Hypoelliptic theory in an analysis of the Schrödinger problem

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Abstract

In this paper we use methods of hypoelliptic analysis to study the linear Schrödinger equation. In order to do that, we implement a regularization procedure to control the singularity in the hyperplane t=0. We then show that the correspondent regularized linear Schrödinger problem can be reduced, by matrix triangulation, to an uncoupled system of two first order equations and give estimates for the eigenvalues of the corresponding arising matrices. Parametrices for the first order system are constructed and used to solve the regularized problem.

Keywords: Schrödinger operator, Hypoelliptic equations, Regularization procedure

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1 Introduction

One of the most important equations in mathematical physics is the Schrödinger equation, as it plays a key role in quantum mechanics. An important feature of this equation is the fact that it is instationary and therefore, one cannot use standard elliptic techniques for its resolution. Moreover, even when compared with the heat operator, the Schrödinger one presents an additional difficulty since its fundamental solution possesses (non-removable) singularities in the hyperplane t = 0. One overcomes this problem by performing a standard regularization procedure (see [10], [11] for details) which allows some degree of control over these singularities and, thus, enables (to some extend) the application of methods for hypoelliptic boundary value problems.

Hypoelliptic theory has its roots in the work of Hörmander in [8], where a necessary and sufficient condition for a solution of a homogeneous boundary value problem to be C^∞up to boundary of the domain was given. His condition, of an algebraic nature, was formulated in terms of behavior of the zeros of the so-called characteristic function of the boundary value problem near the infinity. Roughly speaking, Hörmander's condition is similar to the algebraic condition that characterizes hypoelliptic partial differential operators with constant coefficients. Another characterization of the same type of problem is given by Barros-Neto [4] based on regularity properties of the fundamental kernels associated to the boundary value problem under consideration. With the help of such kernels, he constructed parametrices for the boundary value problem which in turns allows to obtain an explicit solution for the b.v.p. under consideration. We remark, although, that this is a rather complicated method to implement.

The aim of this paper is to apply the hypoelliptic theory to the parametrices arising in the b.v.p. for the regularized Schrödinger operator (therefore, an hypoelliptic operator). At the end of the paper we show that this approach allows not only to obtain existence and uniqueness results, but also provides a representation formula for the solution of the Schrödinger problem.

The paper is structured as follows: in Section 2 we present some necessary notions and results about hypoelliptic analysis. In the following section we introduce the regularization procedure for the Schrödinger operator. In Section 4 we study the regularized Schrödinger operator and we obtain its parametrix. In the last section we will use the obtained parametrix to solve the regularized Schrödinger problem and we will make a reference to the expected solution in the general case.

2 Preliminaries

Consider the partial differential operator with constant coefficients

$$P(D) = \sum_{|p|\leq m} a_p D^p,$$

with $p = (p_1, ..., p_n) \in \mathbb{N}_0^n$, $|p| = p_1 + ... + p_n$, $a_p \in \mathbb{C}$ and $D^p = \dfrac{\partial^{|p|}}{\partial_{x_1}^{p_1}...\partial_{x_n}^{p_n}}$, such that all distributions solutions of the equation $P(D)u = f$ are always smooth functions whenever f is a smooth function.

Definition 2.1. *We say that the differential operator $P(D)$ is hypoelliptic if*

$$P(D)T \in C^\infty(\Omega) \Rightarrow T \in C^\infty(\Omega)$$

for every open set $\Omega \subset \mathbb{R}^n$ and every distribution $T \in \mathcal{D}'(\Omega)$.

Let $\zeta = (\zeta_1, ..., \zeta_n) \in \mathbb{C}^n$ with $1 \leq j \leq n$. The polynomial

$$P(\zeta) = \sum_{|p| \leq m} a_p \zeta^p$$

is called the characteristic polynomial (or symbol) of $P(D)$. Denote by $N = \{\zeta \in \mathbb{C}^n : P(\zeta) = 0\}$ the set of zeros of $P(\zeta)$. For every $\xi \in \mathbb{R}^n$, let

$$d(\xi, N) = \inf_{\zeta \in N} |\xi - \zeta|$$

be the distance from ξ to N.

