Equations over free inverse monoids with idempotent variables

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Abstract. We introduce the notion of idempotent variables for studying equations in inverse monoids. It is proved that it is decidable in singly exponential time (DEXPTIME) whether a system of equations in idempotent variables over a free inverse monoid has a solution. The result is proved by a direct reduction to solve language equations with one-sided concatenation and a known complexity result by Baader and Narendran. Decidability for systems of typed equations over a free inverse monoid with one irreducible variable and at least one unbalanced equation is proved with the same complexity.

Our results improve known complexity bounds by Deis et al. Our results also apply to larger families of equations where no decidability has been previously known. It is also conjectured that DEXPTIME is optimal.

1 Introduction

It is decidable whether equations over free monoids and free groups are solvable. These classical results were proved by Makanin in his seminal papers [10], [11]. A first estimation of the time complexity for deciding solvability was more than triple or four times exponential, but over the years it was lowered. It went down to PSPACE by Plandowski [15, 16] for free monoids. Extending his method Gutiérrez showed that the same complexity bound applies in the setting of free groups [6]. In [7] Jeż used his "recompression technique" and achieved the best known space complexity to date: NTIME($n \log n$). Perhaps even more importantly, he presented the simplest known proof deciding the problem WordEquations. (Re-)compression leads also to an easy-under-stand algorithmic description for the set of all solutions for equations over free monoids and free groups, see Diekert et al. [5]. Actually, the result in [5] is more general and copes with free monoids with involution and rational constraints.

In the present paper we study equations over inverse monoids. Inverse monoids are monoids with involution and constitute the most natural intermediate structure between monoids and groups. They are well-studied and pop-up in various applications, for example when investigating systems which are deterministic and codeterministic. Inverse monoids arise naturally as monoids of injective transformations closed under inversion. Indeed, up to isomorphism, these are all the inverse monoids, as stated in the classical Vagner-Preston representation theorem. This makes inverse monoids ubiquitous in geometry, topology and other fields.

The fifties of the last century boosted the systematic study of inverse monoids. However, the word problem remained unsolved until the early seventies, when Scheiblich [18] and Munn [12] independently provided solutions for free inverse monoids. The next natural step is to consider solvability of equations, i.e., the existential theory. However, Rozenblat's paper [17] destroyed all hope for a general solution: solving equations in free inverse monoids is undecidable. Thus, the best we can hope is to prove decidability for particular subclasses. For almost a decade, the reference paper on this subject has been the paper of Deis, Meakin and Sénizergues [4] which showed decidability for a given equation together with a solution over the free quotient group using Rabin's tree theorem. This resulted in an algorithm which is super-exponential (and at least doubly exponential in their specific setting). In the present paper, we achieve various improvements. The main result lowers the complexity to singly exponential time; and we conjecture that this is optimal. Moreover, we study equations with idempotent variables instead of lifting properties, which leads to a uniform approach and simplified the proof. It also enabled us to generalize some results concerning one-variable equations to a broader setting, thereby leading to new decidability results.

A more precise statement about the progress with respect to [4] is as follows. First, Theorem 3 shows that deciding solvability of systems of equations in idempotent variables over $\mathrm{FIM}(A)$ is possible in DEXPTIME, while the complexity of the algorithm in [4, Theorem 8] is much higher, since the algorithm involves Rabin's Tree Theorem[5]. Second, with respect to unbalanced one-variable equations and [4, Theorem 13], our Theorem 5 admits the presence of arbitrarily many idempotent variables, and the complexity very much improved in view of Theorem 3. Third, our proofs are shorter and easier to understand by a direct reduction to language equations. This enables us to use well-known results from [2] by Baader and Narendran.

2 Preliminaries

We say that a function $f : \mathbb{N} \to \mathbb{N}$ is *polynomial* if $f(n) \in n^{\mathcal{O}(1)}$. It is *singly exponential,* if $f(n) \le 2^{p(n)}$ where p is a polynomial. The complexity classes PSPACE resp. DEXPTIME refer to problems which can be solved on deterministic Turing machines within a polynomial space bound resp. singly exponential time bound. The notation is standard, see for example [13].

We also use the standard notation from combinatorics on words. A (finite) set is called an *alphabet* and an element of an alphabet is called a *letter*. The free monoid generated by an alphabet A is denoted by A^*. It consists of all finite sequences of letters from A. The elements of A^* are called *words*. The empty word is denoted by 1. The *length* of a word u is denoted by $|u|$. We have $|u| = n$ for $u = a_1 \cdots a_n$ where $a_i \in A$. The empty word has length 0, and it is the only word with this property. A word u is a *factor* of a word v if there exist $p, q \in A^*$ such that $puq = v$. A factor u is a *prefix* of v if $uq = v$ for some $q \in A^*$. We also write $u \le v$ if u is a prefix of v and we let $\mathrm{Pref}(v) = \{u \in A^* \mid u \le v\}$. Given a word w, we denote by $\mathrm{last}(w)$ the last letter of w (if $w \ne \varepsilon$) or the word ε (if $w = \varepsilon$). A *language* is a subset of A^*. The notion of prefix extends to languages by $\mathrm{Pref}(L) = \{u \in A^* \mid \exists v \in L : u \le v\}$.

Throughout this paper, the alphabet A is endowed with an involution. An *involution* is a mapping $\overline{}$ such that $\overline{\overline{x}} = x$ for all elements. In particular, an involution is a bijection. We use the following convention. There is a subset $A_+ \subseteq A$ such that such that first, $A = A_+ \cup \{\overline{a} \mid a \in A_+\}$ and second, $A_+ \cap \{\overline{a} \mid a \in A_+\} = \{a \in A \mid \overline{a} = a\}$. Therefore, if the involution on A is without fixed points then $A = A_+ \cup \{\overline{a} \mid a \in A_+\}$ is a disjoint union. Thus, there are no self-involuting letters, which is case of primary interest. If an involution is defined for a monoid then we additionally require that $\overline{xy} = \overline{y}\,\overline{x}$ for all its elements x, y. This applies in particular to a free monoid A^* over a set with involution: for a word $w = a_1 \cdots a_m$ we thus have $\overline{w} = \overline{a_m} \cdots \overline{a_1}$. If $\overline{a} = a$ for all $a \in A$ then $\overline{w}$ simply means to read the word from right-to-left.

A morphism[6]in the category of *sets with involution* is a mapping respecting the involution. Likewise, a morphism between monoids with involution is a monoid-homomorphism respecting the involution.

Consider a monoid M with involution and a mapping $\varphi : A \to M$ respecting the involution. This is a morphism of sets with involution and there is exactly one morphism $\Phi : A^* \to M$ of monoids with involution such that $\Phi(a) = \varphi(a)$ for all $a \in A$. In this sense, A^* is the free monoid with involution on A (w.r.t. to the category of sets with involution).

Every group is a monoid with involution by letting $\overline{x} = x^{-1}$; and a morphism between groups is the same as a homomorphism between groups. The identity is an involution on a monoid if and only if the monoid is commutative. In particular, $\mathbb{N}$ is viewed as a monoid with involution and the length function $A^* \to \mathbb{N}, u \mapsto |u|$ is a morphism. However, if a group is commutative then, by default, we still let $\overline{x} = x^{-1}$. This applies in particular to $\mathbb{Z}$ where $\overline{n} = -n$.

