New canonical triple covers of surfaces

Carlos Rito

Abstract

We construct a surface of general type with canonical map of degree 12 which factors as a triple cover and a bidouble cover of $\mathbb{P}^2$. We also show the existence of a smooth surface with $q = 0$, $\chi = 13$ and $K^2 = 9\chi$ such that its canonical map is either of degree 3 onto a surface of general type or of degree 9 onto a rational surface.
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1 Introduction

Let S be a smooth minimal surface of general type. Denote by $\phi : S \dashrightarrow \mathbb{P}^{p_g-1}$ the canonical map and let $d := \deg(\phi)$. The following Beauville's result is well-known.

Theorem 1 ([Be]). *If the canonical image $\Sigma := \phi(S)$ is a surface, then either:*

(i) $p_g(\Sigma) = 0$, *or*

(ii) Σ *is a canonical surface (in particular $p_g(\Sigma) = p_g(S)$).*

Moreover, in case (i) $d \leq 36$ *and in case* (ii) $d \leq 9$.

Beauville has also constructed families of examples with $\chi(\mathcal{O}_S)$ arbitrarily large for $d = 2, 4, 6, 8$ and $p_g(\Sigma) = 0$. Although this is a classical problem, for $d > 8$ the number of known examples drops drastically: only Tan's example [Ta2, §5] with $d = K^2 = 9$, $\chi = 4$ and Persson's example [Pe] with $d = K^2 = 16$, $\chi = 4$ are known. More recently, Du and Gao [DuGa] claim that if the canonical map is an abelian cover of $\mathbb{P}^2$, then these are the only possibilities for $d > 8$.

In this note we construct an example with $d = K^2 = 12$ which factors as a triple cover and a bidouble cover of $\mathbb{P}^2$.

Known examples for case (ii) with $d = 3$ date to 1991/2: Pardini's example [Pa2] with $K^2 = 27$, $\chi = 6$ and Tan's examples [Ta2] with $K^2 \leq 6\chi$, $5 \leq \chi \leq 9$. Nowadays this case $d = 3$ is still mysterious. On the one hand no one has given a bound for χ, on the other hand there are no examples for other values of the invariants.

More generally, for the case where the canonical map factors through a triple cover of a surface of general type and $d = 6$, we have only the family given in [CiPaTo, Example 3.4] with invariants on the Noether's line $K^2 = 2p_g - 4$.

Here we show the existence of a smooth regular surface S with $\chi = 13$ and $K^2 = 9\chi$ such that its canonical map ϕ factors through a triple cover of a surface of general type. This is the first example on the border line $K^2 = 9\chi$ (recall that, for a surface of general type, one always has $2p_g - 4 \leq K^2 \leq 9\chi$).

This surface is an unramified cover of a fake projective plane. We show that if $\deg(\phi) \neq 3$, then ϕ is of degree 9 onto a rational surface. If this is the case, then one might expect to be able to recover the construction of the rigid surface S has a covering of $\mathbb{P}^2$, which would be interesting. Since it seems very difficult to provide a geometric construction of a fake projective plane, we conjecture that $d = 3$.

Notation

We work over the complex numbers. All varieties are assumed to be projective algebraic. A $(-n)$-curve on a surface is a curve isomorphic to $\mathbb{P}^1$ with self-intersection $-n$. Linear equivalence of divisors is denoted by $\equiv$. The rest of the notation is standard in Algebraic Geometry.

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2 Basics on Galois triple covers

Our references for triple covers are [Ta1], [Ta2] or [Mi].

Let X be a smooth surface. A Galois triple cover $\pi : Y \to X$ is determined by divisors L, M, B, C on X such that $B \in |2L - M|$ and $C \in |2M - L|$. The branch locus of π is $B + C$ and $3L \equiv 2B + C$, $3M \equiv B + 2C$. The surface Y is normal iff $B + C$ is reduced. The singularities of Y lie over the singularities of $B + C$.