Theorem 2.2. *Let $P(\zeta)$ be a constant coefficient polynomial. The following properties are equivalent:*

(H_1) *$\zeta \in N$, $|\zeta| \to +\infty$ implies $|\mathrm{Im}\zeta| \to +\infty$;*

(H_2) *$\xi \in \mathbb{R}^n$, $|\xi| \to +\infty$ implies $d(\xi, N) \to +\infty$;*

(H_3) *for all $n-$tuples $p = (p_1, ..., p_n)$ with $|p| \neq 0$, $\xi \in \mathbb{R}^n$, $|\xi| \to +\infty$ implies*

$$\frac{|(D^p P)(\xi)|}{|1 + P(\xi)|} \to 0.$$

Now, we recall some sufficient conditions for hypoellipticity.

Definition 2.3. *A distribution $E \in D'(\mathbb{R}^n)$ is said to be a parametrix of $P(D)$ if the distribution $R = P(D)E - \delta$ is a integrable function in some open neighborhood of the origin in $\mathbb{R}^n$. The distribution R is called the rest of the parametrix.*

Hypoelliptic operators can be characterized in terms of regularity properties of their fundamental solutions. We have the following result.

Theorem 2.4. *Let $P(D)$ be a partial differential operator with constants coefficients. If $P(D)$ is hypoelliptic then every fundamental solution is C^∞ in $\mathbb{R}^n \setminus \{0\}$. Conversely, if there is a fundamental solution which is C^∞ function in $\mathbb{R}^n \setminus \{0\}$ then $P(D)$ is hypoelliptic.*

A well know theorem proved by Malgrange [9] and Ehrenpreis [6] states that every partial differential operator with constant coefficients possesses a fundamental solution. This result, combined with Theorem 2.4, implies that, in order to show that an operator $P(D)$ is hypoelliptic, it suffices to show that it has at least one fundamental solution which is C^∞ in $\mathbb{R}^n \setminus \{0\}$. When this is the case, all fundamental solutions will be C^∞ in $\mathbb{R}^n \setminus \{0\}$.

In view of Theorem 2.4, in order to show that an operator $P(D)$ is hypoelliptic, it suffices to show that is has at least one C^∞ fundamental solution in $\mathbb{R}^n \setminus \{0\}$ or analogously, that one can construct a parametrix with smooth rest for that operator $P(D)$.

Theorem 2.5. *If a differential operator $P(D)$, with constant coefficients, possesses a C^∞ parametrix in $\mathbb{R}^n \setminus \{0\}$ with rest in $C^\infty(\mathbb{R}^n)$, then $P(D)$ is hypoelliptic.*

3 Regularization procedure

As stated before, fundamental solutions of the Schrödinger operator have non-removable singularities in the whole of the hyperplane $t = 0$. As a consequence, we no longer have convergence, in the classical sense, of the integrals that define the Teodorescu and Cauchy-Bitsadze operators.

One overcomes this problem by implementation a regularization procedure (see [10]): first, we create a family of operators and correspondent fundamental solutions, which are locally integrable in $\mathbb{R}^n \times \mathbb{R}_0^+ \setminus \{(0, \ldots, 0, 0)\}$. Then, we prove that these family converges to the original operator whenever $\epsilon \to 0$.

To this end, we apply the modified Wick rotation $t \to (\epsilon + i)t$ to the heat operator

$$(-\Delta + \mathbf{k}_\epsilon \partial_t)[(\epsilon + i)e(x, (\epsilon + i)t)] = \delta(x)\delta(t), \tag{1}$$

with $\mathbf{k}_\epsilon = \frac{\epsilon+i}{\epsilon^2+1}$. Let us remark that, for each $\epsilon > 0$, $-\Delta + \mathbf{k}_\epsilon \partial_t$ is a hypoelliptic operator and therefore we ensure the good behavior for the associated integral operators. In addition, we get a family of fundamental solutions for this family of operators given by (see [5], [12])

$$\begin{aligned} e^\epsilon(x,t) &= (\epsilon + i)e(x, (\epsilon + i)t) \\ &= (\epsilon + i)\ \frac{H(t)}{(4\pi(\epsilon + i)t)^{\frac{n}{2}}}\ \exp\left(-\frac{(\epsilon + i)|x|^2}{4(\epsilon^2 + 1)t}\right), \quad \epsilon > 0. \end{aligned}$$

4 Decomposition of Pseudodifferential Operators of order 2

From now on, we consider $\Omega = \underline{\Omega} \times [0, T) \subset \mathbb{R}^n \times \mathbb{R}_0^+$, an open (non-empty) domain with a piecewise smooth boundary $\Gamma = \partial\Omega$. One proves that $-\Delta \pm \mathbf{k}_\epsilon \partial_t$ is a pseudodifferential operator with principal symbol in the Hörmander class $S^0_{1,0}$. (for more details see [7]).