[5] The DEXPTIME result was obtained first by the second and third author, but not published. The same improvement for the main result in [4] was discovered later independently the two other authors; and the present paper is the joint outcome of both approaches. In addition, we take the opportunity to correct a mistake in [4] about some special one-variable equations, where a technical assumption (point 2 in Def. 2) was missing.

[6] In our notation a *homomorphism* is a mapping which respects the algebraic structure whereas a *morphism* respects the involution and, depending on the category, it also has to respect the algebraic structure.

A monoid M is said to be *inverse* if for every $x \in M$ there exists exactly one element $\overline{x} \in M$ satisfying $x\overline{x}x = x$ and $\overline{x}x\overline{x} = \overline{x}$. Clearly, $\overline{\overline{x}} = x$ by uniqueness of $\overline{x}$ and, hence, M is a set with involution. The mapping $x \mapsto \overline{x}$ is also called an *inversion*. Idempotents commute in inverse monoids (see e.g., [14]), hence the subset $E(M) = \left\{e \in M \mid e^2 = e\right\}$ is a subsemigroup. Since necessarily $\overline{e} = e$ for $e \in E(M)$ one easily deduces that $\overline{xy} = \overline{y}\,\overline{x}$ for all $x, y \in M$. As a consequence, an inverse monoid is a monoid with involution. Frequently in the literature the notation $\overline{x} = x^{-1}$ is also used for elements of inverse monoids, just as for groups (which constitute a proper subclass of inverse monoids). Just as for groups, by default, the involution on an inverse monoid is supposed to be given by its inversion. We proceed now to describe Scheiblich's construction of free inverse monoids. Let us recall some concepts and fix some notation for free groups. If the involution on A is without fixed points then the free group $\mathrm{FG}(A_+)$ is as usual the quotient monoid of A^* defined by $\{a\overline{a} = 1 \mid a \in A\}$. It satisfies the universal property that every mapping of A_+ to a group G uniquely extends to a homomorphism $\mathrm{FG}(A_+) \to G$. But the same construction works if we allow fixed points for the involution on A. We denote the quotient monoid of A^* defined by $\{a\overline{a} = 1 \mid a \in A\}$ by $F(A)$. Thus, if the involution on A is without fixed points then $F(A) = \mathrm{FG}(A_+)$, otherwise $F(A)$ is a free product of a free group by cyclic groups of order 2. Now, every morphism of a set with involution A to a group G extends uniquely to a homomorphism $F(A) \to G$. This follows because in a group G we have $x = x^{-1}$ if and only if $x^2 = 1$. As a set we can identify $F(A)$ with the subset of reduced words in A^*. As usual, a word is called *reduced* if it does not contain any factor $a\overline{a}$ where $a \in A$. Observe that this embedding of $F(A)$ into A^* is compatible with the involution. In the following we let $\pi : A^* \to F(A)$ be the canonical morphism from A^* onto $F(A)$. It is well-known that every word $u \in A^*$ can be transformed into a unique reduced word $\widehat{u}$ by successively erasing factors of the form $a\overline{a}$ where $a \in A$. This leads to the equivalence

$$\forall u, v \in A^* : \pi(u) = \pi(v) \;\Leftrightarrow\; \widehat{u} = \widehat{v}.$$

As we systematically identify the set $F(A)$ with the subset $\widehat{A^*}$ of A^*, concepts such as length, factor, prefix, and prefix-closure are inherited from free monoids to free groups via reduced words. For the same reason, it makes sense to write $\widehat{u} = \pi(u)$, for $u \in A^*$, because $\pi(u) \in F(A)$ is identified with $\widehat{u} \in A^*$.

Following Scheiblich [18], we represent elements of $\mathrm{FIM}(A)$ as pairs (X, g) where the second component is a group element $g \in F(A)$ and the first component is a finite prefix closed subset X of $F(A)$ such that $g \in X$. In other terms, this means that X is a finite connected subset of the Cayley graph of $F(A)$ (over A) such that $1, g \in X$. Formally, we let

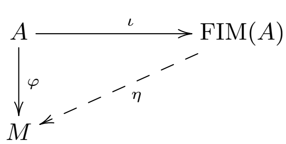
$$\mathrm{FIM}(A) = \{(X, g) \mid |X| < \infty \wedge g \in X = \mathrm{Pref}(X) \subseteq F(A)\}\,.$$

The multiplication on $\mathrm{FIM}(A)$ is defined through

$$(X, g)(Y, h) = (X \cup gY, gh).$$

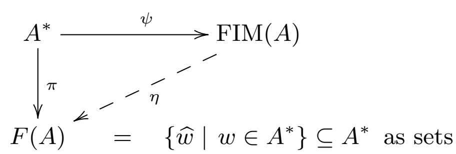
It is easy to see that $\mathrm{FIM}(A)$ is a monoid with identity $(\{1\}, 1)$ and every (X, g) has a unique inverse $(g^{-1}X, g^{-1})$, hence $\mathrm{FIM}(A)$ is an inverse monoid.

Let $\psi : A^* \to \mathrm{FIM}(A)$ be the homomorphism of monoids defined by $\psi(a) = (\{1, a\}, a)$. Then we have $\psi(\overline{a}) = (\{1, \overline{a}\}, \overline{a}) = \overline{(\{1, a\}, a)}$ and ψ is a morphism of monoids with involution. We have again the universal property of being free with respect to sets with involution. Let M be an inverse monoid and $\varphi : A \to M$ a morphism of sets with involution. Then there is exactly one morphism $\Phi : \mathrm{FIM}(A) \to M$ of monoids with involution such that $\Phi(a) = \varphi(a)$ for all $a \in A$. To make this precise, write $\iota = \psi|_A$. Given a mapping $\varphi : A \to M$ respecting the involution where M is an inverse monoid, there exists a unique morphism of inverse monoids $\eta : \mathrm{FIM}(A) \to M$ such that the following diagram commutes.



The monoid $\mathrm{FIM}(A)$ is, up to isomorphism, the unique inverse monoid satisfying this universal property. Therefore $\mathrm{FIM}(A)$ is the *free inverse monoid* over the set A with involution. If the involution is without fixed points then $\mathrm{FIM}(A)$ is the *free inverse monoid* over the set A_+. In particular, $\pi : A^* \to F(A)$ factorizes through η.

The following diagram summarizes our notation.



