rk}(H_1) - 1)(\mathrm{rk}(H_2) - 1) + 1,$$

where n is the maximum of orders $|P \cap (H_1H_2)|$ over all finite subgroups P of G. As a consequence,

$$\mathrm{rk}(H_1 \cap H_2) \leq 6m(\mathrm{rk}(H_1) - 1)(\mathrm{rk}(H_2) - 1) + 1$$

if G has a free subgroup of index m.

How does the upper bound arising from Corolllary 3.3 compare with the upper bounds in Theorem 3.4? In general, for arbitrary free subgroups, the bounds in Theorem 3.4 are smaller since they are linear on m and ours are quadratic. However, we claim that our bound is actually smaller than the second bound in Theorem 3.4 if $m \leq 5$ and H_1, H_2 are noncyclic (if H_1 or H_2 is cyclic, so is $H_1 \cap H_2$ and we have a trivial case anyway). Indeed, the product $p = (\mathrm{rk}(H_1) - 1)(\mathrm{rk}(H_2) - 1)$ is then positive and so

$$\begin{array}{l} m^2(\mathrm{rk}(H_1) - 1)(\mathrm{rk}(H_2) - 1) + m < 6m(\mathrm{rk}(H_1) - 1)(\mathrm{rk}(H_2) - 1) + 1 \\ \Leftrightarrow m^2p + m \leq 6mp \Leftrightarrow mp + 1 \leq 6p \Leftrightarrow mp < 6p \Leftrightarrow m < 6. \end{array}$$

The following example shows that our bound may also beat the first bound provided by Theorem 3.4:

Example 3.5 *Let $A = \{a, b, c\}$ and let C_2 be a cyclic group of order 2. Let $G = F_A \times C_2$ and*

$$H_1 = \langle (a, 1), (bc, 1) \rangle, \quad H_2 = \langle (ab, 1), (c, 0) \rangle.$$

Then:

(i) H_1 and H_2 are free subgroups of rank 2 of the virtually free group G;

(ii) Theorem 3.4 provides the upper bound $\mathrm{rk}(H_1 \cap H_2) \leq 13$*;*

(iii) Theorem 3.2 provides the upper bound $\mathrm{rk}(H_1 \cap H_2) \leq 6$*;*

(iii) actually, $\mathrm{rk}(H_1 \cap H_2) = 1$.

Indeed, $F_A \times \{0\}$ is a free subgroup of index 2 of G, hence G is virtually free.

It is easy to see that projecting H_1 into its first component we get a free group with basis $\{a, bc\}$, and we can deduce from that fact that H_1 is itself free of rank 2. Similarly, H_2 is free of rank 2.

Let $P = \{1\} \times C_2 \leq G$. It is easy to check that $|P \cap H_1H_2| = 2$, e.g.

$$(1, 1) = (a, 1)(bc, 1)(c, 0)^{-1}(ab, 1)^{-1} \in H_1H_2$$

and so we get the upper bound $\mathrm{rk}(H_1 \cap H_2) \leq 13$ from Theorem 3.4. On the other hand, it is immediate that Corollary 3.3 yields the upper bound $\mathrm{rk}(H_1 \cap H_2) \leq 6$.

Finally, with the help of the standard algorithm to compute a basis for the intersection in free groups [14, Proposition 9.4], it is easy to check that

$$\langle a, bc \rangle \cap \langle ab, c \rangle = \langle abc \rangle.$$

It follows easily that

$$H_1 \cap H_2 = \langle ((abc)^2, 0) \rangle$$

and so $\mathrm{rk}(H_1 \cap H_2) = 1$.

We present further applications of Theorem 3.2:

Corollary 3.6 *Let G be a virtually polycyclic group. Then G is strongly Howson and ξ_G is a bounded function.*

Proof. Let P be a polycyclic subgroup of G of index m. Let $n = \mathrm{prk}(G)$. By (6), we have

$$(n_1, n_2)\xi_P \leq n$$

for all $n_1, n_2 \in \mathbb{N}$. By Theorem 3.2, we get $(n_1, n_2)\xi_G \leq n + m - 1$ for all $n_1, n_2 \geq 1$. Thus ξ_G is bounded and G is strongly Howson. □

The general virtually nilpotent case is a bit harder. Note that a non finitely generated nilpotent group is not polycyclic.

Theorem 3.7 *Let G be a virtually nilpotent group. Then G is strongly Howson and*

$$(n_1, n_2)\xi_G \leq \frac{(m(p-1)+1)^{n+1} - m(p-1) - 1}{m(p-1)} + m - 1$$

for all $n_1, n_2 \geq 2$ and $p = \min\{n_1, n_2\}$.