The total symbol of $-\Delta \pm \mathbf{k}_\epsilon \partial_t$ is

$$P_{\mathbf{k}_\epsilon}(x, t, \xi, \tau) := P_{\mathbf{k}_\epsilon}(\xi, \tau) = -|\xi|^2 \pm i\mathbf{k}_\epsilon \tau,$$

where $x = (x_1, \ldots, x_n)$, $\xi = (\xi_1, \ldots, \xi_n)$ and $|\xi|^2 = \xi_1^2 + \ldots + \xi_n^2$.

In our case, condition (H2) guarantees that the symbol $P_{\mathbf{k}_\epsilon}(x, t, \xi, \tau)$ is not singular (see Theorem 2.2). This allows us to conclude that all the elements of the family $-\Delta \pm \mathbf{k}_\epsilon \partial_t$, with symbol $P_{\mathbf{k}_\epsilon}(\xi, \tau)$, are invertible modulo regularizing operators. Moreover, condition (H2) implies that for each compact $K \subset \Omega$ the total symbol $P_{\mathbf{k}_\epsilon}(\xi, \tau)$, as a polynomial of degree 2 in $|\xi|$, has no real zeros for

$|\tau|$ large. Therefore, when $|\tau| > M$, $P_{\mathbf{k}_\epsilon}(\xi,\tau)$ will have the following complex roots

$$z_1 = z_1(\tau) = \sqrt{\pm i\mathbf{k}_\epsilon\tau} \qquad\qquad z_2 = z_2(\tau) = -\sqrt{\pm i\mathbf{k}_\epsilon\tau}.$$

Consequently,

$$P_{\mathbf{k}_\epsilon}(\xi,\tau) = P^-_{\mathbf{k}_\epsilon}(\xi,\tau)\ P^+_{\mathbf{k}_\epsilon}(\xi,\tau),$$

where

$$\begin{aligned} P^-_{\mathbf{k}_\epsilon}(\xi,\tau) &= (|\xi| - z_2(\tau)) = (|\xi| + \sqrt{\pm i\mathbf{k}_\epsilon\tau}), \\ P^+_{\mathbf{k}_\epsilon}(\xi,\tau) &= (|\xi| - z_1(\tau)) = (|\xi| - \sqrt{\pm i\mathbf{k}_\epsilon\tau}), \end{aligned} \tag{2}$$

for $(x,t) \in K$, $|\tau| > M$.

From now until the end of the paper we will restrict our study to the backward case.

Theorem 4.1. *Consider the operator $-\Delta - \mathbf{k}_\epsilon\partial_t$ and its total symbol $P_{\mathbf{k}_\epsilon}(\xi,\tau)$. There exists a pseudodifferential operator $L_{\mathbf{k}_\epsilon}$ invertible modulo regularizing operators, such that*

$$-\Delta - \mathbf{k}_\epsilon\partial_t = L_{\mathbf{k}_\epsilon}(D,\partial_t)P^-_{\mathbf{k}_\epsilon}(D,\partial_t)P^+_{\mathbf{k}_\epsilon}(D,\partial_t) + R_{\mathbf{k}_\epsilon}(D,\partial_t), \tag{3}$$

where $R_{\mathbf{k}_\epsilon}$ is a regularizing term, where $D = (\partial_{x_1},\ldots,\partial_{x_n})$.

