The Lyapunov exponents of zero divergence 3-dimensional vector fields

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Abstract

We prove that for a C^1-generic (dense G_δ) subset of all the conservative vector fields on 3-dimensional compact manifolds without singularities, we have for μ-a.e. point $p \in M$ that either the Lyapunov exponents at p are zero or X is an Anosov vector field where μ is the Lebesgue measure. Then we prove that for a C^1-dense subset of all the conservative vector fields on 3-dimensional compact manifolds, we have for μ-a.e. $p \in M$ that either the Lyapunov exponents at p are zero or p belongs to a compact invariant set with (m_p-)dominated splitting for the linear Poincaré flow.

1 Introduction and statement of the results

Lyapunov exponents measure the exponential behavior of the tangent map of a dynamical system and if they are non-null together with Hölder regularity and the *Pesin Theory* of non-uniformly hyperbolic systems we get a rich information about geometric properties of the system, namely stable/unstable manifold theory for μ-a.e. point in M and this geometric tools are the base of most of the central results on dynamical systems nowadays. So it is of utmost importance detect when do Lyapunov exponents vanish.

A central result in this direction for discrete systems is the *Mañé-Bochi* theorem, which provides a C^1-residual set of area-preserving diffeomorphisms on surfaces where either we have Anosov systems or for μ-a.e. point zero Lyapunov exponents. This theorem was announced in the beginning of the 1980's by Mañé (1948-1995) in [11] but there was only available a sketch of a proof, see [12], the complete proof due to Bochi appeared in [4].

Motivated by these results, Bochi-Viana in [5] extend this result to a large class of discrete systems: volume preserving diffeomorphisms with arbitrary dimension, symplectic maps and also linear cocycles. For a survey of the theory see [6].

Highly inspired by their results we explore here the continuous-time case by following closely the strategy for the proof of discrete case doing the natural adaptations and developing the required techniques for perturbations of vector fields. Our first result is the analogous to the *Mañé-Bochi* theorem

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for volume-preserving vector fields in 3-dimensional compact manifolds without singularities;

Theorem 1 *There is a residual* $\mathfrak{R} \subseteq \mathfrak{X}^1_\mu(M)^*$ *such that if* $X \in \mathfrak{R}$ *then we have:*

(a) X *is Anosov or*

(b) zero Lyapunov exponents for μ*-a.e.* $p \in M$.

Finally developing the ideas of the proof of Theorem 1, jointly with some new observations and again in the 3-dimensional setting we are able to prove the following:

Theorem 2 *There is a dense* $\mathfrak{D} \subseteq \mathfrak{X}^1_\mu(M)$ *such that if* $X \in \mathfrak{D}$*, then there exists* X^t*-invariant sets* D *and* O *verifying* $\mu(D \cup O) = 1$ *and*

(a) For $p \in O$ *we have zero Lyapunov exponents.*

(b) D *is a countable increasing union of compact invariant sets* Λ_{m_n} *admitting a* m_n*-dominated splitting for the linear Poincaré flow.*

2 Preliminaries

2.1 Notation

Given a volume form ω, let μ be the measure associated to ω and we call it Lebesgue measure. We consider vector fields $X : M \to TM$, where M is a 3-dimensional compact, connected, without boundary C^∞ Riemannian manifold M. Given a vector field X we have an associated flow X^t which is the infinitesimal generator of X, i.e., $\frac{dX^t}{dt}|_{t=s}(p) = X(X^s(p))$. We denote by $\mathfrak{X}^1_\mu(M)$ the set of all C^1 vector fields which preserves μ. We assume that $\mathfrak{X}^1_\mu(M)$ is endowed with the C^1 topology. The flow X^t has a tangent map DX^t_p which is the solution of the non-autonomous linear differential equation $\dot{u(t)} = DX_{X^t(p)}.u(u)$ called the *linear variational equation*. The subset of $\mathfrak{X}^1_\mu(M)$ formed by the vector fields without singularities will be denoted by $\mathfrak{X}^1_\mu(M)^*$.