Proof. Suppose that N is a nilpotent group of class n and rank $k \geq 2$. We claim that

$$\mathrm{rk}(H) \leq \frac{k^{n+1} - k}{k - 1} \tag{13}$$

for every $H \leq N$.

Let

$$N = N_0 \rhd N_1 \rhd \ldots \rhd N_n = \{1\} \tag{14}$$

be the lower central series of N. By [9, Corollary 10.3], we have

$$\mathrm{rk}(N_{i-1}/N_i) \leq k^i \tag{15}$$

for $i = 1, \ldots, n$. Since $[N_{i-1}, N_{i-1}] \subseteq [N, N_{i-1}] = N_i$, the quotient N_{i-1}/N_i is abelian. By (15), there exist $x_1, \ldots, x_{k^i} \in N_{i-1}$ such that

$$N_{i-1}/N_i = \langle x_1 N_i, \ldots x_{k^i} N_i \rangle.$$

Let $\pi_i : N_{i-1} \to N_{i-1}/N_i$ be the canonical projection. For $j = 0, \ldots, k^i$, let

$$N_{i,j} = \langle x_1 N_i, \ldots x_j N_i \rangle \pi_i^{-1}.$$

Since $\langle x_1 N_i, \ldots x_{j-1} N_i \rangle \trianglelefteq \langle x_1 N_i, \ldots x_j N_i \rangle$ due to N_{i-1}/N_i being abelian, we get $N_{i,j-1} \trianglelefteq N_{i,j}$ and so we have a chain

$$N_{i-1} = N_{i,k^i} \trianglerighteq \ldots \trianglerighteq N_{i,1} \trianglerighteq N_{i,0} = N_i. \tag{16}$$

Moreover,

$$N_{i,j}/N_{i,j-1} = \langle x_1N_i, \ldots x_jN_i\rangle\pi_i^{-1} \,/\, \langle x_1N_i, \ldots x_{j-1}N_i\rangle\pi_i^{-1} \cong \langle x_1N_i, \ldots x_jN_i\rangle/\langle x_1N_i, \ldots x_{j-1}N_i\rangle$$

and is therefore cyclic since N_{i-1}/N_i is abelian.

Inserting the chains (16) into (14), we obtain a subnormal series for N with length

$$k + k^2 + \ldots + k^n = \frac{k^{n+1} - k}{k-1}$$

and cyclic quotients. In particular, N is polycyclic. Now (13) follows from (6).

Assume now that N is a nilpotent subgroup of G of class n and index m. Let $n_1, n_2 \geq 2$ and suppose that $H_1, H_2 \leq N$ are such that $\mathrm{rk}(H_j) \leq n_j$ for $j = 1, 2$. Since each H_j is also nilpotent of class $\leq n$ and $H_1 \cap H_2 \leq H_j$, (13) yields

$$\mathrm{rk}(H_1 \cap H_2) \leq \frac{n_j^{n+1} - n_j}{n_j - 1}$$

and so, writing $p = \min\{n_1, n_2\}$, we get

$$(n_1, n_2)\xi_N \leq p + p^2 + \ldots + p^n = \frac{p^{n+1} - p}{p-1}.$$

In particular, N is strongly Howson and we may apply Theorem 3.2 to get

$$(n_1, n_2)\xi_G \leq (m(n_1 - 1) + 1, m(n_2 - 1) + 1)\xi_N + m - 1.$$

Since $\min\{m(n_1 - 1) + 1, m(n_2 - 1) + 1\} = m(p - 1) + 1$, we get

$$(n_1, n_2)\xi_G \leq \frac{(m(p-1)+1)^{n+1} - m(p-1) - 1}{m(p-1)} + m - 1.$$

□

Our last application involves graphs of groups, but first we deal with the following particular case:

Theorem 3.8 *Let $G = S_1 * \ldots * S_t$ be a free product of strongly polycyclic groups and let $M = \max\{h(S_1), \ldots, h(S_t)\}$. Then G is strongly Howson and*

$$(n_1, n_2)\xi_G \leq M(n_1 - 1)(n_2 - 1) + M$$

for all $n_1, n_2 \geq 1$.

Proof. In view of (6), we have $\mathrm{rk}(L) \leq M$ for all $i \in \{1, \ldots, t\}$ and $L \leq S_i$.