Proof. Assume that $P_{\mathbf{k}_\epsilon}(\xi,\tau)$, as well as $P^\pm_{\mathbf{k}_\epsilon}(\xi,\tau)$, satisfy conditions (H1) and (H2) in Theorem 2.2. According to [3], we can construct the following first order operators $K^\pm_{\mathbf{k}_\epsilon}(D,\partial_t)$

$$K^\pm_{\mathbf{k}_\epsilon}(D,\partial_t) = \partial_t + P^\pm_{\mathbf{k}_\epsilon}(D,\partial_t),$$

where $P^\pm_{\mathbf{k}_\epsilon}(D,\partial_t)$ are pseudodifferential operators with symbol $P^\pm_{\mathbf{k}_\epsilon}(\xi,\tau)$, resp. According to [1], the operators $K^\pm_{\mathbf{k}_\epsilon}(D,\partial_t)$ will be parametrices for the operators $P^\pm_{\mathbf{k}_\epsilon}(D,\partial_t)$, for $|(\xi,\tau)|$ large, in the sense that $K^\pm_{\mathbf{k}_\epsilon}P^\pm_{\mathbf{k}_\epsilon} = I + \mathcal{R}_{\mathbf{k}_\epsilon} \sim I$, where I represents the identity operator.

Hence

$$\begin{aligned} -\Delta - \mathbf{k}_\epsilon\partial_t &= P_{\mathbf{k}_\epsilon} \\ &= P_{\mathbf{k}_\epsilon}(K^+_{\mathbf{k}_\epsilon}K^-_{\mathbf{k}_\epsilon}(I + \mathcal{R}_{\mathbf{k}_\epsilon})P^-_{\mathbf{k}_\epsilon}P^+_{\mathbf{k}_\epsilon}) \\ &= P_{\mathbf{k}_\epsilon}K^+_{\mathbf{k}_\epsilon}K^-_{\mathbf{k}_\epsilon}P^-_{\mathbf{k}_\epsilon}P^+_{\mathbf{k}_\epsilon} + P_{\mathbf{k}_\epsilon}K^+_{\mathbf{k}_\epsilon}K^-_{\mathbf{k}_\epsilon}\mathcal{R}_{\mathbf{k}_\epsilon}P^-_{\mathbf{k}_\epsilon}P^+_{\mathbf{k}_\epsilon} \\ &= L_{\mathbf{k}_\epsilon}P^-_{\mathbf{k}_\epsilon}P^+_{\mathbf{k}_\epsilon} + R_{\mathbf{k}_\epsilon}, \end{aligned}$$

where $L_{\mathbf{k}_\epsilon} = P_{\mathbf{k}_\epsilon}K^+_{\mathbf{k}_\epsilon}K^-_{\mathbf{k}_\epsilon}$ and $R_{\mathbf{k}_\epsilon} = P_{\mathbf{k}_\epsilon}K^+_{\mathbf{k}_\epsilon}K^-_{\mathbf{k}_\epsilon}\mathcal{R}_{\mathbf{k}_\epsilon}P^-_{\mathbf{k}_\epsilon}P^+_{\mathbf{k}_\epsilon}$ is a regularizing operator. Moreover, the symbol of $R_{\mathbf{k}_\epsilon}$ is

$$R_{\mathbf{k}_\epsilon}(\xi,\tau) = \frac{1}{(|\xi| - z_1(\tau))(|\xi| - z_2(\tau))},$$

i.e., the symbol of the operator $R_{\mathbf{k}_\epsilon}$ is the characteristic function associated to the operator $-\Delta - \mathbf{k}_\epsilon\partial_t$. $\square$

4.1 The hypoelliptic operator $-\boldsymbol{\Delta} - \mathbf{k}_\epsilon \partial_\mathbf{t}$

We now apply the previous factorization of each $-\Delta - \mathbf{k}_\epsilon \partial_t$ to study the equation $(-\Delta - \mathbf{k}_\epsilon \partial_t)u = f$. For that, we rewrite this equation as

$$P^+_{\mathbf{k}_\epsilon} P^-_{\mathbf{k}_\epsilon} u = \tilde{f} - R_{\mathbf{k}_\epsilon} u,$$

where $\tilde{f} = (P_{\mathbf{k}_\epsilon} K^+_{\mathbf{k}_\epsilon} K^-_{\mathbf{k}_\epsilon})^{-1} f$, with $(P_{\mathbf{k}_\epsilon} K^+_{\mathbf{k}_\epsilon} K^-_{\mathbf{k}_\epsilon})^{-1}$ the parametrix of $P_{\mathbf{k}_\epsilon} K^+_{\mathbf{k}_\epsilon} K^-_{\mathbf{k}_\epsilon}$ and $R_{\mathbf{k}_\epsilon}$ a regularizing Tikhonov-type operator.

