DERIVATIONS OF A PARAMETRIC FAMILY OF SUBALGEBRAS OF THE WEYL ALGEBRA

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ABSTRACT. An Ore extension over a polynomial algebra $\mathbb{F}[x]$ is either a quantum plane, a quantum Weyl algebra, or an infinite-dimensional unital associative algebra A_h generated by elements x, y, which satisfy $yx - xy = h$, where $h \in \mathbb{F}[x]$. When $h \neq 0$, the algebra A_h is subalgebra of the Weyl algebra A_1 and can be viewed as differential operators with polynomial coefficients. This paper determines the derivations of A_h and the Lie structure of the first Hochschild cohomology group $\mathsf{HH}^1(\mathsf{A}_h) = \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)/\mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_h)$ of outer derivations over an arbitrary field. In characteristic 0, we show that $\mathsf{HH}^1(\mathsf{A}_h)$ has a unique maximal nilpotent ideal modulo which it is 0 or a direct sum of simple Lie algebras that are field extensions of the one-variable Witt algebra. In positive characteristic, we obtain decomposition theorems for $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$ and $\mathsf{HH}^1(\mathsf{A}_h)$ and describe the structure of $\mathsf{HH}^1(\mathsf{A}_h)$ as a module over the center of A_h.

1. INTRODUCTION

We consider a family of infinite-dimensional unital associative algebras A_h parametrized by a polynomial h in one variable, whose definition is given as follows:

Definition 1.1. *Let $\mathbb{F}$ be a field, and let $h \in \mathbb{F}[x]$. The algebra A_h is the unital associative algebra over $\mathbb{F}$ with generators x, y and defining relation $yx = xy + h$ (equivalently, $[y, x] = h$ where $[y, x] = yx - xy$).*

These algebras arose naturally in considering Ore extensions over a polynomial algebra $\mathbb{F}[x]$. Many algebras can be realized as iterated Ore extensions, and for that reason, Ore extensions have become a mainstay in associative theory. Recall that an Ore extension $\mathsf{A} = \mathsf{R}[y, \sigma, \delta]$ is built from a unital associative (not necessarily commutative) algebra R over a field $\mathbb{F}$, an $\mathbb{F}$-algebra endomorphism σ of R, and a σ-derivation of R, where by a σ-derivation δ we mean that δ is $\mathbb{F}$-linear and $\delta(rs) = \delta(r)s + \sigma(r)\delta(s)$ holds for all $r, s \in \mathsf{R}$. Then $\mathsf{A} = \mathsf{R}[y, \sigma, \delta]$ is the algebra generated by y over R subject to the relation

$$yr = \sigma(r)y + \delta(r) \qquad \text{for all } r \in \mathsf{R}.$$

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Under the assumption that $\mathsf{R} = \mathbb{F}[x]$ and σ is an automorphism of R, the following result holds. (Compare [AVV] and [AD], which have a somewhat different division into cases.)

Lemma 1.2. *Assume $\mathsf{A} = \mathsf{R}[y, \sigma, \delta]$ is an Ore extension with $\mathsf{R} = \mathbb{F}[x]$, a polynomial algebra over a field $\mathbb{F}$ of arbitrary characteristic and σ an automorphism of R. Then A is isomorphic to one of the following:*

(a) *a quantum plane*
(b) *a quantum Weyl algebra*
(c) *an algebra A_h with generators x, y and defining relation $yx = xy + h$ for some polynomial $h \in \mathbb{F}[x]$.*

The algebras A_h result from taking $\mathsf{R} = \mathbb{F}[x]$, σ to be the identity automorphism, and $\delta : \mathsf{R} \to \mathsf{R}$ to be the derivation given by

$$\delta(f) = f'h, \tag{1.3}$$

where f' is the usual derivative of f with respect to x.

Quantum planes and quantum Weyl algebras are examples of generalized Weyl algebras in the sense of [B, 1.1], and as such, have been studied extensively. In [BLO1, BLO2], we determined the center, normal elements, and prime ideals of the algebras A_h, as well as the automorphisms and their invariants, isomorphisms between two algebras A_g and A_h, and the irreducible A_h-modules over any field $\mathbb{F}$. Our aim in this paper is to compute the derivations and first cohomology group of the algebras A_h over an arbitrary field.

When $h = 1$, the algebra A_1 is the Weyl algebra, and Sridharan [Sr] showed that when the characteristic of $\mathbb{F}$ is 0, the Hochschild cohomology of A_1 vanishes in positive degrees. In particular, the derivations of A_1 are all inner when $\mathsf{char}(\mathbb{F}) = 0$, since the first cohomology vanishes (compare [D1] and [D2]). In recent work [GG], Gerstenhaber and Giaquinto have used the fact that the Euler-Poincaré characteristic is invariant under deformation to compute the cohomology of the Weyl algebra, the quantum plane, and the quantum Weyl algebra under the assumption $\mathsf{char}(\mathbb{F}) = 0$.

Progress towards determining the derivations of A_h for arbitrary h has been made in [N], primarily in the characteristic 0 case. Theorem 9.1 of [N] shows that when $\mathsf{char}(\mathbb{F}) = 0$, every derivation is inner if and only if $h \in \mathbb{F}^*$ (in the notation used here). Nowicki also establishes decomposition results (see [N, Thms. 10.1 and 11.2]) for derivations of A_h. These results can be obtained as special cases of Theorem 5.7 below, which gives a direct sum decomposition of $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$. In addition, we derive expressions for the Lie bracket in the quotient $\mathsf{HH}^1(\mathsf{A}_h) = \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)/\mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_h)$ of $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$ modulo the ideal $\mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_h)$ of inner derivations when $\mathsf{char}(\mathbb{F}) = 0$ and use these formulas to understand the structure of the Lie algebra $\mathsf{HH}^1(\mathsf{A}_h)$ (see Theorem 5.13). In Theorem 5.1 and Corollary 5.25, we show that there is a unique maximal nilpotent ideal of $\mathsf{HH}^1(\mathsf{A}_h)$ and explicitly describe the structure of the quotient by this ideal in terms of the one-variable Witt algebra (centerless Virasoro algebra).

When $\mathsf{char}(\mathbb{F}) = p > 0$, not all derivations of A_1 are inner (contrary to the statement in [R]). In Section 3, we introduce two non-inner derivations E_x and E_y of A_1 and use them in Theorem 3.8 to describe $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_1)$ as well as $\mathsf{HH}^1(\mathsf{A}_1)$. Section 6 of the paper is devoted to studying $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$ for arbitrary $h \neq 0$ in the characteristic $p > 0$ case. The restriction map $\mathsf{Res} : \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h) \to \mathsf{Der}_{\mathbb{F}}(\mathsf{Z}(\mathsf{A}_h))$ from derivations of A_h to derivations of the center $\mathsf{Z}(\mathsf{A}_h)$ of A_h is a morphism of Lie algebras, and in the case $h = 1$, this map is surjective with kernel $\mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_1)$. Viewing A_h as a subalgebra of A_1 for $h \neq 0$ and applying results from Section 3 on derivations of A_1, we determine the kernel and image of Res in Proposition 6.9 and Theorem 6.17 respectively. This enables us in Theorem 6.21 to exp licitly determine all derivations of A_h, for arbitrary $h \neq 0$, when $\mathsf{char}(\mathbb{F}) = p > 0$. To illustrate this result, we compute $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$ for $h = x^m$ for any $m \geq 0$ (Corollary 6.24) and for any $h \in \mathbb{F}[x^p]$ (Example 6.26). In Proposition 6.27, we provide a criterion for a derivation of A_h to be inner for general h, and in Theorem 6.29, we present necessary and sufficient conditions on h for $\mathsf{HH}^1(\mathsf{A}_h)$ to be free over $\mathsf{Z}(\mathsf{A}_h)$. Propositions 6.34 and 6.40 give formulas for the Lie brackets in $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$.

Several well-known algebras have the form A_h for some $h \in \mathbb{F}[x]$. For example, A_0 is the polynomial algebra $\mathbb{F}[x, y]$; A_1 is the Weyl algebra; and the algebra A_x is the universal enveloping algebra of the two-dimensional non-abelian Lie algebra (there is only one such Lie algebra up to isomorphism). The algebra A_{x^2} is often referred to as the Jordan plane. It appears in noncommutative algebraic geometry (see for example, [SZ] and [AS]) and exhibits many interesting features such as being Artin-Schelter regular of dimension 2. In a series of articles [S1]–[S3], Shirikov has undertaken an extensive study of the automorphisms, derivations, prime ideals, and modules of the algebra A_{x^2}. Aspects of the theory developed in [S1]–[S3] have been extended by Iyudu [I] to include results on varieties of finite-dimensional modules of A_{x^2} over algebraically closed fields of characteristic 0. Cibils, Lauve, and Witherspoon [CLW] have used quotients of the algebra A_{x^2} and cyclic subgroups of their automorphism groups to construct new examples of finite-dimensional Hopf algebras in prime characteristic which are Nichols algebras.

The universal enveloping algebras $\mathsf{YM}(n)$ of the Yang-Mills algebras form another family of infinite-dimensional associative algebras which have been studied because of their connections with deformation theory. Theorem 5.11 of [HS] determines the Lie structure of the first Hochschild cohomology group of $\mathsf{YM}(n)$ over an algebraically closed field of characteristic 0. This turns out to be finite dimensional and can be described in terms of the orthogonal Lie algebra $\mathfrak{so}(n)$. By contrast, $\mathsf{HH}^1(\mathsf{A}_h)$ generally is infinite dimensional and related to the Witt algebra under the assumption $\mathbb{F}$ has characteristic 0.

There are striking similarities in the behavior of the algebras A_h as h ranges over the polynomials in $\mathbb{F}[x]$. For that reason, we believe that studying them as one family provides much insight into their structure, derivations, automorphisms, and modules.

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2. PRELIMINARIES

In this section, we recall some necessary background from [BLO1] and prove results required for our description of the derivations of A_h. We begin with facts about embeddings.

Lemma 2.1. [BLO1, Sec. 3]

(a) *Suppose that f and g are nonzero elements of $\mathbb{F}[x]$ and $g = fr$ for some $r \in \mathbb{F}[x]$. Regard $\mathsf{A}_f = \langle x, y, 1\rangle$ and $\mathsf{A}_g = \langle x, \tilde{y}, 1\rangle$ with the relations $yx - xy = f$ and $\tilde{y}x - x\tilde{y} = g$ respectively. Then the map $\varepsilon : \mathsf{A}_g \to \mathsf{A}_f$ with $x \mapsto x, \quad \tilde{y} \mapsto yr$ gives an embedding of A_g into A_f.*

(b) *For all $h \in \mathbb{F}[x]$, $h \neq 0$, there is an embedding of the algebra A_h into the Weyl algebra A_1. If x, y are the generators of the Weyl algebra so that $[y, x] = 1$, then A_h can be identified with the subalgebra $\mathsf{A}_h = \langle x, \hat{y}, 1\rangle$ of A_1 generated by x, $\hat{y} = yh$, and 1.*

(c) *Regard $\mathsf{A}_h \subseteq \mathsf{A}_1$ as in* (b)*, and write $\mathsf{R} = \mathbb{F}[x]$. Then*

$$\mathsf{A}_h = \bigoplus_{i\geq 0} \mathsf{R}h^i y^i = \bigoplus_{i\geq 0} y^i h^i \mathsf{R}. \tag{2.2}$$

Because we often use the embedding in Lemma 2.1 (b) as a tool for proving results, and because the structure and derivations of $\mathsf{A}_0 = \mathbb{F}[x, y]$ are very well understood, for the remainder of this paper we adopt the following conventions:

Conventions 2.3.

- $\mathsf{R} = \mathbb{F}[x]$*, and the polynomial $h \in \mathsf{R}$ is nonzero;*
- *the generators of the Weyl algebra A_1 are x, y, 1 and $[y, x] = 1$;*
- *the generators of the algebra A_h are x, $\hat{y}$, 1 and $[\hat{y}, x] = h$;*
- *when A_h is viewed as a subalgebra of A_1, then $\hat{y} = yh$.*

The center of the Weyl algebra A_1 is $\mathbb{F}1$ when $\mathsf{char}(\mathbb{F}) = 0$. When $\mathsf{char}(\mathbb{F}) = p > 0$, the center of A_1 has been described by Revoy in [R] (see also [ML]). The next result describes the center of an arbitrary algebra A_h.

Theorem 2.4. [BLO1, Sec. 5] *Regard $\mathsf{A}_h \subseteq \mathsf{A}_1$ as in Conventions 2.3, and let $\mathsf{Z}(\mathsf{A}_h)$ denote the center of A_h.*

(1) *If $\mathsf{char}(\mathbb{F}) = 0$, then $\mathsf{Z}(\mathsf{A}_h) = \mathbb{F}1$.*

(2) *If $\mathsf{char}(\mathbb{F}) = p > 0$, then $\mathsf{Z}(\mathsf{A}_h)$ is the polynomial subalgebra $\mathbb{F}[x^p, \mathsf{z}_h] = \mathbb{F}[x^p, h^p y^p]$ of A_1, where*

$$\mathsf{z}_h = h^p y^p = y^p h^p = \hat{y}(\hat{y} + h')(\hat{y} + 2h') \cdots (\hat{y} + (p-1)h') = \hat{y}^p - \frac{\delta^p(x)}{h}\hat{y},$$

and δ is the derivation of $\mathsf{R} = \mathbb{F}[x]$ with $\delta(f) = f'h$ for all $f \in \mathsf{R}$. Moreover $\frac{\delta^p(x)}{h} \in \mathsf{Z}(\mathsf{A}_h) \cap \mathbb{F}[x] = \mathbb{F}[x^p]$.

(3) *If $\mathsf{char}(\mathbb{F}) = 0$, then A_h is free over its center $\mathsf{Z}(\mathsf{A}_h)$ with basis $\{x^i\hat{y}^j \mid i,j \in \mathbb{Z}_{\geq 0}\}$. If $\mathsf{char}(\mathbb{F}) = p > 0$, then A_h is free over $\mathsf{Z}(\mathsf{A}_h)$ with basis $\{x^i h^j y^j \mid 0 \leq i,j < p\}$ or with basis $\{x^i\hat{y}^j \mid 0 \leq i,j < p\}$.*

The *centralizer* $\mathsf{C}_{\mathsf{A}_h}(x) = \{a \in \mathsf{A}_h \mid [a,x] = 0\}$ of x in A_h has been calculated in [BLO1], and we summarize the results next.

Lemma 2.5. [BLO1, Lem. 6.3] $\mathsf{C}_{\mathsf{A}_h}(x) = \mathsf{Z}(\mathsf{A}_h)\mathsf{R}$. *Hence,*

$$\mathsf{C}_{\mathsf{A}_h}(x) = \begin{cases} \mathsf{R} = \mathbb{F}[x] & \text{if } \ \mathsf{char}(\mathbb{F}) = 0, \\ \mathbb{F}[x, h^p y^p] & \text{if } \ \mathsf{char}(\mathbb{F}) = p > 0. \end{cases}$$

In particular, $\mathsf{C}_{\mathsf{A}_1}(x) = \mathsf{R}$ when $\mathsf{char}(\mathbb{F}) = 0$, and $\mathsf{C}_{\mathsf{A}_1}(x) = \mathbb{F}[x, y^p]$ when $\mathsf{char}(\mathbb{F}) = p > 0$.

The *normalizer*

$$\mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h) = \{u \in \mathsf{A}_1 \mid [u, \mathsf{A}_h] \subseteq \mathsf{A}_h\} \tag{2.6}$$

of A_h in A_1 is closely related to the derivations of A_h, as

$$u \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h) \iff \mathsf{ad}_u \ \text{ restricts to a derivation of } \ \mathsf{A}_h, \tag{2.7}$$

where ad_u is the inner derivation of A_1 given by $\mathsf{ad}_u(v) = [u,v] = uv - vu$.

We begin with a computational lemma from [BLO1, Lem. 5.2] and then introduce a certain element $\pi_h \in \mathsf{R}$ that depends upon h and plays an essential role in describing $\mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)$.

Lemma 2.8. *Let $h \in \mathsf{R} = \mathbb{F}[x]$, and let $\delta : \mathsf{R} \to \mathsf{R}$ be the derivation with $\delta(f) = f'h$ for all $f \in \mathsf{R}$. Then*

$$[\hat{y}^n, f] = \sum_{j=1}^{n} \binom{n}{j} \delta^j(f)\hat{y}^{n-j} \qquad \text{in } \ \mathsf{A}_h \tag{2.9}$$

$$[y^n, f] = \sum_{j=1}^{n} \binom{n}{j} f^{(j)} y^{n-j} \qquad \text{in } \ \mathsf{A}_1 \tag{2.10}$$

where $f^{(j)} = (\frac{d}{dx})^j(f)$.

Corollary 2.11. *For all $r \in \mathsf{R}$ and all $n \geq 0$,*

$$[ry^n, \hat{y}] = -(rh)'y^n + r\sum_{j=1}^{n+1} \binom{n+1}{j} h^{(j)} y^{n+1-j}. \tag{2.12}$$

Proof. Using (2.10), we have

$$\begin{aligned}[ry^n, \hat{y}] &= [ry^n, yh] = [ry^n, hy] + [ry^n, h'] \\ &= r\sum_{j=1}^{n}\binom{n}{j}h^{(j)}y^{n+1-j} - hr'y^n + r\sum_{j=1}^{n}\binom{n}{j}h^{(j+1)}y^{n-j} \\ &= -(rh)'y^n + r\sum_{j=1}^{n+1}\binom{n+1}{j}h^{(j)}y^{n+1-j}. \end{aligned}$$ □

Lemma 2.13. *Let* $\mathsf{R} = \mathbb{F}[x]$.

(i) *There is a unique monic polynomial* $\pi_h \in \mathsf{R}$ *such that*

$$\forall\, r \in \mathsf{R}, \qquad h \mid h'r \iff \pi_h \mid r.$$

In particular, $\pi_h \mid h$, *and* $\pi_h = 1$ *if* $h' = 0$.

(ii) *If* $h \notin \mathbb{F}$, *write* $h = \lambda \mathsf{u}_1^{\alpha_1}\cdots \mathsf{u}_t^{\alpha_t}$, *where* $\lambda \in \mathbb{F}^*$, $t \geq 1$, $\alpha_i \geq 1$ *for all* i, *and the* u_i *are distinct monic primes in* R.

(a) *If* $\mathsf{char}(\mathbb{F}) = 0$, *then* $\pi_h = \mathsf{u}_1 \cdots \mathsf{u}_t$.

(b) *If* $\mathsf{char}(\mathbb{F}) = p > 0$, *then* $\pi_h = \displaystyle\prod_{i,\, \mathsf{u}_i^{\alpha_i} \notin \mathbb{F}[x^p]} \mathsf{u}_i$, *and if* $h \in \mathbb{F}[x^p]$, *then* $\pi_h = 1$.

Hence, $\pi_h = \frac{h}{\mathsf{gcd}(h,h')}$.

Proof. Let $\mathsf{J} = \{r \in \mathsf{R} \mid\ h \text{ divides } h'r\}$. Then J is an ideal of the principal ideal domain R, so there is a unique monic polynomial $\pi_h \in \mathsf{R}$ that generates J. This proves the existence and uniqueness of π_h. Furthermore, it is clear that $\pi_h \mid h$ since $h \in \mathsf{J}$, and that $\pi_h = 1$ if $h \in \mathbb{F}$ or if $h \in \mathbb{F}[x^p]$, as $h' = 0$.

Assume $h \notin \mathbb{F}$ and $h = \lambda \mathsf{u}_1^{\alpha_1}\cdots \mathsf{u}_t^{\alpha_t}$ as above. Set $\mathsf{u} = \mathsf{u}_1\cdots \mathsf{u}_t$. Then

$$h' = \frac{h}{\mathsf{u}}\sum_{i=1}^{t}\alpha_i \mathsf{u}_1\cdots \mathsf{u}_i'\cdots \mathsf{u}_t.$$

Given $r \in \mathsf{R}$, it is easy to see that h divides $h'r$ if and only if u divides $r\sum_{i=1}^{t}\alpha_i \mathsf{u}_1\cdots \mathsf{u}_i'\cdots \mathsf{u}_t$. The latter occurs if and only if u_j divides $r\sum_{i=1}^{t}\alpha_i \mathsf{u}_1\cdots \mathsf{u}_i'\cdots \mathsf{u}_t$ for every j. This is equivalent to having u_j divide $r\alpha_j \mathsf{u}_1\cdots \mathsf{u}_j'\cdots \mathsf{u}_t$ for every j. Hence, h divides $h'r$ if and only if u_j divides $r\alpha_j\mathsf{u}_j'$ for every j.

If $\mathrm{char}(\mathbb{F}) = 0$, $\alpha_j\mathsf{u}_j' \neq 0$ and has degree smaller than u_j, so u_j divides r for all j. Thus, $\pi_h = \mathsf{u}_1\cdots \mathsf{u}_t$. If $\mathsf{char}(\mathbb{F}) = p > 0$, then $\mathsf{u}_j^{\alpha_j} \in \mathbb{F}[x^p]$ if and only if $\alpha_j\mathsf{u}_j' = 0$, so h divides $h'r$ if and only if u_j divides r for every j such that $\mathsf{u}_j^{\alpha_j} \notin \mathbb{F}[x^p]$. It follows in this case that $\pi_h = \displaystyle\prod_{i,\, \mathsf{u}_i^{\alpha_i} \notin \mathbb{F}[x^p]} \mathsf{u}_i$. □

Definition 2.14. *When* $\mathsf{char}(\mathbb{F}) = 0$, *set* $\varrho_h = 1$. *When* $\mathsf{char}(\mathbb{F}) = p > 0$, *let* $h = \lambda \mathsf{u}_1^{\alpha_1}\cdots \mathsf{u}_t^{\alpha_t}$ *be the factorization of* h, *where the* u_i *are the distinct monic prime factors given in Lemma 2.13, and* $\lambda \in \mathbb{F}^*$. *After possibly renumbering, assume* $\mathsf{u}_i \notin \mathbb{F}[x^p]$ *for* $1 \leq i \leq \ell$ *and* $\mathsf{u}_j \in \mathbb{F}[x^p]$ *for* $\ell < j \leq t$ *(in case* $\ell = 0$, *there are no such u_i, and in case $\ell = t$, there are no such u_j). For each $1 \le i \le \ell$, take $k_i \ge 0$ and $0 \le \overline{\alpha_i} < p$ so that $\alpha_i = k_i p + \overline{\alpha_i}$. Let*

$$\varrho_h = \mathsf{u}_1^{k_1 p} \cdots \mathsf{u}_\ell^{k_\ell p} \mathsf{u}_{\ell+1}^{\alpha_{\ell+1}} \cdots \mathsf{u}_t^{\alpha_t}. \tag{2.15}$$

In the characteristic $p > 0$ case, ϱ_h is the unique monic polynomial of maximal degree in $\mathbb{F}[x^p]$ dividing h, and

$$h = \begin{cases} \lambda \varrho_h & \text{if } \ h \in \mathbb{F}[x^p] \\ \lambda \mathsf{u}_1^{\overline{\alpha_1}} \cdots \mathsf{u}_\ell^{\overline{\alpha_\ell}} \varrho_h & \text{if } \ h \notin \mathbb{F}[x^p]. \end{cases} \tag{2.16}$$

To avoid separating considerations into cases, often we will write $h = \lambda \mathsf{u}_1^{\overline{\alpha_1}} \cdots \mathsf{u}_\ell^{\overline{\alpha_\ell}} \varrho_h$ with the understanding that the product $\mathsf{u}_1^{\overline{\alpha_1}} \cdots \mathsf{u}_\ell^{\overline{\alpha_\ell}}$ should be interpreted as being 1 if $\ell = 0$. Whenever $h \in \mathbb{F}^*$, then h is as in the first option of (2.16) with $\varrho_h = 1$.

Theorem 2.17. *Regard $\mathsf{A}_h \subseteq \mathsf{A}_1$ as in Conventions 2.3. Let $\pi_h \in \mathsf{R} = \mathbb{F}[x]$ be as in Lemma 2.13, and set $a_n = \pi_h h^{n-1} y^n$ for all $n \ge 1$.*

(a) *Assume $a \in \mathsf{A}_1$ and write $a = \sum_{i \ge 0} r_i y^i$ with $r_i \in \mathsf{R}$. Then the following hold:*
  - (i) *If $\mathsf{char}(\mathbb{F}) = 0$, then $a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h) \iff \pi_h h^{i-1} \mid r_i$ for all $i \ge 1$. Hence, $\mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h) = \mathsf{R} \oplus \bigoplus_{n \ge 1} \mathsf{R} a_n$.*
  - (ii) *If $\mathsf{char}(\mathbb{F}) = p > 0$, then $a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h) \iff$*
    - *for all $i \not\equiv 0 \bmod p$, $\pi_h h^{i-1} \mid r_i$*
    - *for all $i \equiv 0 \bmod p$, $i > 0$, $h^{i-1} \mid r_i'$, or equivalently, $r_i \in c_i \varrho_h^{p-1} h^{i-p} + \mathbb{F}[x^p]$ for some $c_i \in \mathsf{R}$ with $c_i' \in \mathsf{R}\left(\frac{h}{\varrho_h}\right)^{p-1}$.*

  In particular, $a = \sum_{i \ge 0} r_i y^i \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)$ if and only if $r_i y^i \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)$ for all $i \ge 0$.

(b) *For all $\mathbb{F}$ and $n \ge 1$, $\mathsf{R} a_n \subset \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)$, and $h' a_n$ and $\frac{h}{\pi_h} a_n$ are in A_h.*

Proof. For (a), suppose $a = \sum_{i \ge 0} r_i y^i$, where $r_i \in \mathsf{R}$ for all i. We will treat the characteristic 0 and p cases together by adopting the convention that $p = 0$ when $\mathsf{char}(\mathbb{F}) = 0$. In that case, the statement $i \not\equiv 0 \bmod p$ simply means $i \ne 0$, while $i \equiv 0 \bmod p$ means $i = 0$.

Now $a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)$ exactly when $[a, x]$ and $[a, \hat{y}]$ are in A_h. In particular,

$$[a, x] \in \mathsf{A}_h \iff \sum_{i \not\equiv 0 \bmod p} i r_i y^{i-1} \in \mathsf{A}_h \iff h^{i-1} \mid r_i \quad \forall\ i \not\equiv 0 \bmod p \tag{2.18}$$

by (2.2). Hence, we may assume $a = \sum_{i \not\equiv 0 \bmod p} s_i h^{i-1} y^i + \sum_{i \equiv 0 \bmod p} r_i y^i$ for some $s_i \in \mathsf{R}$. Since $[a, x] \in \mathsf{A}_h$, it follows that $[a, g] \in \mathsf{A}_h$ for all $g \in \mathsf{R}$. Therefore,

$[a,\hat{y}] = [a, yh] \in \mathsf{A}_h \iff [a, hy] \in \mathsf{A}_h$. Now using Lemma 2.8, we have

$$\begin{aligned}
[a, hy] &= \sum_{i \not\equiv 0 \bmod p} [s_i h^{i-1} y^i, hy] + \sum_{i \equiv 0 \bmod p} [r_i y^i, hy] \\
&= \sum_{i \not\equiv 0 \bmod p} s_i h^{i-1} \sum_{j=1}^{i} \binom{i}{j} h^{(j)} y^{i-j+1} - \sum_{i \not\equiv 0 \bmod p} (s_i h^{i-1})' h y^i \\
&\quad - \sum_{i \equiv 0 \bmod p} r_i' h y^i.
\end{aligned}$$

Since by (2.2) all the terms in the first sum with $j \geq 2$ belong to A_h, we have

$$\begin{aligned}
[a, hy] \in \mathsf{A}_h &\iff \sum_{i \not\equiv 0 \bmod p} s_i h^{i-1} h' y^i - \sum_{i \not\equiv 0 \bmod p} s_i' h^i y^i - \sum_{i \equiv 0 \bmod p} r_i' h y^i \in \mathsf{A}_h \\
&\iff \sum_{i \not\equiv 0 \bmod p} s_i h^{i-1} h' y^i - \sum_{i \equiv 0 \bmod p} r_i' h y^i \in \mathsf{A}_h, \qquad (2.19)
\end{aligned}$$

as $s_i' h^i y^i \in \mathsf{A}_h$ for all $i \not\equiv 0$, again using (2.2).