By the Kurosh subgroup theorem, every subgroup $H \leq G$ is isomorphic to a free product of the form

$$(*_{j\in J} L_j) * F_A,$$

where each L_j is the intersection of H with some conjugate of some S_i. The *Kurosh rank* of H is defined by

$$\mathrm{Krk}(H) = |J| + |A|.$$

It follows from Grushko Theorem on the additivity of ranks in free products [8] that

$$\mathrm{Krk}(H) \leq \mathrm{rk}(H). \tag{17}$$

In general finite Kurosh rank does not imply finite rank. But in the present case, since $\mathrm{rk}(L_j) \leq M$ for every $j \in J$, we have

$$\mathrm{rk}(H) \leq \sum_{j \in J} \mathrm{rk}(L_j) + |A| \leq M|J| + |A| \leq M\mathrm{Krk}(H). \tag{18}$$

Let $H_j \leq G$ satisfy $\mathrm{rk}(H_j) \leq n_j$ for $j = 1, 2$. We may assume that $H_1 \cap H_2$ is nontrivial. Now (17) yields $\mathrm{Krk}(H_j) \leq \mathrm{rk}(H_j) \leq n_j$. Since strongly polycyclic groups are right-orderable [22], it follows from [1, Theorem A] that

$$\mathrm{Krk}(H_1 \cap H_2) \leq (\mathrm{Krk}(H_1) - 1)(\mathrm{Krk}(H_2) - 1) + 1.$$

Hence (17) and (18) yield

$$\begin{aligned} \mathrm{rk}(H_1 \cap H_2) &\leq M\mathrm{Krk}(H_1 \cap H_2) \leq M(\mathrm{Krk}(H_1) - 1)(\mathrm{Krk}(H_2) - 1) + M \\ &\leq M(\mathrm{rk}(H_1) - 1)(\mathrm{rk}(H_2) - 1) + M \leq M(n_1 - 1)(n_2 - 1) + M. \end{aligned}$$

□

Now we prove the following lemma:

Lemma 3.9 *Let G be the fundamental group of a finite graph of groups $\mathcal{G}$ with finite edge groups.*

(i) If $\mathcal{G}$ has virtually polycyclic vertex groups, then G has a finite index normal subgroup which is a finitary free product of strongly polycyclic groups.

(ii) If $\mathcal{G}$ has finitely generated virtually nilpotent vertex groups, then G has a finite index normal subgroup which is a finitary free product of finitely generated nilpotent groups.

Proof. (i) Let

$$\mathcal{G} = ((G_v)_{v \in V}, (G_e)_{e \in E}, (\alpha_e)_{e \in E})$$

be such a graph of groups, built over the finite connected graph

$$\Gamma = (V, E, \alpha, \bar{\ }).$$

Fix a spanning tree T of Γ and let $G = \pi_1(\mathcal{G}, T)$. Since polycyclic groups are residually finite, it follows that each vertex group G_v is residually finite. Now the class of residually finite groups is closed under amalgamated free products with finite amalgamated subgroups and under HNN extensions with finite associated subgroups [3, 6]. Since Γ is a finite graph, we may use the decomposition of G in terms of HNN extensions and amalgamated products over the finite edge groups to deduce that G is itself residually finite.

Let

$$X = (\bigcup_{e \in E} G_e \alpha_e) \setminus \{1\} \subseteq G \setminus \{1\}$$

consist of the image of the edge groups in G through the boundary monomorphisms, with the identity removed. Since both the graph and the edge groups are finite, so is X. Let $x \in X$. Since G is residually finite, there exists some $N_x \lhd G$ of finite index such that $x \notin N_x$. Let

$$N = \bigcap_{x \in X} N_x.$$

Since X is finite, N is still a normal subgroup of G of finite index.

By [15, Corollary 2], since G is the fundamental group of a finite graph of groups with finite edge groups, every finite index $H \leq G$ is itself the fundamental group of a finite graph of groups $\mathcal{G}_H$ where:

- the vertex groups are conjugates of subgroups of the form $H \cap yG_vy^{-1}$ $(v \in V,\ y \in G)$;
- the edge groups are conjugates of subgroups of the form $H \cap y(G_e\alpha_e)y^{-1}$ $(e \in E,\ y \in G)$.

We consider now the case $H = N$. Since $N \cap X = \emptyset$ by construction, we have $N \cap G_e\alpha_e = \{1\}$ for every $e \in E$. Since N is normal, we get

$$N \cap y(G_e\alpha_e)y^{-1} = y(y^{-1}Ny \cap G_e\alpha_e)y^{-1} = y(N \cap G_e\alpha_e)y^{-1} = 1.$$

Thus $\mathcal{G}_N$ has trivial edge groups.

On the other hand, if G' has a polycyclic subgroup F' of index m and $H' \leq G'$, it follows from Proposition 2.1 that $[H' : H' \cap F'] \leq m$. Since $H' \cap F'$ must be itself polycyclic, H' is virtually polycyclic as well.