2.2 Oseledets theorem for 3-flows

The Oseledets theorem [15] is valid in the setting of discrete-time *cocycles* (for a prove see [13]). It holds in particular for any dynamical cocycle over a diffeomorphism $f : M \to M$ defined by a continuous map $F(p, v) = (f(p), Df_p.v)$ which verifies $\Pi \circ F = f \circ \Pi$ where $\Pi : TM \to M$ is the canonical projection and $F(p, .)$ is linear on the fiber T_pM. Oseledets theorem asserts that we have for μ-a.e. point $p \in M$ a splitting $T_pM = E^1_p \oplus ... \oplus E^{k(p)}_p$ (Oseledets splitting) and real numbers $\lambda_1(p) > ... > \lambda_{k(p)}(p)$ (Lyapunov exponents) such that $Df_p(E^i_p) = E^i_{f(p)}$ and

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|Df^n_p.v^i\| = \lambda_i(p)$$

for any $v^i \in E^i_p - \{0\}$ and $i = 1, ..., k(p)$. Consider $X \in \mathfrak{X}^1_\mu(M)$ and the associated flow $X^t : M \to M$. Since Oseledets theorem is an asymptotic result and $DX^r_{(.)}$, for fixed r, is an uniformly bounded operator we may replace the tangent map $DX^t_p = DX^r_{X^n(p)} \circ DX^n_p$ by the least integer time-n map, DX^n_p, and reformulate Oseledets theorem. Oseledets theorem allow us to conclude also that:

$$\lim_{t\to\pm\infty} \frac{1}{t}\log|\det(DX^t_p)| = \sum_{i=1}^{k(p)} \lambda_i(p).dim(E^i_p) \qquad (1)$$

which is related to the sub-exponential decrease of the angle between any subspaces of the Oseledets splitting along μ-a.e. orbits. Since we have $DX^t_p(X(p)) = X(X^t(p))$, we already know one of the Oseledets subspaces, $\mathbb{R}X(p)$, and also that its associated Lyapunov exponent is zero. For the other two, since in the conservative setting on 3-manifolds we have $|\det(DX^t_p)| = 1$, then by (1), we have $\lambda_1(p) + \lambda_3(p) = 0$. Hence either $\lambda_1(p) = -\lambda_3(p) > 0$ or both are zero. The former case gives for μ-a.e. $p \in M$ two directions E^u_p and E^s_p respectively associated to the positive Lyapunov exponent and the negative one with asymptotic exponential behavior. We denote by $\mathfrak{O}(X)$ the Oseledets points, $\mathfrak{O}^+(X) \subseteq \mathfrak{O}(X)$ the points with positive Lyapunov exponent and $\mathfrak{O}^0(X) \subseteq \mathfrak{O}(X)$ the points with all Lyapunov exponents zero. We note that $\mathfrak{O}^+(X) = \mathfrak{O}(X) - \mathfrak{O}^0(X)$. When there is no ambiguity we denote $\mathfrak{O}(X)$ by $\mathfrak{O}$ omitting the vector field.

2.3 The linear Poincaré flow

Let R be the set of regular points for the vector field X, i.e., $X(p) \neq 0$ for all $p \in R$. X induces a decomposition of the tangent bundle in a way that each fiber T_pM has a splitting $N_p \oplus \mathbb{R}X(p)$ where $N_p = \mathbb{R}X(p)^\perp$ is the normal sub-bundle for $p \in R$. Consider the automorphism of vector bundles:

$$\begin{array}{rccc} DX^t: & T_RM & \longrightarrow & T_RM \\ & (p,v) & \longmapsto & (X^t(p), DX^t(p).v) \end{array}$$

In spite of R being X^t-invariant and $\mathbb{R}X(p)$ being DX^t-invariant, there is no reason for the sub-bundle N_R to be DX^t-invariant. So consider the quotient space $\widetilde{N}_R = T_RM/\mathbb{R}X(R)$ of equivalence classes which is isometrically isomorphic to N_R via $\phi : N_R \longrightarrow \widetilde{N}_R$. The restriction map $DX^t\mid_{\widetilde{N}_R}$is DX^t-invariant. There exists an unique map $P^t_X(p) : N_R \longrightarrow N_R$ such that the diagram commutes:

$$\begin{array}{ccc} N_R & \xrightarrow{P^t_X} & N_R \\ \phi\downarrow & & \phi\downarrow \\ \widetilde{N}_R & \xrightarrow{DX^t} & \widetilde{N}_R \end{array}$$