From this we deduce that h^i must divide $s_i h^{i-1} h'$ for all $i \not\equiv 0 \bmod p$; that is, h must divide $s_i h'$ for all such i. By Lemma 2.13, this means that π_h divides s_i for each $i \not\equiv 0 \bmod p$, and in turn this says that $\pi_h h^{i-1}$ divides r_i for all $i \not\equiv 0 \bmod p$. In particular, (i) and the first assertion of (ii) hold.

Now from (2.19), we also see that $h^{i-1} \mid r_i'$ for all $i \equiv 0 \bmod p$, $i > 0$. Note that $h^{i-1} = h^{i-p} h^{p-1} = (\frac{h}{\varrho_h})^{p-1} \varrho_h^{p-1} h^{i-p}$. Hence, we may write $r_i' = d_i v_i$, where $d_i \in \mathsf{R}(\frac{h}{\varrho_h})^{p-1}$ and $v_i = \varrho_h^{p-1} h^{i-p} \in \mathbb{F}[x^p]$. Since $d_i v_i \in \mathsf{im}\frac{d}{dx} = \sum_{j=0}^{p-2} \mathbb{F}[x^p] x^j$ and $v_i \in \mathbb{F}[x^p]$, it follows that $d_i \in \sum_{j=0}^{p-2} \mathbb{F}[x^p] x^j$. Therefore $d_i = c_i'$ for some $c_i \in \mathsf{R}$, and $(c_i v_i)' = c_i' v_i = d_i v_i = r_i'$. This gives $r_i \in c_i v_i + \mathbb{F}[x^p] = c_i \varrho_h^{p-1} h^{i-p} + \mathbb{F}[x^p]$, as in (ii). That $r_i y^i \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)$ for every r_i of this form for $i \equiv 0 \bmod p$, $i > 0$, can be shown by direct computation. This proves the remaining parts of (a).

The first part of (b) is an immediate consequence of (a) except when $n \equiv 0 \bmod p$ and $\mathsf{char}(\mathbb{F}) = p > 0$. For $a_{kp} = \pi_h h^{kp-1} y^{kp}$ with $k \geq 1$, observe that $[r a_{kp}, f] = 0$ for all $r, f \in \mathsf{R}$ since $y^{kp} \in \mathsf{Z}(\mathsf{A}_1)$. Moreover,

$$\begin{aligned}
[r a_{kp}, hy] &= h[r\pi_h h^{kp-1}, y] y^{kp} = -h(r\pi_h h^{kp-1})' y^{kp} \\
&= -(r\pi_h)' h^{kp} y^{kp} + r\pi_h h' h^{kp-1} y^{kp},
\end{aligned}$$

which is in A_h by (2.2) and the fact that h divides $\pi_h h'$ by Lemma 2.13. Now $h' a_n = h' \pi_h h^{n-1} y^n \in \mathsf{A}_h$ is a consequence of that fact too, and $\frac{h}{\pi_h} a_n = h^n y^n \in \mathsf{A}_h$ is clear. □

Remark 2.20. *It follows from Theorem 2.17 that when* $\mathsf{char}(\mathbb{F}) = 0$ *and* $\frac{h}{\pi_h} \in \mathbb{F}^*$, *then* $\mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h) = \mathsf{A}_h$.

If $\mathsf{char}(\mathbb{F}) = p > 0$, *we set*

$$\begin{aligned} \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)_{\not\equiv 0} &= \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h) \cap \Big(\bigoplus_{i \not\equiv 0 \bmod p} \mathsf{R} y^i \Big), \\ \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)_{\equiv 0} &= \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h) \cap \mathsf{C}_{\mathsf{A}_1}(x). \end{aligned} \tag{2.21}$$

Then every $a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)$ *has a unique expression* $a = b + c$ *with* $b \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)_{\not\equiv 0}$ *and* $c \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)_{\equiv 0}$. *In particular, when* $\frac{h}{\pi_h} \in \mathbb{F}^*$, *then* $b \in \mathsf{A}_h$.

3. Derivations of A_1

We will use derivations of A_1 heavily in our investigations of derivations of A_h. In the next result, we provide a quick proof of the known fact that the derivations of A_1 are inner in the $\mathsf{char}(\mathbb{F}) = 0$ case, in part to establish the notation we will adopt later.

3.1. $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_1)$ when $\mathsf{char}(\mathbb{F}) = 0$.

Proposition 3.1. (Cf. [D2, Lem. 4.6.8]). *Assume* $\mathsf{char}(\mathbb{F}) = 0$. *Then every derivation of the Weyl algebra* A_1 *is inner.*

Proof. Suppose $D \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_1)$. Assume that $D(x) = \sum_{i \geq 0} d_i y^i$, where $d_i \in \mathsf{R} = \mathbb{F}[x]$ for all i. Set

$$u = \sum_{i \geq 0} \frac{d_i}{i+1} y^{i+1}.$$

Then $\mathsf{ad}_u(x) = \sum_{i \geq 0} d_i y^i = D(x)$, so that $E = D - \mathsf{ad}_u \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_1)$ has the property that $E(x) = 0$.

Then from $[E(y), x] + [y, E(x)] = E(1) = 0$, we determine that $[E(y), x] = 0$. Thus, $E(y) \in \mathsf{C}_{\mathsf{A}_1}(x) = \mathsf{R}$ by Lemma 2.5. Since $E(y) \in \mathsf{R}$ and $\mathsf{char}(\mathbb{F}) = 0$, there exists a $w \in \mathsf{R}$ such that $w' = -E(y)$. Then $\mathsf{ad}_w(x) = 0 = E(x)$ and $\mathsf{ad}_w(y) = [w, y] = -w' = E(y)$. Therefore $D - \mathsf{ad}_u = E = \mathsf{ad}_w$ and $D = \mathsf{ad}_u + \mathsf{ad}_w \in \mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_1)$. Hence, $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_1) = \mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_1)$. □

3.2. $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_1)$ when $\mathsf{char}(\mathbb{F}) = p > 0$.

3.2.1. *The derivations* E_x *and* E_y.

Over fields of characteristic $p > 0$, the derivations $(\mathsf{ad}_x)^p = \mathsf{ad}_{x^p}$ and $(\mathsf{ad}_y)^p = \mathsf{ad}_{y^p}$ are 0 on the Weyl algebra A_1. However, A_1 has two special derivations E_x and E_y, which are specified by

$$E_x(x) = y^{p-1}, \quad E_x(y) = 0, \quad \text{and} \quad E_y(x) = 0, \quad E_y(y) = x^{p-1}. \tag{3.2}$$

Then zE_x and zE_y are also derivations of A_1 for every $z \in \mathsf{Z}(\mathsf{A}_1) = \mathbb{F}[x^p, y^p]$. Let φ be the anti-automorphism of A_1 defined by

$$\varphi(x) = y, \quad \varphi(y) = x. \tag{3.3}$$

Then

$$\varphi E_x \varphi = \varphi E_x \varphi^{-1} = E_y, \quad \text{and} \quad \varphi E_y \varphi = \varphi E_y \varphi^{-1} = E_x. \tag{3.4}$$

Lemma 3.5. *Assume* A_1 *is the Weyl algebra over* $\mathbb{F}$*, where* $\mathsf{char}(\mathbb{F}) = p > 0$*. Then*

$$\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_1) = \mathsf{Z}(\mathsf{A}_1)E_x + \mathsf{Z}(\mathsf{A}_1)E_y + \mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_1).$$

Proof. The right side is clearly contained in $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_1)$. For the other containment, suppose $D \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_1)$, and assume that $D(x) = \sum_{i\geq 0} d_i y^i$, where $d_i \in \mathsf{R}$ for all i. Set

$$b = \sum_{i \not\equiv -1 \,\mathsf{mod}\, p} \frac{d_i}{i+1} y^{i+1}.$$

Then $\mathsf{ad}_b(x) = \sum_{i\not\equiv -1 \,\mathsf{mod}\, p} d_i y^i$, so that $E = D - \mathsf{ad}_b \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_1)$ has the property that $E(x) = \sum_{i\equiv -1 \,\mathsf{mod}\, p} d_i y^i$.

Suppose that $E(y) = \sum_{j\geq 0} e_j y^j$, where $e_j \in \mathsf{R}$ for all j. Then

$$0 = E(1) = [E(y), x] + [y, E(x)] = \sum_{j\geq 1} j e_j y^{j-1} + \sum_{i\equiv -1 \,\mathsf{mod}\, p} d_i' y^i,$$

from which we determine that $d_i' = 0$ for all $i \equiv -1 \,\mathsf{mod}\, p$, and $e_j = 0$ for all $j \not\equiv 0 \,\mathsf{mod}\, p$. The first implies $d_i \in \mathbb{F}[x^p]$ for all such i, so that $w = \sum_{i\equiv -1 \,\mathsf{mod}\, p} d_i y^{i-(p-1)} \in \mathsf{Z}(\mathsf{A}_1)$ and $E(x) = w y^{p-1} = w E_x(x)$. As a result, $F = E - wE_x$ has the property that $F(x) = 0$ and $F(y) = \sum_{j\equiv 0 \,\mathsf{mod}\, p} e_j y^j$.

Now it is a direct consequence of the decomposition $\mathsf{R} = \bigoplus_{j=0}^{p-1} \mathbb{F}[x^p]x^j$ and the fact that $\mathsf{im}\frac{d}{dx} = \bigoplus_{j=0}^{p-2} \mathbb{F}[x^p]x^j$ that every $e \in \mathsf{R}$ can be expressed as $e = cx^{p-1} - r'$ for some $r \in \mathsf{R}$ and a unique $c \in \mathbb{F}[x^p]$. Applying that result to each e_j, we have that there exist $c_j \in \mathbb{F}[x^p]$ and $r_j \in \mathsf{R}$, so that $e_j = c_j x^{p-1} - r_j'$. Then $F(y) = \sum_{j\equiv 0 \,\mathsf{mod}\, p} e_j y^j = \left(\sum_{j\equiv 0 \,\mathsf{mod}\, p} c_j y^j\right) x^{p-1} - \sum_{j\equiv 0 \,\mathsf{mod}\, p} r_j' y^j$. Setting $z = \sum_{j\equiv 0 \,\mathsf{mod}\, p} c_j y^j$ and $c = \sum_{j\equiv 0 \,\mathsf{mod}\, p} r_j y^j$, we see that $z \in \mathsf{Z}(\mathsf{A}_1)$ and $(F - zE_y - \mathsf{ad}_c)(x) = 0 = (F - zE_y - \mathsf{ad}_c)(y)$. Consequently, $D = wE_x + zE_y + \mathsf{ad}_b + \mathsf{ad}_c \in \mathsf{Z}(\mathsf{A}_1)E_x + \mathsf{Z}(\mathsf{A}_1)E_y + \mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_1)$. □

3.2.2. *The action of* E_x *and* E_y *on* A_1.

The next lemma describes how E_x and E_y act on various elements of A_1.

Lemma 3.6. *Assume* $\mathsf{char}(\mathbb{F}) = p > 0$*. When* $g \in \mathbb{F}[x]$*, let* $g^{(k)} = \left(\frac{d}{dx}\right)^k (g)$*, and when* $g \in \mathbb{F}[y]$*, let* $g^{(k)} = \left(\frac{d}{dy}\right)^k (g)$*. Assume* φ *is the anti-automorphism in* (3.3)*, and let* $\partial_p : \mathbb{F}[x] \to \mathbb{F}[x]$ *be the* $\mathbb{F}$*-linear map defined by*

$$\partial_p\left(\sum_{i=0}^{p-1} r_i x^i\right) = \sum_{i=0}^{p-1} \frac{d}{d(x^p)}(r_i)\, x^i, \qquad \textit{for } r_i \in \mathbb{F}[x^p]. \tag{3.7}$$

Then the following hold in A_1*:*

(a) $E_x(x^n) = \sum_{k=1}^{p} \binom{n}{k} x^{n-k}(y^{p-1})^{(k-1)}$ for $n \geq 1$;

(b) $E_x(g) = \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} g^{(k)} y^{p-k} - \partial_p(g)$ for all $g \in \mathbb{F}[x]$;

(c) $E_x = -\frac{d}{d(x^p)}$ on $\mathbb{F}[x^p]$ and $E_x(g^p) = -(g')^p$ for all $g \in \mathbb{F}[x]$;

(d) $E_y(g) = \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} x^{p-k} g^{(k)} - \varphi\partial_p(g(x))$ for all $g \in \mathbb{F}[y]$;

(e) $E_y(\hat{y}) = E_y(y)h = x^{p-1}h$;

(f) $E_x(\hat{y}) = h'y^p + \sum_{k=1}^{p-2} \frac{(-1)^{k-1}}{(k+1)k} h^{(k+1)} y^{p-k} - \partial_p(h)\, y - \partial_p(h')$.

Proof. Part (a) can be shown using induction on n (the case $n = 1$ saying $E_x(x) = y^{p-1}$). Assume $E_x(x^n) = \sum_{k=1}^{n} \binom{n}{k} x^{n-k}(y^{p-1})^{(k-1)}$, and substitute that expression into $E_x(x^{n+1}) = E_x(x^n)x + x^n E_x(x)$. Applying the fact that $fx = xf + \frac{d}{dy}(f)$ for all $f \in \mathbb{F}[y]$ to the first summand and simplifying gives the desired expression for the $n+1$ case. Since $(y^{p-1})^{(k-1)} = 0$ for all $k > p$, the index of summation need only go up to p.

For (b), we have using $\binom{p-1}{k-1} = (-1)^{k-1}$ and $(p-1)! = -1$ that

$$\begin{aligned} E_x(x^n) &= \sum_{k=1}^{p} \binom{n}{k} x^{n-k}(y^{p-1})^{(k-1)} \\ &= \sum_{k=1}^{p-1} \frac{(x^n)^{(k)}}{k!} \binom{p-1}{k-1} (k-1)! y^{p-k} - \binom{n}{p} x^{n-p} \\ &= \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} (x^n)^{(k)} y^{p-k} - \binom{n}{p} x^{n-p}. \end{aligned}$$

Now if $n = jp + \ell$ with $0 \leq \ell < p$, then $x^n = (x^p)^j x^\ell$ and $\binom{n}{p} = j$, so $\partial_p(x^n) = \binom{n}{p} x^{n-p}$. Thus,

$$E_x(x^n) = \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} (x^n)^{(k)} y^{p-k} - \partial_p(x^n),$$

where ∂_p is as in (3.7). This, together with the linearity of derivations, implies (b).

As a special case of (b), we have $E_x(x^{jp}) = -jx^{(j-1)p}$ for all $j \geq 1$ so that $E_x = -\frac{d}{d(x^p)}$ on $\mathbb{F}[x^p]$. In particular, if $g(x) = \sum_{j \geq 0} \gamma_j x^j$, then, as claimed in (c),

$$E_x(g^p) = \sum_{j\geq 0} \gamma_j^p E_x(x^{jp}) = -\sum_{j\geq 1} j\gamma_j^p x^{(j-1)p} = -\sum_{j\geq 1} j^p\gamma_j^p x^{(j-1)p} = -(g')^p.$$

For (d), applying the anti-automorphism φ in (3.3) which interchanges x and y, and using (3.4), we have $E_y(g(y)) = \varphi E_x \varphi^{-1}(g(y)) = \varphi(E_x(g(x)))$ for $g(y) \in \mathbb{F}[y]$, and so (d) now follows from applying φ to (b).

Part (e) is apparent, and (f) can be derived from the following calculation which uses the relation $[y, \partial_p(f)] = \partial_p(f')$, for $f \in \mathsf{R}$:

$$\begin{aligned} E_x(\hat{y}) &= E_x(yh) = yE_x(h) = y\sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} h^{(k)} y^{p-k} - y\partial_p(h) \\ &= \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} \left(h^{(k)}y + h^{(k+1)}\right) y^{p-k} - \partial_p(h)\, y - \partial_p(h') \\ &= h'y^p + \sum_{k=1}^{p-2} \frac{(-1)^{k-1}}{(k+1)k} h^{(k+1)} y^{p-k} - \partial_p(h)\, y - \partial_p(h'). \qquad \square \end{aligned}$$

We have the following consequence of this result.

Theorem 3.8. *Assume* A_1 *is the Weyl algebra over* $\mathbb{F}$*, where* $\mathsf{char}(\mathbb{F}) = p > 0$*. Then*

(a) $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_1) = \mathsf{Z}(\mathsf{A}_1)E_x \oplus \mathsf{Z}(\mathsf{A}_1)E_y \oplus \mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_1)$, *where* $E_x, E_y \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_1)$ *are given by* $E_x(x) = y^{p-1}$, $E_x(y) = 0$, $E_y(x) = 0$, $E_y(y) = x^{p-1}$.

(b) $\mathsf{HH}^1(\mathsf{A}_1) = \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_1)/\mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_1) \cong \mathsf{Der}_{\mathbb{F}}(\mathbb{F}[t_1, t_2])$ *as Lie algebras, where* $t_1 = x^p$, $t_2 = y^p$.

Proof. In Lemma 3.5, we have established that $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_1)$ is the sum of the terms on the right side of (a). Suppose $D = wE_x + zE_y + \mathsf{ad}_a = 0$ for some $a \in \mathsf{A}_1$ and $z, w \in \mathsf{Z}(\mathsf{A}_1)$. Applying D to x^p and using the fact that x^p is central, we have from Lemma 3.6 (c) that $0 = D(x^p) = -w$. Similarly, applying D to y^p gives $z = 0$. Hence $\mathsf{ad}_a = 0$ also, and the sum in (a) is direct.

The map $\mathsf{Res} : \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_1) \to \mathsf{Der}_{\mathbb{F}}(\mathsf{Z}(\mathsf{A}_1))$ given by restricting a derivation of A_1 to the center $\mathsf{Z}(\mathsf{A}_1) = \mathbb{F}[t_1, t_2]$, where $t_1 = x^p, t_2 = y^p$, is clearly a morphism of Lie algebras. It follows from Lemma 3.6 that $\mathsf{Res}(E_x) = -\frac{d}{dt_1}$ and $\mathsf{Res}(E_y) = -\frac{d}{dt_2}$. Hence $wE_x + zE_y + \mathsf{ad}_a \mapsto -w\frac{d}{dt_1} - z\frac{d}{dt_2}$ for all $w, z \in \mathsf{Z}(\mathsf{A}_1)$, which shows the map is onto. Now $\mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_1)$ is in the kernel. But since every $D \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_1)$ has the form $D = wE_x + zE_y + \mathsf{ad}_a$, we see the kernel is exactly $\mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_1)$. $\square$

Remark 3.9. *It is well known that* $\mathsf{Der}_{\mathbb{F}}(\mathbb{F}[t_1, t_2])$ *is a free* $\mathbb{F}[t_1, t_2]$*-module of rank* 2 *with basis* $\frac{d}{dt_1}, \frac{d}{dt_2}$*. This Lie algebra is often referred to as the* Witt algebra *in* 2 *variables. A. Solotar and M. Suárez-Álvarez have pointed out to us one could alternately use the fact that* A_1 *is Azumaya over its center, combined with a result on the homology of Azumaya algebras in* [CW] *and the Van den Bergh duality between homology and cohomology (see* [Be]*), to conclude that* $\mathsf{HH}^1(\mathsf{A}_1)$ *is free* *of rank 2 over the center* $\mathsf{Z}(\mathsf{A}_1)$ *when* $\mathsf{char}(\mathbb{F}) = p > 0$*. Theorem 3.8, which also establishes this result, identifies explicit generators* E_x *and* E_y *for* $\mathsf{HH}^1(\mathsf{A}_1)$ *over* $\mathsf{Z}(\mathsf{A}_1)$.

3.2.3. *Lie brackets in* $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_1)$ *when* $\mathsf{char}(\mathbb{F}) = p > 0$.

Next we determine the multiplication in $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_1)$.

Lemma 3.10. *Assume* $\mathsf{char}(\mathbb{F}) = p > 0$*. Then* $[E_x, E_y] = \mathsf{ad}_{\varpi}$ *where*

$$\varpi = \sum_{n=1}^{p-1} \frac{(p-1-n)!}{n} x^n y^n. \tag{3.11}$$

Proof. It suffices to compute the action of $[E_x, E_y]$ on x and y. Using (a) of Lemma 3.6 and the fact that $\binom{p-1}{k} = (-1)^k$ for $0 \le k \le p-1$, we have

$$\begin{aligned}
[E_x, E_y](y) = E_x(x^{p-1}) &= \sum_{k=1}^{p-1} \binom{p-1}{k} x^{p-1-k} (y^{p-1})^{(k-1)} \\
&= -\sum_{k=1}^{p-1} (k-1)!\, x^{p-1-k} y^{p-k} = -\sum_{n=1}^{p-1} (p-1-n)!\, x^{n-1} y^n.
\end{aligned}$$

Then

$$[E_x, E_y](x) = -E_y(y^{p-1}) = \sum_{n=1}^{p-1} (p-1-n)!\, x^n y^{n-1}$$

upon applying φ to the relation above. However, if ϖ is as in (3.11), then

$$\mathsf{ad}_{\varpi}(x) = \sum_{n=1}^{p-1} (p-1-n)!\, x^n y^{n-1} \quad \text{and} \quad \mathsf{ad}_{\varpi}(y) = -\sum_{n=1}^{p-1} (p-1-n)!\, x^{n-1} y^n.$$

Thus, $[E_x, E_y] = \mathsf{ad}_{\varpi}$, as desired. □

Products in $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_1)$ can now be described using this result.

Lemma 3.12. *Assume* $\mathsf{char}(\mathbb{F}) = p > 0$*. For all* $D, E \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_1)$*,* $a \in \mathsf{A}_1$*,* $w, z \in \mathsf{Z}(\mathsf{A}_1)$*, we have*

- $[D, \mathsf{ad}_a] = \mathsf{ad}_{D(a)}$,
- $z\mathsf{ad}_a = \mathsf{ad}_{za}$,
- $[wD, zE] = wD(z)E - zE(w)D + wz[D, E]$,
- $[wE_x, zE_y] = wE_x(z)E_y - zE_y(w)E_x + wz\,\mathsf{ad}_{\varpi}$*, with* ϖ *as in* (3.11).

4. Generalities on Derivations of A_h

We turn our attention now to the Lie algebra $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$ of $\mathbb{F}$-linear derivations of A_h for arbitrary $0 \neq h \in \mathsf{R} = \mathbb{F}[x]$ and arbitrary $\mathbb{F}$. Throughout, we view A_h as a subalgebra of A_1 as in Conventions 2.3, and apply the results we have just established in Sections 3.1 and 3.2 on $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_1)$ to derive information about $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$.

We begin by determining when a derivation of A_h extends to one of A_1. We then define the derivations D_e, $e \in \mathsf{C}_{\mathsf{A}_h}(x)$, and introduce the element a_0, which belongs to a localization of A_1 and is a natural extension of the elements $a_n = \pi_h h^{n-1} \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)$ for $n \geq 1$. The main results of this section are Theorem 4.9, which describes a decomposition of $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$ into a sum of Lie subalgebras for arbitrary $\mathbb{F}$, and Theorem 4.15, which gives expressions for various products involving the derivations D_g, $g \in \mathsf{R}$, and ad_{ra_n} for $n \geq 0$ and $r \in \mathsf{R}$. This sets the stage for Section 5, where we show that these derivations along with the inner derivations generate $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$ when $\mathsf{char}(\mathbb{F}) = 0$.

4.1. **Extensions of derivations.**

To determine a necessary and sufficient condition for a derivation of A_h to extend to a derivation of A_1, we require a basic result about derivations of A_h, which can be shown using [GW, Exer. 2ZC].

Lemma 4.1. *Fix $u, v \in \mathsf{A}_h$. Let $d : \mathbb{F}[x] \to \mathsf{A}_h$ be the unique derivation such that $d(x) = u$. There is a derivation $D \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$ such that $D(x) = d(x) = u$ and $D(\hat{y}) = v$ if and only if $[v, x] + [\hat{y}, u] = d(h)$. If such a derivation exists, it is unique.*

In the next result, we will use the fact that $D(h) \in \mathsf{A}_h h = h\mathsf{A}_h$ for every $D \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$. This follows from the computation $D(h) = [D(\hat{y}), x] + [\hat{y}, D(x)]$ and the fact [BLO1, Lem. 6.1] that $[\mathsf{A}_h, \mathsf{A}_h] \subseteq h\mathsf{A}_h$.

Theorem 4.2. *Regard $\mathsf{A}_h \subseteq \mathsf{A}_1$ as in Conventions 2.3.*

(i) *A derivation $D \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$ extends to a derivation $\widetilde{D}$ of A_1 if and only if $D(\hat{y}) \in \mathsf{A}_1 h$. In particular, if $D(\hat{y}) = ah$ and $D(h) = bh$ for $a \in \mathsf{A}_1$ and $b \in \mathsf{A}_h$, then $\widetilde{D}$ is determined by*

$$\widetilde{D}(x) = D(x), \qquad \widetilde{D}(y) = a - yb.$$

(ii) *Suppose that $D, E \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_1)$ restrict to derivations of A_h and $D = E$ as derivations of A_h. Then $D = E$ as derivations of A_1.*

Proof. (i) Assume $D \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$. If D extends to a derivation $\widetilde{D}$ of A_1, then

$$D(\hat{y}) = \widetilde{D}(\hat{y}) = \widetilde{D}(yh) = \widetilde{D}(y)h + yD(h) \in \mathsf{A}_1 h.$$

Conversely, suppose $D(\hat{y}) = ah$ where $a \in \mathsf{A}_1$. We may assume $D(h) = bh$ where $b \in \mathsf{A}_h$. By Lemma 4.1 applied to A_1 (and so with $\widetilde{D}$ replacing D and y replacing $\hat{y}$ in quoting that result) there is a unique derivation $\widetilde{D}$ of A_1 with

$$\widetilde{D}(x) = D(x), \qquad \widetilde{D}(y) = a - yb$$

if and only if $[a - yb, x] + [y, D(x)] = D(1) = 0$. Since A_1 is a domain, it suffices to show that $\big([a - yb, x] + [y, D(x)]\big)h = 0$. For this, we have

$$\begin{aligned}[a - yb, x]h + [y, D(x)]h &= [ah, x] - [ybh, x] + [y, D(x)]h \\ &= [D(\hat{y}), x] - [yD(h), x] + [\hat{y}, D(x)] - y[h, D(x)] \\ &= [D(\hat{y}), x] + [\hat{y}, D(x)] - [y, x]D(h) - y[D(h), x] - y[h, D(x)] \\ &= D([\hat{y}, x]) - D(h) - yD([h, x]) = 0.\end{aligned}$$

Note that $\widetilde{D}$ thus defined restricts to D on A_h.

(ii) Now assume that $D, E \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_1)$ both restrict to derivations of A_h and $D = E$ as derivations of A_h. The assumptions imply that $D(r) = E(r)$ for all $r \in \mathsf{R}$, and $D(yh) = D(\hat{y}) = E(\hat{y}) = E(yh)$. Therefore,

$$D(y)h + yD(h) = E(y)h + yE(h),$$

and so $D(y)h = E(y)h$. Since $h \neq 0$, we have $D(y) = E(y)$. □

For any $a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)$, ad_a is a derivation of A_h, and if a happens to belong to A_h, then $[D, \mathsf{ad}_a] = \mathsf{ad}_{D(a)}$ for any derivation $D \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$. However, if $a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h) \setminus \mathsf{A}_h$, then $D(a)$ may not be defined. This can be remedied in the following way.

Recall from [BLO1, Cor. 4.3] that

$$\Sigma = \{h^m \mid m \geq 0\} \tag{4.3}$$

is a left and a right Ore set in both A_1 and $\mathsf{A}_h \subseteq \mathsf{A}_1$, and the corresponding localizations $\mathsf{A}_1\Sigma^{-1} = \mathsf{A}_h\Sigma^{-1}$ are equal. It is well known that derivations extend under localization. In particular, if $D \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$, then D extends uniquely to a derivation $\widetilde{D}$ of $\mathsf{A}_h\Sigma^{-1} = \mathsf{A}_1\Sigma^{-1}$, with $\widetilde{D}(h^{-1}) = -h^{-1}D(h)h^{-1}$.