The equation $P^-_{\mathbf{k}_\epsilon} P^+_{\mathbf{k}_\epsilon} u = \tilde{f} - R_{\mathbf{k}_\epsilon} u$ is equivalent to the following system of equations

$$\left\{ \begin{array}{lcl} P^+_{\mathbf{k}_\epsilon} u & = & v \\ P^-_{\mathbf{k}_\epsilon} v & = & \tilde{f} - R_{\mathbf{k}_\epsilon} u \end{array} \right. , \tag{4}$$

and consequently we reduce our equation to a system of first order equations. Introducing the operator

$$Q_{\mathbf{k}_\epsilon}(D, \partial_t) = \mathbf{k}_\epsilon \partial_t + P^+_{\mathbf{k}_\epsilon}(\xi, \partial_t) D + P^-_{\mathbf{k}_\epsilon}(\xi, \partial_t) D, \tag{5}$$

we reduce our previous system to

$$\left\{ \begin{array}{lcl} Q_{\mathbf{k}_\epsilon}(D, \partial_t) w_1 & = & v \\ Q_{\mathbf{k}_\epsilon}(D, \partial_t) w_2 & = & \tilde{f} - R_{\mathbf{k}_\epsilon} u \end{array} \right. .$$

Letting $w_2 = D w_1$ we can write (5) as a matricial equation

$$(\partial_t - \mathcal{A}_{\mathbf{k}_\epsilon}(t))\mathbf{w} = \mathbf{g}, \tag{6}$$

where $\mathbf{w} = (w_1, w_2)^T$, $g = (0, g)^T$ and $\mathcal{A}_{\mathbf{k}_\epsilon}$ is the matrix

$$\left[\begin{array}{cc} 0 & 1 \\ P^-_{\mathbf{k}_\epsilon} & P^+_k \end{array} \right]. \tag{7}$$

$\mathcal{A}_{\mathbf{k}_\epsilon}$ is a matrix-pseudodifferential operator whose symbol, $\sigma(\mathcal{A}_{\mathbf{k}_\epsilon}) = a(\xi, \tau)$, belongs to $S^1_{0,1}$.

We also remark that, since $\det(\tau I - \sigma(\mathcal{A}_{\mathbf{k}_\epsilon})) = Q_{\mathbf{k}_\epsilon}(\xi, \tau)$, the eigenvalues of $\sigma(\mathcal{A}_{\mathbf{k}_\epsilon})$ are the roots of $Q_{\mathbf{k}_\epsilon}(\xi, \tau)$.

4.2 Localization of the eigenvalues of the matrix $\mathcal{A}_{\mathbf{k}_\epsilon}$

Let K be a compact subset of Ω. For every $(x, t) \in K$, we consider the polynomials $Q_{\mathbf{k}_\epsilon}(x, t, \xi, \tau) := Q_{\mathbf{k}_\epsilon}(\xi, \tau)$ and define the set of zeros

$$N_{\mathbf{k}_\epsilon}(K) = \left\{ (\xi, \tau) \in \mathbb{C}^{m+1} : \ Q_{\mathbf{k}_\epsilon}(\xi, \tau) = 0 \right\},$$

for $(x, t) \in K$. For $(\xi, \tau) \in \mathbb{R}^{m+1}$, let $d((\xi, \tau), N_{\mathbf{k}_\epsilon}(K))$ be the distance from (ξ, τ) to $N_{\mathbf{k}_\epsilon}(K)$.

Lemma 4.2. *Consider the pseudodifferential operator (5). For each compact set $K \subset \Omega$, there exists a constant $C = C(K) > 0$ such that*

$$C^{-1} \leq d \sum_{j=1}^{2} \left(\frac{|\partial_{\xi_j} Q_{\mathbf{k}_\epsilon}(\xi,\tau)|}{|Q_{\mathbf{k}_\epsilon}(\xi,\tau)|} \right)^{\frac{1}{j}} \leq C, \tag{8}$$

for $d := d((\xi,\tau), N_{\mathbf{k}_\epsilon}(K))$, $(\xi,\tau) \in \mathbb{R}^{m+1}$, $(x,t) \in K$ and $Q_{\mathbf{k}_\epsilon}(\xi,\tau) \neq 0$.