If $B+C$ is smooth, we have

$$\chi(\mathcal{O}_Y) = 3\chi(\mathcal{O}_X) + \frac{1}{2}\left(L^2 + K_X L\right) + \frac{1}{2}\left(M^2 + K_X M\right), \quad (1)$$

$$K_Y^2 = 3K_X^2 + 4\left(L^2 + K_X L\right) + 4\left(M^2 + K_X M\right) - 4LM, \quad (2)$$

$$q(Y) = q(X) + h^1(X, \mathcal{O}_X(K_X + L)) + h^1(X, \mathcal{O}_X(K_X + M)), \quad (3)$$

$$p_g(Y) = p_g(X) + h^0(X, \mathcal{O}_X(K_X + L)) + h^0(X, \mathcal{O}_X(K_X + M)). \quad (4)$$

Now suppose that $\sigma : X \to X'$ is the minimal resolution of a normal surface X' with a set $s = \{s_1, \ldots, s_n\}$ of ordinary cusps (singularities of type A_2). If the (-2)-curves A_i, A_i' satisfying $\sigma^{-1}(s_i) = A_i + A_i'$ can be labelled such that

$$\sum_1^n (2A_i + A_i') \equiv 3J,$$

for some divisor J, then we say that s is a 3-*divisible* set of cusps.

Proposition 2. *Let X' be a minimal surface of general type containing a 3-divisible set as above as only singularities. Let $\phi : Y \to X$ be a Galois triple cover with branch locus $\sum_1^n (A_i + A_i')$.*

If $n = 3\chi(\mathcal{O}_X)$, then $\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_X)$ and $K_{Y'}^2 = 3K_X^2$, where Y' is the minimal model of Y.

Proof:

Let $\widetilde{X} \to X$ be the blow-up at the singular points of $\sum_1^n (A_i + A_i')$. Denote by $\widehat{A_i}, \widehat{A_i'}$ the (-3)-curves which are the strict transforms of A_i, A_i', $i = 1, \ldots, n$. The surface Y' is the minimal model of the Galois triple cover of $\widetilde{X}$ with branch locus $\sum_1^n (\widehat{A_i} + \widehat{A_i'})$. The result follows from (1) and (2) (notice that $K_{\widetilde{X}}^2 = K_X^2 - n$ and $K_{Y'}^2 = 3K_{\widetilde{X}}^2 + 3n$).

Remark 3. *Note that the cusps induce smooth points on the covering surface, i.e. the pullback of the divisor $\sum_1^n (A_i + A_i')$ is contracted to smooth points of Y'.*

3 A surface with canonical map of degree 12

The following result has been shown by Tan [Ta3, Thm 6.2.1], using codes. Here we give an alternative proof.

Lemma 4. *Let X be a double cover of $\mathbb{P}^2$ ramified over a quartic curve with 3 cusps. Then the 3 cusps of X are 3-divisible.*

Proof: Let $B \subset \mathbb{P}^2$ be a quartic curve with 3 cusps at points p_1, p_2, p_3 (it is well known that such a curve exists; it is unique up to projective equivalence). Consider the canonical resolution $X' \to X$. The strict transform of the lines through p_1p_2, p_2p_3 and p_1p_3 is an union of disjoint (-1)-curves $E_i, E'_i \subset X'$, $i = 1, 2, 3$. These curves and the (-2)-curves A_i, A'_i, $i = 1, 2, 3$, which contract to the cusps of X can be labelled such that the intersection matrix of the curves $A_1, A'_1, A_2, A'_2, A_3, A'_3, E_1, E_2$ is

```
[-2  1  0  0  0  0  1  0]
[ 1 -2  0  0  0  0  0  0]
[ 0  0 -2  1  0  0  1  0]
[ 0  0  1 -2  0  0  0  1]
[ 0  0  0  0 -2  1  0  0]
[ 0  0  0  0  1 -2  0  1]
[ 1  0  1  0  0  0 -1  0]
[ 0  0  0  1  0  1  0 -1]
```

This matrix has determinant zero. Since

$$b_2(X') = 12\chi(\mathcal{O}_{X'}) - K^2_{X'} + 4q(X') - 2 = 8,$$

these 8 curves are dependent in $\mathrm{Num}(X')$, and this relation has to be expressed in the nullspace of the matrix. Using computer algebra we get that this nullspace has basis

```
( 2  1  1 -1 -1 -2  3 -3).
```

Thus $2A_1 + A'_1 + A_2 - A'_2 - A_3 - 2A'_3 + 3E_1 - 3E_2 = 0$ in $\mathrm{NS}(X')$ (notice that X' has no non-trivial torsion). One has $\mathrm{NS}(X') = \mathrm{Pic}(X')$ for regular surfaces X' (Castelnuovo), hence there exists a divisor L such that