Lemma 4.4. *Suppose $D \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$, and let $\widetilde{D}$ be the extension of D to a derivation of $\mathsf{A}_1\Sigma^{-1}$. Then $[D, \mathsf{ad}_a] = \mathsf{ad}_{\widetilde{D}(a)}$ for all $a \in \mathsf{N}_{\mathsf{A}_1\Sigma^{-1}}(\mathsf{A}_h)$, and $\widetilde{D}(a) \in \mathsf{N}_{\mathsf{A}_1\Sigma^{-1}}(\mathsf{A}_h)$. In particular, $\widetilde{D}(a) \in \mathsf{N}_{\mathsf{A}_1\Sigma^{-1}}(\mathsf{A}_h)$ for all $a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)$.*

Proof. Assume $b \in \mathsf{A}_h \subseteq \mathsf{A}_1$ and $a \in \mathsf{N}_{\mathsf{A}_1\Sigma^{-1}}(\mathsf{A}_h)$. Then $[a, b] \in \mathsf{A}_h$ and $D([a, b]) = \widetilde{D}([a, b]) = [\widetilde{D}(a), b] + [a, D(b)]$ so that

$$[D, \mathsf{ad}_a](b) = D([a, b]) - [a, D(b)] = [\widetilde{D}(a), b] = \mathsf{ad}_{\widetilde{D}(a)}(b). \tag{4.5}$$

Since $[\widetilde{D}(a), b] = [D, \mathsf{ad}_a](b) \in \mathsf{A}_h$, it is clear that $\widetilde{D}(a) \in \mathsf{N}_{\mathsf{A}_1\Sigma^{-1}}(\mathsf{A}_h)$. □

4.2. The derivations D_e.

Lemma 4.1 implies that for each $e \in \mathsf{C}_{\mathsf{A}_h}(x)$ there is a unique derivation D_e of A_h with $D_e(x) = 0$ and $D_e(\hat{y}) = e$. Such a derivation satisfies $D_e(f) \in \mathsf{C}_{\mathsf{A}_h}(x)$ for all $f \in \mathsf{C}_{\mathsf{A}_h}(x)$, since $0 = D_e([x, f]) = [x, D_e(f)]$. These derivations play a prominent role in our investigations and also can be used to construct automorphisms of A_h.

Proposition 4.6. *Assume* $e, f \in \mathsf{C}_{\mathsf{A}_h}(x) = \mathsf{Z}(\mathsf{A}_h)\mathsf{R}$*. Then*

(i) $[D_e, D_f] = D_c$*, where* $c = D_e(f) - D_f(e) \in \mathsf{C}_{\mathsf{A}_h}(x)$*, so that* $\mathcal{D}_\mathsf{C} = \{D_e \mid e \in \mathsf{C}_{\mathsf{A}_h}(x)\}$ *is a Lie subalgebra of* $\mathsf{Der}_\mathbb{F}(\mathsf{A}_h)$*.*

(ii) $D_{\delta(g)} = -\mathsf{ad}_g$ *for all* $g \in \mathsf{R}$*, where* $\delta(g) = g'h$*. In particular,* $D_h = -\mathsf{ad}_x$*.*

(iii) *When* $\mathsf{char}(\mathbb{F}) = 0$*, then* $\mathcal{D}_\mathsf{C} = \{D_g \mid g \in \mathsf{R}\}$*. Moreover,*

(a) $\mathcal{D}_\mathsf{C}$ *is abelian, and* D_g *is locally nilpotent for all* $g \in \mathsf{R}$*.*

(b) *For any* $g \in \mathsf{R}$*,* $\phi_g = \exp(D_g) = \sum_{n=0}^{\infty} \frac{(D_g)^n}{n!}$ *is an automorphism of* A_h *with inverse* $\phi_{-g} = \exp(-D_g)$*, and* $\{\phi_g \mid g \in \mathsf{R}\}$ *is an abelian subgroup of* $\mathsf{Aut}_\mathbb{F}(\mathsf{A}_h)$ *isomorphic to* $(\mathsf{R}, +)$*.*

Remark 4.7. *The automorphism* ϕ_g *satisfies* $\phi_g(x) = x$ *and* $\phi_g(\hat{y}) = \hat{y} + g$*, and* $\phi_f \circ \phi_g = \phi_{f+g}$ *holds for all* $f, g \in \mathsf{R}$*. In* [BLO1, Thm. 8.3 (iv)] *it is shown that if* ϕ_g *is defined by these expressions for the algebra* A_h *over any field, then* $\{\phi_g \mid g \in \mathsf{R}\}$ *forms a normal subgroup of* $\mathsf{Aut}_\mathbb{F}(\mathsf{A}_h)$ *isomorphic to* $(\mathsf{R}, +)$*.*

Every derivation ad_c, with $c \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)_{\equiv 0}$ as in (2.21), can be realized as a derivation in $\mathcal{D}_\mathsf{C}$ as follows.

Lemma 4.8. *Assume* $\mathsf{char}(\mathbb{F}) = p > 0$ *and* $c \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)_{\equiv 0}$*. Then there is* $f \in \mathsf{C}_{\mathsf{A}_h}(x)$ *such that* $\mathsf{ad}_c = D_f$*.*

Proof. Set $f = \mathsf{ad}_c(\hat{y})$. Then $f \in \mathsf{A}_h$ because $c \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)$. Moreover, as $c \in \mathsf{C}_{\mathsf{A}_1}(x)$, it follows that $[f, x] = [\mathsf{ad}_c(\hat{y}), x] = \mathsf{ad}_c([\hat{y}, x]) = 0$, so $f \in \mathsf{C}_{\mathsf{A}_h}(x)$. This implies $\mathsf{ad}_c = D_f$, as required. □

The derivations D_g with $g \in \mathsf{R}$ can be used to give a decomposition of $\mathsf{Der}_\mathbb{F}(\mathsf{A}_h)$, as the next result shows.

Theorem 4.9. *Assume* $\mathbb{F}$ *is arbitrary, and regard* $\mathsf{A}_h \subseteq \mathsf{A}_1$*. Then*

$$\mathcal{D}_\mathsf{R} = \{D_g \mid g \in \mathsf{R}\} \quad \text{and} \quad \mathcal{E} = \{F \in \mathsf{Der}_\mathbb{F}(\mathsf{A}_1) \mid F(\mathsf{A}_h) \subseteq \mathsf{A}_h\} \tag{4.10}$$

are Lie subalgebras of $\mathsf{Der}_\mathbb{F}(\mathsf{A}_h)$*,* $\mathcal{D}_\mathsf{R}$ *is abelian, and* $\mathsf{Der}_\mathbb{F}(\mathsf{A}_h) = \mathcal{D}_\mathsf{R} + \mathcal{E}$*.*

Proof. It is clear that $\mathcal{D}_\mathsf{R}$ and $\mathcal{E}$ are Lie subalgebras of $\mathsf{Der}_\mathbb{F}(\mathsf{A}_h)$, and $\mathcal{D}_\mathsf{R}$ is abelian (compare Proposition 4.6 (i)). Assume $D \in \mathsf{Der}_\mathbb{F}(\mathsf{A}_h)$. Then $D(\hat{y}) = \sum_{j \geq 0} r_j \hat{y}^j$, where $r_j \in \mathsf{R}$ for each j. Now $D - D_{r_0} \in \mathsf{Der}_\mathbb{F}(\mathsf{A}_h)$, and

$$(D - D_{r_0})(\hat{y}) = \sum_{j \geq 1} r_j \hat{y}^j = \sum_{j \geq 1} r_j \hat{y}^{j-1} y h \in \mathsf{A}_1 h.$$

Thus by Theorem 4.2, the derivation $D - D_{r_0} \in \mathsf{Der}_\mathbb{F}(\mathsf{A}_h)$ extends to a derivation $E \in \mathsf{Der}_\mathbb{F}(\mathsf{A}_1)$ such that $D = D_{r_0} + E$, where E belongs to $\mathcal{E}$. □

The derivations D_g extend to derivations of $\mathsf{A}_1\Sigma^{-1}$, as the next result shows.

Lemma 4.11. *For* $g \in \mathsf{R}$*, the derivation* $D_g \in \mathsf{Der}_\mathbb{F}(\mathsf{A}_h)$ *extends uniquely to a derivation* $\widetilde{D}_g$ *of* $\mathsf{A}_1\Sigma^{-1}$ *with* $\widetilde{D}_g(\mathsf{R}\Sigma^{-1}) = 0$*,* $\widetilde{D}_g(y) = gh^{-1}$*, and* $[D_g, \mathsf{ad}_a] = \mathsf{ad}_{\widetilde{D}_g(a)}$*, for all* $a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)$*, where* $\widetilde{D}_g(a) \in \mathsf{N}_{\mathsf{A}_1\Sigma^{-1}}(\mathsf{A}_h)$*.*

Proof. It is clear that D_g extends uniquely to a derivation $\widetilde{D}_g$ of $\mathsf{A}_1\Sigma^{-1}$, and $\widetilde{D}_g(h^{-1}) = -h^{-1}D_g(h)h^{-1} = 0$. Then it follows that

$$\widetilde{D}_g(y) = \widetilde{D}_g(\hat{y}h^{-1}) = \widetilde{D}_g(\hat{y})h^{-1} = D_g(\hat{y})h^{-1} = gh^{-1}. \tag{4.12}$$

The final assertion is a direct consequence of Lemma 4.4. □

4.3. **The element $a_0 = \pi_h h^{-1}$ in $\mathsf{N}_{\mathsf{A}_1\Sigma^{-1}}(\mathsf{A}_h)$.**

Let $\widetilde{D}_1$ be the extension of the derivation D_1 to $\mathsf{A}_1\Sigma^{-1}$, and let $a_0 = \widetilde{D}_1(a_1) = \pi_h h^{-1} \in \mathsf{N}_{\mathsf{A}_1\Sigma^{-1}}(\mathsf{A}_h)$. This definition fits naturally with the definition of the elements $a_n = \pi_h h^{n-1}y^n \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)$ for $n \geq 1$. Observe that in general $\mathsf{ad}_{ra_0} \notin \mathcal{E} = \{F \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_1) \mid F(\mathsf{A}_h) \subseteq \mathsf{A}_h\}$. Now since $\delta(r) = r'h$ for all $r \in \mathsf{R}$, the derivation δ extends to a derivation (again denoted by δ) on $\mathsf{R}\Sigma^{-1}$ with $\delta(h^{-1}) = -h'h^{-1}$. The linear transformation given by

$$\delta_0 : \mathsf{R} \to \mathsf{R}, \quad r \mapsto \delta(ra_0) = (ra_0)'h = (r\pi_h h^{-1})'h = (r\pi_h)' - r\frac{\pi_h h'}{h} \tag{4.13}$$

will play a special role in what follows. Since h divides $\pi_h h'$ by Lemma 2.13, it is evident that $\delta_0(\mathsf{R}) = \delta(\mathsf{R}a_0) \subseteq \mathsf{R}$.

Lemma 4.14. *For all $r \in \mathsf{R}$, let $\delta_0(r) = \delta(ra_0)$ as in* (4.13)*, where $a_0 = \pi_h h^{-1} \in \mathsf{N}_{\mathsf{A}_1\Sigma^{-1}}(\mathsf{A}_h)$.*

(a) *Then $\mathsf{ad}_{ra_0} = -D_{\delta(ra_0)} = -D_{\delta_0(r)} \in \mathcal{D}_{\mathsf{R}}$ for all $r \in \mathsf{R}$. In particular, $\mathsf{ad}_{a_0} = -D_{\delta(a_0)} = -D_{\delta_0(1)}$ and $\deg(\delta(a_0)) < \deg h$.*

(b) *$\delta_0(rs) = \delta(rsa_0) = r\delta_0(s) + r's\pi_h$. In particular, $\delta_0(r) = r\delta_0(1) + r'\pi_h$, where $\delta_0(1) = \pi_h' - \frac{\pi_h h'}{h}$.*

Proof. For any $r \in \mathsf{R}$, $\mathsf{ad}_{ra_0}(x) = 0$ and

$$\mathsf{ad}_{ra_0}(\hat{y}) = [ra_0, y]h = -(ra_0)'h = -\delta(ra_0) = -\delta_0(r) \in \mathsf{R}.$$

Thus, $\mathsf{ad}_{ra_0} = -D_{\delta(ra_0)} = -D_{\delta_0(r)} \in \mathcal{D}_{\mathsf{R}}$, as these two derivations agree on a generating set of A_h. It can be seen from (4.13) that $\deg(\delta(a_0)) = \deg(\delta_0(1)) < \deg \pi_h \leq \deg h$. Part (b) follows directly from the definitions. □

4.4. **Main result on products.**

We can now state our main result on the Lie brackets in $\mathsf{HH}^1(\mathsf{A}_h)$. Since $\mathsf{C}_{\mathsf{A}_h}(x) = \mathsf{Z}(\mathsf{A}_h)\mathsf{R}$, and $D_{zg} = zD_g$ for $z \in \mathsf{Z}(\mathsf{A}_h), g \in \mathsf{R}$, we will focus on products involving the derivations D_g for $g \in \mathsf{R}$. This suffices when $\mathsf{char}(\mathbb{F}) = 0$, since $\mathsf{Z}(\mathsf{A}_h) = \mathbb{F}1$ in that case. When $\mathsf{char}(\mathbb{F}) = p > 0$, more general products will be considered in Section 6.7.

Theorem 4.15. *Set $a_{-1} = 0$ and let $a_0 = \pi_h h^{-1}$. For all $r \in \mathsf{R}$, let $\delta_0(r) = \delta(ra_0) = (r\pi_h h^{-1})'h$ as in* (4.13)*.*

(a) *For all $g, r \in \mathsf{R}$ and $n \geq 0$, we have $[D_g, \mathsf{ad}_{ra_n}] = n\mathsf{ad}_{gra_{n-1}} = n\mathsf{ad}_{ca_{n-1}}$ in $\mathsf{HH}^1(\mathsf{A}_h) = \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)/\mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_h)$, where c is the remainder of the division in R of gr by $\frac{h}{\pi_h}$.*

(b) *For all* $r, s \in \mathsf{R}$ *and all* $m, n \geq 0$, $[\mathsf{ad}_{ra_m}, \mathsf{ad}_{sa_n}] = \mathsf{ad}_{qa_{m+n-1}} = \mathsf{ad}_{da_{m+n-1}}$ *in* $\mathsf{HH}^1(\mathsf{A}_h)$*, where* $q = mr\delta_0(s) - ns\delta_0(r)$*, and* d *is the remainder of the division in* R *of* q *by* $\frac{h}{\pi_h}$.

Our proof of this theorem, which we complete in Section 4.7, will be the culmination of a series of computational results.

4.5. **The product** $[D_g, \mathsf{ad}_a]$ **for** $g \in \mathsf{R}$ **and** $a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)$**.**

Lemma 4.16. *Assume* $D \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_1\Sigma^{-1})$ *has the property that* $D(x) = 0$ *and* $D(y) = f$*, where* $f \in \mathsf{R}\Sigma^{-1}$*. Then*

$$D(y^n) = \sum_{k=1}^{n} \binom{n}{k} f^{(k-1)} y^{n-k}$$

for all $n \geq 1$*, where* $f^{(k-1)}$ *denotes* $(\frac{d}{dx})^{k-1}(f)$ *and* $f^{(0)} = f$.

Proof. The assertion holds for $n = 1$ since $D(y) = f$. For larger n, it follows by induction using the fact that $ys = sy + s'$ for $s \in \mathsf{R}\Sigma^{-1}$. □

Next we compute $\widetilde{D}_g$ on certain elements. Ultimately, this will enable us to calculate $[D_g, \mathsf{ad}_{ra_n}]$.

Corollary 4.17. *Let* $g, r \in \mathsf{R}$ *and assume* $a_n = \pi_h h^{n-1} y^n$ *for* $n \geq 1$*. Let* $\widetilde{D}_g$ *be the extension of* D_g *to* $\mathsf{A}_1\Sigma^{-1}$ *as in Lemma 4.11. Then*

(a) $\widetilde{D}_g(ry^n) = r \sum_{k=1}^{n} \binom{n}{k} (gh^{-1})^{(k-1)} y^{n-k}$.
(b) $\widetilde{D}_g(ra_n) = r\pi_h (gh^{-1})^{(n-1)} h^{n-1} + \sum_{k=1}^{n-1} \binom{n}{k} (gh^{-1})^{(k-1)} h^k r a_{n-k}$.
(c) *Assume* $\mathsf{char}(\mathbb{F}) = p > 0$*. Then* $D_g(\mathsf{z}_h) = \left(gh^{p-1}\right)^{(p-1)}$*, where* $\mathsf{z}_h = h^p y^p \in \mathsf{Z}(\mathsf{A}_h)$.

Proof. Part (a) is immediate from Lemma 4.16, since $\widetilde{D}_g(x) = 0$ and $\widetilde{D}_g(y) = gh^{-1}$ by (4.12). For (b), we have from part (a)

$$\begin{aligned}
\widetilde{D}_g(ra_n) &= r\pi_h h^{n-1} \sum_{k=1}^{n} \binom{n}{k} (gh^{-1})^{(k-1)} y^{n-k} \\
&= r\pi_h (gh^{-1})^{(n-1)} h^{n-1} + \sum_{k=1}^{n-1} \binom{n}{k} (gh^{-1})^{(k-1)} h^k r \pi_h h^{n-k-1} y^{n-k} \\
&= r\pi_h (gh^{-1})^{(n-1)} h^{n-1} + \sum_{k=1}^{n-1} \binom{n}{k} (gh^{-1})^{(k-1)} h^k r a_{n-k}.
\end{aligned}$$

Item (c) is a consequence of the calculation

$$D_g(\mathsf{z}_h) = h^p \sum_{k=1}^{p} \binom{p}{k} (gh^{-1})^{(k-1)} y^{p-k} = h^p (gh^{-1})^{(p-1)} = (gh^{p-1})^{(p-1)}. \quad \square$$

Lemma 4.18. *Let $g \in \mathsf{R}$ and $k \geq 0$. Then, there exist $r_1, \dots, r_{k+1} \in \mathsf{R}$ such that $\left(gh^{-1}\right)^{(k)} = \sum_{i=1}^{k+1} r_i h^{-i}$, with $r_1 = g^{(k)}$ and $r_{k+1} = (-1)^k k! g(h')^k$. In particular, for every $k \geq 0$, there exists $s_k \in \mathsf{R}$ such that*

$$\left(gh^{-1}\right)^{(k)} h^k = s_k + (-1)^k k! g(h')^k h^{-1}. \tag{4.19}$$

Proof. This follows from the identity $(gh^{-1})^{(k)} = \sum_{j=0}^{k} \binom{k}{j} g^{(k-j)} (h^{-1})^{(j)}$. □

Proposition 4.20. *Assume $g, r \in \mathsf{R}$. Then for $a_n = \pi_h h^{n-1} y^n$ the following hold:*

(a) *If $n \geq 2$, there exists $s \in \mathsf{A}_h$ so that $\widetilde{D}_g(ra_n) = s + ngra_{n-1} \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)$. Thus, $[D_g, \mathsf{ad}_{ra_n}] = \mathsf{ad}_{\widetilde{D}_g(ra_n)} \in \{\mathsf{ad}_b \mid b \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)\}$ and*

$$[D_g, \mathsf{ad}_{ra_n}] = n\mathsf{ad}_{gra_{n-1}} \mod \mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_h).$$

(b) $[D_g, \mathsf{ad}_{ra_1}] = \mathsf{ad}_{gra_0} = -D_{\delta_0(gr)}$ *where* $\delta_0(gr) = \delta(gra_0) = \left(gr\pi_h h^{-1}\right)' h.$

(c) $[D_g, \mathsf{ad}_r] = 0.$

Proof. For every $k \geq 0$, let $s_k \in \mathsf{R}$ be given by (4.19). Assume $k, n \geq 2$. Then

$$(gh^{-1})^{(n-1)} h^{n-1} r\pi_h = s_{n-1} r\pi_h + (-1)^{n-1}(n-1)! g(h')^{n-1} h^{-1} r\pi_h, \tag{4.21}$$

$$(gh^{-1})^{(k-1)} rh^k = s_{k-1} rh + (-1)^{k-1}(k-1)! g(h')^{k-1} r. \tag{4.22}$$

The expression in (4.21) is in R since h divides $\pi_h h'$. Now if (4.22) is multiplied by a_{n-k} (where $2 \leq k \leq n-1$), the right side is

$$s_{k-1} rha_{n-k} + (-1)^{k-1}(k-1)! g(h')^{k-1} r\pi_h h^{n-k-1} y^{n-k},$$

which is in A_h by (b) of Theorem 2.17. Hence, by Corollary 4.17, we have (a). Part (b) follows from Corollary 4.17 and Lemma 4.14 (a). Part (c) is clear. □

4.6. The product $[\mathsf{ad}_{ra_m}, \mathsf{ad}_{sa_n}]$ for $r, s \in \mathsf{R}$.

Here we focus on the commutators $[\mathsf{ad}_{ra_m}, \mathsf{ad}_{sa_n}]$. As before, $f^{(k)}$ denotes $\left(\frac{d}{dx}\right)^k (f)$ for any $f \in \mathsf{R}$. Our starting point is a fact about the terms $(r\pi_h h^\ell)^{(k)}$ for $r \in \mathsf{R}$.

Lemma 4.23. *Fix $\ell \geq 0$ and let $r \in \mathsf{R}$. If $k \geq 2$, then*

$$(r\pi_h h^\ell)^{(k)} \in \mathsf{R}h^{\ell+2-k} + \mathsf{R}h^{\ell+1-k} h'. \tag{4.24}$$

Proof. Consider first the case $k = 2$. Then

$$(r\pi_h h^\ell)^{(2)} = (r\pi_h)'' h^\ell + 2\ell(r\pi_h)' h^{\ell-1} h' + \ell(\ell-1) r\pi_h h^{\ell-2}(h')^2 + \ell r\pi_h h^{\ell-1} h''. \tag{4.25}$$

Since h divides $\pi_h h'$, it follows that $\ell(\ell-1) r\pi_h h^{\ell-2}(h')^2 \in \mathsf{R}h^{\ell-1}h'$. We may suppose $\pi_h h' = dh$ for $d \in \mathsf{R}$ and then take the derivative of both sides to get $\pi_h h'' = d'h + dh' - \pi_h' h'$. From that we deduce $\ell r\pi_h h^{\ell-1} h''$ belongs to $\mathsf{R}h^\ell + \mathsf{R}h^{\ell-1}h'$, which is the right-hand side of (4.24) when $k = 2$. The first two summands of (4.25) also clearly belong to the right-hand side of (4.24), so the result holds when $k = 2$.

The inductive step follows from the fact that for $r, s \in \mathsf{R}$

$$(rh^{\ell+2-k})' \in \mathsf{R}h^{\ell+2-(k+1)} \quad \text{and}$$
$$(sh^{\ell+1-k}h')' \in \mathsf{R}h^{\ell+2-(k+1)} + \mathsf{R}h^{\ell+1-(k+1)}h'. \qquad \square$$

The proof of the next lemma will use the fact that $[\mathsf{R}, \mathsf{R}] = 0$ and the relation $[y^m, f] = \sum_{k=1}^{m} \binom{m}{k} f^{(k)} y^{m-k}$ in A_1 from Lemma 2.8.

Lemma 4.26. *Let $r, s \in \mathsf{R}$, and let $m, n \geq 1$. In the Lie algebra $\mathsf{HH}^1(\mathsf{A}_h)$,*

$$[\mathsf{ad}_{ra_m}, \mathsf{ad}_{sa_n}] = \mathsf{ad}_{[ra_m, sa_n]} = \mathsf{ad}_{qa_{m+n-1}}, \quad \text{where } q = mr\delta_0(s) - ns\delta_0(r).$$

Proof. We first compute $[ra_m, sa_n]$ in $\mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)$ and then argue that certain elements are 0 in the factor Lie algebra $\mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)/\mathsf{A}_h$. For all $r, s \in \mathsf{R}$,

$$\begin{aligned}
[ra_m, sa_n] &= r\pi_h h^{m-1}[y^m, s\pi_h h^{n-1}]y^n - s\pi_h h^{n-1}[y^n, r\pi_h h^{m-1}]y^m \\
&= r\pi_h h^{m-1} \sum_{k=1}^{m} \binom{m}{k} (s\pi_h h^{n-1})^{(k)} y^{m+n-k} \\
&\quad - s\pi_h h^{n-1} \sum_{k=1}^{n} \binom{n}{k} (r\pi_h h^{m-1})^{(k)} y^{m+n-k}.
\end{aligned}$$

For $k \geq 2$, Lemma 4.23 implies that $\binom{m}{k}(s\pi_h h^{n-1})^{(k)} = uh^{n-1+2-k} + vh^{n-1+1-k}h'$ for some $u, v \in \mathsf{R}$ (which depend on k and m). Observe that

$$\begin{aligned}
r\pi_h h^{m-1} u h^{n+1-k} y^{m+n-k} &= ru\pi_h h^{m+n-k} y^{m+n-k} \in \mathsf{A}_h, \quad \text{and also} \\
r\pi_h h^{m-1} v h^{n-k} h' y^{m+n-k} &= rv\pi_h h' h^{m+n-1-k} y^{m+n-k} \in \mathsf{A}_h
\end{aligned}$$

because $\pi_h h'$ is divisible by h. Similar reasoning applies to the terms in the second summation. It follows that the terms coming from the above sums can be nonzero in $\mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)/\mathsf{A}_h$ only when $k = 1$. Thus, modulo A_h,

$$\begin{aligned}
[ra_m, sa_n] &= mr\pi_h h^{m-1}(s\pi_h h^{n-1})' y^{m+n-1} - ns\pi_h h^{n-1}(r\pi_h h^{m-1})' y^{m+n-1} \\
&= \left(mrh^{m-1}(s\pi_h h^{-1}h^n)' - nsh^{n-1}(r\pi_h h^{-1}h^m)'\right) \pi_h y^{m+n-1} \\
&= \left(mr\delta_0(s)h^{m+n-2} - ns\delta_0(r)h^{m+n-2}\right) \pi_h y^{m+n-1} \\
&= \left(mr\delta_0(s) - ns\delta_0(r)\right) a_{m+n-1},
\end{aligned}$$

where $\delta_0 : \mathsf{R} \to \mathsf{R}$ is as in (4.13). Hence, in $\mathsf{HH}^1(\mathsf{A}_h)$ we have $[\mathsf{ad}_{ra_m}, \mathsf{ad}_{sa_n}] = \mathsf{ad}_{[ra_m, sa_n]} = \mathsf{ad}_{qa_{m+n-1}}$, where $q = mr\delta_0(s) - ns\delta_0(r)$, as desired. $\square$

4.7. **Proof of Theorem 4.15.**

Take $g \in \mathsf{R}$. By Proposition 4.20, we have the following products in $\mathsf{HH}^1(\mathsf{A}_1)$: $[D_g, \mathsf{ad}_{ra_n}] = \mathsf{ad}_{\tilde{D}_g(ra_n)} = n\mathsf{ad}_{gra_{n-1}}$ if $n \geq 2$, and $[D_g, \mathsf{ad}_{ra_1}] = -D_{\delta_0(gr)} = \mathsf{ad}_{gra_0}$. By Lemma 4.14 (a) and Theorem 4.9, $[D_g, \mathsf{ad}_{ra_0}] = -[D_g, D_{\delta_0(r)}] = 0$, which shows that (a) holds for $n = 0$ as well. Since $\frac{h}{\pi_h} a_n \in \mathsf{A}_h$ for all n, the rest of part (a) follows from applying the division algorithm.

For $m, n \geq 1$, part (b) is a consequence of Lemma 4.26. Given the skew-symmetry of the formula in (b), it suffices to consider the case $m = 0$. By Lemma 4.14 (a) and Proposition 4.20, we have in $\mathsf{HH}^1(\mathsf{A}_h)$,

$$[\mathsf{ad}_{ra_0}, \mathsf{ad}_{sa_n}] = -[D_{\delta_0(r)}, \mathsf{ad}_{sa_n}] = -n\,\mathsf{ad}_{\delta_0(r)sa_{n-1}} = -\mathsf{ad}_{ns\delta_0(r)a_{n-1}},$$

which implies (b). □

4.8. **Properties of δ_0.**

We conclude this section with a few results on the map δ_0 that will be used in the next two sections. Their statements require the element ϱ_h in (2.15).

Lemma 4.27. *Assume $\mathbb{F}$ is arbitrary, and let $\delta_0 : \mathsf{R} \to \mathsf{R}$, $\delta_0(r) = \delta(ra_0)$, be as in (4.13). For all $r \in \mathsf{R}$, $\frac{h}{\pi_h \varrho_h}$ divides $\delta_0(r)$ if and only if $\frac{h}{\pi_h \varrho_h}$ divides r.*

Proof. Let $\hat{h} = \frac{h}{\varrho_h}$. Then $\pi_{\hat{h}} = \pi_h$ and $\varrho_{\hat{h}} = 1$. Let $\hat{\delta}(r) = r'\hat{h}$, and let $\hat{a}_0 = \pi_{\hat{h}}\hat{h}^{-1} = \varrho_h a_0$. Then $\frac{h}{\pi_h \varrho_h} = \frac{\hat{h}}{\pi_{\hat{h}}}$ and

$$\hat{\delta}(r\hat{a}_0) = (r\hat{a}_0)'\,\hat{h} = (ra_0)'\,\varrho_h\hat{h} = (ra_0)'\,h = \delta(ra_0).$$

Thus, it is no loss of generality to assume that $\varrho_h = 1$.