Thus each group $H \cap yG_vy^{-1}$ is virtually polycyclic and so $\mathcal{G}_N$ has virtually polycyclic vertex groups.

But then N, being the fundamental group of $\mathcal{G}_N$, is a free product of finitely many virtually polycyclic groups and a free group of finite rank. Since a free group of finite rank is the free product of finitely many cyclic groups (hence polycyclic), it follows that N is indeed a free product of finitely many virtually polycyclic groups, say $N = K_1 * \ldots * K_t$, with the K_i nontrivial.

Since K_i is indeed virtually strongly polycyclic for each i, it contains a strongly polycyclic subgroup P_i of finite index for $i = 1, \ldots, t$. Since a subgroup of P_i must be still strongly polycyclic, we may assume that $P_i \trianglelefteq K_i$. Let

$$\varphi : N \to K_1/P_1 \times \ldots \times K_t/P_t$$

be the canonical epimorphism. Then $\mathrm{Ker}\,(\varphi)$ is a finite index subgroup of N. Since $[G : N] < \infty$, we have $[G : \mathrm{Ker}\,(\varphi)] < \infty$ as well. Let

$$Q = \bigcap_{g \in G} g(\mathrm{Ker}\,(\varphi))g^{-1}.$$

Since $[G : \mathrm{Ker}\,(\varphi)] < \infty$, Q is a finite index normal subgroup of G.

Since $Q \subseteq N = K_1 * \ldots * K_t$, it follows from the Kurosh subgroup theorem [17] that Q is isomorphic to a free product of the form

$$(*_{j \in J} L_j) * F_A,$$

where each L_j is the intersection of Q with some conjugate of some K_i. Now $L_j = Q \cap y_j K_i y_j^{-1}$ implies

$$L_j = y_j(y_j^{-1}Qy_j \cap K_i)y_j^{-1} = y_j(Q \cap K_i)y_j^{-1} \subseteq y_j(\mathrm{Ker}\,(\varphi) \cap K_i)y_j^{-1} \subseteq y_j P_i y_j^{-1}$$

and so L_j, being a subgroup of a strongly polycyclic group, is also strongly polycyclic. Since F_A is a free product of cyclic groups, it follows that Q is a finite index normal subgroup of G which is a free product of strongly polycyclic groups.

(ii) Each vertex group is a finite extension of a finitely generated nilpotent group, therefore the vertex groups are virtually polycyclic. Thus we only need to perform minimal adaptations to the proof of (i) which we proceed to enhance:

- Since the class of nilpotent groups is closed under taking subgroups, the same happens with the class of finitely generated nilpotent groups (since they are polycyclic and in view of (6)) and therefore with the class of finitely generated virtually nilpotent groups (in view of Proposition 2.1). Thus $\mathcal{G}_N$ has finitely generated virtually nilpotent vertex groups.
- N is the free product of finitely many finitely generated virtually nilpotent groups and a free group of finite rank. Since $\mathbb{Z}$ is nilpotent, then N is the free product of finitely many finitely generated virtually nilpotent groups.
- We choose the P_i to be finitely generated nilpotent. The free factors of Q are then finitely generated nilpotent groups.

□

We can finally prove the following:

Theorem 3.10 *Let G be the fundamental group of a finite graph of groups with virtually polycyclic vertex groups and finite edge groups. Then G is strongly Howson and there exists some constant $M > 0$ such that:*

$$(n_1, n_2)\xi_G \leq M(n_1 - 1)(n_1 - 1) + M$$

for all $n_1, n_2 \geq 1$.

Proof. By Lemma 3.9(i), G has a finite index normal subgroup F which is a finitary free product of strongly polycyclic groups. By Theorem 3.8, there exists a constant $M' > 0$ such that

$$(n_1, n_2)\xi_F \leq M'(n_1 - 1)(n_2 - 1) + M'$$

for all $n_1, n_2 \geq 1$. Let $m = [G : F]$ and $M = M'm^2$. By Theorem 3.2, we get

$$\begin{aligned}(n_1, n_2)\xi_G &\leq (m(n_1 - 1) + 1, m(n_2 - 1) + 1)\xi_F + m - 1\\ &\leq M'(m(n_1 - 1))(m(n_2 - 1)) + M' + m - 1\\ &= M'm^2(n_1 - 1)(n_1 - 1) + M' + m - 1\\ &\leq M(n_1 - 1)(n_1 - 1) + M\end{aligned}$$

and we are done. □

4 Takahasi's Theorem

We recall, from the introduction, that a group G is a *Takahasi group* if every ascending chain

$$H_1 \leq H_2 \leq H_3 \leq \dots$$

of finitely generated subgroups of G with bounded rank is stationary.