$$2A_1 + A'_1 + A_2 + 2A'_2 + 2A_3 + A'_3 \equiv 3L.$$

□

Let $Q_1, Q_2 \subset \mathbb{P}^2$ be quartic curves with 3 cusps each such that $Q_1 + Q_2$ has 6 cusps and 16 nodes. Let V be the bidouble cover of $\mathbb{P}^2$ defined by the divisors Q_1, Q_2, $Q_3 := 0$ (for information on bidouble covers see e.g. [Ca] or [Pa1]). Consider the divisors J_1, J_2, J_3 such that $2J_1 \equiv Q_2 + Q_3$, $2J_2 \equiv Q_1 + Q_3$, $2J_3 \equiv Q_1 + Q_2$. We have

$$p_g(V) = p_g\left(\mathbb{P}^2\right) + \sum_1^3 h^0\left(\mathbb{P}^2, K_{\mathbb{P}^2} + J_i\right) = 3,$$

$$\chi(\mathcal{O}_V) = 4\chi\left(\mathcal{O}_{\mathbb{P}^2}\right) + \frac{1}{2}\sum_1^3 J_i\left(K_{\mathbb{P}^2} + J_i\right) = 4$$

and V has 12 ordinary cusps and no other singularities.

Denote by W_1, W_2, W_3 the double covers of $\mathbb{P}^2$ with branch curves Q_2, Q_1, $Q_1 + Q_2$, respectively (the intermediate surfaces of the bidouble cover). The canonical map of V factors through maps $V \to W_i$, $i = 1, 2, 3$, hence it is of degree 4. We get from Lemma 4 that the 3 cusps of W_1 and the 3 cusps of W_2 are 3-divisible, therefore the 12 cusps of V are also 3-divisible.

Now let $S \to V$ be the Galois triple cover ramified over the 12 cusps. We **claim** that $p_g(S) = p_g(V) = 3$. This implies that the canonical map of S factors through the triple cover, thus it is of degree 12. From Proposition 2, $q(S) = 0$ and $K_S^2 = 12$.

So it remains to prove the claim. Let $\widetilde{V}$ be the smooth minimal resolution of V. The cusps of V correspond to configurations of (-2)-curves $A_i + A_i' \subset \widetilde{V}$, $i = 1, \ldots, 12$. These can be labelled such that there exist divisors L, M satisfying $2B + C \equiv 3L$, $B + 2C \equiv 3M$, where $B := \sum A_i$ and $C := \sum A_i'$.

Below we use the notation $D \geq 0$ for $h^0\left(\widetilde{V}, \mathcal{O}_{\widetilde{V}}(D)\right) > 0$.

From (4) we need to show that $K_{\widetilde{V}} + L \not\geq 0$ and $K_{\widetilde{V}} + M \not\geq 0$. Suppose first that $K_{\widetilde{V}} + L \geq 0$. Then

$$(K_{\widetilde{V}} + L)A_i = -1, \ \forall i \implies K_{\widetilde{V}} + L - B \geq 0$$

and

$$(K_{\widetilde{V}} + L - B)A_i' = -1, \ \forall i \implies K_{\widetilde{V}} + L - B - C \geq 0.$$

From $3K_{\widetilde{V}} + 3L - 3B - 3C \geq 0$ and $3(L+M) \equiv 3(B+C)$ one gets $3K_{\widetilde{V}} - 3M \geq 0$, i.e. $3K_{\widetilde{V}} - B - 2C \geq 0$. This implies the existence of an element in the linear system $|3K_V|$ having multiplicity > 1 at each of the 12 cusps of V.

Let q_1, q_2 be the defining equations of Q_1, Q_2. The surface V is given by equations $w^2 = q_1$, $t^2 = q_2$ in the weighted projective space $\mathbb{P}(x^1, y^1, z^1, w^2, t^2)$. It is easy to see that the linear system of polynomials of degree 3 has no element with multiplicity > 1 at the cusps of V (for instance using computer algebra).

The case $K_{\widetilde{V}} + M \geq 0$ is analogous. □

4 A surface with $K^2 = 9\chi$

Based on the work of Prasad and Yeung [PrYe], Cartwright and Steger [CaSt] constructed a fake projective plane F with an automorphism j of order 3 such that F/j is a surface of general type with $\chi = 1$, $p_g = 0$, $K^2 = 3$ and fundamental group $\mathbb{Z}_{13}$. This surface has a set of three 3-divisible cusps (cf. [Ke]). Denote by B the unit ball in $\mathbb{C}^2$ and let P, H and G be the groups such that $F = B/P$, $F/j = B/H$ and the universal cover of F/j is B/G. We have the following commutative diagram, where the vertical arrows denote unramified $\mathbb{Z}_{13}$ covers and the horizontal arrows denote $\mathbb{Z}_3$ covers ramified over cusps.