For $r \in \mathsf{R}$, $\delta\left(r\frac{h}{\pi_h}a_0\right) = \delta(r) = r'h$ is divisible by h, and therefore by $\frac{h}{\pi_h}$, and this establishes one of the implications. For the direct implication, let u be a prime divisor of h, and write $h = u^\alpha v$, where $\alpha \geq 1$ and $\mathsf{gcd}(u, v) = 1$. Since $\varrho_h = 1$, we may also assume that $\alpha < p$ when $\mathsf{char}(\mathbb{F}) = p > 0$. It follows that $\pi_h = u\pi_v$. Write $r = u^k s$, where $k \geq 0$ and $\mathsf{gcd}(u, s) = 1$. We will show that if $u^{\alpha-1}$ divides $\delta(ra_0)$, then $u^{\alpha-1}$ divides r. Since u is an arbitrary prime divisor of h, it will follow from this that $\frac{h}{\pi_h}$ divides r, provided it divides $\delta(ra_0)$.

With this notation, we have

$$\begin{aligned}\delta_0(r) = \delta(ra_0) = \left(r\pi_h h^{-1}\right)' h &= \left(u^{k+1-\alpha}s\pi_v v^{-1}\right)' u^\alpha v\\ &= (k+1-\alpha)u^k u' s\pi_v + u^{k+1}v\,\left(s\pi_v v^{-1}\right)'.\end{aligned}$$

Assume $u^{\alpha-1}$ divides $\delta_0(r)$. It is enough to argue that $k \geq \alpha - 1$. Supposing the contrary, we have $k < \alpha - 1$, so $k + 1 \leq \alpha - 1$, which implies that u^{k+1} divides $\delta_0(r)$. Now $v\left(s\pi_v v^{-1}\right)' \in \mathsf{R}$, so u divides $(k+1-\alpha)u's\pi_v$. Note that $u' \neq 0$, because we are assuming $\varrho_h = 1$. As u', s, and v are coprime to u, this implies $k = \alpha - 1$ when $\mathsf{char}(\mathbb{F}) = 0$, which is a contradiction. When $\mathsf{char}(\mathbb{F}) = p > 0$, then $k \equiv \alpha - 1 \bmod p$, but since $1 \leq \alpha < p$, we again have the contradiction $k = \alpha - 1$. Thus, indeed $k \geq \alpha - 1$. □

Lemma 4.28. *Assume $\mathbb{F}$ is arbitrary. Then the following hold.*

(a) $\mathsf{ker}\delta_0 = (\mathsf{R} \cap \mathsf{Z}(\mathsf{A}_h))\,\frac{h}{\pi_h \varrho_h}$.

(b) $\mathsf{dim}\left\{\delta_0(r) \mid r \in \mathsf{R},\ \mathsf{deg}\,r < \mathsf{deg}\,\frac{h}{\pi_h \varrho_h}\right\} = \mathsf{deg}\,\frac{h}{\pi_h \varrho_h}$.

(c) *When* $\mathsf{char}(\mathbb{F}) = 0$, *then* $\mathsf{ker}\delta_0 = \mathbb{F}\frac{h}{\pi_h}$ *and*
$\mathsf{dim}\left\{\delta_0(r) \mid r \in \mathsf{R},\ \mathsf{deg}\,r < \mathsf{deg}\,\frac{h}{\pi_h}\right\} = \mathsf{deg}\,\frac{h}{\pi_h}$.

(d) *For* $s \in \mathsf{R}$, $\left(\frac{s}{h}\right)' = 0$ *if and only if* $s \in (\mathsf{R} \cap \mathsf{Z}(\mathsf{A}_h))\, \frac{h}{\varrho_h}$.

Proof. (a) Let $c \in \mathsf{R} \cap \mathsf{Z}(\mathsf{A}_h)$ and note that

$$\delta_0\left(c\frac{h}{\pi_h\varrho_h}\right) = \left(c\frac{h}{\pi_h\varrho_h}\pi_h h^{-1}\right)' h = (c\varrho_h^{-1})'h = 0.$$

Therefore, $(\mathsf{R} \cap \mathsf{Z}(\mathsf{A}_h))\, \frac{h}{\pi_h\varrho_h} \subseteq \ker\delta_0$.

For the other containment, suppose that $\delta_0(r) = 0$. Then Lemma 4.27 implies that we may write $r = \tilde{r}\frac{h}{\pi_h\varrho_h}$ for $\tilde{r} \in \mathsf{R}$. Then applying Lemma 4.14 (b) we have

$$0 = \delta_0\left(\tilde{r}\frac{h}{\pi_h\varrho_h}\right) = \tilde{r}\delta_0\left(\frac{h}{\pi_h\varrho_h}\right) + \tilde{r}'\frac{h}{\pi_h\varrho_h}\pi_h = \tilde{r}'\frac{h}{\pi_h\varrho_h}\pi_h,$$

which forces $\tilde{r}' = 0$, and thus $r = \tilde{r}\frac{h}{\pi_h\varrho_h} \in (\mathsf{R} \cap \mathsf{Z}(\mathsf{A}_h))\, \frac{h}{\pi_h\varrho_h}$.

For (b), every $r \in \ker\delta_0 = (\mathsf{R} \cap \mathsf{Z}(\mathsf{A}_h))\frac{h}{\pi_h\varrho_h}$ is divisible by $\frac{h}{\pi_h\varrho_h}$, so r must be 0 or have degree greater than or equal to the degree of $\frac{h}{\pi_h\varrho_h}$. Thus, the linear map

$$\left\{r \in \mathsf{R} \;\Big|\; \deg r < \deg \tfrac{h}{\pi_h\varrho_h}\right\} \longrightarrow \left\{\delta_0(r) \;\Big|\; \deg r < \deg \tfrac{h}{\pi_h\varrho_h}\right\} \tag{4.29}$$

is an isomorphism. Part (c) is immediate from (b) and the fact that $\mathsf{Z}(\mathsf{A}_h) = \mathbb{F}1$ and $\varrho_h = 1$ when $\mathsf{char}(\mathbb{F}) = 0$.

For (d), it is clear that $\left(\frac{s}{h}\right)' = 0$ if $s \in (\mathsf{R} \cap \mathsf{Z}(\mathsf{A}_h))\, \frac{h}{\varrho_h}$. For the other direction, suppose that $\left(\frac{s}{h}\right)' = 0$. Then $s'h = sh'$, so h divides sh' and it follows that π_h divides s. Moreover,

$$\delta_0\left(\frac{s}{\pi_h}\right) = h\left(\frac{s}{h}\right)' = 0,$$

and this implies that $\frac{s}{\pi_h} \in \ker\delta_0 = (\mathsf{R} \cap \mathsf{Z}(\mathsf{A}_h))\, \frac{h}{\pi_h\varrho_h}$, thus establishing the claim that $s \in (\mathsf{R} \cap \mathsf{Z}(\mathsf{A}_h))\, \frac{h}{\varrho_h}$. □

5. $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$ WHEN $\mathsf{char}(\mathbb{F}) = 0$

The one-variable *Witt algebra* (also known as the centerless Virasoro algebra) is the derivation algebra $\mathsf{W} = \mathsf{Der}_{\mathbb{F}}(\mathbb{F}[t]) = \mathsf{span}_{\mathbb{F}}\{w_n = t^{n+1}\frac{d}{dt} \mid n \geq -1\}$, where $[w_m, w_n] = (n-m)w_{m+n}$ for $m, n \geq -1$, $(w_{-2} = 0)$. When $\mathbb{F}$ is the complex field, W is the Lie algebra of vector fields on the unit circle, so it has played an important role in many areas of mathematics and physics. Our aim in this section is to show the following result about $\mathsf{HH}^1(\mathsf{A}_h) = \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)/\mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_h)$ for fields of characteristic 0, which we prove in Section 5.5.

Theorem 5.1. *Let* $\mathsf{char}(\mathbb{F}) = 0$*, and assume* $h \neq 0$ *and* $a_n = \pi_h h^{n-1}$ *for all* $n \geq 0$*. Then* $\mathsf{HH}^1(\mathsf{A}_h) = \mathsf{Z}(\mathsf{HH}^1(\mathsf{A}_h)) \oplus [\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)]$*;*

$$\mathcal{N} = \mathsf{span}_{\mathbb{F}}\{\mathsf{ad}_{ra_n} \mid r \in \mathsf{R}\pi_{(h/\pi_h)},\ n \geq 0\} \tag{5.2}$$

is the unique maximal nilpotent ideal of $[\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)]$*; and*

$$\mathsf{HH}^1(\mathsf{A}_h)/\mathcal{N} = \mathsf{Z}(\mathsf{HH}^1(\mathsf{A}_h)) \oplus [\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)]/\mathcal{N}, \quad \text{where}$$

(i) $\mathsf{Z}(\mathsf{HH}^1(\mathsf{A}_h)) \cong \{D_{r\frac{h}{\pi_h}} \mid \deg r < \deg \pi_h\}$.

(ii) $[\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)]/\mathcal{N} \cong \big((\mathsf{R}/\mathsf{R}\pi_{(h/\pi_h)}) \otimes \mathsf{W}\big)$*, and* $\mathsf{W} = \mathsf{span}_{\mathbb{F}}\{w_i \mid i \geq -1\}$ *is the Witt algebra.*

(iii) $\big(\mathsf{R}/\mathsf{R}\pi_{(h/\pi_h)}\big) \otimes \mathsf{W} \cong \big((\mathsf{R}/\mathsf{R}\mathsf{u}_1) \otimes \mathsf{W}\big) \oplus \cdots \oplus \big((\mathsf{R}/\mathsf{R}\mathsf{u}_k) \otimes \mathsf{W}\big)$*, a direct sum of simple Lie algebras, where* $\mathsf{u}_1, \ldots, \mathsf{u}_k$ *are the monic prime factors of* h *with multiplicity* > 1*, and each summand is a field extension of* W.

We start by describing the decomposition $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h) = \mathcal{D}_{\mathsf{R}} + \mathcal{E}$ in Theorem 4.9 more explicitly and prove Theorem 5.1 in a series of results. We conclude the section by interpreting Theorem 5.1 in some cases of special interest.

Theorem 5.3. *Assume* $\mathsf{char}(\mathbb{F}) = 0$*, and regard* $\mathsf{A}_h \subseteq \mathsf{A}_1$*. Then* $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h) = \mathcal{D} \oplus \mathcal{E}$ *where* $\mathcal{D} = \{D_g \mid g \in \mathsf{R},\ \deg g < \deg h\}$ *and* $\mathcal{E} = \{\mathsf{ad}_a \mid a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)\}$.

Proof. We know from Theorem 4.9 that $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h) = \mathcal{D}_{\mathsf{R}} + \mathcal{E}$, where $\mathcal{D}_{\mathsf{R}} = \{D_g \mid g \in \mathsf{R}\}$ and $\mathcal{E} = \{F \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_1) \mid F(\mathsf{A}_h) \subseteq \mathsf{A}_h\}$. Since every derivation of A_1 is inner (see Proposition 3.1), $\mathcal{E} = \{\mathsf{ad}_a \mid a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)\}$. Assume $D_f \in \mathcal{D}_{\mathsf{R}}$ and write $f = qh + g$, where $\deg g < \deg h$. When $\mathsf{char}(\mathbb{F}) = 0$, there exists $r \in \mathsf{R}$ so that $r' = -q$. Then $(D_f - \mathsf{ad}_r)(x) = 0$, and $(D_f - \mathsf{ad}_r)(\hat{y}) = f + [\hat{y}, r] = f + r'h = f - qh = g$. Therefore $D_f - \mathsf{ad}_r = D_g$ and $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h) = \mathcal{D} + \mathcal{E}$, where $\mathcal{E} = \{\mathsf{ad}_a \mid a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)\}$ and $\mathcal{D} = \{D_g \mid g \in \mathsf{R},\ \deg g < \deg h\}$.

Suppose now that $D \in \mathcal{D} \cap \mathcal{E}$. Then $D(\mathsf{R}) = 0$ and $D(\hat{y}) = g$ for some $g \in \mathsf{R}$ with $\deg g < \deg h$ since $D \in \mathcal{D}$. But then $D(y)h = D(\hat{y}) = g \in \mathsf{R} \subset \mathsf{A}_h$. This implies $D(y) \in \mathsf{R}$, and since $\deg g < \deg h$, it must be that $g = 0$, and hence $D = 0$. □

Example 5.4. *When* $\mathsf{char}(\mathbb{F}) = 0$ *and there are no repeated prime factors in* h*, we have* $\frac{h}{\pi_h} \in \mathbb{F}^*$*. In this situation,* $\mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h) = \mathsf{A}_h$ *(compare Remark 2.20). Then* $\mathcal{E} = \mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_h)$*, and* $\mathsf{HH}^1(\mathsf{A}_h) \cong \mathcal{D} = \{D_g \mid g \in \mathsf{R}, \deg g < \deg h\}$ *is an abelian Lie algebra of dimension* $\deg h$.

In light of this result, it is tempting to think that the subalgebra $\mathcal{E}$ might be an ideal of $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$. However, that is not true in general as the next example illustrates.

Example 5.5. *Let* $\mathsf{char}(\mathbb{F}) = 0$ *and* $h = x^m$ *for* $m \geq 2$*. Then* $\pi_h = x$*, and according to Proposition 4.20(b),* $[D_1, \mathsf{ad}_{a_1}] = \mathsf{ad}_{a_0} = -D_{\delta(a_0)}$*, where* $\delta(a_0) = \big(\pi_h h^{-1}\big)' h = 1 - m$*. Thus,* $[D_1, \mathsf{ad}_{a_1}] = (m-1)D_1 \notin \mathcal{E}$.

Lemma 5.6. *Let* $\mathsf{char}(\mathbb{F}) = 0$ *and* $h \neq 0$ *be arbitrary. Assume* $g \in \mathsf{R}$ *with* $\deg g < \deg h$*, and* $r_n \in \mathsf{R}$ *with* $\deg r_n < \deg \frac{h}{\pi_h}$ *for all* $n \geq 0$.

(i) *If* $D_g + \sum_{n\geq 1} \mathsf{ad}_{r_n a_n} \in \mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_h)$*, then* $g = 0 = r_n$ *for all* $n \geq 1$.

(ii) *If* $\sum_{n\geq 0} \mathsf{ad}_{r_n a_n} \in \mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_h)$*, then* $r_n = 0$ *for all* $n \geq 0$.

Proof. (i) Write $D_g + \sum_{n\geq 1} \mathsf{ad}_{r_n a_n} = \mathsf{ad}_a$ for some $a \in \mathsf{A}_h$. Then

$$D_g = \mathsf{ad}_a - \sum_{n\geq 1} \mathsf{ad}_{r_n a_n} \in \mathcal{D} \cap \mathcal{E} = 0,$$

by Theorem 5.3. It follows that $g = 0$ and $\mathsf{ad}_b = 0$, where $b = a - \sum_{n\geq 1} r_n a_n$. Thus, $b \in \mathsf{A}_1$ centralizes A_h. By Lemma 2.5, $b \in \mathsf{R} \subset \mathsf{A}_h$, so in fact $b \in \mathbb{F}$, as it commutes with $\hat{y}$. In particular, we have $\sum_{n\geq 1} r_n a_n \in \mathsf{A}_h$. Since $a_n = \pi_h h^{n-1} y^n$, we conclude from part (c) of Lemma 2.1 that h divides $r_n \pi_h$ for all $n \geq 1$; that is, $r_n \in \mathsf{R}\frac{h}{\pi_h}$ for all $n \geq 1$. But since $\deg r_n < \deg \frac{h}{\pi_h}$, it must be that $r_n = 0$ for all $n \geq 1$.

(ii) Assume $\sum_{n\geq 0} \mathsf{ad}_{r_n a_n} \in \mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_h)$. By Proposition 4.14 (a), $\mathsf{ad}_{r_0 a_0} = -D_{\delta_0(r_0)}$. As $\deg r_0 < \deg \frac{h}{\pi_h}$, we have that $\deg \delta_0(r_0) < \deg h$. Therefore, by (i) we know that $r_n = 0$ for all $n \geq 1$, and $\delta_0(r_0) = 0$. This implies $r_0 \in \ker\delta_0 = (\mathsf{R} \cap \mathsf{Z}(\mathsf{A}_h))\frac{h}{\pi_h} = \mathbb{F}\frac{h}{\pi_h}$ by Lemma 4.28. But then $\deg r_0 < \deg \frac{h}{\pi_h}$ forces $r_0 = 0$ to hold. □

5.1. **The structure of $\mathcal{E}$.**

Recall from Theorem 5.3 that $\mathcal{E} = \{\mathsf{ad}_a \mid a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)\}$ when $\mathsf{char}(\mathbb{F}) = 0$. The next theorem, a key result in our paper, clarifies the relationship between $\mathcal{E}$ and $\mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_h)$ and provides more detailed information about $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$ and $\mathsf{HH}^1(\mathsf{A}_h) = \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)/\mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_h)$.

Theorem 5.7. *Assume* $\mathsf{char}(\mathbb{F}) = 0$. *Then as vector spaces over* $\mathbb{F}$,

(i) $\mathcal{E} = \mathsf{span}_{\mathbb{F}}\{\mathsf{ad}_{ra_n} \mid r \in \mathsf{R}, \deg r < \deg \frac{h}{\pi_h},\ n \geq 1\} \oplus \mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_h)$.
(ii) $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h) = \mathcal{D} \oplus \mathsf{span}_{\mathbb{F}}\{\mathsf{ad}_{ra_n} \mid r \in \mathsf{R}, \deg r < \deg \frac{h}{\pi_h},\ n \geq 1\} \oplus \mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_h)$, *where* $\mathcal{D} = \{D_g \mid g \in \mathsf{R},\ \deg g < \deg h\}$.
(iii) $\mathsf{HH}^1(\mathsf{A}_h) \cong \mathcal{D} \oplus \mathsf{span}_{\mathbb{F}}\{\mathsf{ad}_{ra_n} \mid r \in \mathsf{R}, \deg r < \deg \frac{h}{\pi_h},\ n \geq 1\}$.

Remark 5.8. *In the statement of Theorem 5.7 (iii) and in what follows, we identify the derivations* D_g $(\deg g < \deg h)$ *and the derivations* ad_{ra_n} $(\deg r < \deg \frac{h}{\pi_h},\ n \geq 1)$ *with their image in* $\mathsf{HH}^1(\mathsf{A}_h) = \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)/\mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_h)$ *and use the same notation for both.*

Proof of Theorem 5.7. Clearly $\mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_h) \subseteq \mathcal{E} = \{\mathsf{ad}_a \mid a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)\}$. Moreover, the sum $\mathsf{span}_{\mathbb{F}}\{\mathsf{ad}_{ra_n} \mid r \in \mathsf{R}, \deg r < \deg \frac{h}{\pi_h},\ n \geq 1\} + \mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_h)$ is direct by Lemma 5.6 (ii).

To show $\mathcal{E}$ equals this direct sum, assume $b \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)$. By Theorem 2.17(a)(i), we may suppose $b = r_0 + \sum_{n\geq 1} r_n a_n$, where $r_n \in \mathsf{R}$ for all n. For $n \geq 1$, write $r_n = q_n \frac{h}{\pi_h} + \tilde{r}_n$, with $q_n, \tilde{r}_n \in \mathsf{R}$ and $\deg \tilde{r}_n < \deg \frac{h}{\pi_h}$. Then,

$$b = r_0 + \sum_{n\geq 1} q_n \frac{h}{\pi_h} a_n + \sum_{n\geq 1} \tilde{r}_n a_n.$$

Since $\frac{h}{\pi_h} a_n = h^n y^n \in \mathsf{A}_h$ for all $n \geq 1$, we have $a = r_0 + \sum_{n\geq 1} q_n \frac{h}{\pi_h} a_n \in \mathsf{A}_h$. Thus, $\mathsf{ad}_b = \sum_{n\geq 1} \mathsf{ad}_{\tilde{r}_n a_n} + \mathsf{ad}_a$ is an element of $\mathsf{span}_{\mathbb{F}}\{\mathsf{ad}_{ra_n} \mid r \in \mathsf{R}, \deg r < \deg \frac{h}{\pi_h},\ n \geq 1\} \oplus \mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_h)$. Combining that with Theorem 5.3 gives (ii), and hence (iii). □

5.2. **The commutator ideal** $[\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)]$.

Proposition 5.9. *Assume* $\mathsf{char}(\mathbb{F}) = 0$. *Then*

$$[\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)] = \mathsf{span}_{\mathbb{F}}\{\mathsf{ad}_{ra_n} \mid r \in \mathsf{R},\ \deg r < \deg \tfrac{h}{\pi_h},\ n \geq 0\}. \tag{5.10}$$

Moreover, $\mathsf{HH}^1(\mathsf{A}_h)/[\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)]$ *is an abelian Lie algebra of dimension* $\deg \pi_h$.

Proof. Assume $r \in \mathsf{R}$, $\deg r < \deg \frac{h}{\pi_h}$, and $n \geq 0$. Then by Lemma 4.15 (a),

$$\mathsf{ad}_{ra_n} = \tfrac{1}{n+1}[D_1, \mathsf{ad}_{ra_{n+1}}]$$

in $\mathsf{HH}^1(\mathsf{A}_h)$, which proves the right side of (5.10) is contained in the left. The reverse containment follows from Theorem 5.7 (iii), Lemma 4.15, and the fact that $\mathcal{D}$ is abelian (Theorem 4.9).

Consider the linear map

$$\rho : \{g \in \mathsf{R} \mid \deg g < \deg h\} \longrightarrow \mathsf{HH}^1(\mathsf{A}_h)/[\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)], \tag{5.11}$$

with $\rho(g) = D_g + [\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)]$. By Theorem 5.7 (iii) and (5.10), ρ is surjective.

Now suppose $g \in \mathsf{R}$ with $\deg g < \deg h$, and $\rho(g) = 0$. Then there exist $r_n \in \mathsf{R}$ with $\deg r_n < \deg \frac{h}{\pi_h}$, so that $D_g = \sum_{n\geq 0} \mathsf{ad}_{r_n a_n} = \mathsf{ad}_{r_0 a_0} + \sum_{n\geq 1} \mathsf{ad}_{r_n a_n}$. Hence, by Lemma 4.14 (a), $D_{g+\delta_0(r_0)} - \sum_{n\geq 1} \mathsf{ad}_{r_n a_n} = 0$. Thus, $g = -\delta_0(r_0)$ by Lemma 5.6 (i). Conversely, if $g = -\delta_0(r_0)$ for some $r_0 \in \mathsf{R}$ with $\deg r_0 < \deg \frac{h}{\pi_h}$, then $\rho(g) = 0$. Therefore,

$$\ker\rho = \left\{\delta_0(q) \mid \deg q < \deg \tfrac{h}{\pi_h}\right\}, \tag{5.12}$$

and $\dim \ker\rho = \deg \frac{h}{\pi_h}$, by Lemma 4.28 (c). Consequently,

$$\dim \left(\mathsf{HH}^1(\mathsf{A}_h)/[\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)]\right) = \deg h - \deg \tfrac{h}{\pi_h} = \deg \pi_h. \qquad \square$$

5.3. **The center of** $\mathsf{HH}^1(\mathsf{A}_h)$.

Theorem 5.13. *Assume* $\mathsf{char}(\mathbb{F}) = 0$. *Then*

$$\mathsf{HH}^1(\mathsf{A}_h) = \mathsf{Z}(\mathsf{HH}^1(\mathsf{A}_h)) \oplus [\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)], \quad \text{where} \tag{5.14}$$

(5.15)
$$\mathsf{Z}(\mathsf{HH}^1(\mathsf{A}_h)) = \left\{ D_{r\frac{h}{\pi_h}} \;\middle|\; \deg r < \deg \pi_h \right\} \text{and } \dim \mathsf{Z}\left(\mathsf{HH}^1(\mathsf{A}_h)\right) = \deg \pi_h.$$

Proof. Let $z \in \mathsf{Z}(\mathsf{HH}^1(\mathsf{A}_h))$. By Theorem 5.7 (iii), we may write $z = D_g + \sum_{n=1}^{\ell} \mathsf{ad}_{r_n a_n}$, with $g, r_n \in \mathsf{R}$, $\deg g < \deg h$ and $\deg r_n < \deg \frac{h}{\pi_h}$ for all n. Then by Lemma 4.15 (a), $0 = [D_1, z] = \sum_{n=1}^{\ell} n\, \mathsf{ad}_{r_n a_{n-1}}$. By Lemma 5.6 (ii), $r_n = 0$ for all $1 \leq n \leq \ell$ and $z = D_g$. But then $0 = [D_g, \mathsf{ad}_{a_1}] = \mathsf{ad}_{g a_0}$, so $\frac{h}{\pi_h}$ divides g. This proves one direction of the inclusion in (5.15).

Conversely, for all $g, r, s \in \mathsf{R}$ and $n \geq 1$, we have in $\mathsf{HH}^1(\mathsf{A}_h)$,

$$\left[D_{r\frac{h}{\pi_h}}, \mathsf{ad}_{sa_n}\right] = n\mathsf{ad}_{\frac{h}{\pi_h}rsa_{n-1}} = 0 = \left[D_{r\frac{h}{\pi_h}}, D_g\right],$$

showing that $D_{r\frac{h}{\pi_h}} \in \mathsf{Z}(\mathsf{HH}^1(\mathsf{A}_h))$ and implying that (5.15) holds.

To verify the sum in (5.14) is direct, suppose

$$z \in \mathsf{Z}(\mathsf{HH}^1(\mathsf{A}_h)) \cap [\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)].$$

By (5.15), there is a $g \in \mathsf{R}\frac{h}{\pi_h}$ with $\deg g < \deg h$ such that $z = D_g$. But then $g \in \ker\rho$, where ρ is as in (5.11), and hence $g = \delta_0(q)$ for some q with $\deg q < \deg\frac{h}{\pi_h}$ by (5.12). Hence, $\frac{h}{\pi_h}$ divides $\delta_0(q)$. But when $\mathsf{char}(\mathbb{F}) = 0$, Lemma 4.27 implies that $\frac{h}{\pi_h}$ divides q. Since $\deg q < \deg \frac{h}{\pi_h}$, it follows that $q = 0$, so that $z = 0$.

We know now that the map

$$\iota : \mathsf{Z}(\mathsf{HH}^1(\mathsf{A}_h)) \to \mathsf{HH}^1(\mathsf{A}_h)/[\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)],$$

given by restriction of the canonical epimorphism is injective. By Proposition 5.9 and (5.15), both algebras have dimension $\deg\pi_h$, so ι is in fact an isomorphism. In particular,

$$\mathsf{HH}^1(\mathsf{A}_h) = \mathsf{Z}(\mathsf{HH}^1(\mathsf{A}_h)) + [\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)],$$

which finishes the proof. □

5.4. The structure of $[\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)]$.

Let $\mathsf{char}(\mathbb{F}) = 0$, and assume as before $h = \lambda \mathsf{u}_1^{\alpha_1}\cdots\mathsf{u}_t^{\alpha_t}$, $\pi_h = \mathsf{u}_1\cdots\mathsf{u}_t$, where the u_i are the distinct monic prime factors of h and $\lambda \in \mathbb{F}^*$. Let

$$\varsigma = \delta_0(1) = \pi_h' - \frac{\pi_h h'}{h} = \sum_{i=1}^{t}(1-\alpha_i)\mathsf{u}_1\cdots\widehat{\mathsf{u}}_i\cdots\mathsf{u}_t\mathsf{u}_i'. \tag{5.16}$$

Observe that $\frac{h}{\pi_h} = \lambda\prod_{i,\ \alpha_i\geq 2}\mathsf{u}_i^{\alpha_i-1}$, so that $\pi_{(h/\pi_h)} = \prod_{i,\ \alpha_i\geq 2}\mathsf{u}_i$ is the product of the distinct prime factors of h having multiplicity > 1, and $\mathsf{gcd}(\varsigma, \pi_{(h/\pi_h)}) = 1$.