Clearly, every subgroup of a Takahasi group is itself a Takahasi group. We can prove the following partial converse:

Theorem 4.1 *Every finite extension of a Takahasi group is a Takahasi group.*

Proof. Let G have a Takahasi subgroup F of index m and let $H_1 \leq H_2 \leq \dots$ be an ascending chain of subgroups of G with $\mathrm{rk}(H_j) \leq r$ for every $j \geq 1$.

We may assume that G has a standard decomposition (3). Write $K_j = H_j \cap F$. By Proposition 2.1, there exist $I_j \subseteq \{2, \dots, m\}$ and $x_i^{(j)} \in F$ $(i \in I_j)$ such that

$$H_j = K_j \cup (\bigcup_{i \in I_j} K_j x_i^{(j)} b_i). \tag{19}$$

Hence we have an ascending chain $K_1 \leq K_2 \leq \dots$ of subgroups of F. By (19), we have $[H_j : K_j] \leq m$ for every $j \geq 1$. Since $\mathrm{rk}(H_j) \leq r$, it follows from Proposition 3.1 that

$$\mathrm{rk}(K_j) \leq m(r-1) + 1.$$

Since F is a Takahasi group, there exists some $p \in \mathbb{N}$ such that $K_p = K_{p+1} = \dots$.

On the other hand, we have necessarily

$$I_1 \subseteq I_2 \subseteq \dots \subseteq \{2, \dots, m\},$$

hence there exists some $q \geq p$ and some $I \subseteq \{2, \dots, m\}$ such that

$$H_j = K_p \cup (\bigcup_{i \in I} K_p x_i^{(j)} b_i)$$

for every $j \geq q$. Moreover, for every $i \in I$, we have

$$K_p x_i^{(q)} \subseteq K_p x_i^{(q+1)} \subseteq \dots$$

Since two right cosets $K_p x, K_p y$ must be disjoint or equal, we get $K_p x_i^{(q)} = K_p x_i^{(q+1)} = \dots$ and so $H_q = H_{q+1} = \dots$ $\square$

In view of Theorem 1.2, we immediately get:

Corollary 4.2 *Every virtually free group is a Takahasi group.*

We note that, if we fix H_1, the length of a chain $H_1 \leq H_2 \leq \dots$ with subgroups of equal rank cannot be bounded, even in the free group case:

Example 4.3 *Let $A = \{a, b, c, d, e\}$ and let F be the free group on A. Let*

$$H_1 = \langle acb^{-1}, ac^{-1}b^{-1}, adb^{-1}, ad^{-1}b^{-1} \rangle.$$

Fix $n \geq 2$ and define

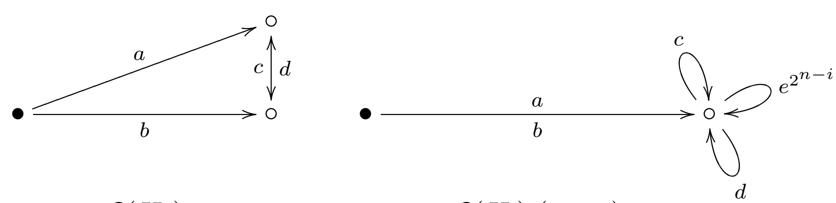
$$H_i = \langle H_1, ab^{-1}, ae^{2^{n-i}}b^{-1} \rangle$$

for $i = 2, \ldots, n$. Then

$$H_1 < H_2 < \ldots < H_n$$

and all subgroups H_i have rank 4.

Indeed, the Stallings automata of the H_i are of the form



$\mathcal{S}(H_1)$ $\qquad$ $\mathcal{S}(H_i)$ $(i > 2)$

It follows easily from (1) that $H_1 < H_2 < \ldots < H_n$. On the other hand, (2) yields $\text{rk}(H_1) = 6 - 3 + 1$ and

$$\text{rk}(H_i) = (4 + 2^{n-i}) - (1 + 2^{n-i}) + 1 = 4$$

for $i = 2, \ldots, n$.

In the free group case, bounds can be obtained in relation with concepts such as *fringe*, *overgroup* or *algebraic extension* (see [19]), but it is not clear how they could be efficiently generalized to more general classes of groups.