Denoting by $\Pi_{X^t(p)}$ the canonical projection on $N_{X^t(p)}$, the linear map $P^t_X(p) : N_p \longrightarrow N_{X^t(p)}$ is defined by $P^t_X(p).v = \Pi_{X^t(p)} \circ DX^t(p).v$. The linear map is also a flow since $P^{t+s}_X(p) = P^t_X(X^s(p)) \circ P^s_X(p)$. We call $\{P^t_X(p)\}_{t\in\mathbb{R}}$ the *linear Poincaré flow* at p associated to the vector field X and this notion was first introduced by Doering in [8] to prove the hyperbolicity of robustly transitive 3-dimensional flows. In our setting, if we have an Oseledets point p with $X(p) \neq 0$ and $p \in \mathfrak{O}^+$, the Oseledets splitting on T_pM induces a $P^t_X(p)$-invariant splitting on N_p, say $\Pi_p(E^\sigma_p) = N^\sigma_p$ for $\sigma = u, s$. If $p \in \mathfrak{O}^0$, then the P^t_X-invariant splitting will be trivial, i.e., it is just the normal sub-bundle. In the next lemma we show that the dynamics remains the same.

Lemma 2.1 *The Lyapunov exponents of $P_X^t(p)$ associated to the subspaces N_p^u and N_p^s are respectively $0 \leq \lambda_u(p)$ and $\lambda_s(p) \leq 0$.*

Proof: Let $n^u \in N_p^u$ and denote by $\theta_t = \angle(X(X^t(p)), E^u_{X^t(p)})$ then,

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|P_X^t(p).n^u\| = \lim_{t\to\pm\infty} \frac{1}{t} \log \|\Pi_{X^t(p)} \circ DX_p^t.(\alpha.X(p) + v^u)\|,$$

for some $\alpha \in \mathbb{R}$ and $v^u \in E_p^u$. But then,

$$\begin{aligned}
&\lim_{t\to\pm\infty} \frac{1}{t} \log \|\alpha.\Pi_{X^t(p)} \circ DX_p^t.(X(p)) + \Pi_{X^t(p)} \circ DX_p^t.v^u)\| = \\
= &\lim_{t\to\pm\infty} \frac{1}{t} \log \|\alpha.\Pi_{X^t(p)} \circ X(X^t(p)) + \Pi_{X^t(p)} \circ DX_p^t.v^u)\| = \\
= &\lim_{t\to\pm\infty} \frac{1}{t} \log(sin(\theta_t).\|DX_p^t.v^u\|) = \\
= &\lim_{t\to\pm\infty} [\frac{1}{t} \log sin(\theta_t) + \frac{1}{t} \log \|DX_p^t.v^u\|] = \lambda_u(p),
\end{aligned}$$

and analogously for N_p^s. $\square$

Therefore to decrease the Lyapunov exponents associated to the tangent flow we decrease the Lyapunov exponents associated to the linear Poincaré flow.

In this conservative context we may restate the Oseledets theorem for the linear Poincaré flow as;

Theorem 2.2 *Let $X \in \mathfrak{X}^1_\mu(M)$. For μ-a.e. $p \in M$ there exists the upper Lyapunov exponent $\lambda^+(X,p)$ defined by the limit $\underset{t\to+\infty}{lim} \frac{1}{t} log\|P_X^t(p)\|$ that is a non-negative measurable function of p. For μ-a.e. point $p \in \mathfrak{O}^+$ there is a splitting of the normal bundle $N_p = N_p^u \oplus N_p^s$ which varies measurably with p such that:*

If $\vec{0} \neq v \in N_p^u$, then $\underset{t\to\pm\infty}{lim} \frac{1}{t} log\|P_X^t(p).v\| = \lambda^+(X,p)$.

If $\vec{0} \neq v \in N_p^s$, then $\underset{t\to\pm\infty}{lim} \frac{1}{t} log\|P_X^t(p).v\| = -\lambda^+(X,p)$.