Recall from Proposition 5.9 that

$$[\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)] = \mathsf{span}_{\mathbb{F}}\{\mathsf{ad}_{ra_n} \mid r \in \mathsf{R},\ \deg r < \deg\tfrac{h}{\pi_h},\ n \geq 0\},$$

where $a_n = \pi_h h^{n-1}y^n$ for all $n \geq 0$, and $a_n \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)$ for all $n \geq 1$. For $m, n \geq 0$ and $r, s \in \mathsf{R}$, by Lemma 4.15 (b) we have $[\mathsf{ad}_{ra_m}, \mathsf{ad}_{sa_n}] = \mathsf{ad}_{qa_{m+n-1}} = \mathsf{ad}_{da_{m+n-1}}$ in $\mathsf{HH}^1(\mathsf{A}_h)$, where $q = mr\delta_0(s) - ns\delta_0(r)$ and d is the remainder when q is divided by $\frac{h}{\pi_h}$ in R.

Using (5.14) and the fact that $\delta_0(r) = r\delta_0(1) + r'\pi_h$ and π_h is divisible by $\pi_{(h/\pi_h)}$, we have that

$$\mathcal{N} = \mathsf{span}_{\mathbb{F}}\{\mathsf{ad}_{ra_n} \mid r \in \mathsf{R}\pi_{(h/\pi_h)},\ n \geq 0\}$$

is an ideal of $\mathsf{HH}^1(\mathsf{A}_h)$ contained in $[\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)]$. Our immediate goal is to demonstrate several important properties of the ideal $\mathcal{N}$ and to understand the Lie algebra

$$\mathcal{L} = [\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)]/\mathcal{N}.$$

For $g \in \mathsf{R}$ and $m \geq -1$, set

$$e_{g,m} = -\mathsf{ad}_{ga_{m+1}} + \mathcal{N}. \tag{5.17}$$

Then for $r \in \mathsf{R}$, we have

$$g = r \mod \mathsf{R}\pi_{(h/\pi_h)} \implies e_{g,m} = e_{r,m}. \tag{5.18}$$

Theorem 4.15 (b) shows that the elements ad_{ra_m} have a multiplication very similar to that of $\mathsf{R} \otimes \mathsf{W}$, where W is the Witt algebra. This motivates the next result.

Lemma 5.19. *Assume* $\mathsf{char}(\mathbb{F}) = 0$, *and let* $\mathsf{W} = \mathsf{span}_{\mathbb{F}}\{w_n \mid n \geq -1\}$ *be the Witt algebra so that* $[w_m, w_n] = (n-m)w_{m+n}$ *for* $m, n \geq -1$ $(w_{-2} = 0)$. *Then* $\mathcal{L} = [\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)]/\mathcal{N} \cong \big(\mathsf{R}/\mathsf{R}\pi_{(h/\pi_h)}\big) \otimes \mathsf{W}$, *and* $\mathcal{L}$ *is simple if* $\pi_{(h/\pi_h)}$ *is a prime polynomial.*

Proof. In proving this lemma, we will use r to denote both an element of R and the coset it determines in $\mathsf{R}/\mathsf{R}\pi_{(h/\pi_h)}$, which is permissible to do by (5.18).

The elements $e_{x^j,m}$, with $0 \leq j < \deg \pi_{(h/\pi_h)}$ and $m \geq -1$, generate $\mathcal{L}$ by (5.18). To show they form a basis of $\mathcal{L}$, suppose $\sum_{j,m} \gamma_{j,m} e_{x^j,m} = 0$, for scalars $\gamma_{j,m}$, $0 \leq j < \deg \pi_{(h/\pi_h)}$ and $m \geq -1$. Let $r_m = \sum_j \gamma_{j,m} x^j$. Thus, $\sum_{m \geq -1} \mathsf{ad}_{r_m a_{m+1}} \in \mathcal{N}$, which by Lemma 5.6 (ii) implies that $r_m \in \mathsf{R}\pi_{(h/\pi_h)}$ for all $m \geq -1$, since by construction, $\deg r_m < \deg \pi_{(h/\pi_h)} \leq \deg \frac{h}{\pi_h}$. Hence, it must be that $r_m = 0$ and $\gamma_{j,m} = 0$, for all $0 \leq j < \deg \pi_{(h/\pi_h)}$ and $m \geq -1$.

Assume $\upsilon \in \mathsf{R}$ satisfies $\upsilon\varsigma = 1 \mod \mathsf{R}\pi_{(h/\pi_h)}$, and consider the linear map

$$\big(\mathsf{R}/\mathsf{R}\pi_{(h/\pi_h)}\big) \otimes \mathsf{W} \to \mathcal{L}, \quad r \otimes w_m \mapsto e_{r\upsilon,m}.$$

Now

$$[r \otimes w_m, s \otimes w_n] = (n-m)(rs \otimes w_{m+n}) \mapsto (n-m)e_{rs\upsilon,m+n}.$$

However, in $\mathcal{L}$ we have by Lemma 4.15 (b) (as $\pi_{(h/\pi_h)}$ divides π_h) that

$$\begin{aligned} [e_{r\upsilon,m}, e_{s\upsilon,n}] &= (m-n)\mathsf{ad}_{rs\upsilon^2\varsigma a_{m+n+1}} + \mathcal{N} \\ &= (m-n)\mathsf{ad}_{rs\upsilon a_{m+n+1}} + \mathcal{N} = (n-m)e_{rs\upsilon,m+n}. \end{aligned}$$

Thus, this map is a Lie homomorphism with inverse map given by $e_{r,m} \mapsto r\varsigma \otimes w_m$ for $r \in \mathsf{R}$, $\deg r < \deg \pi_{(h/\pi_h)}$, so that $\mathcal{L} \cong \big(\mathsf{R}/\mathsf{R}\pi_{(h/\pi_h)}\big) \otimes \mathsf{W}$.

Suppose now that $\pi_{(h/\pi_h)}$ is a prime polynomial. We argue that $\mathbb{K} \otimes \mathsf{W}$ is simple, where $\mathbb{K}$ denotes the field $\mathsf{R}/\mathsf{R}\pi_{(h/\pi_h)}$. Let Ω denote a nonzero ideal of $\mathbb{K} \otimes \mathsf{W}$, and let $0 \neq \omega = \sum_{n=-1}^{\ell} \xi_n \otimes w_n \in \Omega$, where ω is chosen so that $\ell \geq -1$ is minimal. Then

$$0 \neq [1 \otimes w_{-1}, \omega] = \sum_{n=0}^{\ell} [1 \otimes w_{-1}, \xi_n \otimes w_n] = \sum_{n=0}^{\ell} (n+1)\xi_n \otimes w_{n-1} \in \Omega.$$

This contradicts the minimality of ℓ, unless $\ell = -1$. Hence, we may suppose $0 \neq \xi \otimes w_{-1} \in \Omega$ for some $0 \neq \xi \in \mathbb{K}$. From this it follows that Ω contains

$$[\xi \otimes w_{-1}, \kappa \otimes w_{m+1}] = (m+2)\xi\kappa \otimes w_m$$

for every $\kappa \in \mathbb{K}$ and $m \geq -1$, and consequently $\mathbb{K} \otimes \mathsf{W} \subseteq \Omega$. □

Assume there are $k \geq 0$ distinct monic prime factors of h with multiplicity > 1. If $k = 0$, then $\frac{h}{\pi_h} \in \mathbb{F}^*$ and $\pi_{(h/\pi_h)} = 1$. In this case, $\mathsf{R}/\mathsf{R}\pi_{(h/\pi_h)} = 0$ and $\mathcal{L} = [\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)]/\mathcal{N} = 0$. If $k \geq 1$, then after possibly renumbering the factors, we may suppose that $\mathsf{u}_1, \dots \mathsf{u}_k$ are the distinct monic primes occurring with multiplicity > 1 in h. In other words, $\pi_{(h/\pi_h)} = \mathsf{u}_1 \cdots \mathsf{u}_k$. Then

$$\mathsf{R}/\mathsf{R}\pi_{(h/\pi_h)} = \mathsf{R}/\mathsf{R}\mathsf{u}_1 \cdots \mathsf{u}_k \cong \mathsf{R}/\mathsf{R}\mathsf{u}_1 \oplus \cdots \oplus \mathsf{R}/\mathsf{R}\mathsf{u}_k, \tag{5.20}$$

so it follows that

$$\left(\mathsf{R}/\mathsf{R}\pi_{(h/\pi_h)}\right) \otimes \mathsf{W} \cong ((\mathsf{R}/\mathsf{R}\mathsf{u}_1) \otimes \mathsf{W}) \oplus \cdots \oplus ((\mathsf{R}/\mathsf{R}\mathsf{u}_k) \otimes \mathsf{W}). \tag{5.21}$$

By Lemma 5.19, each of the summands $(\mathsf{R}/\mathsf{R}\mathsf{u}_i) \otimes \mathsf{W}$ corresponds to a simple ideal of $\mathcal{L}$, so $\mathcal{L}$ is semisimple in this case.

Corollary 5.22. *Assume* $\mathsf{char}(\mathbb{F}) = 0$ *and* $h = \lambda \mathsf{u}_1^{\alpha_1} \cdots \mathsf{u}_t^{\alpha_t}$, *where* $\lambda \in \mathbb{F}^*$, *the* u_i *are the distinct monic prime factors of* h, *and for* $k \geq 0$, $\mathsf{u}_1, \dots, \mathsf{u}_k$ *are the ones which occur with multiplicity* > 1. *(When* $k = 0$, *no factor has multiplicity* > 1.*) Let* $\mathcal{N} = \mathsf{span}_{\mathbb{F}}\{\mathsf{ad}_{ra_n} \mid r \in \mathsf{R}\pi_{(h/\pi_h)},\ n \geq 0\} \subseteq [\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)]$. *Then the following hold:*

(i) $\mathcal{N}$ *is the unique maximal nilpotent ideal of* $[\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)]$ *and the quotient* $[\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)]/\mathcal{N}$ *is the direct sum of* k *simple Lie algebras*

$$[\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)]/\mathcal{N} \cong ((\mathsf{R}/\mathsf{R}\mathsf{u}_1) \otimes \mathsf{W}) \oplus \cdots \oplus ((\mathsf{R}/\mathsf{R}\mathsf{u}_k) \otimes \mathsf{W}), \tag{5.23}$$

where W *is the Witt algebra.*

(ii) *If* $\alpha_i \leq 2$ *for all* $1 \leq i \leq t$, *then* $\mathcal{N} = 0$.
   (a) *If* $\alpha_i = 1$ *for all* i, *then* $[\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)] = 0$.
   (b) *If some* $\alpha_i = 2$, *then* $[\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)]$ *is the direct sum of simple Lie algebras (compare* (5.23)*).*

(iii) *If there is* i *such that* $\alpha_i \geq 3$, *then* $\mathcal{N} \neq 0$, *and* $[\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)]$ *is neither nilpotent nor semisimple.*

Proof. By Lemma 5.19 and the above, $\mathcal{L} = [\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)]/\mathcal{N}$ is a direct sum of $k \geq 0$ simple Lie algebras of the form $\left(\mathsf{R}/\mathsf{R}\mathsf{u}_i\right) \otimes \mathsf{W}$, where $i \leq k$ and W is the Witt algebra.

To show that $\mathcal{N}$ is nilpotent, let $\mathcal{N}_j \subseteq \mathcal{N}$ for $j \geq 1$ be defined by

$$\mathcal{N}_j = \mathsf{span}_{\mathbb{F}}\{\mathsf{ad}_{ra_n} \mid r \in \mathsf{R}\big(\pi_{(h/\pi_h)}\big)^j,\ n \geq 0\}. \tag{5.24}$$

Then it is easy to see, using Lemma 4.15 and the fact that $\pi_{(h/\pi_h)}$ divides π_h, that $\mathcal{N}_j$ is an ideal of $[\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)]$ and $[\mathcal{N}, \mathcal{N}_j] \subseteq \mathcal{N}_{j+1}$. As $\frac{h}{\pi_h}$ divides $(\pi_{(h/\pi_h)})^n$ for some n, it follows that $\mathcal{N}_n = 0$ and $\mathcal{N}$ is nilpotent.

For any nilpotent ideal $\mathcal{J}$ of $[\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)]$, $(\mathcal{J} + \mathcal{N})/\mathcal{N}$ is a nilpotent ideal of $\mathcal{L}$. Since $\mathcal{L}$ is either 0 or a direct sum of simple ideals, it has no nonzero nilpotent ideals. Hence, $\mathcal{J} \subseteq \mathcal{N}$, which proves the claim that $\mathcal{N}$ is the unique maximal nilpotent ideal of $[\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)]$.

If all prime factors of h have multiplicity at most 2, then $\pi_{(h/\pi_h)} = \frac{h}{\lambda\pi_h}$ and $\mathcal{N} = 0$. Thus, $[\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)] = \mathcal{L}$ and part (ii) follows. If there is a prime factor of h with multiplicity greater than 2, then $\frac{h}{\pi_h}$ does not divide $\pi_{(h/\pi_h)}$, so $\mathcal{N} \neq 0$. In particular, $[\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)]$ is not semisimple, as it has a nonzero nilpotent ideal. However, if $[\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)]$ were nilpotent, then $\mathcal{N} = [\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)]$ and thus $\pi_{(h/\pi_h)} = 1$, so $\frac{h}{\pi_h} \in \mathbb{F}^*$, which contradicts our hypothesis. Therefore, $[\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)]$ is not nilpotent either. □

We now have all the pieces to assemble the proof of Theorem 5.1.

5.5. **Proof of Theorem 5.1.**
By Theorem 5.13, $\mathsf{HH}^1(\mathsf{A}_h) = \mathsf{Z}(\mathsf{HH}^1(\mathsf{A}_h)) \oplus [\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)]$ if $\mathsf{char}(\mathbb{F}) = 0$, where $\mathsf{Z}(\mathsf{HH}^1(\mathsf{A}_h)) = \left\{D_{r\frac{h}{\pi_h}} \mid \deg r < \deg \pi_h\right\}$ and $\dim \mathsf{Z}\left(\mathsf{HH}^1(\mathsf{A}_h)\right) = \deg \pi_h$. Then Corollary 5.22 tells us that $\mathcal{N} = \mathsf{span}_{\mathbb{F}}\{\mathsf{ad}_{ra_n} \mid r \in \mathsf{R}\pi_{(h/\pi_h)},\ n \geq 0\}$ is the unique maximal nilpotent ideal of $[\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)]$ and $[\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)]/\mathcal{N} \cong ((\mathsf{R}/\mathsf{R}\mathsf{u}_1) \otimes \mathsf{W}) \oplus \cdots \oplus ((\mathsf{R}/\mathsf{R}\mathsf{u}_k) \otimes \mathsf{W})$, a direct sum of simple Lie algebras, where W is the Witt algebra; $\mathsf{u}_1, \ldots, \mathsf{u}_k$ are the monic prime factors of h with multiplicity > 1; and each summand is a field extension of W. This establishes all the assertions in Theorem 5.1 and concludes the proof. □

Corollary 5.25. *Assume* $\mathsf{char}(\mathbb{F}) = 0$. *Then*

(a) $\mathsf{Z}(\mathsf{HH}^1(\mathsf{A}_h)) \oplus \mathcal{N}$ *is the unique maximal nilpotent ideal of* $\mathsf{HH}^1(\mathsf{A}_h)$.
(b) $\mathsf{HH}^1(\mathsf{A}_h)$ *is a nilpotent Lie algebra if and only if* $\frac{h}{\pi_h} \in \mathbb{F}^*$.
(c) [Example 5.4 revisited] *If* $\frac{h}{\pi_h} \in \mathbb{F}^*$, *then* $\pi_{(h/\pi_h)} = 1$, *which implies* $[\mathsf{HH}^1(\mathsf{A}_h), \mathsf{HH}^1(\mathsf{A}_h)] = 0 = \mathcal{N}$ *and*

$$\mathsf{HH}^1(\mathsf{A}_h) \cong \{D_g \mid \deg g < \deg \pi_h = \deg h\},$$

an abelian Lie algebra of dimension $\deg h$.

It is a consequence of Theorem 5.1 that $\mathsf{HH}^1(\mathsf{A}_h)$ modulo its unique maximal nilpotent ideal $\mathsf{Z}(\mathsf{HH}^1(\mathsf{A}_h)) \oplus \mathcal{N}$ is either 0 or a direct sum of ideals that are simple Lie algebras of the form $\mathsf{R}_f \otimes \mathsf{W}$, where $f \in \mathsf{R} = \mathbb{F}[x]$, $\mathsf{R}_f = \mathsf{R}/\mathsf{R}f$, and W is the Witt algebra. Proposition 5.28 below gives a criterion for two such algebras R_f and R_g to be isomorphic.

Recall that the *centroid* of an $\mathbb{F}$-algebra $\mathcal{A}$ is

$$\mathsf{Ctd}_{\mathbb{F}}(\mathcal{A}) = \{\chi \in \mathsf{End}_{\mathbb{F}}(\mathcal{A}) \mid a\chi(b) = \chi(ab) = \chi(a)b \text{ for all } a, b \in \mathcal{A}\}. \tag{5.26}$$

If two algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic via an isomorphism η, then $\mathsf{Ctd}_{\mathbb{F}}(\mathcal{A}_1)$ is isomorphic to $\mathsf{Ctd}_{\mathbb{F}}(\mathcal{A}_2)$ via the isomorphism $\chi \mapsto \eta\chi\eta^{-1}$.

Now it follows from [BN, Cor. 2.23] that if $\mathcal{A}$ and $\mathcal{B}$ are algebras over a field $\mathbb{F}$, $\mathcal{B}$ is perfect and finitely generated as a module over its algebra of multiplication operators, and $\mathcal{A}$ is unital, then

$$\mathsf{Ctd}_{\mathbb{F}}(\mathcal{A} \otimes \mathcal{B}) \cong \mathsf{Ctd}_{\mathbb{F}}(\mathcal{A}) \otimes \mathsf{Ctd}_{\mathbb{F}}(\mathcal{B}). \tag{5.27}$$

(The roles of $\mathcal{A}$ and $\mathcal{B}$ are reversed here from what is in [BN] to make this compatible with our expressions.) We will apply this result to compute the centroid of the Lie algebra $\mathsf{R}_f \otimes \mathsf{W}$, which we can do since W is perfect and generated by w_{-1}, w_2, and then use this to show

Proposition 5.28. $\mathsf{R}_f \otimes \mathsf{W} \cong \mathsf{R}_g \otimes \mathsf{W}$ *if and only if* $\mathsf{R}_f = \mathsf{R}/\mathsf{R}f$ *and* $\mathsf{R}_g = \mathsf{R}/\mathsf{R}g$ *are isomorphic.*

Proof. If $\chi \in \mathsf{Ctd}_{\mathbb{F}}(\mathsf{W})$, then $n\chi(w_n) = \chi([w_0, w_n]) = [w_0, \chi(w_n)]$, which implies that $\chi(w_n)$ lives in the eigenspace $\mathbb{F}w_n$ of ad_{w_0} corresponding to n. Thus, $\chi(w_n) = \lambda_n w_n$ for some $\lambda_n \in \mathbb{F}$. But then the above calculation says: $n\lambda_n w_n = \chi([w_0, w_n]) = [\chi(w_0), w_n] = n\lambda_0 w_n$, which forces $\lambda_n = \lambda_0$ for all n. Hence, $\chi = \lambda_0\, \mathsf{id}_{\mathsf{W}}$ and $\mathsf{Ctd}_{\mathbb{F}}(\mathsf{W}) = \mathbb{F}\mathsf{id}_{\mathsf{W}}$. (Compare [BN, Ex. 2.25].)

Any $\chi \in \mathsf{Ctd}_{\mathbb{F}}(\mathsf{R}_f)$ satisfies $\chi(r) = \chi(1)r$ for all r. Thus, if $s_\chi = \chi(1)$, we have $\chi(r) = s_\chi r$, and the map $\chi \mapsto s_\chi$ shows that $\mathsf{Ctd}_{\mathbb{F}}(\mathsf{R}_f) \cong \mathsf{R}_f$.

Now if $\mathsf{R}_f \otimes \mathsf{W} \cong \mathsf{R}_g \otimes \mathsf{W}$, then their centroids are isomorphic. Hence,

$$\begin{aligned} \mathsf{Ctd}_{\mathbb{F}}(\mathsf{R}_f \otimes \mathsf{W}) \;&\cong\; \mathsf{Ctd}_{\mathbb{F}}(\mathsf{R}_g \otimes \mathsf{W}) \quad \Longleftrightarrow \\ \mathsf{Ctd}_{\mathbb{F}}(\mathsf{R}_f) \otimes \mathsf{Ctd}_{\mathbb{F}}(\mathsf{W}) \;&\cong\; \mathsf{Ctd}_{\mathbb{F}}(\mathsf{R}_g) \otimes \mathsf{Ctd}_{\mathbb{F}}(\mathsf{W}) \;\Longleftrightarrow \\ \mathsf{R}_f \otimes \mathbb{F}\mathsf{id}_{\mathsf{W}} \;&\cong\; \mathsf{R}_g \otimes \mathbb{F}\mathsf{id}_{\mathsf{W}} \quad \Longleftrightarrow \quad \mathsf{R}_f \cong \mathsf{R}_g. \end{aligned}$$

Conversely, if $\psi : \mathsf{R}_f \to \mathsf{R}_g$ is an isomorphism, then $\psi \otimes \mathsf{id}_{\mathsf{W}} : \mathsf{R}_f \otimes \mathsf{W} \to \mathsf{R}_g \otimes \mathsf{W}$ is an isomorphism, with inverse $\psi^{-1} \otimes \mathsf{id}_{\mathsf{W}}$. □

5.6. **Special cases.**

In this concluding subsection, we summarize the derivation results for the well-known examples A_1 (Weyl algebra), A_x (universal enveloping algebra of the two-dimensional non-abelian Lie algebra), and A_{x^2} (Jordan plane). As mentioned earlier, the result for the Weyl algebra goes back to Sridaran [Sr] and can be found in [D2, Sec. 4.6] (see also Proposition 3.1 above). In Theorem 4.6 ($\mathsf{char}(\mathbb{F}) = 0$), Theorem 4.10 ($\mathsf{char}(\mathbb{F}) = p > 2$), and Theorem 4.16 ($\mathsf{char}(\mathbb{F}) = 2$) of [S1], Shirikov has computed the derivations of the Jordan plane A_{x^2}. The results for A_{x^2} in [S1] (see also [S3]) are stated in a different form from what is given in Theorem 5.29 below and in the next section for prime characteristics. The assertions about $\mathsf{HH}^1(\mathsf{A}_h)$ in the next theorem follow from Section 5.4.

Theorem 5.29. *Assume* $\mathsf{char}(\mathbb{F}) = 0$*, and for* $g \in \mathsf{R}$*, let* D_g *denote the derivation of* A_h *with* $D_g(x) = 0$ *and* $D_g(\hat{y}) = g$*. Then*

(i) *For* A_1, $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_1) = \mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_1)$*, so* $\mathsf{HH}^1(\mathsf{A}_1) = 0$.

(ii) *For* A_x, $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_x) = \mathbb{F}D_1 \oplus \mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_x)$*, so* $\mathsf{HH}^1(\mathsf{A}_x)$ *is a one-dimensional Lie algebra with basis* $\{D_1\}$.

(iii) *For* A_{x^m} *with* $m \geq 2$, $\pi_h = x$*, and*

$$\begin{aligned} \mathsf{HH}^1(\mathsf{A}_{x^m})/\mathcal{N} &= \mathsf{Z}(\mathsf{HH}^1(\mathsf{A}_{x^m})) \oplus [\mathsf{HH}^1(\mathsf{A}_{x^m}), \mathsf{HH}^1(\mathsf{A}_{x^m})]/\mathcal{N} \\ &= \mathbb{F}D_{x^{m-1}} \oplus [\mathsf{HH}^1(\mathsf{A}_{x^m}), \mathsf{HH}^1(\mathsf{A}_{x^m})]/\mathcal{N} \\ &\cong\; \mathbb{F}D_{x^{m-1}} \oplus \mathsf{W} \end{aligned}$$

where $\mathsf{W} = \mathsf{span}_{\mathbb{F}}\{w_i \mid i \geq -1\}$ *is the Witt algebra. The ideal* $\mathcal{N}$ *is nilpotent of index* $\leq m-1$*. In particular,* $\mathcal{N} = 0$ *when* $m = 2$.

6. $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$ WHEN $\mathsf{char}(\mathbb{F}) = p > 0$

Throughout we assume that the field $\mathbb{F}$ has characteristic $p > 0$, $h \neq 0$, and ϱ_h is as in Definition 2.14. Our main results in this section are Theorem 6.21 and Corollary 6.23, which give direct sum decompositions for $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$ as a module over the center $\mathsf{Z}(\mathsf{A}_h)$ of A_h, and Theorem 6.29, which gives necessary and sufficient conditions for $\mathsf{HH}^1(\mathsf{A}_h)$ to be a free $\mathsf{Z}(\mathsf{A}_h)$-module. In the final subsection, we determine the Lie brackets in $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$.

6.1. **The derivations D_g and the decomposition.**

From Theorem 4.9, we know that for every $D \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$ there exist $E \in \mathcal{E} = \{F \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_1) \mid F(\mathsf{A}_h) \subseteq \mathsf{A}_h\}$ and $g \in \mathsf{R}$ so that $D = D_g + E$, where D_g is the derivation of A_h given by $D_g(x) = 0$ and $D_g(\hat{y}) = g$. The main problem is to determine conditions for $E \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_1)$ to restrict to a derivation of A_h. Theorem 3.8 tells us that every derivation of A_1 has the form $wE_x + zE_y + \mathsf{ad}_a$ where $w, z \in \mathsf{Z}(\mathsf{A}_1)$, $a \in \mathsf{A}_1$ and E_x, E_y are as in (3.2). However, it is not generally true that wE_x and zE_y restrict to A_h for arbitrary elements w, z of $\mathsf{Z}(\mathsf{A}_1) = \mathbb{F}[x^p, y^p]$.

6.2. **Derivations of the form wE_x.**

Lemma 6.1. *Let $\mathsf{char}(\mathbb{F}) = p > 0$, and assume $E = wE_x + zE_y + \mathsf{ad}_a \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_1)$ restricts to a derivation of A_h, where $w, z \in \mathsf{Z}(\mathsf{A}_1)$ and $a \in \mathsf{A}_1$. Then $w \in \mathsf{Z}(\mathsf{A}_h)$.*

Proof. Derivations map the center to itself, so by Theorem 2.4 and Lemma 3.6 we know that $E(x^p) = -w \in \mathsf{Z}(\mathsf{A}_1) \cap \mathsf{A}_h = \mathsf{Z}(\mathsf{A}_h)$. □

We will provide necessary and sufficient conditions on $w \in \mathsf{Z}(\mathsf{A}_h)$ for wE_x to restrict to a derivation of A_h, but this will require the next lemma.

Lemma 6.2. *Let ϱ_h be as in (2.15), and assume $v \in \mathsf{R}$. Then $vh^{p-1} \in \mathbb{F}[x^p]$ if and only if $v'h = vh'$ if and only if $v \in \mathbb{F}[x^p]\frac{h}{\varrho_h}$.*

Proof.

$$vh^{p-1} \in \mathbb{F}[x^p] \iff (vh^{p-1})' = 0 \iff v'h = vh' \iff \big(vh^{-1}\big)' = 0$$
$$\iff v \in (\mathsf{R} \cap \mathsf{Z}(\mathsf{A}_h))\frac{h}{\varrho_h} = \mathbb{F}[x^p]\frac{h}{\varrho_h} \text{ by Lemma 4.28 (d).} \qquad \square$$

Proposition 6.3. *Assume $\mathsf{char}(\mathbb{F}) = p > 0$ and let $w \in \mathsf{Z}(\mathsf{A}_h)$. The following are equivalent.*

(i) *wE_x restricts to a derivation of A_h;*
(ii) *$w \in \mathsf{Z}(\mathsf{A}_h)\frac{h^p}{\varrho_h}$;*
(iii) *$wE_x(x) \in \mathsf{A}_h$;*
(iv) *$wE_x \in \mathsf{Z}(\mathsf{A}_h)\breve{E}_x$, where $\breve{E}_x = \frac{h^p}{\varrho_h}E_x$.*

Proof. Since $w \in \mathsf{Z}(\mathsf{A}_h)$, we may assume $w = \sum_{i \equiv 0 \bmod p} s_i h^i y^i$, where $s_i \in \mathbb{F}[x^p]$ for all i. Now $wE_x(x) = \sum_{i \equiv 0 \bmod p} s_i h^i y^{i+p-1} \in \mathsf{A}_h \iff h^{p-1}$ divides s_i for each $i \iff$ for each i, $s_i = w_i \frac{h}{\varrho_h} h^{p-1} = w_i \frac{h^p}{\varrho_h} \in \mathbb{F}[x^p]$ for some $w_i \in \mathbb{F}[x^p]$, by Lemma 6.2. Therefore, (ii) and (iii) are equivalent.