3 Language equations over free monoids

Henceforth A denotes an alphabet with involution of constants. We use $a, b, c, \ldots$ to denote letters of A, whereas variables are denoted by capital letters $X, Y, Z \ldots$ or by small letters $x, y, z, \ldots$ and small letters for variables refer to elements in the group $F(A)$. Our complexity results for solving certain equations over free inverse monoids depend on a result of Baader and Narendran [2]. They show that satisfiability of language equations with one-sided concatenation can be decided for free monoids in deterministic exponential time, i.e., the problem is in DEXPTIME. As we need the corresponding result for free groups as well, we define the notion of language equation in a more general framework. In this section let M denote either the free monoid A^* or the group $F(A)$. In particular, we have $A \subseteq M$ and A generates M as a monoid. The set of subsets of M with union as operation forms a commutative idempotent monoid denoted by 2^M. We therefore write $L + K$ instead of $L \cup K$. A *language equation over* M (with one-sided concatenation) has the form

$$L_I + \sum_{i \in I} w_i X_i = L_J + \sum_{j \in J} w_j X_j. \tag{1}$$

Here I and J are finite (disjoint) index sets, L_I, L_J are finite subsets of A^*, $w_k \in A^*$ are words and $X_k \in \Omega$ for $k \in I \cup J$, where Ω is a set of variables. The *size* of an equation $\mathcal{E}$ as in (1) is defined as

$$\|\mathcal{E}\| = |I \cup J| + \sum_{w \in L_I \cup L_J} |w| + \sum_{k \in I \cup J} |w_k|.$$

Consider the following language equation:

$$\{aa\} + aaX + bbY = \{bb\} + bbX + aaY. \tag{2}$$

If we let $L_{\{1,2\}} = \{aa\}$, $L_{\{3,4\}} = \{bb\}$, $X_1 = X_3 = X$, $X_2 = X_4 = Y$, $w_1 = w_4 = aa$, and $w_2 = w_3 = bb$ then Equation (2) has size 16 and is written in the syntactic form of Equation (1):

$$L_{\{1,2\}} + w_1 X_1 + w_2 X_2 = L_{\{3,4\}} + w_3 X_3 + w_4 X_4.$$

A *system of language equations over* M is a finite set $\mathcal{S}$ of language equations. A *solution* is a substitution of each $X \in \Omega$ by some finite subset $\sigma(X) \subseteq M$ (i.e., $\sigma(X)$ is a finite language) such that

$$L_I + \sum_{i \in I} w_i \sigma(X_i) = L_J + \sum_{j \in J} w_j \sigma(X_j)$$

becomes an identity in 2^M for all equations of $\mathcal{S}$. For example, $\sigma(X_1) = \sigma(X_2) = \{1\}$ solves Equation (2) which is a system with a single equation.

The *size* of a system $\mathcal{S} = \{\mathcal{E}_s \mid s \in S\}$ is defined as $\|\mathcal{S}\| = \sum_{s \in S} \|\mathcal{E}_s\|$.

Theorem 1 ([2], Thm. 6.1). *The following problem can be decided in* DEXPTIME.
Input: A system $\mathcal{S}$ of language equations over the free monoid A^*.
Question: Does $\mathcal{S}$ have a solution?

Remark 1. Theorem 7.6 of [2] shows that the problem above is DEXPTIME-complete (for $|A| \geq 2$). We do not use DEXPTIME-hardness here, and it is actually open whether the DEXPTIME-hardness transfers to our setting.

It should also be noted that [2] states Theorem 1 for a single equation, only. However the reduction to a single equation is simple. Indeed, assume that a system $\mathcal{S}$ of language equations over the free monoid A^* has n equations.

Without restriction we have $|A| \geq 2$. Let $k = \lceil \log_2 n \rceil$. Choose n pairwise different words $p_1, \ldots p_n \in A^*$ of length exactly k and for $1 \leq m \leq n$ replace the m-th equation $L_I + \sum_{i \in I} w_i X_i = L_J + \sum_{j \in J} w_j X_j$ by

$$p_m L_I + \sum_{i \in I} p_m w_i X_i = p_m L_J + \sum_{j \in J} p_m w_j X_j.$$

Summing all left-hand sides and all right-hand sides yields the desired reduction to a single equation. The reduction works since $\{p_1, \ldots, p_n\}$ is a prefix code.

4 Typed equations over free inverse monoids

An *equation* over $\mathrm{FIM}(A)$ is a pair (U, V) of words over $A \cup \mathcal{X}$, sometimes written as $U = V$. Here A is an alphabet of *constants*and $\mathcal{X}$ is a set of variables. Variables $X \in \mathcal{X}$ represent elements in $\mathrm{FIM}(A)$ and therefore $\mathcal{X}$ is an alphabet with involution, too. Without restriction we may assume $X \neq \overline{X}$ for all $X \in \mathcal{X}$. A *solution* σ of $U = V$ is a mapping $\sigma : \mathcal{X} \to A^*$ such that $\sigma(\overline{X}) = \overline{\sigma(X)}$ for all $X \in \mathcal{X}$ and such that the replacement of variables by the substituted words in U and in V give the same element in $\mathrm{FIM}(A)$, i.e., $\psi(\sigma(U)) = \psi(\sigma(V))$ in $\mathrm{FIM}(A)$, where σ is extended to a morphism $\sigma : (A \cup \mathcal{X})^* \to A^*$ leaving the constants invariant. Clearly, we may specify σ also by a mapping from $\mathcal{X}$ to $\mathrm{FIM}(A)$. For the following it is convenient to have two more types of variables which are used to represent specific elements in $\mathrm{FIM}(A)$. We let Ω be a set of *idempotent variables* and Γ be a set of *reduced variables*. Both sets are endowed with an involution. We let $\overline{Z} = Z$ for idempotent variables, $\overline{x} \neq x$ for all reduced variables, and $\overline{X} \neq X$ for all variables from $\mathcal{X}$. Thus, idempotent variables are the only variables which are self-involuting. We also insist that A, $\mathcal{X}$, Ω, and Γ are pairwise disjoint. A *typed equation* over $\mathrm{FIM}(A)$ is now a pair (U, V) of words over $A \cup \Omega \cup \Gamma$. A *solution* σ of $U = V$ is given by a mapping respecting the involution from $\Omega \cup \Gamma$ to A^* such that the following conditions hold.

1. $\psi(\sigma(Z))$ is idempotent for all $Z \in \Omega$.
2. $\sigma(x)$ is a reduced word for all $x \in \Gamma$.
3. Extending the mapping σ (as usual) to a homomorphism $\sigma : (A \cup \Omega \cup \Gamma)^* \to A^*$ respecting the involution and letting the letters of A invariant we have $\psi(\sigma(U)) = \psi(\sigma(V))$.

Lemma 1. *Let (U, V) be an (untyped) equation over $\mathrm{FIM}(A)$. For each $X, \overline{X} \in \mathcal{X}$ choose a fresh idempotent variable $Z_X \in \Omega$ and fresh reduced variables $x_X, \overline{x}_X \in \Gamma$. Let τ be the word-substitution (i.e. monoid homomorphism) which replaces each $X, \overline{X} \in \Omega$ by $Z_X x_X$ and $\overline{x}_X Z_X$ respectively. If σ is a solution of (U, V) then a solution σ' for $(\tau(U), \tau(V))$ can be defined as follows. For $\sigma(X) = (P, g)$, where g is represented by a reduced word, we let $\sigma'(Z_X) = (P, 1)$ and $\sigma'(x_X) = (Pref(g), g)$.*
Conversely, if σ' solves $(\tau(U), \tau(V))$ with $\sigma'(Z_X) = (P, 1)$ and $\sigma'(x_X) = (Pref(g), g)$ then $\sigma(X) = (P \cup Pref(g), g)$ defines a solution for (U, V).

Proof. Trivial. ◻

Note that the word-substitution τ' which replaces each $X, \overline{X} \in \Omega$ by $x_X Z_X$ and $Z_X \overline{x}_X$ respectively, has similar properties.

By Lemma 1 we can reduce the satisfiability of equations in $\mathrm{FIM}(A)$ to typed equations. The framework of typed equations is more general; and it fits better to our formalism. Let (U, V) be a typed equation, by the *underlying group equation* we mean the pair $(\pi(U), \pi(V))$ which is obtained by erasing all idempotent variables. Clearly, if (U, V) is satisfiable then $(\pi(U), \pi(V))$ must be solvable in the free group $F(A)$. This leads to the idea of *lifting* a solution of a group equation to a solution of (U, V) in $\mathrm{FIM}(A)$. It has been known by [4] that it is decidable whether a lifting is possible. The following result improves decidability by giving a deterministic exponential time bound.