If $\vec{0} \neq v \notin N_p^u, N_p^s$, then
$\underset{t\to+\infty}{lim} \frac{1}{t} log\|P_X^t(p).v\| = \lambda^+(X,p)$ and $\underset{t\to-\infty}{lim} \frac{1}{t} log\|P_X^t(p).v\| = -\lambda^+(X,p)$.

Next we recall a Lemma due to Doering [8] that relates the hyperbolicity of the linear Poincaré flow with the hyperbolicity of the tangent flow. Here the compactness of Λ plays an important role.

Lemma 2.3 *Let Λ be a X^t-invariant and compact set. Then Λ is hyperbolic for the flow if and only if the linear Poincaré flow is hyperbolic on Λ.*

2.4 Dominated Splitting for the Linear Poincaré flow

Let Λ be a X^t-invariant subset of M. A splitting of the normal bundle $N = N^1 \oplus N^2$ has *m-dominated splitting* for the linear Poincaré flow if it is P^t_X-invariant and we may find an *uniform* $m \in \mathbb{N}$ such that for any point $p \in \Lambda$ the following inequality holds:

$$\Delta(p, m) = \frac{\|P^m_X(p)|N^1_p\|}{\|P^m_X(p)|N^2_p\|} \leq \frac{1}{2}. \tag{2}$$

A few words about this definition; If we take $p \in \mathfrak{O}^+$ an Oseledets regular point for X^t, with a non-trivial splitting (i.e. not all Lyapunov exponents zero) positive Lyapunov exponents only guarantee that we will see expansion for large iterates, say for $P_X^{m(p)}(p)$, but this function $m(.)$ varies from point to point, and possibly is not bounded along the orbit of p. So the information given by the Oseledets theorem is blind to uniformity. If Λ has a dominated splitting, then it varies continuously from point to point and we may extend the splitting to the closure. Moreover, the decomposition is unique and the angle between the two subspaces is bounded away from zero on Λ. Next we define some usefull X^t-invariant sets:
$\Lambda_m(X) := \{p \in \mathfrak{O}^+ : p$ has m-dominated splitting for the linear Poincaré flow$\}$;
$\Gamma_m(X) := M - \Lambda_m(X)$;
$\Gamma^+_m(X) := \mathfrak{O}^+(X) - \Lambda_m(X)$;
$\Gamma^*_m(X) := \{p \in \Gamma^+_m(X) : p \notin \mathrm{Per}(X^t)\}$.
The set of points in $\mathfrak{O}^+$ where (2) does not hold we denote by $\Delta_m(X)$. Clearly their orbits do not have m-dominated splitting, nevertheless for some $p \in \Delta_m(X)$ maybe there exists some iterate $X^t(p)$ where (2) holds. We consider the saturated set $\underset{t\in\mathbb{R}}{\cup} X^t(\Delta_m(X))$ that is equal to $\Gamma^+_m(X)$.

2.5 Ergodicity of sets with dominated splitting for vector fields in $\mathfrak{X}^2_\mu(M)^*$

In this section we prove a result which is similar to a classical theorem of Bowen. Our result says that any hyperbolic set, not necessarily a basic piece, of a non-Anosov conservative 3-flow of class C^2 has zero measure. For that we adapt Theorem 14 in [6] to our setting.

Proposition 2.4 *Let $X \in \mathfrak{X}^2_\mu(M)^*$ and Λ_m be a X^t-invariant set with m-dominated splitting for the linear Poincaré flow. Then $\mu(\overline{\Lambda_m}) = 0$ or X is Anosov.*

For the proof we will need the following lemma which says that dominated splitting on the conservative setting without singularities is tantamount to hyperbolicity.

Lemma 2.5 *For $X \in \mathfrak{X}^1_\mu(M)^*$, if $\Lambda_m \neq \emptyset$ then $\overline{\Lambda_m}$ is a hyperbolic set.*