The implication (i) $\Longrightarrow$ (iii) is clear. Now assume $wE_x(x) \in \mathsf{A}_h$. Then by the equivalence of (ii) and (iii), we may suppose that $w = u\frac{h^p}{\varrho_h}$ for some $u \in \mathsf{Z}(\mathsf{A}_h)$. Now Lemma 3.6 (f) implies that $E_x(\hat{y}) \in h'y^p + \sum_{i=0}^{p-1} \mathsf{R}y^i$, so $wE_x(\hat{y}) = u\frac{h^p}{\varrho_h}E_x(\hat{y}) \in u\frac{h^p}{\varrho_h}h'y^p + \sum_{i=0}^{p-1} \mathsf{R}u\frac{h^p}{\varrho_h}y^i$, which belongs to A_h since ϱ_h divides h'. Thus, (ii) implies (i).

It is clear that (ii) and (iv) are equivalent, as $E_x \neq 0$ and A_1 is a domain. □

Theorem 6.4. *Assume* $\mathsf{char}(\mathbb{F}) = p > 0$*, and let* $E = wE_x + zE_y + \mathsf{ad}_a \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_1)$ *with* $w, z \in \mathsf{Z}(\mathsf{A}_1) = \mathbb{F}[x^p, y^p]$*, and* $a \in \mathsf{A}_1$*. If* $E \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$*, then* $wE_x \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$ *and* $w \in \mathsf{Z}(\mathsf{A}_h)\frac{h^p}{\varrho_h}$.

Proof. Since $E(x) \in \mathsf{A}_h$, we have $wy^{p-1} + [a, x] \in \mathsf{A}_h$. Observe that

$$wy^{p-1} \in \bigoplus_{i \equiv -1 \bmod p} \mathsf{R}y^i \qquad \text{and} \qquad [a, x] \in \bigoplus_{i \not\equiv -1 \bmod p} \mathsf{R}y^i.$$

Thus $wy^{p-1} \in \mathsf{A}_h$ and $[a, x] \in \mathsf{A}_h$. This implies that $wE_x(x) = wy^{p-1} \in \mathsf{A}_h$, and the result now follows from Lemma 6.1 and Proposition 6.3. □

6.3. **Derivations of the form** $D = zE_y + \mathsf{ad}_a$**.**

In view of Theorems 3.8, 4.9, and 6.4, we know that every derivation of A_h has the form $D_g + u\breve{E}_x + zE_y + \mathsf{ad}_a$, where $g \in \mathsf{R}$, D_g and $u\breve{E}_x$ are derivations of A_h, $u \in \mathsf{Z}(\mathsf{A}_h)$, $z \in \mathsf{Z}(\mathsf{A}_1)$, $a \in \mathsf{A}_1$, and $\breve{E}_x = \frac{h^p}{\varrho_h}E_x$. Moreover, every $D_g + u\breve{E}_x$ with $g \in \mathsf{R}$ and $u \in \mathsf{Z}(\mathsf{A}_h)$ gives a derivation of A_h. For that reason, we may assume that $D = zE_y + \mathsf{ad}_a$ is a derivation of A_1 that restricts to a derivation of A_h.

Lemma 6.5. *Let* $D = zE_y + \mathsf{ad}_a \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_1)$ *for some* $z \in \mathsf{Z}(\mathsf{A}_1)$ *and* $a \in \mathsf{A}_1$*, and suppose* $D \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$*. Then* $a = b + c$*, where* $b \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)_{\not\equiv 0}$ *and* $c \in \mathsf{C}_{\mathsf{A}_1}(x) = \mathbb{F}[x, y^p]$ *as in Remark 2.20, and both* ad_b *and* $zE_y + \mathsf{ad}_c$ *are derivations of* A_1 *that restrict to derivations of* A_h*. Moreover, if* $a = \sum_{i \geq 0} r_i y^i$ *and* $z = \sum_{i \equiv 0 \bmod p} c_i y^i$*, where* $r_i \in \mathsf{R}$ *and* $c_i \in \mathbb{F}[x^p]$ *for all* i*, then* $zE_y + \mathsf{ad}_a = D_f + \tilde{z}E_y + \mathsf{ad}_{\tilde{c}} + \mathsf{ad}_b$*, where* $\tilde{z} = \sum_{i \equiv 0 \bmod p, i > 0} c_i y^i$*,* $\tilde{c} = \sum_{i \equiv 0 \bmod p, i > 0} r_i y^i$ *and* $f = c_0 h x^{p-1} - \delta(r_0) \in \mathsf{R}$*, and* $\tilde{z}E_y + \mathsf{ad}_{\tilde{c}} \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$.

Proof. Let a and z be as in the statement of the lemma. Since $zE_y(x) = 0$, we have $D(x) \in \mathsf{A}_h$ if and only if $[a, x] \in \mathsf{A}_h$. As in (2.18), $[a, x] \in \mathsf{A}_h \iff r_i \in \mathsf{R}h^{i-1}$ for all $i \not\equiv 0 \bmod p$. Thus, we write $r_i = s_i h^{i-1}$ for each such i, where $s_i \in \mathsf{R}$.

Now $D(hy) = D(\hat{y}) - D(h') \in \mathsf{A}_h$, and we reason as in (2.19) that

$$D(hy) \in \mathsf{A}_h \iff zhx^{p-1} + \sum_{i \not\equiv 0 \bmod p} s_i h^{i-1} h' y^i - \sum_{i \equiv 0 \bmod p} r_i' h y^i \in \mathsf{A}_h$$

$$\text{(6.6)} \qquad \iff \sum_{i \equiv 0 \bmod p} \left(c_i x^{p-1} - r_i'\right) h y^i \in \mathsf{A}_h \text{ and } \sum_{i \not\equiv 0 \bmod p} s_i h^{i-1} h' y^i \in \mathsf{A}_h$$

$$\iff h^{i-1} \mid (c_i x^{p-1} - r_i') \text{ for all } i \equiv 0 \bmod p,\ i > 0, \text{ and}$$

$$h \mid s_i h' \text{ for all } i \not\equiv 0 \bmod p.$$

Hence, if $D \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$, then $h \mid s_i h'$ for all $i \not\equiv 0 \bmod p$ by (6.6), and we know by Lemma 2.13 that π_h divides each such s_i. Then there exist $b_i \in \mathbb{F}[x]$ so that $r_i = b_i \pi_h h^{i-1}$ for each $i \not\equiv 0 \bmod p$, and $b = \sum_{i \not\equiv 0 \bmod p} b_i \pi_h h^{i-1} y^i \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)_{\not\equiv 0}$ by Theorem 2.17 (b). Then ad_b and D belong to $\mathcal{E} = \{F \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_1) \mid F(\mathsf{A}_h) \subseteq \mathsf{A}_h\}$. Setting $c = a - b = \sum_{i \equiv 0 \bmod p} r_i y^i \in \mathsf{C}_{\mathsf{A}_1}(x)$, we have that $zE_y + \mathsf{ad}_c = D - \mathsf{ad}_b \in \mathcal{E}$. Thus both ad_b and $zE_y + \mathsf{ad}_c$ are derivations of A_1 that restrict to derivations of A_h.

From $E_y(x) = 0$ and $E_y(\hat{y}) = x^{p-1}h$ (Lemma 3.6 (e)), we see that $E_y = D_{x^{p-1}h} \in \mathcal{D}_{\mathsf{R}} \subseteq \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$. Also, from Proposition 4.6 (ii), we have $\mathsf{ad}_r = -D_{\delta(r)} \in \mathcal{D}_{\mathsf{R}}$ for all $r \in \mathsf{R}$. As a result, if z, a, b, c are as above, then $zE_y + \mathsf{ad}_a = D_f + \tilde{z}E_y + \mathsf{ad}_{\tilde{c}} + \mathsf{ad}_b$, where $\tilde{z} = \sum_{i \equiv 0 \bmod p, i>0} c_i y^i$, $\tilde{c} = \sum_{i \equiv 0 \bmod p, i>0} r_i y^i$ and $f = c_0 h x^{p-1} - \delta(r_0) \in \mathsf{R}$, and $\tilde{z}E_y + \mathsf{ad}_{\tilde{c}} \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$. □

6.4. The restriction map $\mathsf{Res} : \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h) \to \mathsf{Der}_{\mathbb{F}}(\mathsf{Z}(\mathsf{A}_h))$.

When $\mathsf{char}(\mathbb{F}) = p > 0$, $\mathsf{Z}(\mathsf{A}_h) = \mathbb{F}[x^p, \mathsf{z}_h]$, where $\mathsf{z}_h = h^p y^p = \hat{y}^p - \frac{\delta^p(x)}{h}\hat{y}$. The map $\mathsf{Res} : \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h) \to \mathsf{Der}_{\mathbb{F}}(\mathsf{Z}(\mathsf{A}_h))$ given by restricting a derivation to $\mathsf{Z}(\mathsf{A}_h)$ is a morphism of Lie algebras. In this section, we investigate this map and describe its kernel and image. This will enable us to determine $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$ in the next section. The derivation δ^p plays a significant role. As δ^p sends x to $\delta^p(x)$, then $\delta^p = \delta^p(x)\frac{d}{dx}$ and

$$\delta^p(r) = \delta^p(x) r' \qquad \text{for all } r \in \mathsf{R}. \tag{6.7}$$

Lemma 6.8. *Let $\mathsf{z}_h = h^p y^p \in \mathsf{Z}(\mathsf{A}_h)$, and write $h^{p-1} = \sum_{i=0}^{p-1} \overline{h}_i x^i$ with $\overline{h}_i \in \mathbb{F}[x^p]$ for all i.*

(a) *For any $r \in \mathsf{R}$, $D_r(\mathsf{z}_h) = \delta^{p-1}(r) - \frac{\delta^p(x)}{h} r = \left(r h^{p-1}\right)^{(p-1)}$.*

(b) *$\delta^p(x) = -\left(h^{p-1}\right)^{(p-1)} h = \overline{h}_{p-1} h$ so that $\delta^p = \overline{h}_{p-1}\delta$ and $D_1(\mathsf{z}_h) = -\overline{h}_{p-1}$.*

Proof. (a) For any $r \in \mathsf{R}$, we have

$$\begin{aligned} D_r(\mathsf{z}_h) &= D_r(\hat{y}^p - \tfrac{\delta^p(x)}{h}\hat{y}) = \textstyle\sum_{n=0}^{p-1} \hat{y}^n r \hat{y}^{p-1-n} - \tfrac{\delta^p(x)}{h} r \\ &= \sum_{n=0}^{p-1} \sum_{j=0}^{n} \binom{n}{j} \delta^j(r) \hat{y}^{p-1-j} - \tfrac{\delta^p(x)}{h} r \\ &= \sum_{j=0}^{p-1} \left(\sum_{n=j}^{p-1} \binom{n}{j} \right) \delta^j(r) \hat{y}^{p-1-j} - \tfrac{\delta^p(x)}{h} r \;=\; \delta^{p-1}(r) - \tfrac{\delta^p(x)}{h} r. \end{aligned}$$

The fact that $D_r(\mathsf{z}_h) = (r h^{p-1})^{(p-1)}$ comes from (c) of Corollary 4.17.

(b) Taking $r = 1$ in part (a) yields $\left(h^{p-1}\right)^{(p-1)} = \delta^{p-1}(1) - \frac{\delta^p(x)}{h} = -\frac{\delta^p(x)}{h}$, and thus $\delta^p(x) = -\left(h^{p-1}\right)^{(p-1)} h$. Since $\left(x^i\right)^{(p-1)} = 0$ for $0 \le i < p-1$ and $\left(x^{p-1}\right)^{(p-1)} = -1$, it follows that $\left(h^{p-1}\right)^{(p-1)} = \left(\sum_{i=0}^{p-1} \overline{h}_i x^i\right)^{(p-1)} = -\overline{h}_{p-1}$. Hence, $\delta^p(x) = \overline{h}_{p-1} h$, and $\delta^p = \delta^p(x)\frac{d}{dx} = \overline{h}_{p-1} h \frac{d}{dx} = \overline{h}_{p-1}\delta$ by (6.7). □

Proposition 6.9. *The kernel of the restriction map* $\mathsf{Res} : \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h) \to \mathsf{Der}_{\mathbb{F}}(\mathsf{Z}(\mathsf{A}_h))$ *is*

$$\ker \mathsf{Res} = \mathcal{D}_\Theta + \{\mathsf{ad}_a \mid a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)\},$$

where $\mathcal{D}_\Theta = \{D_r \mid r \in \Theta\}$ *and* $\Theta = \left\{ r \in \mathsf{R} \mid \delta^{p-1}(r) = \frac{\delta^p(x)}{h} r \right\}$.

Proof. The right side is contained $\ker \mathsf{Res}$ by (a) of Lemma 6.8 and the fact that $\mathsf{Z}(\mathsf{A}_h) \subseteq \mathsf{Z}(\mathsf{A}_1)$. For the other direction, suppose that $D \in \ker \mathsf{Res}$. In view of Lemma 6.5, we may suppose $D = D_r + u\breve{E}_x + \tilde{z}E_y + \mathsf{ad}_b + \mathsf{ad}_{\tilde{c}}$ for some $r \in \mathsf{R}$, $u \in \mathsf{Z}(\mathsf{A}_h)$, $\tilde{z} = \sum_{i \equiv 0 \bmod p, i>0} c_i y^i \in \mathsf{Z}(\mathsf{A}_1)$ with $c_i \in \mathbb{F}[x^p]$, $b \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)_{\not\equiv 0}$, and $\tilde{c} \in \sum_{i \equiv 0 \bmod p, i>0} \mathsf{R}y^i$. Since $\mathsf{ad}_b \in \ker \mathsf{Res}$, we can assume that $E = D_r + u\breve{E}_x + \tilde{z}E_y + \mathsf{ad}_{\tilde{c}} \in \ker \mathsf{Res}$. Applying E to x^p, we see that $u = 0$. Since $\mathsf{ad}_{\tilde{c}}(\mathsf{z}_h) = 0$, we have

$$\begin{aligned} 0 &= \big(D_r + \tilde{z}E_y\big)(\mathsf{z}_h) = \delta^{p-1}(r) - \tfrac{\delta^p(x)}{h} r + \tilde{z}E_y(h^p y^p) \\ &= \delta^{p-1}(r) - \tfrac{\delta^p(x)}{h} r - \tilde{z}h^p \\ &= \delta^{p-1}(r) - \tfrac{\delta^p(x)}{h} r - \textstyle\sum_{i \equiv 0 \bmod p, i>0} c_i h^p y^i. \end{aligned}$$

From this we deduce that $\tilde{z} = 0$ and $\delta^{p-1}(r) = \frac{\delta^p(x)}{h} r$. Therefore, $\mathsf{ad}_{\tilde{c}} = E - D_r \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$, $r \in \Theta$, and $D \in \mathcal{D}_\Theta + \{\mathsf{ad}_a \mid a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)\}$. $\square$

In light of Proposition 6.9, we would like to determine more information about Θ.

Proposition 6.10. *Let* $h^{p-1} = \sum_{i=0}^{p-1} \overline{h}_i x^i$, *with* $\overline{h}_i \in \mathbb{F}[x^p]$ *for all* i, *as in Lemma 6.8, and let* $\mathsf{Res} : \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h) \to \mathsf{Der}_{\mathbb{F}}(\mathsf{Z}(\mathsf{A}_h))$ *be the restriction map.*

(a) *Let* $\vartheta : \mathsf{R} \to \mathbb{F}[x^p]$ *be the* $\mathbb{F}[x^p]$*-module map given by* $\vartheta(r) = D_r(\mathsf{z}_h)$. *Then*

$$\begin{aligned} \Theta &= \{r \in \mathsf{R} \mid \delta^{p-1}(r) = \tfrac{\delta^p(x)}{h} r\} = \{r \in \mathsf{R} \mid \delta^{p-1}(r) = \overline{h}_{p-1} r\} \\ &= \ker\vartheta = \{r \in \mathsf{R} \mid D_r \in \ker \mathsf{Res}\,\} \\ &= \{r \in \mathsf{R} \mid (rh^{p-1})^{(p-1)} = 0\} \\ &= \left\{r \in \mathsf{R} \mid rh^{p-1} \in \operatorname{im} \tfrac{d}{dx}\right\} \;=\; \{r \in \mathsf{R} \mid rh^p \in \operatorname{im} \delta\}. \end{aligned}$$

In particular, Θ *contains* $\operatorname{im} \delta$.

(b) Θ *is a free* $\mathbb{F}[x^p]$*-module of rank* $p-1$ *and* $\delta^{p-1} \neq 0$. *If* $\delta^p = 0$ *then* $\mathbb{F}[x^p] \subseteq \Theta$*; if* $\delta^p \neq 0$ *then* $\mathbb{F}[x^p] \cap \Theta = 0$.

(c) $\operatorname{im} \vartheta = \{D_r(\mathsf{z}_h) \mid r \in \mathsf{R}\} = \mathbb{F}[x^p]\overline{h}$, *where* $\overline{h}$ *is the greatest common divisor in* $\mathbb{F}[x^p]$ *of* $\{\overline{h}_i \mid 0 \le i < p\}$. *Hence,* $\mathsf{Res}(\mathcal{D}_{\mathsf{R}}) = \mathbb{F}[x^p]\overline{h}\frac{d}{d\mathsf{z}_h}$.

(d) *Let* $\breve{q}_i \in \mathbb{F}[x^p]$ *be such that* $\overline{h} = \sum_{i=0}^{p-1} \breve{q}_i \overline{h}_i$, *and set* $\breve{q} = -\sum_{i=0}^{p-1} \breve{q}_i x^{p-1-i}$. *Then* $\mathsf{Res}(D_{\breve{q}}) = \overline{h}\frac{d}{d\mathsf{z}_h}$ *and* $\mathsf{R} = \mathbb{F}[x^p]\breve{q} \oplus \Theta$.

(e) *For all* $f \in \mathsf{R}$, $\left(f' f^{p-1}\right)^{(p-1)} = -(f')^p$. *In particular,* $D_{\frac{h'}{\varrho_h}}(\mathsf{z}_h) = -\frac{(h')^p}{\varrho_h}$.

Proof. (a) Let $r \in \mathsf{R}$. Then by Lemma 6.8 (a),

$$\begin{aligned} r \in \Theta \quad &\Longleftrightarrow \quad \left(rh^{p-1}\right)^{(p-1)} = 0 \\ &\Longleftrightarrow \quad rh^{p-1} \in \sum_{i=0}^{p-2} \mathbb{F}[x^p]x^i = \operatorname{im} \frac{d}{dx} \\ &\Longleftrightarrow \quad rh^p \in \operatorname{im} \delta. \end{aligned}$$

In particular, $\delta(r)h^p = \delta(rh^p) \in \operatorname{im} \delta$ for all $r \in \mathsf{R}$, so (a) holds.

(b) and (c) For the $\mathbb{F}[x^p]$-module map $\vartheta : \mathsf{R} \to \mathbb{F}[x^p]$ given by $\vartheta(r) = \left(rh^{p-1}\right)^{(p-1)}$, $\operatorname{im} \vartheta$ is the ideal of $\mathbb{F}[x^p]$ generated by $\{\vartheta(x^j) \mid 0 \le j < p\}$. Note that $x^j h^{p-1} = \sum_{i=0}^{p-1} \overline{h}_i x^{i+j}$, so $\vartheta(x^j) = -\overline{h}_{p-1-j}$. Since $h \neq 0$, we cannot have $\overline{h}_i = 0$ for all $0 \le i < p$, thus $\operatorname{im} \vartheta = \mathbb{F}[x^p]\,\overline{h}$, where $0 \neq \overline{h} \in \mathbb{F}[x^p]$ is the greatest common divisor of $\{\overline{h}_i \mid 0 \le i < p\}$. In particular, $\operatorname{im} \vartheta$ is a free $\mathbb{F}[x^p]$-module of rank one, and it follows that $\Theta = \ker \vartheta$ is free of rank $p-1$.

If $\delta^{p-1} = 0$, then $\delta^p = 0$ and $\Theta = \mathsf{R}$, which is a contradiction, as R has rank p as an $\mathbb{F}[x^p]$-module. Thus $\delta^{p-1} \neq 0$. Suppose that $\delta^p = 0$. Then $\Theta = \{r \in \mathsf{R} \mid \delta^{p-1}(r) = 0\}$, and it is clear that $\mathbb{F}[x^p] \subseteq \Theta$. Suppose now that $\delta^p \neq 0$. Then, $\delta^p(x) \neq 0$. If $r \in \mathbb{F}[x^p] \cap \Theta$, then $0 = \delta^{p-1}(r) = \frac{\delta^p(x)}{h} r$, so $r = 0$ and $\mathbb{F}[x^p] \cap \Theta = 0$, as asserted in (b).

(d) As $\vartheta(x^{p-1-i}) = -\overline{h}_i$, we have $\operatorname{Res}(D_{x^{p-1-i}}) = -\overline{h}_i \frac{d}{d\mathsf{z}_h}$ for $0 \le i < p$. Now if $\breve{q}_i \in \mathbb{F}[x^p]$, $0 \le i < p$, are taken so that $\overline{h} = \sum_{i=0}^{p-1} \breve{q}_i \overline{h}_i$, then for $\breve{q} = -\sum_{i=0}^{p-1} \breve{q}_i x^{p-1-i}$, it follows that $D_{\breve{q}} = -\sum_{i=0}^{p-1} \breve{q}_i D_{x^{p-1-i}}$ and $\operatorname{Res}(D_{\breve{q}}) = \left(\sum_{i=0}^{p-1} \breve{q}_i \overline{h}_i\right) \frac{d}{d\mathsf{z}_h} = \overline{h} \frac{d}{d\mathsf{z}_h}$.

Suppose $r \in \mathsf{R}$. Then by (c), there exists $u \in \mathbb{F}[x^p]$ such that $\operatorname{Res}(D_r) = u\operatorname{Res}(D_{\breve{q}})$. Hence, $\operatorname{Res}(D_{r-u\breve{q}}) = 0$, $r - u\breve{q} = t \in \Theta$, and $r = u\breve{q} + t$. This shows that $\mathsf{R} = \mathbb{F}[x^p]\breve{q} + \Theta$. Since $\vartheta(u\breve{q}) = u\overline{h} \neq 0$ for all nonzero $u \in \mathbb{F}[x^p]$, it is apparent the sum is direct.

It remains to prove part (e). We assume the stated equality holds for $f, g \in \mathsf{R}$ and show it for $f+g$. Now

$$\begin{aligned} (f+g)'(f+g)^{p-1} &= f' \sum_{k=0}^{p-1} (-1)^k f^k g^{p-1-k} + g' \sum_{k=0}^{p-1} (-1)^k f^k g^{p-1-k} \\ &= f'f^{p-1} + g'g^{p-1} + f' \sum_{k=0}^{p-2} (-1)^k f^k g^{p-1-k} + g' \sum_{k=1}^{p-1} (-1)^k f^k g^{p-1-k} \\ &= f'f^{p-1} + g'g^{p-1} + \sum_{k=0}^{p-2} (-1)^k \left(f'f^k g^{p-1-k} - f^{k+1} g' g^{p-2-k}\right) \\ &= f'f^{p-1} + g'g^{p-1} + \sum_{k=0}^{p-2} (-1)^k \frac{1}{k+1} \left(f^{k+1} g^{p-1-k}\right)'. \end{aligned}$$

Since $\left(\mathsf{im}\frac{d}{dx}\right)^{(p-1)} = 0$, we see that $f \mapsto (f' f^{p-1})^{(p-1)}$ is an additive mapping on R. Hence, it will be enough to show that $\left(f' f^{p-1}\right)^{(p-1)} = -(f')^p$ for $f = \gamma x^m$, with $m \geq 0$ and $\gamma \in \mathbb{F}$. This is immediate from

$$\begin{aligned} \left(f' f^{p-1}\right)^{(p-1)} &= \left(\gamma^p m x^{mp-1}\right)^{(p-1)} = \gamma^p m x^{(m-1)p} \left(x^{p-1}\right)^{(p-1)} \\ &= -\gamma^p m x^{(m-1)p} = -\left(\gamma m x^{m-1}\right)^p \\ &= -(f')^p, \end{aligned}$$

so the equality in (e) holds for all $f \in \mathsf{R}$. Taking $f = h$ gives

$$D_{\frac{h'}{\varrho_h}}(\mathsf{z}_h) = \left(\frac{h'}{\varrho_h} h^{p-1}\right)^{(p-1)} = \frac{1}{\varrho_h}\left(h' h^{p-1}\right)^{(p-1)} = -\frac{(h')^p}{\varrho_h}. \qquad \square$$

Remark 6.11. *The map $\vartheta : \mathsf{R} \to \mathbb{F}[x^p]$, $r \mapsto \left(r h^{p-1}\right)^{(p-1)}$, can be thought of as an inner product with $-(\bar{h}_{p-1}, \dots, \bar{h}_0)$: If we identify $r = \sum_{k=0}^{p-1} r_k x^k \in \bigoplus_{k=0}^{p-1} \mathbb{F}[x^p] x^k$ with the tuple $(r_0, \dots, r_{p-1})$, we can view ϑ as the map $(r_0, \dots, r_{p-1}) \mapsto -\sum_{i=0}^{p-1} r_i \bar{h}_{p-1-i}$. Then Θ is the orthogonal complement of the line generated by $(\bar{h}_{p-1}, \dots, \bar{h}_0)$.*

Example 6.12. *Assume $h = g^m$, where $m \geq 0$ and $g = x - \gamma$ for some $\gamma \in \mathbb{F}$. Then $\mathsf{R} = \bigoplus_{i \geq 0} \mathbb{F} g^i$, and*

$$\mathsf{im}\,\delta = \bigoplus_{i=0}^{p-2} \mathbb{F}[g^p] g^{m+i} = \bigoplus_{\substack{j \geq m \\ j \not\equiv m-1 \bmod p}} \mathbb{F} g^j.$$

Now for $r = \sum_{i \geq 0} r_i g^i$ with $r_i \in \mathbb{F}$ for all i,

$$\begin{aligned} r \in \Theta &\iff r h^p = \sum_{i \geq 0} r_i g^{i+mp} \in \mathsf{im}\,\delta = \bigoplus_{\substack{j \geq m \\ j \not\equiv m-1 \bmod p}} \mathbb{F} g^j \\ &\iff r_i = 0 \ \text{ for } \ i \equiv m-1 \bmod p. \end{aligned}$$

Hence,

$$\Theta = \bigoplus_{\substack{j \geq 0 \\ j \not\equiv m-1 \bmod p}} \mathbb{F} g^j.$$

Recall $\delta_0(r) = \delta(r\pi_h h^{-1}) = (r\pi_h h^{-1})' h$. If $p \nmid m$, then $\pi_h = g$ and from this we see $\delta_0(g^j) = \delta(g^{j+1-m}) = (j+1-m)g^j$, so that $g^j \in \mathsf{im}\,\delta_0$ exactly when $j \not\equiv m-1 \bmod p$. If $p \mid m$, then $\pi_h = 1$ and $\delta_0(g^j) = \delta(g^{j-m}) = j g^{j-1} = \frac{d}{dx}(g^j)$, so $\mathsf{im}\,\delta_0 = \mathsf{im}\,\frac{d}{dx}$. In either event, we have

$$\Theta = \mathsf{im}\,\delta_0 = \bigoplus_{\substack{j \geq 0 \\ j \not\equiv m-1 \bmod p}} \mathbb{F} g^j = \left(\bigoplus_{\substack{0 \leq j < m \\ j \not\equiv m-1 \bmod p}} \mathbb{F} g^j \right) \oplus \mathsf{im}\,\delta.$$

Some cases of special interest are

- *for $h = 1$, $\Theta = \mathsf{im}\,\delta = \bigoplus_{j=0}^{p-2} \mathbb{F}[x^p] x^j = \mathsf{im}\,\frac{d}{dx}$;*

- *for* $h = x$, $\Theta = \operatorname{im}\delta = \bigoplus_{j=1}^{p-1} \mathbb{F}[x^p]x^j$;
- *for* $h = x^n$ *with* $2 \le n < p$, $\Theta = \left(\bigoplus_{j=0}^{n-2} \mathbb{F}x^j\right) \oplus \operatorname{im}\delta$.

In view of Proposition 6.9, we investigate the following.