$$\begin{array}{ccc} B/(G\cap P) & \xrightarrow{p} & B/G \\ {\scriptstyle 13:1}\downarrow & & \downarrow{\scriptstyle 13:1} \\ B/P & \xrightarrow{3:1} & B/H \end{array}$$

We show that the surface $S := B/(G \cap P)$ is regular, hence $p_g(S) = p_g(B/G)$ and then the canonical map ϕ of S factors through the triple cover p. Since $G \cap P$ is the fundamental group of S, the commutator quotient $(G \cap P)/[G \cap P, G \cap P]$ is isomorphic to $H_1(S, \mathbb{Z})$. The first Betti number $b_1(S) = 2q(S)$ is the minimal number of generators of $H_1(S, \mathbb{Z})$ modulo elements of finite order. Thus $q(S) = 0$ if $(G \cap P)/[G \cap P, G \cap P]$ is finite. This is shown in the Appendix, where we use computational GAP [GAP4] data from Cartwright and Steger to compute $G \cap P$.

Now notice that ϕ is not composed with a pencil. In fact otherwise the canonical map of B/G is composed with a pencil and then

$$39 = K^2_{B/G} \geq 4\chi(\mathcal{O}_{B/G}) - 10 = 42,$$

from [Zu, Theorem A] (see also [Ko, Corollary 3.4]).

Finally we prove that if $d := \deg(\phi) \neq 3$, then $d = 9$ and the canonical image $\phi(S)$ is a rational surface. As in the proof of Proposition 4.1 of [Be], we have

$$9\chi(\mathcal{O}_S) \geq K^2_S \geq d \deg(\phi(S)) \geq nd(p_g(S) - 2)$$

where $n = 2$ if $\phi(S)$ is not ruled and $n = 1$ otherwise. This gives $d < 6$ if $\phi(S)$ is not ruled and $d < 12$ otherwise. Since $d \equiv 0 \pmod 3$, then $d = 6$ or 9 and $\phi(S)$ is a rational surface. If $d = 6$, the canonical map of B/G is of degree 2 and then B/G has an involution. But, as seen in the Appendix, the automorphism group of B/G is the semidirect product $\mathbb{Z}_{13} : \mathbb{Z}_3$, so there is no involution on B/G.

Appendix: GAP code

```
#We are using data from
#http://www.maths.usyd.edu.au/u/donaldc/fakeprojectiveplanes/
#C18p3/C18p3-0-FP.gap
#namely the groups 'index9aFP', 'index3aFP' and
#the functions 'FundGp', 'AutGp'.

P:=index9aFP;
H:=index3aFP;
#B/P is a Fake projective plane.
#B/H is a quotient of the fake p.p. B/P by an order 3 automorphism.
#B/H is a surface with p_g=0 and K^2=3 having 3 cusps.
```

```
#Using the Cartwright-Steger function 'FundGp', we see that the
#fundamental group of B/H is "C13". We want to find the group G such
#that B/G is the universal cover of B/H. This is also computed by the
#function 'FundGp'.
#The following function is equal to 'FundGp' except that outputs G.

Grp:=function(G,FOList)
  local e1,e2,e3,GFO,G0,G0FCA;
  e1:=GeneratorsOfGroup(GammaBarFP);
  e2:=Concatenation(e1,List(e1,Inverse));
  Add(e2,One(G));
  e3:=ListX(e2,e2,\*);
  GFO:=Filtered(FOList,fo->fo in G);
  G0:=Group(ListX(e3,GFO,function(elt,fo) return fo^elt; end));
  Print(ForAll(GFO,fo->fo in G0),"\n");
  Print(IsNormal(G,G0),"\n");
  return G0;
end;

G:=Grp(H,FOList);
GP:=Intersection(G,P);

#The "commutative diagram":
StructureDescription(FactorGroup(P,GP)) = "C13";
StructureDescription(FactorGroup(H,G)) = "C13";
StructureDescription(FactorGroup(G,GP)) = "C3";
StructureDescription(FactorGroup(H,P)) = "C3";

#The first Betti number b_1(B/GP) = 0:
Index(GP,CommutatorSubgroup(GP,GP)) = 2916; #It is finite.

#The automorphism group of B/G:
StructureDescription(AutGp(G)) = "C13 : C3";
```

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Carlos Rito
Departamento de Matemática, Faculdade de Ciências
Rua do Campo Alegre 687, Apartado 1013
4169-007 Porto, Portugal
e-mail: crito@fc.up.pt