The proof of this result follows the same steps as in the proof of Lemma 4.1 in [1] and, therefore, we will omit it.

Theorem 4.3. *For every (x,t) in an arbitrary compact $K \subset \Omega$, there exists positive constants $M(K)$, C_1, C_2 such that, whenever $|\tau| > M$, the set of zeros $N_{\mathbf{k}_\epsilon}(K)$ of $Q_{\mathbf{k}_\epsilon}(\xi,\tau)$ is contained in the subset of the complex plane defined by*

$$|\xi| \leq C_1(1+|\tau|), \qquad\qquad |\mathrm{Im}(\xi)| \geq C_2|\tau|. \tag{9}$$

Proof. Using a well know estimate for the zeros of a polynomial of one variable (see [1]), we have

$$|\xi| \leq 1 + \max\{|P^-_{\mathbf{k}_\epsilon}(\xi,\tau)|,\ |P^+_{\mathbf{k}_\epsilon}(\xi,\tau)|\}. \tag{10}$$

Since $P^{\pm}_{\mathbf{k}_\epsilon}(\xi,\tau)$ belongs to $S^1_{1,0}$, we obtain the first inequality of our result. It follows from (H1) in Theorem 2.2 and (8) that for each compact set K there is a constant $C = C(K) > 0$ such that for $|\tau| > M(K)$

$$d = d(\xi,\tau) \geq C(1+|\tau|). \tag{11}$$

If $(\zeta,\tau) \in N_{\mathbf{k}_\epsilon}(K)$, then $d(\mathrm{Re}(\zeta,\tau), N_{\mathbf{k}_\epsilon}(K)) \leq |\mathrm{Im}(\zeta,\tau)|$.
It follows from (11) that

$$|\mathrm{Re}(\zeta)| \leq C|\mathrm{Im}(\zeta,\tau)|.$$

Hence for $|\tau| > M(K)$ and $(\xi,\tau) \in N_{\mathbf{k}_\epsilon}(K)$

$$|\tau| \leq C|\mathrm{Im}\xi|, \tag{12}$$

which is our second inequality. □

4.3 Parametrix of $-\Delta - \mathbf{k}_\epsilon \partial_t$

The aim of this section is to obtain a representation formula for the parametrix of the regularized Schrödinger operator and its symbol. Recall the time interval $[0,T[$ where our operator $-\Delta - \mathbf{k}_\epsilon \partial_t$ is valid. Taking into account the definition of parametrix for a hypoelliptic operator, we can say that for every fixed $t' \in [0,T[$ the pseudodifferential operator

$$U_{\mathbf{k}_\epsilon}(t,t') : \mathcal{E}'(\Omega,\mathbb{C}^2) \to \mathcal{D}'(\underline{\Omega},\mathbb{C}^2)$$

(which depends smoothly on $t \in [t', T[)$ is the parametrix for the operator $\mathbf{k}_\epsilon \partial_t - \mathcal{A}_{\mathbf{k}_\epsilon}(t)$, if

$$\begin{cases} \mathbf{k}_\epsilon \dfrac{dU_{\mathbf{k}_\epsilon}(t,t')}{dt} - \mathcal{A}_{\mathbf{k}_\epsilon}(t) \circ U_{\mathbf{k}_\epsilon}(t,t') \sim 0 & \text{in} \quad \Omega \\ U_{\mathbf{k}_\epsilon}(t,t')|_{t=t'} \sim I & \text{in} \quad \underline{\Omega} \end{cases} . \tag{13}$$

We remark that $U_{\mathbf{k}_\epsilon}(t,t')$ is defined modulo regularizing operators on $\underline{\Omega}$. For our case we can prove the existence of the parametrix as follows.