Proposition 6.13. *Suppose* $D_r + \mathsf{ad}_a \in \mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_h)$ *for some* $r \in \mathsf{R}$ *and* $a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)$. *Then* $r \in \operatorname{im}\delta$, $a \in \mathsf{A}_h + \mathsf{Z}(\mathsf{A}_1)$, *and* $\mathsf{ad}_a, D_r \in \mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_h)$. *Consequently,*

$$\mathcal{D}_\Theta \cap \{\mathsf{ad}_a \mid a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)\} = \mathcal{D}_{\operatorname{im}\delta},$$

where $\mathcal{D}_\Theta = \{D_r \mid r \in \Theta\}$ *and* $\mathcal{D}_{\operatorname{im}\delta} = \{D_r \mid r \in \operatorname{im}\delta\}$.

Proof. For the first statement, suppose that $D_r + \mathsf{ad}_a = \mathsf{ad}_v$ for some $v \in \mathsf{A}_h$. Then it follows from $D_r = \mathsf{ad}_{v-a}$ that $v - a \in \mathsf{C}_{\mathsf{A}_1}(x)$. Writing $v - a = \sum_{i \equiv 0 \bmod p} w_i y^i$, where $w_i \in \mathsf{R}$ for all i, we have $r = D_r(\hat{y}) = [v - a, \hat{y}] = \sum_{i\equiv 0 \bmod p}[w_i y^i, yh] = -\sum_{i \equiv 0 \bmod p} w_i' h y^i$. As a result, $r = -w_0' h \in \operatorname{im}\delta$ and $w_i' = 0$ for all $i > 0$. Hence, $w_i \in \mathbb{F}[x^p]$ for all $i > 0$ and $w = \sum_{i \equiv 0 \bmod p, i>0} w_i y^i \in \mathsf{Z}(\mathsf{A}_1)$. Now $a = (v - w_0) - w \in \mathsf{A}_h + \mathsf{Z}(\mathsf{A}_1)$, which implies that $\mathsf{ad}_a = \mathsf{ad}_{v-w_0}$ and D_r are in $\mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_h)$.

The assertion about $\mathcal{D}_\Theta$ follows from what we have just shown and the fact that $D_{\delta(g)} = -\mathsf{ad}_g$ for all $g \in \mathsf{R}$ by (ii) of Proposition 4.6. □

From Proposition 6.13, we can conclude the following:

Corollary 6.14. *The kernel of the induced map* $\overline{\mathsf{Res}} : \mathsf{HH}^1(\mathsf{A}_h) \to \mathsf{Der}_{\mathbb{F}}(\mathsf{Z}(\mathsf{A}_h))$ *is*

$$\begin{aligned} \ker \overline{\mathsf{Res}} &= \big(\mathcal{D}_\Theta + \{\mathsf{ad}_a \mid a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)\}\big)/\mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_h) \\ &\cong \big(\mathcal{D}_\Theta/\mathcal{D}_{\operatorname{im}\delta}\big) \oplus \big(\{\mathsf{ad}_a \mid a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)\}/\mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_h)\big) \\ &\cong \big(\Theta/\operatorname{im}\delta\big) \oplus \big(\mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)/\big(\mathsf{A}_h + \mathsf{Z}(\mathsf{A}_1)\big)\big), \end{aligned}$$

where the isomorphisms are as $\mathbb{F}[x^p]$*-modules.*

Next, we investigate the image of the map Res. Recall from Proposition 6.10 (c) that $\mathsf{Res}(\mathcal{D}_\mathsf{R}) = \mathbb{F}[x^p]\bar{h}\frac{d}{d\mathsf{z}_h} = \mathbb{F}[x^p]\mathsf{Res}(D_{\breve{q}})$, where $\breve{q}$ is as in (d) of that proposition. Now using Lemma 3.6 (c) and $\breve{E}_x(\mathsf{z}_h) = \frac{1}{\varrho_h}E_x(h^p)\mathsf{z}_h = -\frac{(h')^p}{\varrho_h}\mathsf{z}_h$, we have

$$\breve{E}_x(x^{jp}) = -\frac{h^p}{\varrho_h} j x^{(j-1)p} \quad \text{and} \quad \breve{E}_x(\mathsf{z}_h^k) = -k\mathsf{z}_h^k\frac{(h')^p}{\varrho_h}, \tag{6.15}$$

and thus,

$$\mathsf{Res}(\breve{E}_x) = -\frac{1}{\varrho_h}\left(h^p\frac{d}{d(x^p)} + (h')^p\mathsf{z}_h\frac{d}{d\mathsf{z}_h}\right).$$

In particular, for

$$\breve{F} = \mathsf{z}_h D_{\frac{h'}{\varrho_h}} - \breve{E}_x, \quad \text{we have} \quad \mathsf{Res}(\breve{F}) = \frac{h^p}{\varrho_h}\frac{d}{d(x^p)} \tag{6.16}$$

by Proposition 6.10 (e).

Theorem 6.17. *Assume* $\mathsf{char}(\mathbb{F}) = p > 0$*, and let* $\mathsf{Res} : \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h) \to \mathsf{Der}_{\mathbb{F}}(\mathsf{Z}(\mathsf{A}_h))$ *be the restriction map and* $\overline{\mathsf{Res}} : \mathsf{HH}^1(\mathsf{A}_h) \to \mathsf{Der}_{\mathbb{F}}(\mathsf{Z}(\mathsf{A}_h))$ *be the induced map. Then the following hold.*

(a) $\mathsf{im}\,\mathsf{Res} = \mathsf{im}\,\overline{\mathsf{Res}}$ *is a free* $\mathsf{Z}(\mathsf{A}_h)$*-submodule of* $\mathsf{Der}_{\mathbb{F}}(\mathsf{Z}(\mathsf{A}_h))$ *of rank 2 generated over* $\mathsf{Z}(\mathsf{A}_h)$ *by* $\frac{h^p}{\varrho_h}\frac{d}{d(x^p)}$ *and* $\overline{h}\frac{d}{d\mathsf{z}_h}$*, where* $\overline{h}$ *is as in Proposition 6.10 (c).*

(b) *If* $t_1 = x^p$*,* $t_2 = \mathsf{z}_h$*, and if* $\mathsf{Z}(\mathsf{A}_h)$ *is identified with* $\mathbb{F}[t_1, t_2]$*, then* $\mathsf{im}\,\mathsf{Res}$ *is isomorphic to the subalgebra of the Witt algebra* $\mathsf{Der}_{\mathbb{F}}(\mathbb{F}[t_1, t_2])$ *generated over* $\mathbb{F}[t_1, t_2]$ *by* $\mathsf{d}_1 = \frac{h^p}{\varrho_h}\frac{d}{dt_1}$*,* $\mathsf{d}_2 = \overline{h}\frac{d}{dt_2}$*, where*

$$[\mathsf{d}_1, \mathsf{d}_2] = \frac{d}{dt_1}\left(\overline{h}\right)\frac{h^p}{\varrho_h \overline{h}}\,\mathsf{d}_2.$$

Proof. By the above and Proposition 6.10, for part (a) it suffices to show that

$$\mathsf{im}\,\mathsf{Res} \subseteq \mathsf{Z}(\mathsf{A}_h)\mathsf{Res}(\mathcal{D}_{\mathsf{R}}) + \mathsf{Z}(\mathsf{A}_h)\mathsf{Res}(\breve{E}_x).$$

Given $D \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$, we have established that there exist $g \in \mathsf{R}$, $u \in \mathsf{Z}(\mathsf{A}_h)$, $z \in \mathsf{Z}(\mathsf{A}_1)$, $b \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)_{\not\equiv 0}$ and $c \in \mathsf{C}_{\mathsf{A}_1}(x)$, as in Lemma 6.5, such that $D = D_g + u\breve{E}_x + \mathsf{ad}_b + E$, where $E = zE_y + \mathsf{ad}_c$ and $D_g, u\breve{E}_x, \mathsf{ad}_b, E \in \mathsf{Der}_{\mathbb{F}}\mathsf{A}_h$. Clearly, $\mathsf{Res}(D_g)$, $\mathsf{Res}(u\breve{E}_x)$, and $\mathsf{Res}(\mathsf{ad}_b) = 0$ belong to $\mathsf{Z}(\mathsf{A}_h)\mathsf{Res}(\mathcal{D}_{\mathsf{R}}) + \mathsf{Z}(\mathsf{A}_h)\mathsf{Res}(\breve{E}_x)$, so it remains to argue that the same holds for $\mathsf{Res}(E)$. Note that $E(x) = 0$, so $[E(\hat{y}), x] = 0$, showing that $E(\hat{y}) \in \mathsf{C}_{\mathsf{A}_h}(x) = \mathsf{Z}(\mathsf{A}_h)\mathsf{R}$. Thus, $E \in \mathsf{Z}(\mathsf{A}_h)\mathcal{D}_{\mathsf{R}}$ and $\mathsf{Res}(E) \in \mathsf{Z}(\mathsf{A}_h)\mathsf{Res}(\mathcal{D}_{\mathsf{R}})$.

For part (b), observe that

$$(6.18)\quad \begin{aligned}[][\mathsf{Res}(\breve{F}), \mathsf{Res}(D_{\breve{q}})] &= \left[\frac{h^p}{\varrho_h}\frac{d}{d(x^p)}, \overline{h}\frac{d}{d\mathsf{z}_h}\right] \\ &= \frac{d}{d(x^p)}\left(\overline{h}\right)\frac{h^p}{\varrho_h\overline{h}}\,\overline{h}\frac{d}{d\mathsf{z}_h} = \frac{d}{d(x^p)}\left(\overline{h}\right)\frac{h^p}{\varrho_h\overline{h}}\,\mathsf{Res}(D_{\breve{q}}).\end{aligned}$$

The result is apparent from that, since $\mathsf{d}_1 = \frac{h^p}{\varrho_h}\frac{d}{dt_1} = \mathsf{Res}(\breve{F})$ and $\mathsf{d}_2 = \overline{h}\frac{d}{dt_2} = \mathsf{Res}(D_{\breve{q}})$, where $t_1 = x^p$, $t_2 = \mathsf{z}_h$. □

Example 6.19. *Assume* $h = x^m$*, with* $m \geq 0$*. Write* $m = kp + n$ *with* $k \geq 0$ *and* $0 \leq n < p$*, and set* $t_1 = x^p$ *and* $t_2 = \mathsf{z}_h$*, so that* $\mathsf{Z}(\mathsf{A}_h) = \mathbb{F}[t_1, t_2]$*. Then* $\frac{h^p}{\varrho_h} = t_1^{m-k}$ *and*

$$(6.20)\qquad \overline{h} = \begin{cases} t_1^{m-k} & \text{if } n = 0 \\ t_1^{m-k-1} & \text{if } n \neq 0. \end{cases}$$

Thus, $\mathsf{im}\,\mathsf{Res}$ *is the Lie subalgebra of* $\mathsf{Der}_{\mathbb{F}}(\mathbb{F}[t_1, t_2])$ *generated over* $\mathbb{F}[t_1, t_2]$ *by*

$$t_1^{m-k}\frac{d}{dt_1} \quad \textit{and} \quad t_1^{m-k}\frac{d}{dt_2} \qquad \text{if } n = 0$$
$$t_1^{m-k}\frac{d}{dt_1} \quad \textit{and} \quad t_1^{m-k-1}\frac{d}{dt_2} \qquad \text{if } n \neq 0.$$

Special cases of this result are displayed in the table below:

h	m	k	n	*generators*
1	0	0	0	$\frac{d}{dt_1},\ \frac{d}{dt_2}$
x	1	0	1	$t_1\frac{d}{dt_1},\ \frac{d}{dt_2}$
x^2 $(p>2)$	2	0	2	$t_1^2\frac{d}{dt_1},\ t_1\frac{d}{dt_2}$
x^2 $(p=2)$	2	1	0	$t_1\frac{d}{dt_1},\ t_1\frac{d}{dt_2}$

When $h=1$*, then* $\overline{\mathsf{Res}}$ *is surjective, and by Corollary 6.14 we also know* $\overline{\mathsf{Res}}$ *is injective, as* $\Theta = \mathsf{im}\,\delta$*, so we retrieve a previously established result: the induced map* $\overline{\mathsf{Res}} : \mathsf{HH}^1(\mathsf{A}_1) \to \mathsf{Der}_{\mathbb{F}}(\mathsf{Z}(\mathsf{A}_1))$ *is an isomorphism (see Theorem 3.8 (b)).*

6.5. **Main theorems about derivations.**

Assume $\bar{h} \in \mathbb{F}[x^p]$ and $\breve{q} \in \mathsf{R}$ are as in Proposition 6.10, so that under the restriction map, $\mathsf{Res}(D_{\breve{q}}) = \bar{h}\frac{d}{d\mathsf{z}_h}$. Recall from (6.16) that the derivation $\breve{F} = \mathsf{z}_h D_{\frac{h'}{\varrho_h}} - \breve{E}_x \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$ has the property that $\mathsf{Res}(\breve{F}) = \frac{h^p}{\varrho_h}\frac{d}{d(x^p)}$. Then Res maps $\mathsf{Z}(\mathsf{A}_h)D_{\breve{q}} \oplus \mathsf{Z}(\mathsf{A}_h)\breve{F}$ isomorphically onto $\mathsf{im}\,\mathsf{Res}$ as $\mathsf{Z}(\mathsf{A}_h)$-modules by Theorem 6.17, which leads to our main result on derivations.

Theorem 6.21. *Assume* $\mathsf{char}(\mathbb{F}) = p > 0$*. Then as a* $\mathsf{Z}(\mathsf{A}_h)$*-module,*

$$\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h) = \mathsf{Z}(\mathsf{A}_h)D_{\breve{q}} \oplus \mathsf{Z}(\mathsf{A}_h)\breve{F} \oplus \Big(\mathcal{D}_\Theta + \{\mathsf{ad}_a \mid a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)\}\Big), \tag{6.22}$$

where

(i) $D_r(x) = 0,\ D_r(\hat{y}) = r,$ *for all* $r \in \mathsf{R}$*;*
(ii) $\mathcal{D}_\Theta = \{D_r \mid r \in \Theta\}$ *and* $\Theta = \{r \in \mathsf{R} \mid \mathsf{Res}(D_r) = 0\}$ *as in Proposition 6.10 (a);*
(iii) $D_{\breve{q}}$ *is as in Proposition 6.10 (d);*
(iv) $\breve{F} = \mathsf{z}_h D_{\frac{h'}{\varrho_h}} - \breve{E}_x = \mathsf{z}_h D_{\frac{h'}{\varrho_h}} - \frac{h^p}{\varrho_h}E_x$*. Hence,* $\breve{F}(x) = -\frac{h^p}{\varrho_h}y^{p-1}$*, and*
$$\breve{F}(\hat{y}) = \frac{h^p}{\varrho_h}\sum_{k=1}^{p-2}\frac{(-1)^k}{(k+1)k}h^{(k+1)}y^{p-k} + \frac{h^p}{\varrho_h}\left(\partial_p(h)y + \partial_p(h')\right),$$
where ∂_p *is as in* (3.7).

Proof. Suppose $D \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$. Then there exist $u, v \in \mathsf{Z}(\mathsf{A}_h)$ such that $\mathsf{Res}(D) = u\bar{h}\frac{d}{d\mathsf{z}_h} + v\frac{h^p}{\varrho_h}\frac{d}{d(x^p)} = u\mathsf{Res}(D_{\breve{q}}) + v\mathsf{Res}(\breve{F}) = \mathsf{Res}(uD_{\breve{q}} + v\breve{F})$. Consequently, $D - uD_{\breve{q}} - v\breve{F}$ belongs to $\ker \mathsf{Res}$, which is $\mathcal{D}_\Theta + \{\mathsf{ad}_a \mid a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)\}$ by Proposition 6.9. This implies that D belongs to the right-hand side of (6.22). But since the right-hand side is clearly contained in $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$, we have the result. The action of $\breve{F}$ on x and $\hat{y}$ is a consequence of Lemma 3.6. □

Corollary 6.23. *There exists a finite-dimensional subspace* S *of* R *such that* $\Theta = \mathsf{S} \oplus \mathsf{im}\,\delta$ *and*

$$\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h) = \mathsf{Z}(\mathsf{A}_h)D_{\breve{q}} \oplus \mathsf{Z}(\mathsf{A}_h)\breve{F} \oplus \Big(\mathcal{D}_\mathsf{S} \oplus \{\mathsf{ad}_a \mid a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)\}\Big)$$

as a $\mathsf{Z}(\mathsf{A}_h)$*-module, where* $\mathcal{D}_\mathsf{S} = \{D_s \mid s \in \mathsf{S}\}$ *and* $\mathsf{S} = 0$ *if* $\Theta = \operatorname{im}\delta$.

The information in Examples 6.12 and 6.19, coupled with Theorem 6.21, enables us to determine $\mathsf{Der}_\mathbb{F}(\mathsf{A}_h)$ explicitly for any $h = x^m$.

Corollary 6.24. *Let* $h = x^m$*, where* $m = kp + n$*,* $k \geq 0$*, and* $0 \leq n < p$*. Then*

(i) $\mathsf{Der}_\mathbb{F}(\mathsf{A}_h) = \mathsf{Z}(\mathsf{A}_h)D_{x^{p-1}} \oplus \mathsf{Z}(\mathsf{A}_h)x^{m(p-1)}E_x \oplus \mathcal{D}_\mathsf{S} \oplus \{\mathsf{ad}_a \mid a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)\}$ *if* $n = 0$*, and*

(ii) $\mathsf{Der}_\mathbb{F}(\mathsf{A}_h) = \mathsf{Z}(\mathsf{A}_h)D_{x^{n-1}} \oplus \mathsf{Z}(\mathsf{A}_h)x^{(m-k)p}E_x \oplus \mathcal{D}_\mathsf{S} \oplus \{\mathsf{ad}_a \mid a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)\}$ *if* $1 \leq n < p$,

where $\mathsf{S} = \mathsf{span}_\mathbb{F}\{x^i \mid 0 \leq i < m,\ i \not\equiv n-1 \bmod p\}$ *in* (i) *and* (ii).

Proof. (i) If $n = 0$, then as in (6.20) we have $\overline{h} = (x^p)^{m-k} = h^{p-1}$, and so $\breve{q} = -x^{p-1}$. Since $h' = 0$, $\breve{F} = -\frac{h^p}{\varrho_h}E_x = -x^{m(p-1)}E_x$.

(ii) If $n \neq 0$, $h^{p-1} = (x^p)^{m-k-1} \cdot x^{p-n}$, $\overline{h} = (x^p)^{m-k-1}$, and $\breve{q} = -x^{n-1}$. Since $h' = nx^{m-1}$ and $\varrho_h = x^{kp}$, we have $\breve{F} = \mathsf{z}_h D_{\frac{h'}{\varrho_h}} - \breve{E}_x = n\mathsf{z}_h D_{x^{n-1}} - x^{(m-k)p}E_x$.

In both (i) and (ii), the subspace S can be determined from Example 6.12. □

Here are a few particular instances of these results.

Example 6.25.

- *When* $h = 1$*, then* $\breve{q} = -x^{p-1}$*,* $D_{\breve{q}} = -E_y$*, and* $\breve{F} = -E_x$*, so that*

$$\mathsf{Der}_\mathbb{F}(\mathsf{A}_1) = \mathsf{Z}(\mathsf{A}_1)E_x \oplus \mathsf{Z}(\mathsf{A}_1)E_y \oplus \mathsf{Inder}_\mathbb{F}(\mathsf{A}_1) \qquad \textit{(Theorem 3.8).}$$

- *When* $h = x$*, then* $\breve{q} = -1$*,* $D_{\breve{q}} = -D_1$*,* $\breve{F} = \mathsf{z}_h D_1 - x^p E_x$*, and*

$$\mathsf{Der}_\mathbb{F}(\mathsf{A}_x) = \mathsf{Z}(\mathsf{A}_x)D_1 \oplus \mathsf{Z}(\mathsf{A}_x)x^pE_x \oplus \mathsf{Inder}_\mathbb{F}(\mathsf{A}_x).$$

(That $\{\mathsf{ad}_a \mid a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_x)\} = \mathsf{Inder}_\mathbb{F}(\mathsf{A}_x)$ *follows from Theorem 6.29 below, or this could be deduced from Theorem 2.17.)*

- *When* $h = x^n$*,* $2 \leq n < p$*, then* $\mathsf{S} = \mathsf{span}_\mathbb{F}\{x^i \mid 0 \leq i \leq n-2\}$ *and*

$$\mathsf{Der}_\mathbb{F}(\mathsf{A}_{x^n}) = \mathsf{Z}(\mathsf{A}_{x^n})D_{x^{n-1}} \oplus \mathsf{Z}(\mathsf{A}_{x^n})x^{np}E_x \oplus \mathcal{D}_\mathsf{S} \oplus \{\mathsf{ad}_a \mid a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_{x^n})\}.$$

The next example generalizes the $n = 0$ case above.

Example 6.26. *Assume* $h \in \mathbb{F}[x^p]$*. Then* $\overline{h} = h^{p-1}$*;* $\breve{q} = -x^{p-1}$*;* $\Theta = \{r \in \mathsf{R} \mid rh^{p-1} \in \operatorname{im}\frac{d}{dx}\} = \operatorname{im}\frac{d}{dx}$ *as* $h^{p-1} \in \mathbb{F}[x^p]$ *and* $r'h^{p-1} = (rh^{p-1})'$*. Since* $\delta_0(r) = (rh^{-1})'h = r' \in \operatorname{im}\frac{d}{dx}$*, we have* $\operatorname{im}\delta_0 = \operatorname{im}\frac{d}{dx} = \Theta$*. Now* $\breve{F} = \mathsf{z}_h D_{\frac{h'}{\varrho_h}} - \breve{E}_x = -\lambda h^{p-1}E_x$*, where* λ *is the leading coefficient of* h*. Thus,*

$$\mathsf{Der}_\mathbb{F}(\mathsf{A}_h) = \mathsf{Z}(\mathsf{A}_h)D_{x^{p-1}} \oplus \mathsf{Z}(\mathsf{A}_h)h^{p-1}E_x \oplus \mathcal{D}_\mathsf{S} \oplus \{\mathsf{ad}_a \mid a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)\},$$

where $\mathsf{S} = \mathsf{span}_\mathbb{F}\{x^i \mid 0 \leq i < \deg h,\ i \not\equiv -1 \bmod p\}$.

Proposition 6.27. *Suppose $D = uD_{\breve q} + v\breve F + D_r + \mathsf{ad}_a \in \mathsf{Inder}_{\mathbb F}(\mathsf A_h)$, where $u, v \in \mathsf Z(\mathsf A_h)$, $r \in \Theta$, and $a \in \mathsf N_{\mathsf A_1}(\mathsf A_h)$. Then $u = 0 = v$, $r \in \mathsf{im}\,\delta$ and $a \in \mathsf A_h + \mathsf Z(\mathsf A_1)$. Thus, $\mathsf{HH}^1(\mathsf A_h) = \mathsf{Der}_{\mathbb F}(\mathsf A_h)/\mathsf{Inder}_{\mathbb F}(\mathsf A_h) \cong \mathsf Z(\mathsf A_h)D_{\breve q} \oplus \mathsf Z(\mathsf A_h)\breve F \oplus \mathcal H$, where*

$$\begin{aligned}\mathcal H &= \ker \overline{\mathsf{Res}} = \Big(\mathcal D_\Theta + \{\mathsf{ad}_a \mid a \in \mathsf N_{\mathsf A_1}(\mathsf A_h)\}\Big)/\Big(\mathcal D_{\mathsf{im}\,\delta} + \{\mathsf{ad}_a \mid a \in \mathsf A_h\}\Big),\\ &\cong \big(\Theta/\mathsf{im}\,\delta\big) \ \oplus\ \big(\mathsf N_{\mathsf A_1}(\mathsf A_h)/\big(\mathsf A_h + \mathsf Z(\mathsf A_1)\big)\big),\end{aligned}$$

and this decomposition of $\mathcal H$ is as an $\mathbb F[x^p]$-module.

Proof. Applying D to $\mathsf Z(\mathsf A_h)$ shows that $u = 0 = v$. The remaining assertions come directly from Proposition 6.13. □

6.6. $\mathsf{HH}^1(\mathsf A_h)$ as a $\mathsf Z(\mathsf A_h)$-module.

Proposition 6.27 gives a $\mathsf Z(\mathsf A_h)$-module decomposition of $\mathsf{HH}^1(\mathsf A_h)$, since $\overline{\mathsf{Res}}$ is a $\mathsf Z(\mathsf A_h)$-module map. The main result of this section is Theorem 6.29, which provides necessary and sufficient conditions for $\mathsf{HH}^1(\mathsf A_h)$ to be a free $\mathsf Z(\mathsf A_h)$-module. Our proof of this result uses the map $\delta_0 : \mathsf R \to \mathsf R$ with $\delta_0(r) = \delta(ra_0)$, where $a_0 = \pi_h h^{-1}$, along with the properties in Section 4.8 that δ_0 satisfies.

Lemma 6.28. *Let $\Theta = \{r \in \mathsf R \mid \mathsf{Res}(D_r) = 0\}$ as in Proposition 6.10 (a). Then*

(i) $\mathsf{im}\,\delta \subseteq \mathsf{im}\,\delta_0 \subseteq \Theta$*;*
(ii) $\delta_0(1) = 0$ *if and only if* $\frac{h}{\pi_h \varrho_h} \in \mathbb F^*$*;*
(iii) $\mathsf{im}\,\delta_0$ *is a free $\mathbb F[x^p]$-submodule of $\mathsf R$ of rank $p-1$;*
(iv) *If $\frac{h}{\pi_h\varrho_h} \in \mathbb F^*$, then $\mathsf{im}\,\delta_0 = \Theta$, and $\mathsf R = \mathbb F[x^p]\breve q \oplus \Theta = \mathbb F[x^p]\breve q \oplus \mathsf{im}\,\delta_0$, where $\breve q$ is as in (d) of Proposition 6.10.*

Proof. (i) Recall from (a) of Lemma 4.14 that $D_{\delta_0(r)} = -\mathsf{ad}_{ra_0}$ for $r \in \mathsf R$. This implies that $\mathsf{Res}(D_{\delta_0(r)}) = 0$, where Res is the restriction to $\mathsf Z(\mathsf A_h)$, and hence that $\mathsf{im}\,\delta_0 \subseteq \Theta$. That $\mathsf{im}\,\delta \subseteq \mathsf{im}\,\delta_0$ follows easily from the fact $\delta(r) = \delta(r\frac{h}{\pi_h}\frac{\pi_h}{h}) = \delta_0(r\frac{h}{\pi_h})$ for all $r \in \mathsf R$.

(ii) By Lemma 4.28 (a), $\delta_0(1) = 0$ if and only if $1 \in \ker\delta_0 = (\mathsf R \cap \mathsf Z(\mathsf A_h))\frac{h}{\pi_h\varrho_h} = \mathbb F[x^p]\frac{h}{\pi_h\varrho_h}$; whence $\delta_0(1) = 0$ if and only if $\frac{h}{\pi_h\varrho_h} \in \mathbb F^*$.

(iii) The identity $\delta_0(rs) = r\delta_0(s) + r' s\pi_h = r\delta_0(s)$, which holds for all $r \in \mathbb F[x^p]$ by (b) of Lemma 4.14, implies that $\mathsf{im}\,\delta_0$ is an $\mathbb F[x^p]$-submodule of the free $\mathbb F[x^p]$-module $\mathsf R$. As $\mathbb F[x^p]$ is a Dedekind domain, it is hereditary, so $\mathsf{im}\,\delta_0$ is free, and the short exact sequence

$$0 \to \ker\delta_0 \to \mathsf R \xrightarrow{\delta_0} \mathsf{im}\,\delta_0 \to 0$$

splits. Since $\ker\delta_0 = \mathbb F[x^p]\frac{h}{\pi_h\varrho_h}$ has rank 1, it follows that $\mathsf{im}\,\delta_0$ has rank $p-1$.

(iv) Assume $\frac{h}{\pi_h\varrho_h} \in \mathbb F^*$. Let us first dispose of the case that $h \in \mathbb F[x^p]$. Then $\pi_h = 1$, $\frac{h}{\varrho_h} \in \mathbb F^*$, and $\delta_0 = \frac{d}{dx}$, so that $\mathsf{im}\,\delta_0 = \mathsf{im}\,\frac{d}{dx}$. From Example 6.26, we have $\breve q = -x^{p-1}$, $\Theta = \mathsf{im}\,\frac{d}{dx}$, and $\mathsf R = \mathbb F[x^p]\breve q \oplus \mathsf{im}\,\frac{d}{dx} = \mathbb F[x^p]\breve q \oplus \mathsf{im}\,\delta_0$.