Theorem 2. *The following problem can be decided in* DEXPTIME.

Input: A system $\mathcal{S}$ of equations over $\mathrm{FIM}(A)$ and a fixed solution $\sigma' : \Gamma \to F(A)$ of the system $\pi(\mathcal{S})$ of underlying group equations.

Question: Does $\mathcal{S}$ have a solution $\sigma : \mathcal{X} \to \mathrm{FIM}(A)$ such that $\sigma' = \eta \circ \sigma$?

Proof. Due to Lemma 1 we first transform the system into a new system with variables in $\Omega \cup \Gamma$. Next we replace every reduced variable $x \in \Gamma$ by $(\mathrm{Pref}(\sigma'(x)), \sigma'(x))$. Since the solution is part of the input this increases the size of $\mathcal{S}$ at most quadratic. We obtain a system of equations in idempotent variables and we apply Theorem 3 in Section 5 below. □

The next result combines Theorem 2 and a recent complexity result for systems of equations over free groups [5]. This leads to the following new result:

Corollary 1. *Let $\mathcal{S}$ be a system of equations over the free inverse monoid* $\mathrm{FIM}(A)$ *and $\pi(\mathcal{S})$ the system of underlying group equations.*

1. *On input $\mathcal{S}$ it can be decided in polynomial space whether the system $\pi(\mathcal{S})$ of group equations has at most finitely many solutions. If so, then every solution has at most doubly exponential length.*
2. *On input $\mathcal{S}$ and the promise that $\pi(\mathcal{S})$ has at most finitely many solutions it can be decided in deterministic triple exponential time whether $\mathcal{S}$ has a solution.*

Proof. The statement 1 follows from [5]. In particular, the size of the set of all solutions is at most triple exponential. Since the square of a triple exponential function is triple exponential again, the statement 2 follows from Theorem 2. □

5 Solving equations in idempotent variables

This section shows how to solve equations in idempotent variables. In particular, we obtain the result used in the proof of Theorem 2.

We make use of the following easy observation.

Lemma 2. *Let $P \subseteq A^*$ be prefix closed and $\widehat{P} = \{\widehat{p} \mid p \in P\}$ the corresponding set of reduced words. Then $\widehat{P}$ is prefix closed.*

Proof. Let $p \in P$ and $\widehat{p} \in \widehat{P}$ its reduced form. We have to show that every prefix of $\widehat{p}$ belongs to $\widehat{P}$. For $p = 1$ this is trivial. Hence, let $p = qa$ with $a \in A$ and $\widehat{q}$ the reduced form of q. We have $q \in P$ and, by induction, every prefix of $\widehat{q}$ belongs to $\widehat{P}$. Now, if $\widehat{p}$ is a prefix of $\widehat{q}$, we are done. In the other case we have $\widehat{p} = \widehat{q}a$. Since $\widehat{q}, \widehat{p} \in \widehat{P}$ we are done again. □

Theorem 3. *The following problem can be decided in* DEXPTIME*.*
Input: A system $\mathcal{S}$ of equations in idempotent variables (i.e., without any reduced variable).
Question: Does $\mathcal{S}$ have a solution in $\mathrm{FIM}(A)$*?*

Proof. Every equation $(U, V) \in \mathcal{S}$ can be written as

$$w_0 X_1 w_1 \cdots X_g w_g = w_{g+1} X_{g+2} w_{g+2} \cdots X_d w_d, \tag{3}$$

where $w_i \in A^*$ are words and $X_i \in \Omega$ are the idempotent variables. In linear time we check that for all equations in (3) we have $w_0 w_1 \cdots w_g = w_{g+1} w_{g+2} \cdots w_d$ in the group $F(A)$. If one of these equalities is violated then $\mathcal{S}$ is not solvable and we can stop.

Thus, without restriction we are in the other case, i.e., $w_0 w_1 \cdots w_g = w_{g+1} w_{g+2} \cdots w_d \in F(A)$. Now, it is enough to solve language equations over the group $F(A)$: assume that each X_i represents a finite prefix closed set in $F(A)$. Let us show that we can calculate in polynomial time a subset $L + \sum_{0 \le i \le m} u_i X_i \subseteq F(A)$ which corresponds to an expression $v_0 X_1 v_1 \cdots X_m v_m$. (Actually the time complexity is quadratic, only.)

To see this, let $v_0 X_1 v_1 \cdots X_m v_m$ appear on the left or right of some equation in $\mathcal{S}$. Let p_i be the prefix of $v_0 v_1 \cdots v_m$ having length i. The set $P = \{p_i \mid 0 \le i \le |v_0 v_1 \cdots v_m|\}$ is prefix closed by definition. We replace each $p \in P$ by its reduced form $\widehat{p}$; and we obtain a prefix closed language $L = \widehat{P}$ of reduced words by Lemma 2. Now let

u_i be the reduced form of $v_0 \cdots v_i$ for $0 \leq i < m$. Then we have $u_i \in L$ for $0 \leq i < m$. Writing unions as sums we see that $v_0 X_1 v_1 \cdots X_m v_m$ yields the desired form:

$$L + \sum_{0 \leq i < m} u_i X_i.$$

Recall that in this expression L is represented as a prefix closed subset of reduced words and each u_i is a reduced word belonging to L. Doing this transformation everywhere, we obtain a system of language equations over the free group $F(A)$. Instead of (3) every equation has now the form:

$$L_I + \sum_{i \in I} u_i X_i = L_J + \sum_{j \in J} u_j X_j \tag{4}$$

Here, I, J are finite (disjoint) sets of indices, each L_K is given by a finite prefix closed set of reduced words in A^* and $u_k \in L_K = \mathrm{Pref}(L_K)$ for $k \in K \in \{I, J\}$. By abuse of language we call this system $\mathcal{S}$ again. A solution is now a mapping σ from variables to subsets of A^* such that $\sigma(X)$ is a finite nonempty prefix closed subset of A^* and such that all equations hold as language equations over $F(A)$. We say that a solution σ is *strong* if $\sigma(X)$ consists of reduced words, only. Clearly, $\mathcal{S}$ has a solution if and only if it has a strong solution.

Next, we transform in deterministic polynomial time the system $\mathcal{S}$ into a system $\mathcal{S}_0$ where the equations have a simple syntactic form. This reduction is an intermediate step, only. (Actually, the transformation from $\mathcal{S}$ to $\mathcal{S}_0$ can be done in logspace, but this is not important.) We begin by introducing a fresh variable X_0 and an equation $X_0 = 1$. Moreover, we replace all other variables X by $1 + X$. This allows to drop the restriction that $\sigma(X) \neq \emptyset$. In a second phase, we replace each equation E of type

$$L_I + \sum_{i \in I} u_i X_i = L_J + \sum_{j \in J} u_j X_j.$$

by two equations using a fresh variable X_E and, since each $u_k \in L_K = \mathrm{Pref}(L_K)$ as well as $X_0 = 1$, we may define these equations as follows:

$$X_E = \sum_{u \in L_I} (u X_0 + \mathrm{Pref}(u)) + \sum_{i \in I} (u_i X_i + \mathrm{Pref}(u_i)),$$
$$X_E = \sum_{v \in L_J} (v X_0 + \mathrm{Pref}(v)) + \sum_{j \in J} (u_j X_j + \mathrm{Pref}(u_i)).$$

Thus, there is an equation of the form $X = 1$ and a bunch of equations which have the form

$$X = \sum_{i \in I} (u_i X_i + \mathrm{Pref}(u_i)) \text{ with } I \neq \emptyset.$$

With the help of polynomially many additional fresh variables, it is now obvious that we can transform $\mathcal{S}$ (w.r.t. satisfiability) into an equivalent system $\mathcal{S}_0$ containing only three types of equations:

1. $X = 1$,
2. $X = Y + Z$,
3. $X = uY + \mathrm{Pref}(u)$, where u is a reduced word.