We present now another application of Theorem 4.1 which generalizes Corollary 4.2:

Corollary 4.4 *The fundamental group of a finite graph of groups with virtually polycyclic vertex groups and finite edge groups is a Takahasi group.*

Proof. Let G be such a group. By Lemma 3.9(i), there exists a finite index $N \trianglelefteq G$ which is a free product of strongly polycyclic groups. Since G is finitely generated, it follows from Proposition 3.1 that N is also finitely generated. By Grushko Theorem, we may write $N = S_1 * \ldots * S_t$ for some strongly polycyclic groups $S_1, \ldots, S_t$.

Since every subgroup of a polycyclic group is finitely generated by (6), it follows from [30, Corollary 6.3] that every ascending chain of subgroups of bounded Kurosh rank of a free product of polycyclic groups is stationary.

Now every ascending chain of subgroups of bounded rank of $N = S_1 * \ldots * S_t$ has also bounded Kurosh rank by (17) and is therefore stationary. Thus N is a Takahasi group. By Theorem 4.1, G is also a Takahasi group. $\square$

5 Periodic subgroups

In this section we combine Theorem 4.1 with theorems on fixed subgroups to get results on the periodic subgroups.

Given a group G, we denote by $\mathrm{End}(G)$ the endomorphism monoid of G. Given $\varphi \in \mathrm{End}(G)$, the *fixed subgroup* of φ is defined by

$$\mathrm{Fix}(\varphi) = \{x \in G \mid x\varphi = x\}$$

and the *periodic subgroup* of φ is defined by

$$\mathrm{Per}(\varphi) = \bigcup_{n \geq 1} \mathrm{Fix}(\varphi^n).$$

Given $x \in \mathrm{Per}(\varphi)$, the *period* of x is the least $n \geq 1$ such that $x\varphi^n = x$.

Theorem 5.1 *Let G be the fundamental group of a finite graph of groups with finitely generated virtually nilpotent vertex groups and finite edge groups. Then there exists a constant $M > 0$ such that*

$$\mathrm{rk}(\mathrm{Per}(\varphi)) \leq M$$

for every $\varphi \in \mathrm{End}(G)$.

Proof. By Lemma 3.9(ii), G has a finite index normal subgroup N which is a finitary free product of finitely generated nilpotent groups, say $N = K_1 * \ldots * K_t$. Let $n = [G : N]$. By [12, Lemma 2.2], the intersection F of all subgroups of G of index $\leq n$ is a fully invariant subgroup of G, in the sense that $F\varphi \subseteq F$ for every $\varphi \in \mathrm{End}(G)$. Moreover, since G is finitely generated, we have $[G : F] < \infty$. Since $F \leq N$, it follows from the Kurosh subgroup theorem that F is isomorphic to a free product of the form

$$(*_{j \in J} H_j) * F_A,$$

where each H_j is the intersection of F with some conjugate of some K_i. Since G is finitely generated, it follows from Proposition 3.1 that F is finitely generated and so has finite Kurosh rank by (17). Similarly to the proof of Lemma 3.9(ii), it follows easily that F is a finitary free product of finitely generated nilpotent groups, say $F = L_1 * \ldots * L_s$. By [29, Theorem 7], we have

$$\mathrm{Krk}(\mathrm{Fix}(\psi)) \leq s$$

for every $\psi \in \mathrm{End}(F)$. Since each L_i is polycyclic, it follows from (6) that there exists some constant $M' > 0$ such that

$$\mathrm{rk}(P) \leq M'$$

for all $i \in \{1, \ldots, s\}$ and $P \leq L_i$. Hence we may apply (18) to get

$$\mathrm{rk}(\mathrm{Fix}(\psi)) \leq M'\mathrm{Krk}(\mathrm{Fix}(\psi)) \leq M's$$

for every $\psi \in \mathrm{End}(F)$.

Write $M = M's + [G : F] - 1$. Let $\varphi \in \mathrm{End}(G)$ and let $\psi = \varphi|_F$. Since F is a fully invariant subgroup of G, we have $\psi \in \mathrm{End}(F)$. Moreover, $\mathrm{Fix}(\varphi) \cap F = \mathrm{Fix}(\psi)$. By Proposition 2.1, we get

$$[\mathrm{Fix}(\varphi) : \mathrm{Fix}(\psi)] = [\mathrm{Fix}(\varphi) : \mathrm{Fix}(\varphi) \cap F] \leq [G : F],$$

hence

$$\mathrm{rk}(\mathrm{Fix}(\varphi)) \leq \mathrm{rk}(\mathrm{Fix}(\psi)) + [G : F] - 1 \leq M's + [G : F] - 1 = M. \tag{20}$$

We note that

$$m|m' \quad \Rightarrow \quad \mathrm{Fix}(\varphi^m) \leq \mathrm{Fix}(\varphi^{m'}) \tag{21}$$

for all $m, m' \geq 1$: Indeed, if $m' = mk$ and $u \in \mathrm{Fix}(\varphi^m)$, then

$$u\varphi^{m'} = u\varphi^{mk} = u\varphi^m\varphi^{m(k-1)} = u\varphi^{m(k-1)} = \ldots = u\varphi^m = u$$

and so $u \in \mathrm{Fix}(\varphi^{m'})$.