The operator $U_{\mathbf{k}_\epsilon}(t,t')$ is defined by

$$U_{\mathbf{k}_\epsilon}(t,t')u = (2\pi)^{-m} \int e^{ix\xi} \mathcal{U}_{\mathbf{k}_\epsilon}(t,t') \hat{u}(\xi) d\xi, \tag{14}$$

for all $u \in C_c^\infty(\underline{\Omega})$, where $\mathcal{U}_{\mathbf{k}_\epsilon}(t,t')$ is the symbol of $U_{\mathbf{k}_\epsilon}(t,t')$. We construct a formal symbol (for more details see [1])

$$\mathcal{U}_{\mathbf{k}_\epsilon}(t,t') = \sum_{j=0}^{\infty} (\mathcal{U}_{\mathbf{k}_\epsilon})_j(t,t')$$

from which a true symbol can be constructed by use of cut-off functions in the standard way. Proceeding formally, we write

$$(\mathbf{k}_\epsilon \partial_t - \mathcal{A}_{\mathbf{k}_\epsilon}(t)) U_{\mathbf{k}_\epsilon}(t,t')u = (2\pi)^{-m} \int e^{ix\xi} (\mathbf{k}_\epsilon \partial_t - a(D+\xi,\tau)) \mathcal{U}_{\mathbf{k}_\epsilon}(t,t') \hat{u}(\xi) d\xi,$$

and we require for each $0 \le t < T$ that

$$(\mathbf{k}_\epsilon \partial_t - a(D+\xi,\tau)) \mathcal{U}_{\mathbf{k}_\epsilon}(t,t') = 0 \tag{15}$$

and $\mathcal{U}_{\mathbf{k}_\epsilon}(t,t') = I$ (identity matrix).

Let $\lambda(\tau) = (1 + |\tau|)$, and consider the expression

$$zI - \lambda^{-1} a(\tau) = \lambda^{-1}(z\lambda I - a(\tau)).$$

It follows from Theorem 4.3 that for each compact set $K \subset \Omega$ there exists a positive constants M, C_1, C_2 such that if $(x,t) \in K$ and $|\tau| \ge M$, the eigenvalues of the matrix $\lambda^{-1} a(\xi,\tau)$ lie in $\mathbb{C}^+$ inside the circle

$$|z| \quad \le \quad C_1(1 + |\tau|),$$

and in the half plane-plane $\mathrm{Im} z \ge C_2$. For any $R \le M$ and $R \le |\tau| \le R+1$, let Γ_R be a contour in the upper half-plane that encircles the eigenvalues of the matrix $\lambda^{-1} a(\tau)$ for $(x,t) \in K$. In view of the previous remarks we could take the length of Γ_R to be less than $2\pi(R+2)$. We are going to represent $\mathcal{U}_{\mathbf{k}_\epsilon}$ as

$$\mathcal{U}_{\mathbf{k}_\epsilon}(t,t') = \frac{1}{2\pi i} \oint_{\Gamma_R} e^{i\lambda(t-t')z} k(\tau; z) \, dz,$$

where k is a suitable formal symbol $\sum_{j=1}^{\infty} k_j$.

Since $k(\tau; z)$ is going to be a holomorphic function of z, it follows that $\mathcal{U}_{\mathbf{k}_\epsilon}$ remains the same if the contour Γ_R is changed but still encircles the eigenvalues.

To finalize, we want to estimate the symbols

$$(\mathcal{U}_{\mathbf{k}_\epsilon})_j(t, t') = \frac{1}{2\pi i} \oint_{\Gamma_R} e^{i\lambda(t-t')z} k_j(\tau; z)\, dz. \tag{16}$$

Theorem 4.4. *For every $K \subset \Omega$, there is a constant $c > 0$ such that to every pair of n-tuples $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, ..., \beta_n) \in \mathbb{N}^n$ and to every pair of integers r and N, there exists a constant $C_1 = C_1(\alpha, \beta, r, K)$ such that*

$$|D_x^\beta D_\xi^\alpha \partial_t^r (\mathcal{U}_{\mathbf{k}_\epsilon})_j(t, t')| \leq C_1 (t - t')^{-N} (1 + |\tau|)^{1-|\beta|+r-2N}.$$

for all $(x, t) \in K$ and $|\tau| > c$.