Phrased differently, without restriction $\mathcal{S}$ is of the form $\mathcal{S}_0$ at the very beginning. At this point we start a nondeterministic polynomial time reduction. This means, if S has a solution then at least one outcome of the nondeterministic procedure yields a solvable system $\mathcal{S}'$ of language equations. If none of the possible outcomes is solvable then S is not solvable. During this procedure we are going to mark some equations and this forces us to define the notion of solution for systems with marked equations. A *(strong) solution* is defined as a mapping σ such that each $\sigma(X)$ is given by a prefix closed set of (reduced) words in A^* such that all equations hold as language equations over $F(A)$, but all marked equations hold as language equations over A^* as well. (Thus, we have a stronger condition for marked equations.) We can think of an "evolution" of language equations over $F(A)$ to language equations over the free monoid A^*, and in the middle during the evolution we have a mixture of both interpretations.

Initially we mark all equations of type $X = 1$ or $X = Y + Z$. This is possible because we may start with a strong solution if $\mathcal{S}$ is solvable.

Now we proceed in rounds until all equations are marked. We start a round, if some of the equations $X = uY + \mathrm{Pref}(u)$ is not yet marked. If $u = 1$ is the empty word we simply mark that equation, too. Hence we may assume $u \neq 1$ and we may write $u = va$ with $a \in A$. Nondeterministically we guess whether there exists a strong solution σ such that $\overline{a} \in \sigma(Y)$.

If our guess is "$\overline{a} \notin \sigma(Y)$", then we mark the equation $X = vaY + \mathrm{Pref}(va)$. If the guess is true then marking is correct because then vaw is reduced for all $w \in \sigma(Y)$. Whether or not $\overline{a} \notin \sigma(Y)$ is true, marking an equation never introduces new solutions. Thus, a wrong guess does not transform an unsatisfiable system into a satisfiable one.

Hence, it is enough to consider the other case that the guess is "$\overline{a} \in \sigma(Y)$" for some strong solution σ. In this case we introduce two fresh variables Y', Y'' and a new marked equation

$$Y = Y' + \overline{a}Y'' + \mathrm{Pref}(\overline{a}).$$

If $\overline{a} \in \sigma(Y)$ is correct then we can extend the strong solution so that $\overline{a} \notin \sigma(Y')$. If $\overline{a} \in \sigma(Y)$ is false then, again, this step does not introduce any new solution.

Finally, we replace the equation $X = vaY + \mathrm{Pref}(va)$ by the following three equations, the first two of them are marked and the variables X', X'' are fresh

$$\begin{aligned} X &= X' + X'' && \text{(marked)},\\ X' &= vaY' + \mathrm{Pref}(va) && \text{(marked)},\\ X'' &= vY'' + \mathrm{Pref}(v). \end{aligned}$$

If the guess "$\overline{a} \in \sigma(Y)$" was correct, then the new system has a strong solution. If the new system has any solution then the old system has a solution because $X'' = vY'' + \mathrm{Pref}(v)$ is unmarked as long as $v \neq 1$. After polynomial many rounds all equations are marked. This defines the new system $\mathcal{S}'$. If $\mathcal{S}'$ has a solution σ' then the restriction of σ' to the original variables is also a solution of the original system $\mathcal{S}$. If all our guesses were correct with respect to a strong solution σ of $\mathcal{S}$ then $\mathcal{S}'$ has a strong solution σ' such that σ is the restriction of σ' to the original variables. Hence, $\mathcal{S}$ has a solution if and only if $\mathcal{S}'$ has a solution.

It is therefore enough to consider the system $\mathcal{S}'$ of language equations over A^*. All the equations are still of one of the three types above. Let σ be any mapping from variables in $\mathcal{S}'$ to finite languages of A^*, i.e., $\sigma(X) \subseteq A^*$ denotes an arbitrary finite language for all variables. Then we have the following implications.

1. $\sigma(X) = 1$ implies $\mathrm{Pref}(\sigma(X)) = 1$,
2. $\sigma(X) = \sigma(Y) + \sigma(Z)$ implies $\mathrm{Pref}(\sigma(X)) = \mathrm{Pref}(\sigma(Y)) + \mathrm{Pref}(\sigma(Z))$,
3. $\sigma(X) = u\sigma(Y) + \mathrm{Pref}(u)$ implies $\mathrm{Pref}(\sigma(X)) = u\mathrm{Pref}(\sigma(Y)) + \mathrm{Pref}(u)$.

Thus, the system $\mathcal{S}'$ of language equations over A^* has a solution if and only if $\mathcal{S}'$ has a language solution in finite and prefix closed sets.

In order to finish the proof, let us briefly repeat what we have done so far. The input has been a system $\mathcal{S}$ of equations over $\mathrm{FIM}(A)$ in idempotent variables. If $\mathcal{S}$ has a solution then it has a strong solution and making all guesses correct we end up with a system $\mathcal{S}'$ of language equations over A^* which has a strong solution in finite and prefix closed sets. Conversely, consider some system $\mathcal{S}'$ which is obtained by the nondeterministic choices. (Note that the number of different systems $\mathcal{S}'$ is bounded by a singly exponential function and DEXPTIME is enough time to calculate a list containing all $\mathcal{S}'$.) Assume that $\mathcal{S}'$ has a solution σ' in finite subsets of A^*. Due to the syntactic structure of $\mathcal{S}'$ there is also a solution σ in finite prefix closed subsets of A^*. This is due to the three implications above. Moreover, σ solves $\mathcal{S}$ as a system of language equations over the group $F(A)$. Using Lemma 2 we see that σ solves the original system over the free inverse monoid $\mathrm{FIM}(A)$. Thus, since the square of a singly exponential function is singly exponential, it is enough to apply the result in [2], see Theorem 1 above. □

The above result improves [4, Theorem 8] which was derived from Rabin's Tree Theorem leading to a super-exponential complexity.

6 One-variable equations

Throughout this section we assume that the involution on A is without fixed points, i.e., $F(A)$ is equal to the free group $\mathrm{FG}(A_+)$ in the standard terminology. It is open whether we can remove this restriction.

The following notation is defined for any alphabet Σ and any nonempty word $p \in \Sigma^+$. For $u \in \Sigma^*$ we let $|u|_p$ be the number of occurrences of p as a factor in u. Formally:

$$|u|_p = \left|\left\{u' \mid u'p \le u\right\}\right|.$$

The following equation is trivial since p may occur across the border between u and v at most $|p| - 1$ times.

$$0 \le |uv|_p - |u|_p - |v|_p \le |p| - 1. \tag{5}$$

Next, assuming that Σ is equipped with an involution, we define a "difference" function $\delta_p : \Sigma^* \to \mathbb{Z}$ by

$$\delta_p(u) = |u|_p - |u|_{\overline{p}}.$$

Since $\delta_p(u) = \delta_{\overline{p}}(\overline{u})$ we have $\delta_p(u) = -\delta_p(\overline{u})$, and the mapping δ_p respects the involution.