Hence we have an ascending chain of subgroups of G of the form

$$\mathrm{Fix}(\varphi) \leq \mathrm{Fix}(\varphi^{2!}) \leq \mathrm{Fix}(\varphi^{3!}) \leq \ldots$$

By (20), we have $\mathrm{rk}(\mathrm{Fix}(\varphi^{m!})) \leq M$ for every $m \geq 1$. Since every finitely generated nilpotent group is polycyclic, G is a Takahasi group by Corollary 4.4 and so there exists some $k \geq 1$ such that $\mathrm{Fix}(\varphi^{m!}) = \mathrm{Fix}(\varphi^{k!})$ for every $m \geq k$. In view of (21), we get

$$\mathrm{Per}(\varphi) = \bigcup_{m \geq 1} \mathrm{Fix}(\varphi^m) = \bigcup_{m \geq 1} \mathrm{Fix}(\varphi^{m!}) = \mathrm{Fix}(\varphi^{k!}).$$

Therefore $\mathrm{rk}(\mathrm{Per}(\varphi)) = \mathrm{rk}(\mathrm{Fix}(\varphi^{k!})) \leq M$ by (20). $\square$

Corollary 5.2 *Let G be the fundamental group of a finite graph of groups with finitely generated virtually nilpotent vertex groups and finite edge groups. Let $\varphi \in \mathrm{End}(G)$. Then there exists a constant $R_\varphi > 0$ such that every $x \in \mathrm{Per}(\varphi)$ has period $\leq R_\varphi$.*

Proof. By Theorem 5.1, we have $\mathrm{rk}(\mathrm{Per}(\varphi)) < \infty$. Assume that $\mathrm{Per}(\varphi) = \langle x_1, \ldots, x_r \rangle$. Let R_φ denote the least common multiple of the periods of the elements $x_1, \ldots, x_r$. Let $x \in \mathrm{Per}(\varphi)$. Then there exist $i_1, \ldots, i_n \in \{1, \ldots, r\}$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$ such that $x = x_{i_1}^{\varepsilon_1} \ldots x_{i_n}^{\varepsilon_n}$. It follows that

$$\begin{aligned} x\varphi^{R_\varphi} &= (x_{i_1}^{\varepsilon_1} \ldots x_{i_n}^{\varepsilon_n})\varphi^{R_\varphi} = (x_{i_1}\varphi^{R_\varphi})^{\varepsilon_1} \ldots (x_{i_n}\varphi^{R_\varphi})^{\varepsilon_n} \\ &= x_{i_1}^{\varepsilon_1} \ldots x_{i_n}^{\varepsilon_n} = x, \end{aligned}$$

hence x has period $\leq R_\varphi$. $\square$

Note that, in particular, the preceding results hold for finitely generated virtually free groups.

We remark also that we cannot get any analogue of Theorem 5.1 involving direct products. In fact, by [23, Theorem 4.1], there exist automorphisms φ of $F_2 \times \mathbb{Z}$ such that neither $\mathrm{Fix}(\varphi)$ nor $\mathrm{Per}(\varphi)$ is finitely generated.

Acknowledgements

The first author is partially supported by CNPq, PRONEX-Dyn.Syst. and FAPESB (Brazil).

The second author acknowledges support from:

- CNPq (Brazil) through a BJT-A grant (process 313768/2013-7);

- the European Regional Development Fund through the programme COMPETE and the Portuguese Government through FCT (Fundação para a Ciência e a Tecnologia) under the project PEst-C/MAT/UI0144/2011.

The third author acknowledges support from a GSRT/Greece excellence grant, cofunded by the ESF/EU and National Resources.

References

[1] Y. Antolín, A. Martino and I. Schwabrow, Kurosh rank of intersections of subgroups of free products of right-orderable groups, preprint, arXiv:1109.0233v2, 2013.

[2] L. Bartholdi and P. V. Silva, Rational subsets of groups, Chapter 23 of the handbook *AutoMathA* (to appear), arXiv:1012.1532, 2010.

[3] G. Baumslag, On the residual finiteness of generalized free products of nilpotent groups, *Trans. Amer. Math. Soc.* 106 (1963), 193–209.