Proof. For $z \in \Gamma_R$, we have

$$|D_\xi^\alpha \partial_t^r (e^{i\lambda(t-t')z})| \leq C (t - t')^{-N} (1 + |\tau|)^{-|\alpha|} \lambda^{r-N}. \tag{17}$$

Using Leibniz formula we can write $D_x^\beta\ D_\xi^\alpha\ \partial_t^r (e^{i\lambda(t-t')z} k_j(\tau; z))$ as a linear combination of products of the type $D_x^{\beta'} D_\xi^{\alpha'} \partial_t^r (k_j) D_\xi^{\alpha''} \partial_t^{r''} (e^{i\lambda(t-t')z})$, each of which can be estimated by

$$C (t - t')^{-N} (1 + |\tau|)^{1-|\beta'|-|r''|} \lambda^{\alpha''-N}. \tag{18}$$

Since $\lambda = (1 + |\tau|)$, it follows

$$\lambda^{\alpha''-N} \leq (1 + |\tau|)^{r-N}. \tag{19}$$

Combining (18) and (19) we have

$$|D_x^\beta D_\xi^\alpha \partial_t^r (e^{i\lambda(t-t')z} k_j(\tau)) \leq C (t - t')^{-N} (1 + |\tau|)^{1-|\beta|+r-2N},$$

and hence the following estimate for our symbol

$$|D_x^\beta D_\xi^\alpha \partial_t^r (\mathcal{U}_{\mathbf{k}_\epsilon})_j(t, t')| \leq C (t - t')^{-N} (1 + |\tau|)^{1-|\beta|+r-2N} \oint_{\Gamma_R} |dz|.$$

Since $C \oint_{\Gamma_R} |dz| \leq C(2\pi(R + 2)) = C_1$ we finally obtain

$$|D_x^\beta D_\xi^\alpha \partial_t^r (\mathcal{U}_{\mathbf{k}_\epsilon})_j(t, t')| \quad \leq \quad C_1 (t - t')^{-N} (1 + |\tau|)^{1-|\beta|+r-2N}.$$

□

5 The initial boundary problem

In this section we aim to solve the initial value problem associated to the operator $-\Delta - \mathbf{k}_\epsilon \partial_t$ by means of the parametrices representation previously obtained. Consider the following initial value problem

$$\begin{cases} (-\Delta - \mathbf{k}_\epsilon \partial_t)u(x,t) = f(x,t) & \text{in} \quad \Omega \\ u(x,0) = h(x) & \text{in} \quad \underline{\Omega} \end{cases}, \tag{20}$$

where $f \in C^\infty(\Omega)$ and $h \in C^\infty(\underline{\Omega})$. By Definition 2.3 in [2] we have that the operator in (20) is hypoelliptic. Hence to solve (20) it is sufficient to study the homogeneous problem

$$\begin{cases} (-\Delta - \mathbf{k}_\epsilon \partial_t)u(x,t) = 0 & \text{in} \quad \Omega \\ u(x,0) = h(x) & \text{in} \quad \underline{\Omega} \end{cases}, \tag{21}$$

in fact, if u_1 is any solution of $(-\Delta - \mathbf{k}_\epsilon \partial_t)u = f$ obtained via convolution with the kernel of the operator applied to f, and u_2 is a solution of the homogeneous problem (21) with h substituted for $h - u_1$, then $u = u_1 + u_2$ satisfies (20). In view of (3) this is equivalent modulo a Tikhonov operator to solve the problem

$$\begin{cases} P^+_{\mathbf{k}_\epsilon} u(x,t) = 0 & \text{in} \quad \Omega \\ u(x,0) = h(x) & \text{in} \quad \underline{\Omega} \end{cases}. \tag{22}$$

Since we had constructed the parametrix of the operator $-\Delta - \mathbf{k}_\epsilon \partial_t$ in (14), we have that the solution u of (22) is equal to $U_{\mathbf{k}_\epsilon} h$.

Taking into account [2], we can guarantee that the solution obtained previously for the $-\Delta - \mathbf{k}_\epsilon \partial_t-$problem is unique. From the ideas presented in [5], we conclude that the unique solution of the $-\Delta - \mathbf{k}_\epsilon \partial_t$ problem converges, when $\epsilon \to 0$, to the unique solution of the classical problem. Combining this two facts we can ensure that $U_{\mathbf{k}_\epsilon} h$ converges, when $\epsilon \to 0$, to the solution of the following problem

$$\begin{cases} (-\Delta - i\partial_t)u(x,t) = 0 & \text{in} \quad \Omega \\ u(x,0) = h(x) & \text{in} \quad \underline{\Omega} \end{cases},$$

where $\Omega = \underline{\Omega} \times [0,T) \subset \mathbb{R}^n \times \mathbb{R}^+_0$, $f \in C^\infty(\Omega)$ and $h \in C^\infty(\underline{\Omega})$.

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