By definition, we have

$$\delta_p(uv) - \delta_p(u) - \delta_p(v) = (|uv|_p - |u|_p - |v|_p) - (|uv|_{\overline{p}} - |u|_{\overline{p}} - |v|_{\overline{p}})$$

Hence, we may use Equation (5) to conclude:

$$|\delta_p(uv) - \delta_p(u) - \delta_p(v)| \le |p| - 1. \tag{6}$$

As we identify $F(\Sigma)$ with the subset of reduced words in Σ^*, the mapping δ_p is defined from $F(\Sigma)$ to $\mathbb{Z}$, too. The next lemma shows that its deviation from being a homomorphism can be upper bounded. The next lemma will be applied to a primitive word p, only. Let us remind that a word is defined to be primitive if it cannot be written in the form v^i for some word v with $i > 1$ and it is not empty . Every nonempty word u has a *primitive root*: it is the uniquely defined primitive word p such that $u \in p^+$.

Lemma 3. *Let $u_1, \ldots, u_n, p$ be reduced words with $p \neq 1$. Let w be the uniquely defined reduced word such that w is equal to $u_1 \cdots u_n$ in the group $F(\Sigma)$. Then we have:*

$$|\delta_p(w) - \delta_p(u_1) - \cdots - \delta_p(u_n)| \le 3(|p| - 1)(n - 1). \tag{7}$$

Proof. Clearly, Equation (7) holds for $n = 1$. Hence, let $n \ge 2$. Let u be the reduced word such that $u_1 \cdots u_{n-1}$ reduces to u. By induction, we have $|\delta_p(u) - \delta_p(u_1) - \cdots - \delta_p(u_{n-1})| \le 3(|p| - 1)(n - 2)$. Let $v = u_n$. By triangle inequality it is enough to show

$$|\delta_p(w) - \delta_p(u) - \delta_p(v)| \le 3(|p| - 1). \tag{8}$$

To see this write $u = u'r$ and $v = \overline{r}v'$ such that $w = u'v'$.

$$\begin{aligned} \delta_p(w) - \delta_p(u) - \delta_p(v) = {} & \delta_p(w) - \delta_p(u') - \delta_p(v') \\ & + \delta_p(u') + \delta_p(r) - \delta_p(u) \\ & + \delta_p(\overline{r}) + \delta_p(v') - \delta_p(v) \end{aligned}$$

The result follows by Equation (6) and triangle inequality. ◻

We will apply Lemma 3 in the following equivalent form.

$$\delta_p(u_1) + \cdots + \delta_p(u_n) - 3(|p| - 1)(n - 1) \le \delta_p(w) \le \delta_p(u_1) + \cdots + \delta_p(u_n) + 3(|p| - 1)(n - 1). \tag{9}$$

The following lemma is easy to prove. It is however here where we use $a \neq \overline{a}$ for all $a \in A$. Let us recall that a word q is *cyclically reduced* if qq is reduced. In other words if a is the first letter of q, the last letter of q is different from $\overline{a}$.

Lemma 4. *Let $n \in \mathbb{Z}$ and $q \in F(A)$ be a primitive and cyclically reduced word. Then we have $\delta_q(q^n) = n$.*

Proof. We may assume without loss of generality that $n > 0$. Clearly, $|q^n|_q \geq n$. Suppose that $|q^n|_q > n$. Then q is a proper factor of qq, hence we may write $q = q_1 q_2 = q_2 q_1$ in reduced products with $q_1, q_2 \neq 1$. It is well known (see e.g. [9]) that this contradicts the primitivity of q. Thus, $|q^n|_q = n$.

Suppose now that $\overline{q}$ is a proper factor of qq. Then we may write $q = q_1 q_2$ as a reduced product with $\overline{q} = q_2 q_1$ since q is cyclically reduced. Moreover, since $\overline{q} = \overline{q}_2 \overline{q}_1$ we get $q_2 = \overline{q}_2$ and $q_1 = \overline{q}_1$. Hence $q_1 = q_2 = 1$ because q_1, q_2 are reduced and $a \neq \overline{a}$ for all $a \in A$. Thus, $|q^n|_{\overline{q}} = 0$ and so $\delta_q(q^n) = |q^n|_q - |q^n|_{\overline{q}} = |q^n|_q = n$. ◻

An (untyped) equation (U, V) is called a *one-variable equation*, if we can write $UV \in (A \cup \{X, \overline{X}\})^*$. More generally, we also consider systems of typed equations with at most one reduced variable x (and $\overline{x}$), i.e., every equation (U, V) in the system satisfies $UV \in (A \cup \Omega \cup \{x, \overline{x}\})^*$. Let us fix some more notation, we let $\Sigma = A \cup \Omega \cup \Gamma$ with $\Gamma = \{x, \overline{x}\}$. In particular, we have $\overline{X} = X$ for all $X \in \Omega$ and $\alpha \neq \overline{\alpha}$ for all $\alpha \in A \cup \Gamma$.

Definition 1. *Let $u, v \in \Gamma^*$. We say that (u, v) is* unbalanced *if $u \neq v$ in the free inverse monoid* $\mathrm{FIM}(\Gamma)$.

Otherwise we say that (u, v) is balanced.

Remark 2. Using the well-known structure of $\mathrm{FIM}(\Gamma)$, a pair (u, v) as in Definition 1 is balanced if and only if the following three conditions are satisfied.

- $\delta_x(u) = \delta_x(v)$.
- $\max\{\delta_x(u') \mid u' \leq u\} = \max\{\delta_x(v') \mid v' \leq v\}$.
- $\min\{\delta_x(u') \mid u' \leq u\} = \min\{\delta_x(v') \mid v' \leq v\}$.

We extend the notion defined in Definition 1 to an untyped one-variable equation. In the following we let $\pi_{A,\Gamma}$ be the morphism from $(A \cup \Omega \cup \Gamma)^*$ to $F(A \cup \Gamma)$ which is induced by cancelling the symbols in Ω.

Definition 2. *Let (U, V) be an untyped one-variable equation with $\mathcal{X} = \{X, \overline{X}\}$. We say that (U, V) is* unbalanced *if it fulfills both conditions:*
1- (u, v) is unbalanced as a word over Γ where u (resp. v) is obtained from U (resp. V) by replacing X by x (and $\overline{X}$ by $\overline{x}$) and erasing all other symbols.
2- $\pi_{A,\Gamma}(U) \neq \pi_{A,\Gamma}(V)$ in the free group $F(A \cup \Gamma)$.

The following definition is a bit more technical, but it will lead to better results.

Definition 3. *Let U, V be words over $A \cup \Omega \cup \Gamma$. We say that (U, V) is* strongly unbalanced *if $\pi_{A,\Gamma}(U) \neq \pi_{A,\Gamma}(V)$ in the free group $F(A \cup \Gamma)$ and at least one of the following conditions is satisfied.*

(SU1) $\delta_x(U) \neq \delta_x(V)$.
(SU2) For all $z \in \Omega \cup \{1\}$ and all prefixes $V'z$ of V there exists some prefix $U'z$ of U such that $\delta_x(U') > \delta_x(V')$.
(SU3) For all $z \in \Omega \cup \{1\}$ and all prefixes $V'z$ of V there exists some prefix $U'z$ of U such that $\delta_{\overline{x}}(U') > \delta_{\overline{x}}(V')$.