Henceforth, we assume $h \notin \mathbb F[x^p]$. Suppose we can show that in this case there exists $\kappa \in \mathsf R$ such that $\mathsf R = \mathbb F[x^p]\kappa \oplus \mathsf{im}\,\delta_0$. Then since $\mathsf{im}\,\delta_0 \subseteq \Theta$ by (i), and $\mathsf R \neq \Theta$ by Proposition 6.10, it follows that $\kappa \notin \Theta$. Any $r \in \Theta$ must have trivial projection onto $\mathbb{F}[x^p]\kappa$, as $\mathsf{Res}(D_r) = 0$. Hence, $\Theta \subseteq \mathsf{im}\,\delta_0$, equality would hold, and (iv) would follow from Proposition 6.10.

By (iii), it will be enough to show that the $\mathbb{F}[x^p]$-module $\mathsf{R}/\mathsf{im}\,\delta_0$ is torsion free, as this will imply it is free, so that the natural epimorphism $\mathsf{R} \to \mathsf{R}/\mathsf{im}\,\delta_0$ will yield the decomposition $\mathsf{R} = \mathsf{K} \oplus \mathsf{im}\,\delta_0$, for some rank-one free $\mathbb{F}[x^p]$-submodule $\mathsf{K} = \mathbb{F}[x^p]\kappa$.

Claim: The $\mathbb{F}[x^p]$*-module* $\mathsf{R}/\mathsf{im}\,\delta_0$ *is torsion free.*

Proof of the claim: We will show that whenever $s \in \mathsf{R}$, $0 \neq w \in \mathbb{F}[x^p]$, and $ws \in \mathsf{im}\,\delta_0$, then $s \in \mathsf{im}\,\delta_0$. We can assume $w \notin \mathbb{F}$.

First notice that $\mathsf{R} = \mathbb{F}[x^p]x^{p-1} \oplus \mathsf{im}\frac{d}{dx}$, so that $\mathsf{R}/\mathsf{im}\frac{d}{dx}$ is a torsion-free $\mathbb{F}[x^p]$-module. This means that if $w \in \mathbb{F}[x^p]$ divides r', for some $r \in \mathsf{R}$, then $r' = w\tilde{r}\,'$ for some $\tilde{r} \in \mathsf{R}$.

By assumption $\frac{h}{\pi_h \varrho_h} \in \mathbb{F}^*$, so we have that $\delta_0(r) = r\delta_0(1) + r'\pi_h = r'\pi_h$ by (ii). Thus, we need to show that $w \mid r'\pi_h$ implies $w \mid r'$, for all $r \in \mathsf{R}$. Since we are in the case $h \notin \mathbb{F}[x^p]$, we can assume $\pi_h = \mathsf{u}_1 \cdots \mathsf{u}_{\ell'}$, where the u_i are distinct monic prime factors of h in R and $\mathsf{u}_i \notin \mathbb{F}[x^p]$ for all $i = 1, \ldots, \ell'$. Suppose that $w \mid r'\pi_h$ for some $r \in \mathsf{R}$. Let v be a prime factor of w in R, and let $\alpha \geq 1$ be the largest power of v that divides w. Since $w \in \mathbb{F}[x^p]$, this implies that $v^\alpha \in \mathbb{F}[x^p]$. The claim will be proved if we show that v^α divides r'. This is clear if v and u_i are coprime for all i, so we can assume, without loss of generality, that $v = \mathsf{u}_1$. Since $\mathsf{u}_1 \notin \mathbb{F}[x^p]$, it follows that $p \mid \alpha$, say $\alpha = pn$ for some $n \geq 1$, and u_1^{pn-1} divides r'. In particular, $\mathsf{u}_1^{p(n-1)} \in \mathbb{F}[x^p]$ divides r', so by the above there exists $\tilde{r} \in \mathsf{R}$ so that $r' = \mathsf{u}_1^{p(n-1)}\tilde{r}\,'$. Moreover, u_1^{p-1} divides $\tilde{r}\,'$. We will finish the proof of the claim by showing that this implies that u_1^p divides $\tilde{r}\,'$. This will be accomplished in three steps:

Step 1: Assume $\mathsf{u}_1 = x$. Then $tx^{p-1} = \tilde{r}\,'$, for some $t \in \mathsf{R}$. In particular, $tx^{p-1} \in \mathsf{im}\frac{d}{dx} = \bigoplus_{i=0}^{p-2} \mathbb{F}[x^p]x^i$, so $t \in \bigoplus_{i=1}^{p-1} \mathbb{F}[x^p]x^i$. Hence x divides t, and $\mathsf{u}_1^p = x^p$ divides $\tilde{r}\,'$.

Step 2: Assume $\deg \mathsf{u}_1 = 1$. Then there is $\xi \in \mathbb{F}$ so that $\mathsf{u}_1 = x - \xi$. Note that the automorphism $\sigma_\xi : \mathsf{R} \to \mathsf{R}$ given by $x \mapsto x + \xi$ commutes with the derivation $\frac{d}{dx}$, as $(x + \xi)' = 1$. Thus, if we apply σ_ξ to the relation $\tilde{r}\,' = \mathsf{u}_1^{p-1}t$ we obtain

$$\sigma_\xi(\tilde{r})\,' = \sigma_\xi(\tilde{r}\,') = \sigma_\xi(\mathsf{u}_1)^{p-1}\sigma_\xi(t) = x^{p-1}\sigma_\xi(t).$$

Then by **Step 1** we have that $\sigma_\xi(\tilde{r}\,') = x^p\tilde{t}$, for some $\tilde{t} \in \mathsf{R}$. Applying $\sigma_\xi^{-1} = \sigma_{-\xi}$ to that relation, we obtain $\tilde{r}' = (x - \xi)^p\,\sigma_{-\xi}(\tilde{t})$, so that u_1^p divides $\tilde{r}\,'$.

Step 3: The general case. Consider the factorization $\mathsf{f}_1^{\beta_1} \cdots \mathsf{f}_k^{\beta_k}$ of u_1 into linear factors over the algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$. As $\mathsf{u}_1 \notin \mathbb{F}[x^p]$, we have that $\mathsf{u}_1{}' \neq 0$, so u_1 and u_1' are coprime. This implies that $\beta_j = 1$ for all j, and thus $\mathsf{u}_1^{p-1} = \mathsf{f}_1^{p-1} \cdots \mathsf{f}_k^{p-1}$. Since $\deg \mathsf{f}_j = 1$, we can apply **Step 2** to conclude that for all j, f_j^p divides $\tilde{r}\,'$ in $\overline{\mathbb{F}}[x]$. Hence, u_1^p divides $\tilde{r}\,'$, and this occurs in $\mathbb{F}[x]$, as u_1^p and $\tilde{r}\,'$ are in $\mathbb{F}[x]$.

Thus, the claim is established, and there is $\kappa \in \mathsf{R}$ so that $\mathsf{R} = \mathbb{F}[x^p]\kappa \oplus \mathsf{im}\,\delta_0$. As we have argued earlier, this is sufficient to give the assertions in (iv). □

Theorem 6.29. *Assume* $\mathsf{char}(\mathbb{F}) = p > 0$*, and let* $D_{\breve{q}}$ *and* $\breve{F}$ *be as in Theorem 6.21. Then* $\mathsf{HH}^1(\mathsf{A}_h) = \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)/\mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_h)$ *is a free* $\mathsf{Z}(\mathsf{A}_h)$*-module if and only if* $\frac{h}{\pi_h} \in \mathbb{F}^*$*. When* $\frac{h}{\pi_h} \in \mathbb{F}^*$*, then*

$$\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h) = \mathsf{Z}(\mathsf{A}_h)D_{\breve{q}} \oplus \mathsf{Z}(\mathsf{A}_h)\breve{F} \oplus \mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_h),$$

so that $\mathsf{HH}^1(\mathsf{A}_h)$ *is a free* $\mathsf{Z}(\mathsf{A}_h)$*-module of rank* 2 *with* $\mathsf{Z}(\mathsf{A}_h)$*-basis* $\{D_{\breve{q}}, \breve{F}\}$.

Proof. Suppose first that $\mathsf{HH}^1(\mathsf{A}_h)$ is a free $\mathsf{Z}(\mathsf{A}_h)$-module. As $\mathsf{Z}(\mathsf{A}_h)$ is a domain, $\mathsf{HH}^1(\mathsf{A}_h)$ is torsion free over $\mathsf{Z}(\mathsf{A}_h)$. Note that $h^p\mathsf{ad}_{a_1} = \mathsf{ad}_{h^p a_1} = \mathsf{ad}_{h^p\pi_h y} \in \mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_h)$, so $\mathsf{ad}_{a_1} \in \mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_h)$, because $h^p \in \mathsf{Z}(\mathsf{A}_h)$. This implies that $\pi_h = [\pi_h y, x] = \mathsf{ad}_{a_1}(x) \in [\mathsf{A}_h, \mathsf{A}_h] \subseteq h\mathsf{A}_h$, by [BLO1, Lem. 6.1]. Hence h divides π_h and $\frac{h}{\pi_h} \in \mathbb{F}^*$.

Conversely, assume $\frac{h}{\pi_h} = \lambda \in \mathbb{F}^*$. Then $a_0 = \pi_h h^{-1} = \lambda^{-1}$, and $\delta_0(r) = \delta(\lambda^{-1}r)$ for all $r \in \mathsf{R}$. Therefore, $\mathsf{im}\,\delta = \mathsf{im}\,\delta_0 = \Theta$, where the last equality follows from (iv) of Lemma 6.28. By (a) of Corollary 6.23, $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h) = \mathsf{Z}(\mathsf{A}_h)D_{\breve{q}} \oplus \mathsf{Z}(\mathsf{A}_h)\breve{F} \oplus \{\mathsf{ad}_a \mid a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)\}$. Now suppose $a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)$. As in Remark 2.20, $a = b + c$ where $b \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)_{\not\equiv 0}$, and $c \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)_{\equiv 0}$. Because $\frac{h}{\pi_h} \in \mathbb{F}^*$, we know $b \in \mathsf{A}_h$. By Lemma 4.8, $\mathsf{ad}_c = D_f$ for some $f \in \mathsf{C}_{\mathsf{A}_h}(x) = \mathsf{Z}(\mathsf{A}_h)\mathsf{R}$. As $\mathsf{R} = \mathbb{F}[x^p]\breve{q} \oplus \Theta = \mathbb{F}[x^p]\breve{q} \oplus \mathsf{im}\,\delta$, it follows that $\mathsf{C}_{\mathsf{A}_h}(x) = \mathsf{Z}(\mathsf{A}_h)\breve{q} \oplus \mathsf{Z}(\mathsf{A}_h)\mathsf{im}\,\delta$. We may assume $f = u\breve{q} + \sum_i v_i\delta(r_i)$ for some $u, v_i \in \mathsf{Z}(\mathsf{A}_h)$ and $r_i \in \mathsf{R}$. But then $\mathsf{ad}_c = D_f = uD_{\breve{q}} + \sum_i v_i D_{\delta(r_i)} = uD_{\breve{q}} - \sum_i v_i\mathsf{ad}_{r_i}$ by (ii) of Proposition 4.6. The directness of the decomposition in Theorem 6.21 forces $u = 0$, and $\mathsf{ad}_c = -\sum_i v_i\mathsf{ad}_{r_i} = -\sum_i \mathsf{ad}_{v_i r_i} \in \mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_h)$. This shows that $\{\mathsf{ad}_a \mid a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)\} = \mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_h)$ and completes the proof. □

Remark 6.30. *When* $h = x$*, then* $\frac{h}{\pi_h} \in \mathbb{F}^*$*, so Theorem 6.29 gives the result* $\{\mathsf{ad}_a \mid a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_x)\} = \mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_x)$ *mentioned in Example 6.25.*

Remark 6.31. *When* $\frac{h}{\pi_h} \in \mathbb{F}^*$*, it follows from Theorem 6.29 and Proposition 6.27 that* $\mathcal{H} = \ker\overline{\mathsf{Res}} = 0$*. Hence, in this case,* $\mathsf{HH}^1(\mathsf{A}_h)$ *is isomorphic via the map* $\overline{\mathsf{Res}}$ *to the subalgebra of the Witt algebra* $\mathsf{Der}_{\mathbb{F}}(\mathbb{F}[t_1, t_2])$ *generated over* $\mathbb{F}[t_1, t_2]$ *by the derivations* $\mathsf{d}_1 = h^p\frac{d}{dt_1}, \mathsf{d}_2 = \overline{h}\frac{d}{dt_2}$*, where* $t_1 = x^p$ *and* $t_2 = \mathsf{z}_h$*, (see Theorem 6.17 for details).*

6.7. Products in $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$.

Suppose $u, v \in \mathsf{Z}(\mathsf{A}_h)$ and $D, E \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$. Then

$$[uD, vE] = uD(v)E - vE(u)D + uv[D, E]. \tag{6.32}$$

Equation (6.32) tells us that to compute products in $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$, it suffices to know the action of the restriction $\mathsf{Res}(D)$ on $\mathsf{Z}(\mathsf{A}_h) = \mathbb{F}[x^p, \mathsf{z}_h]$ for all derivations D in $\mathcal{B} = \left\{D_{\breve{q}}, \breve{F}, D_r, \mathsf{ad}_a \mid r \in \Theta, a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)\right\}$, where $D_{\breve{q}}$ and $\breve{F} = \mathsf{z}_h D_{\frac{h'}{\varrho_h}} - \breve{E}_x$

are as in Theorem 6.21, and the commutator $[D, E]$ for all pairs $D \neq E$ in $\mathcal{B}$. The first part is easy, since

$$\begin{aligned} \mathsf{Res}(D_{\breve{q}}) = \bar{h}\frac{d}{d\mathsf{z}_h}, \qquad \mathsf{Res}(\breve{F}) = \frac{h^p}{\varrho_h}\frac{d}{d(x^p)}, \quad \text{and} \\ \mathsf{Res}(D_r) = 0 = \mathsf{Res}(\mathsf{ad}_a) \ \ \forall\, r \in \Theta,\, a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h). \end{aligned} \tag{6.33}$$

Now it follows from Theorem 2.17 that any $a \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)$ has the form $a = b + c$, where $b \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)_{\not\equiv 0}$, $c \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)_{\equiv 0}$, and b is a sum of terms of the form ra_n with $a_n = \pi_h h^{n-1}y^n$ for $n \geq 1$, and $r \in \mathsf{R}$. Lemma 4.8 says that $\mathsf{ad}_c = D_f = \sum_i z_i D_{r_i}$ for some $f = \sum_i z_i r_i \in \mathsf{C}_{\mathsf{A}_h}(x) = \mathsf{Z}(\mathsf{A}_h)\mathsf{R}$. Hence, we are able to reduce our considerations to products of the form in (a)-(e) below, so that the commutator of any pair of derivations in $\mathcal{B}$ can be deduced from the next proposition.

Proposition 6.34. *Let $a_n = \pi_h h^{n-1}y^n$ for all $n \geq 0$, and assume $a_{-1} = 0$. The Lie brackets in $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$ satisfy the following, where $\delta_0(r) = (r\pi_h h^{-1})'h$, as in (4.13).*

(a) $[D_f, D_g] = 0$ *for all* $f, g \in \mathsf{R}$.

(b) $[D_g, \mathsf{ad}_{ra_n}] = n\mathsf{ad}_{gra_{n-1}} = n\mathsf{ad}_{ca_{n-1}}$ *in* $\mathsf{HH}^1(\mathsf{A}_h)$*, where c is the remainder of the division of gr by $\frac{h}{\pi_h}$ in* R.

(c) $[\mathsf{ad}_{ra_m}, \mathsf{ad}_{sa_n}] = \mathsf{ad}_{qa_{m+n-1}} = \mathsf{ad}_{da_{m+n-1}}$ *in* $\mathsf{HH}^1(\mathsf{A}_h)$ *for all $r, s \in \mathsf{R}$ and all $m, n \geq 0$, where $q = mr\delta_0(s) - ns\delta_0(r)$, and d is the remainder of the division in R of q by $\frac{h}{\pi_h}$.*

(d) *Assume $r \in \mathsf{R}$ and $m = kp + n$, where $k \geq 0$ and $0 \leq n < p$. Then in* $\mathsf{HH}^1(\mathsf{A}_h)$,

$$[\breve{E}_x, \mathsf{ad}_{ra_m}] = \mathsf{z}_h^k\,[\breve{E}_x, \mathsf{ad}_{ra_n}] = \begin{cases} \mathsf{z}_h^{k+1}\mathsf{ad}_{\zeta_n a_{n-1}} & \text{if } 1 \leq n < p, \\ \mathsf{z}_h^k\,[D_{\delta_0(r)}, \breve{E}_x] & \text{if } n = 0, \end{cases} \tag{6.35}$$

where $\zeta_n = \frac{h}{\pi_h \varrho_h}\delta_0(r) + nr\frac{h'}{\varrho_h}$, and the product $[D_{\delta_0(r)}, \breve{E}_x]$ can be computed using (e).

(e) *For $g \in \mathsf{R}$, $[D_g, \breve{E}_x] = D_e + \mathsf{ad}_b$, where $b = b_1 + b_2$ with*

$$b_1 = \frac{gh^{p-1}}{\varrho_h}y^{p-1} \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h), \quad b_2 = \sum_{k=2}^{p-1}(-1)^k\frac{(gh^{-1})^{(k-1)}h^p}{\varrho_h}\frac{y^{p-k}}{p-k} \in \mathsf{A}_h,$$

and $e = \Big([D_g, \breve{E}_x] - \mathsf{ad}_b\Big)(\hat{y}) \in \mathsf{C}_{\mathsf{A}_h}(x)$.

Proof. Part (a) is clear, and parts (b) and (c) are immediate from Lemma 4.15. For (d), we have $a_m = \mathsf{z}_h^k a_n$ so that

$$\begin{aligned} [\breve{E}_x, \mathsf{ad}_{ra_m}] &= [\breve{E}_x, \mathsf{z}_h^k\mathsf{ad}_{ra_n}] = \breve{E}_x(\mathsf{z}_h^k)\mathsf{ad}_{ra_n} + \mathsf{z}_h^k\,[\breve{E}_x, \mathsf{ad}_{ra_n}] \\ &= -k\mathsf{z}_h^k\,\mathsf{ad}_{r\frac{(h')^p}{\varrho_h}a_n} + \mathsf{z}_h^k\,[\breve{E}_x, \mathsf{ad}_{ra_n}] = \mathsf{z}_h^k\,[\breve{E}_x, \mathsf{ad}_{ra_n}] \end{aligned} \tag{6.36}$$

by (6.15), where the last equality holds because $h'a_n \in \mathsf{A}_h$ (see Theorem 2.17(b)). In particular, when $n = 0$, then $[\breve{E}_x, \mathsf{ad}_{ra_m}] = \mathsf{z}_h^k\,[\breve{E}_x, \mathsf{ad}_{ra_0}] = \mathsf{z}_h^k[D_{\delta_0(r)}, \breve{E}_x]$ as claimed in (d), since $\mathsf{ad}_{ra_0} = -D_{\delta_0(r)}$.

Assume $1 \le n < p$. Then the equalities $[\breve{E}_x, \mathsf{ad}_{ra_n}] = \frac{h^p}{\varrho_h}\mathsf{ad}_{E_x(r\pi_h h^{n-1})y^n}$ and

$$\frac{h^p}{\varrho_h}E_x(r\pi_h h^{n-1})y^n = \frac{1}{\varrho_h}\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k}(r\pi_h h^{n+p-1})^{(k)}y^{n+p-k} - \frac{h^p}{\varrho_h}\partial_p(r\pi_h h^{n-1})y^n$$

follow directly from Lemma 3.6. By Lemma 4.23, we have that $(r\pi_h h^{n+p-1})^{(k)} \in \mathsf{R}h^{n+p-k+1} + \mathsf{R}h^{n+p-k}h'$ for all $k \ge 2$, so that

$$\frac{1}{\varrho_h}\sum_{k=2}^{p-1}\frac{(-1)^{k-1}}{k}(r\pi_h h^{n+p-1})^{(k)}y^{n+p-k} \in \mathsf{A}_h,$$

as ϱ_h divides both h and h'. Since $n < p$, $\frac{h^p}{\varrho_h}\partial_p(r\pi_h h^{n-1})y^n \in \mathsf{A}_h$. Thus, modulo A_h we have

$$\frac{h^p}{\varrho_h}E_x(r\pi_h h^{n-1})y^n = \frac{1}{\varrho_h}(r\pi_h h^{n+p-1})'y^{n+p-1} = \mathsf{z}_h\,\zeta_n a_{n-1},$$

where $\zeta_n = \frac{h}{\pi_h\varrho_h}\delta_0(r) + nr\frac{h'}{\varrho_h}$. This combined with (6.36) gives (d) for $n \neq 0$.

To compute $[D_g, \breve{E}_x]$ in part (e), note that since $D_g(x) = 0$, Lemma 4.16 implies

$$\begin{aligned}[D_g, \breve{E}_x](x) &= \frac{h^p}{\varrho_h}\sum_{k=1}^{p-1}\binom{p-1}{k}(gh^{-1})^{(k-1)}y^{p-1-k} \\ &= \sum_{k=1}^{p-1}(-1)^k\frac{(gh^{-1})^{(k-1)}h^p}{\varrho_h}y^{p-1-k}.\end{aligned}$$

Let

$$b = \sum_{k=1}^{p-1}(-1)^k\frac{(gh^{-1})^{(k-1)}h^p}{\varrho_h}\frac{y^{p-k}}{p-k} \in \mathsf{A}_1. \tag{6.37}$$

Observe that $\mathsf{ad}_b(x) = [D_g, \breve{E}_x](x) \in \mathsf{A}_h$, and

$$b_1 = \frac{gh^{p-1}}{\varrho_h}y^{p-1} = g\frac{h}{\pi_h\varrho_h}(\pi_h h^{p-2}y^{p-1}) = g\frac{h}{\pi_h\varrho_h}a_{p-1} \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h). \tag{6.38}$$

It is easy to deduce from Lemma 4.18 that $\frac{(gh^{-1})^{(k-1)}h^k}{\varrho_h} \in \mathsf{R}$ for all $k \ge 2$, and thus

$$b_2 = \sum_{k=2}^{p-1}(-1)^k\frac{(gh^{-1})^{(k-1)}h^p}{\varrho_h}\frac{y^{p-k}}{p-k} \in \mathsf{A}_h.$$

As a result, $b = b_1 + b_2 \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)$.

Now $G = [D_g, \breve{E}_x] - \mathsf{ad}_b \in \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$ satisfies $G(x) = 0$ so that $0 = [G(\hat{y}), x]$. This shows that $e = G(\hat{y}) \in \mathsf{C}_{\mathsf{A}_h}(x)$. But then $(D_e - G)(x) = 0 = (D_e - G)(\hat{y})$, which implies that $G = D_e$. Consequently, $[D_g, \breve{E}_x] = D_e + \mathsf{ad}_b$, as desired. $\square$

It remains to determine the expression for $e = \big([D_g, \breve{E}_x] - \mathsf{ad}_b\big)(\hat{y})$ in part (e) of Proposition 6.34. We do so by considering the terms of $[D_g, \breve{E}_x](\hat{y})$ that centralize x. Define the projection map $\mathsf{P} : \mathsf{A}_1 \to \mathsf{C}_{\mathsf{A}_1}(x)$ by $\mathsf{P}(ry^k) = ry^k$ if $p \mid k$ and $\mathsf{P}(ry^k) = 0$ otherwise. Note that $\mathsf{P}(\mathsf{A}_h) = \mathsf{C}_{\mathsf{A}_h}(x)$ and $\mathsf{P}(ra) = r\mathsf{P}(a)$ for all $r \in \mathsf{R}$ and $a \in \mathsf{A}_1$.

Lemma 6.39. *Let $g, r \in \mathsf{R}$. Then*

(a) $\mathsf{P}(D_g(h^n y^n)) = h^n \left(gh^{-1}\right)^{(n-1)}$ *for $1 \leq n \leq p$;*
(b) $\mathsf{P}([ry^n, \hat{y}]) = rh^{(n+1)}$ *for $1 \leq n < p$ and $\mathsf{P}([r, \hat{y}]) = -r'h$.*

Proof. Corollary 4.17 (a) implies $D_g(h^n y^n) = \sum_{k=1}^{n} \binom{n}{k} h^n \left(gh^{-1}\right)^{(k-1)} y^{n-k}$ for all $1 \leq n \leq p$, and (a) is a direct consequence of this. Now (2.12) says $[ry^n, \hat{y}] = -(rh)'y^n + \sum_{k=1}^{n+1} \binom{n+1}{k} rh^{(k)} y^{n+1-k}$. Applying the map P to that yields (b). □

Proposition 6.40. *For $g \in \mathsf{R}$, write $[D_g, \breve{E}_x] = D_e + \mathsf{ad}_b$, with $e \in \mathsf{C}_{\mathsf{A}_h}(x)$ and $b \in \mathsf{N}_{\mathsf{A}_1}(\mathsf{A}_h)$ as in Proposition 6.34. Assume ∂_p is as in* (3.7). *Then*
(6.41)
$$e = \frac{1}{\varrho_h}\left(\sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k}(gh^{p-1})^{(k)} h^{(p-k)}\right) + \frac{h^{p-1}}{\varrho_h}\Big(h\partial_p(g) - g\partial_p(h)\Big) \in \mathsf{R}.$$

Proof. Note that $\mathsf{P}\left((D_e + \mathsf{ad}_b)(\hat{y})\right) = \mathsf{P}(e + [b, \hat{y}]) = e + \mathsf{P}([b, \hat{y}])$, so by (6.37) and Lemma 6.39, we have

$$\begin{aligned}
\mathsf{P}\left((D_e + \mathsf{ad}_b)(\hat{y})\right) &= e + \frac{1}{\varrho_h}\sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} \mathsf{P}\left(\Big[(gh^{p-1})^{(k-1)} y^{p-k}, \hat{y}\Big]\right) \\
&= e + \frac{1}{\varrho_h}\sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k}(gh^{p-1})^{(k-1)} h^{(p+1-k)} \\
&= e + \frac{1}{\varrho_h}\sum_{k=1}^{p-2} \frac{(-1)^{k}}{k+1}(gh^{p-1})^{(k)} h^{(p-k)}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(D_e + \mathsf{ad}_b)(\hat{y}) = [D_g, \breve{E}_x](\hat{y}) = D_g(\breve{E}_x(\hat{y})) - \breve{E}_x(g) \\
&\quad = \frac{1}{\varrho_h} D_g(h'h^p y^p) + \frac{1}{\varrho_h}\sum_{k=1}^{p-2} \frac{(-1)^{k-1}}{(k+1)k} h^{(k+1)} h^k\, D_g(h^{p-k} y^{p-k}) \\
&\qquad - \frac{h^{p-1}}{\varrho_h}\partial_p(h)\, D_g(hy) - \frac{h^p}{\varrho_h}\sum_{k=0}^{p-2} \frac{(-1)^k}{k+1}\, g^{(k+1)} y^{p-1-k} + \frac{h^p}{\varrho_h}\partial_p(g).
\end{aligned}$$

Hence,

$$\begin{aligned}\mathsf{P}\left((D_e+\mathsf{ad}_b)(\hat{y})\right) &= \frac{1}{\varrho_h}\sum_{k=1}^{p-2}\frac{(-1)^{k-1}}{(k+1)k}h^{(k+1)}(gh^{p-1})^{(p-k-1)} \\ &\quad +\frac{1}{\varrho_h}h'(gh^{p-1})^{(p-1)}+\frac{h^{p-1}}{\varrho_h}\big(h\partial_p(g)-g\partial_p(h)\big).\end{aligned}$$

Equating both expressions for $\mathsf{P}\left((D_e+\mathsf{ad}_b)(\hat{y})\right)$ gives

$$\begin{aligned}\varrho_h e &= h'(gh^{p-1})^{(p-1)}+h^{p-1}\left(h\partial_p(g)-g\partial_p(h)\right) \\ &\quad+\sum_{k=1}^{p-2}\frac{(-1)^{k-1}}{(k+1)k}h^{(p-k)}(gh^{p-1})^{(k)}+\sum_{k=1}^{p-2}\frac{(-1)^{k-1}}{k+1}(gh^{p-1})^{(k)}h^{(p-k)} \\ &=\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k}(gh^{p-1})^{(k)}h^{(p-k)}+h^{p-1}\big(h\partial_p(g)-g\partial_p(h)\big). \qquad \square\end{aligned}$$

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