[4] J. Berstel, *Transductions and Context-free Languages*, Teubner, Stuttgart, 1979.

[5] O. Bogopolski and K.-U. Bux, Subgroup conjugacy separability for surface groups, preprint, 2013.

[6] D. E. Cohen, Residual finiteness and Britton's lemma, *J. London Math. Soc.* s2-16(2) (1977), 232–234.

[7] J. Friedman, Sheaves on graphs, their homological invariants, and a proof of the Hanna Neumann conjecture: with an appendix by Warren Dicks, *Memoirs Amer. Math. Soc.* (to appear).

[8] I. A. Grushko, On the bases of a free product of groups, *Mat. Sbornik* 8 (1940), 169–182.

[9] M. Hall, Jr., The Theory of Groups, Macmillan Co., New York, 1959.

[10] K. Hirsch, On infinite soluble groups (III), *Proc. London Math. Soc.* s2-49(1) (1946), 184–194.

[11] A. G. Howson, On the intersection of finitely generated free groups, *J. London Math. Soc.* 29 (1954), 428–434.

[12] T. Hsu and D. T. Wise, Ascending HNN extensions of polycyclic groups are residually finite, *J. Pure Appl. Algebra* 182 (2003) 65–78.

[13] I. Kapovich, Amalgamated products and the Howson property, *Canad. Math. Bull.* 40 (1997), 330–340.

[14] I. Kapovich and A. Miasnikov, Stallings foldings and subgroups of free groups, *J. Algebra* 248 (2002), 608–668.

[15] A. Karrass, A. Pietrowski and D. Solitar, An improved subgroup theorem for HNN groups with some applications, *Can. J. Math.* 26(1) (1974), 214–224.

[16] A. Karrass, A. Pietrowski and D. Solitar, Finitely generated groups with a free subgroup of finite index, *J. Austral. Math. Soc.* 16 (1973), 458–466.

[17] A. G. Kurosh, Die Untergruppen der freien Produkte von beliebigen Gruppen, *Math. Ann.* 109 (1934), 647–660.

[18] R. C. Lyndon and P. E. Schupp, Combinatorial Group Theory, Springer-Verlag, 1977.

[19] A. Miasnikov, E. Ventura and P. Weil, Algebraic extensions in free groups, in: *Algebra and Geometry in Geneva and Barcelona*, Trends in Mathematics, Birkhäuser, Basel, 2007, pp. 225–253.

[20] I. Mineyev, Groups, graphs and the Hanna Neumann Conjecture, *J. Topol. Anal.* 4(1) (2012), 1–12.

[21] H. Neumann, On the intersection of finitely generated free groups, *Publ. Math.* 4 (1956), 186–189; Addendum, *Publ.Math.* 5 (1957), 128.

[22] A. H. Rhemtulla, Polycyclic right ordered groups, in: *Algebra, Carbondale 1980* (ed. R. K. Amayo), Lecture Notes in Mathematics 848, Springer, 1981, pp. 230–234.

[23] E. Rodaro, P. V. Silva and M. Sykiotis, Fixed points of endomorphisms of graph groups, *J. Group Theory* 16(4) (2013), 573–583.

[24] J. Sakarovitch, *Elements of Automata Theory*, Cambridge University Press, 2009.

[25] O. Schreier, Die Untergruppen der freien Gruppen, *Abh. Math. Sem. Univ. Hamburg* 5 (1927), 161–183.

[26] P. Scott and T. Wall, Topological methods in group theory, in: *Homological Group Theory* (eds. T. Wall and D. Johnson), London Math. Soc. Lecture Notes Series 36, 1979, pp. 137–203.

[27] J.-P. Serre, *Arbres, amalgames, SL2* (rédigé avec la collaboration de Hyman Bass), Astérisque 46, Société Mathématique de France, 1977.

[28] J. Stallings. Topology of finite graphs, *Invent. Math.* 71 (1983), 551–565.

[29] M. Sykiotis, Fixed subgroups of endomorphisms of free products, *J.Algebra* 315 (2007), 274–278.

[30] M. Sykiotis, On subgroups of finite complexity in groups acting on trees, *J. Pure Appl. Algebra* 200 (2005), 1–23.

[31] M. Takahasi, Note on locally free groups, *J. Inst. Polytechnics, Osaka City Univ. ser. A* 1(2) (1950), 65–70.

[32] B. A. F. Wehrfritz, *Group and Ring Theoretic Properties of Polycyclic Groups*, Springer, 2009.

[33] A. Zakharov, On the rank of the intersection of free subgroups in virtually free groups, preprint, arXiv:1301.3115v2, 2013.