The following result improves the complexity in the corresponding statement of [4]. (Note that the condition $\pi_{A,\Gamma}(U) \neq \pi_{A,\Gamma}(V)$ was missing in [4], but the proof is not valid without this additional requirement.)

Theorem 4. *The following problem can be decided in* DEXPTIME.

Input: A system $\mathcal{S}$ of one-variable equations over $\mathcal{X} = \{X, \overline{X}\}$ where at least one equation (U, V) is unbalanced according to Definition 2.

Question: Does $\mathcal{S}$ have a solution in $\mathrm{FIM}(A)$*?*

Proof. Suppose that (U, V) is unbalanced. The pair (U, V) must then contradict one of the three conditions of Remark 2. Let us distinguish cases and, in each case, reduce the given unbalanced equation into a *strongly* unbalanced *typed* equation.
In all cases, we introduce a fresh idempotent variable Z, a fresh reduced variable x, and use the word-substitutions τ' (or τ) defined in Lemma 1: $\tau'(X) = xZ, \tau'(\overline{X}) = \overline{Z}\overline{x}, \tau(X) = Zx, \tau(\overline{X}) = \overline{x}\overline{Z}$ or the trivial substitution $\theta(X) = x, \theta(\overline{X}) = \overline{x}$.

Case 1: $\delta_X(U) \neq \delta_X(V)$.
In this case $(\theta(U), \theta(V))$ fulfills condition $(SU1)$.
Case 2: $\max\{\delta_X(U') \mid U' \leq U\} > \max\{\delta_X(V') \mid V' \leq V\}$.
There is some prefix $U' \leq U$ such that for all prefixes $V' \leq V$ we have $\delta_X(U') > \delta_X(V')$ and, in particular, $\delta_X(U') > \delta_X(1) = 0$. We choose $\delta_X(U')$ to be maximal and, since $\delta_X(U')$ is positive, we may choose U' such that $X = \mathrm{last}(U')$, so that $\mathrm{last}(\tau'(U')) = Z$. Now, for every $z \in \{Z, 1\}$,

$$\begin{aligned}\delta_x(\tau'(U')) = \delta_X(U') > \max\left\{\delta_X(V') \mid V' \leq V\right\} &= \max\left\{\delta_x(W) \mid W \leq \tau'(V)\right\} \\ &\geq \max\left\{\delta_x(W'z) \mid W'z \leq \tau'(V)\right\}.\end{aligned}$$

This prefix $\tau'(U')$ shows that $(\tau'(U), \tau'(V))$ fulfills condition $(SU2)$ (this is actually a stronger requirement than asked by Definition 3, because this single prefix $\tau'(U')$ serves for all $W'z$). **Case 2'**: $\max\{\delta_X(U') \mid U' \leq U\} < \max\{\delta_X(V') \mid V' \leq V\}$.
By Case 2 the typed equation $(\tau'(V), \tau'(U))$ fulfills condition $(SU2)$.
Case 3: $\min\{\delta_X(U') \mid U' \leq U\} > \min\{\delta_X(V') \mid V' \leq V\}$.
We may assume that $\delta_X(U) = \delta_X(V) = k$. If $U = U'U''$ and $V = V'V''$, we have $\delta_{\overline{X}}(\overline{U''}) = \delta_X(U'') = k - \delta_X(U')$ and $\delta_{\overline{X}}(\overline{V''}) = \delta_X(V'') = k - \delta_X(V')$, thus $(\overline{U}, \overline{V})$ fulfills that $\max\left\{\delta_{\overline{X}}(U') \mid U' \leq \overline{U}\right\} < \max\left\{\delta_{\overline{X}}(V') \mid V' \leq \overline{V}\right\}$.
By a reasoning similar to that of case 2, one can show that $(\tau(\overline{V}), \tau(\overline{U}))$ fulfills condition $(SU3)$.
Case 3': $\min\{\delta_X(U') \mid U' \leq U\} < \min\{\delta_X(V') \mid V' \leq V\}$.
By Case 3 the typed equation $(\tau(\overline{U}), \tau(\overline{V}))$ fulfills condition $(SU3)$.
We have thus reduced Theorem 4 above to Theorem 5 below. ◻

Theorem 5. *The following problem can be decided in* DEXPTIME.

Input: A system $\mathcal{S}$ of typed equations with at most one reduced variable (i.e., $\Gamma = \{x, \overline{x}\}$) where at least one equation $(U, V) \in \mathcal{S}$ is strongly unbalanced.

Question: Does $\mathcal{S}$ have a solution in FIM(A)*?*

The proof of Theorem 5 relies on the following combinatorial observation.

Lemma 5. *Let (U, V) be a strongly unbalanced equation with $U, V \in (A \cup \Omega \cup \{x, \overline{x}\})^*$ and $n = max\{|U|, |V|\}$. Let $k \in \mathbb{Z}$ be an integer and σ be a solution to (U, V) such that $\sigma(x) = (Pref(p^k), p^k)$ for some nonempty cyclically reduced word $p \in A^*$. Then we have $|k| \leq 6n\,|p|$.*

Proof. Without restriction, p is primitive and $k > 1$. (Replace p by its primitive root and interchange the role of p and $\overline{p}$, if necessary.) For a word $W \in (A \cup \Omega \cup \{x, \overline{x}\})^*$ we write $\sigma(W) = (\sigma_1(W), \sigma_2(W))$ where $\sigma_1(W) \subseteq A^*$ is a prefix closed set of reduced words and $\sigma_2(W) \in F(A)$. Choose m maximal such that $\delta_p(w) = m$ for some $w \in \sigma_1(V)$. We fix $w \in A^*$ and we observe that we have $m \geq 0$ and for every word $u \in \sigma_1(U) = \sigma_1(V)$, we have

$$\delta_p(u) \leq m \tag{10}$$

Case (SU2): (U, V) fulfills condition (SU2).
We choose a prefix V' of V of minimal length with respect to the property $w \in \sigma_1(V')$. We consider two subcases.

<u>Subcase *Ω*:</u> $\mathrm{last}(V') \in \Omega$.

Let $z := \mathrm{last}(V')$. The word V' thus decomposes as $V' = V''z$. Since $\sigma_1(V''z) = \sigma_1(V'') \cup \sigma_2(V'')\sigma_1(z)$, it follows from the minimality of $V''z$ that $w \in \sigma_2(V'')\sigma_1(z)$. Since $\sigma_2(Z) = 1$ for every $Z \in \Omega$ and $|V'| \leq n - 1$, it follows that w is the product of at most $n - 1$ reduced words $v_1 \dots v_t$ in $A \cup \{\sigma_2(x), \sigma_2(\overline{x})\}$ by some $z' \in \sigma_1(z)$.

For each letter a of A, $\delta_p(a) \leq 1$ and, since p is primitive, by Lemma 4, $\delta_p(\sigma_2(x)) = k, \delta_p(\sigma_2(\overline{x})) = -k$. We thus get

$$\sum_{i=1}^{t} \delta_p(v_i) \leq k\delta_x(V') + n - 1. \tag{11}$$

Since $w = v_1 \dots v_t z'$, we obtain the following upper bound:

$$\begin{aligned} m = & \quad \delta_p(w) \\ \leq & \quad \sum_{i=1}^{t} \delta_p(v_i) + \delta_p(z') + 3(|p|-1)(n-1) \quad \text{by (9)} \\ \leq & \quad k\delta_x(V') + \delta_p(z') + n - 1 + 3(|p|-1)(n-1) \quad \text{by (11)} \end{aligned} \tag{12}$$

By (SU2) there exists a prefix U' of U such that $\delta_x(U') > \delta_x(V')$ and $\text{last}(U') = z$. The word U' thus decomposes as $U' = U''z$. Let us define $u := \sigma_2(U'')z'$. We remark that $u \in \sigma_1(U)$, hence it fulfills Equation (10). Using similar arguments based on Equation (9) and Lemma 4 we obtain:

$$k\delta_x(U') + \delta_p(z') - (n-1) - 3(|p|-1)(n-1) \leq \delta_p(u). \tag{13}$$

Combining the above inequalities we obtain:

$$\begin{aligned} k \leq & \quad k(\delta_x(U') - \delta_x(V')) & \text{since } \delta_x(U') > \delta_x(V') \\ \leq & \quad -\delta_p(z') + (n-1) + 3(|p|-1)(n-1) + \delta_p(u) - k\delta_x(V') & \text{by (13)} \\ \leq & \quad -\delta_p(z') + (n-1) + 3(|p|-1)(n-1) + m - k\delta_x(V') & \text{by (10)} \\ \leq & \quad 2(n-1) + 6(|p|-1)(n-1) & \text{by (12)} \\ \leq & \quad 6n(|p|) \end{aligned} \tag{14}$$

Subcase 1: $\text{last}(V') \notin \Omega$.

We just need to perform some adaptations to the preceding case. The word w is the product of at most n reduced words $v_1 \dots v_t$ in $A \cup \{\sigma_2(x), \sigma_2(\overline{x})\}$, and by similar methods we obtain

$$m = \delta_p(w) \leq k\delta_x(V') + n + 3(|p|-1)(n-1). \tag{15}$$

By (SU2) (where we choose $z := 1$) there exists a prefix U' of U such that $\delta_x(U') > \delta_x(V')$. Let us define $u := \sigma_2(U')$. We get

$$k\delta_x(U') - n - 3(|p|-1)(n-1) \leq \delta_p(u). \tag{16}$$

Since $u = \sigma_2(U') \in \sigma_1(U)$, here also u fulfills (10). Hence, putting (16) (10) and (15) together we obtain the desired result:

$$k \leq k(\delta_x(U') - \delta_x(V')) \leq 6(|p|-1)(n-1) + 2n \leq 6n\,|p|\,. \tag{17}$$

Case (SU3): (U, V) fulfills condition (SU3).
This case is dealt with in a similar manner.
Case (SU1): (U, V) fulfills condition (SU1).
By symmetry in U and V, we may assume without restriction $\delta_x(U) > \delta_x(V)$. Let us choose $V' := V, w := \sigma_2(V), m := \delta_p(w), U' := U, u := \sigma_2(U)$. The arguments of Case (SU2), Subcase 1, apply on these choices for V', w, m, U', u. (In fact, an argument provided by James Howie in [19] shows that in this case the solution $\sigma(x)$ is unique). ◻

Proof of Theorem 5. Let n be the size of the system $\mathcal{S}$, it is defined as

$$\|\mathcal{S}\| = \sum_{(U,V)\in\mathcal{S}} |UV|\,.$$

Since $\pi_{A,\Gamma}(U) \neq \pi_{A,\Gamma}(V)$ for at least one equation in the system, the set of solutions for the underlying group equations is never equal to $F(A)$. By [1, 8], the set of solutions of a one-variable free group equation is therefore a finite union of sets of the form

$$\left\{ rq^k s \;\middle|\; k \in \mathbb{Z} \right\}, \tag{18}$$

where q is cyclically reduced and both products rqs and $r\overline{q}s$ are reduced. A self-contained proof of this fact has been given in [3].

In the description above $q = 1$ is possible. Moreover, [3] shows $|rqs| \in \mathcal{O}(n)$. Hence, as we aim for DEXPTIME there is time enough to consider all possible candidates for r and s. This means we can fix r and s; and it is enough to consider a single set $S = \{ rq^k s \mid k \in \mathbb{Z} \}$, only. Next we replace in $\mathcal{S}$ all occurrences of x by rxs (and $\overline{x}$ by $\overline{s}\,\overline{x}\,\overline{r}$). This leads to a new system which we still denote by $\mathcal{S}$ and without restriction we have $S = \{ q^k \mid k \in \mathbb{Z} \}$. The new size m of $\mathcal{S}$ is at most quadratic in n.

Now, we check if $k = 0$ leads to a solution of $\mathcal{S}$. This means that we simply cancel x and $\overline{x}$ everywhere. We obtain a system over idempotent variables; and we can check satisfiability by Theorem 3. Note that this includes the case $q = 1$. Thus, henceforth we may assume that q is a primitive cyclically reduced word. By Lemma 5 we see that it is enough to replace S by $S' = \{ q^k \mid |k| \leq 6m\,|q| \}$. Since $|q| \in \mathcal{O}(m)$ we obtain a cubic bound for the maximal length of words in S', this means the length of each word in S' is bounded by $\mathcal{O}(n^6)$. This is small enough to check satisfiability of the original system $\mathcal{S}$ in DEXPTIME by Theorem 3. □

Conclusion and directions for future research

The notion of "idempotent variable" unifies the approach the study of equations in free inverse monoids. As the general situation is undecidable, progress is possible only by improving complexities in classes where decidability is known and/or to enlarge the class of equations where decidability is possible. We have achieved progress in both fields. For equations in idempotent variables we lowered the complexity down to DEXPTIME. Actually have the following conjecture.

Conjecture 6 *Let $A = \{ a, \overline{a}, b, \overline{b} \}$. The problem to decide whether an equation in* $\mathrm{FIM}(A)$ *has a solution is* DEXPTIME-*complete, provided all variables are idempotent.*

Using a recent result in [5] that it is decidable in PSPACE whether an equation in free groups has only finitely many solutions, we could derive a "promise result" in Corollary 1 with triple exponential time complexity. We don't think that this is optimal, because we believe that solving equations in free groups is in NP. But this fundamental conjecture is wide open and resisted all known techniques.

More concretely, let us resume some interesting and specific questions on equations in free inverse monoids which are left open:

- Is the decision problem solved here DEXPTIME-hard? We conjecture: yes! (See Remark 1 and Conjecture 6.)
- Is the (other) special kind of equations solved by Theorem 23 of [4] also solvable in DEXPTIME?
- Is it possible to remove *Assumption 2* in the definition of an "unbalanced" equation (it asserts that the image of the left-hand side and right-hand side are different in the free group), and still maintain decidability of the system of equations?
- What happens if the underlying equation in the free group is true for all elements in the free group? This means the statement is a tautology the free group.
- What *more general* kinds of one-variable equations in the free inverse monoid are algorithmically solvable (possibly all of them)?
- Does Jeż' recompression technique apply to language equations? If yes, then this would open a new approach to tackle equations over free inverse monoids.

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