Proof: Any $p \in \Lambda_m$ has m-dominated splitting for the linear Poincaré flow. Since we do not have singularities and we have constant dimensions of the subbundles this splitting extends to the closure and we get $\Delta(p, m) \leq \frac{1}{2}$ for every $p \in \overline{\Lambda_m}$. Of course for any $i \in \mathbb{N}$ we have $\Delta(p, i.m) \leq \frac{1}{2^i}$. For every $t > 0$ we write $t = i.m + r$, where $i \in \mathbb{N}$ and $r \in [0, i)$. Since $\|P_X^r\|$ is bounded, say by M, we take $\hat{C} = 2^{\frac{r}{m}} M^2$ and $\sigma = 2^{-\frac{1}{m}}$ to get $\Delta(p,t) \leq \hat{C}\sigma^t$ for every $p \in \Lambda_m$ and $t > 0$. We denote by α_t the angle $\angle(N^u_{X^t(p)}, N^s_{X^t(p)})$ and it is well known that, if Λ_m has m-dominated splitting, then for all $p \in \Lambda_m$ we have $\alpha_t \geq \beta > 0$. Since we do not have singularities there exists $K > 1$ such that for all $p \in M$, $K^{-1} \leq \|X(p)\| \leq K$. Since the flow is conservative we have:

$$sin(\alpha_0) = \|P_X^t(p)|_{N_p^u}\|.\|P_X^t(p)|_{N_p^s}\|.sin(\alpha_t).\frac{\|X(X^t(p))\|}{\|X(p)\|}.$$

So;

$$\begin{aligned} \|P_X^t(p)|_{N_p^s}\|^2 &= \frac{sin(\alpha_0)}{sin(\alpha_t)}.\frac{\|X(p)\|}{\|X(X^t(p))\|}.\Delta(p,t) \leq \\ &\leq \Delta(p, i.m+r).sin(\beta)^{-1}K^2 \leq \sigma^t \hat{C} sin(\beta)^{-1} K^2 \end{aligned}$$

Analogously we get

$$\begin{aligned} \|P_X^{-t}(p)|_{N_p^u}\|^2 &= \frac{sin(\alpha_t)}{sin(\alpha_0)}.\frac{\|X(X^t(p))\|}{\|X(p))\|}.\Delta(p,t) \leq \\ &\leq \Delta(p, i.m+r).sin(\beta)^{-1}K^2 \leq \sigma^t \hat{C} sin(\beta)^{-1} K^2, \end{aligned}$$

and we have $\overline{\Lambda_m}$ hyperbolic for the linear Poincaré flow, with constants σ and $C := \hat{C} sin(\beta)^{-1} K^2$. Now by Lemma 2.3 we conclude that $\overline{\Lambda_m}$ is hyperbolic for the flow. $\square$

Now we use standard smooth ergodic and hyperbolic dynamics theory to prove Proposition 2.4. Denote by μ_u the 1-dimensional Lebesgue measure on the unstable manifold. Suppose that the norm is adapted to get the hyperbolicity constants $C = 1$ and $\sigma \in (0,1)$. In what follows we assume that $\mu(\Lambda) > 0$.

Lemma 2.6 *There exists a segment of orbit $(x_t)_{t>0}$ on Λ such that $\mu_u(W^u_\epsilon(x_t) - \Lambda) \underset{t\to\infty}{\to} 0$.*

Proof: There exists $x \in \Lambda$ such that $\mu_u(W^u_\epsilon(x) \cap \Lambda) > 0$, otherwise $\forall x \in \Lambda$ $\mu_u(W^u_\epsilon(x) \cap \Lambda) = 0$ and since X^t is twice differentiable we get absolute continuity along unstable manifolds and by a Fubinni disintegration argument we contradict $\mu(\Lambda) > 0$. We take this point x and since $\mu_u(W^u_\epsilon(x) \cap \Lambda) > 0$ there exists $y \in \Lambda$ with density one on $W^u_\epsilon(x) \cap \Lambda$. We define $x_n = X^n(y)$ and we get $\mu_u(X^{-n}(W^u_\epsilon(x_n))) \underset{n\to+\infty}{\to} 0$. Therefore,

$$\frac{\mu_u(X^{-n}(W^u_\epsilon(x_n)) - \Lambda)}{\mu_u(W^u_\epsilon(x_n))} \underset{t\to\infty}{\to} 0$$

Claim 2.1 *Let $x_1, x_2 \in W^u_\epsilon(x_n)$ such that $d_u(x_1, x_2) < D$. There exists $K > 0$ such that for all $t \geq 0$ we have $\frac{\|DX^{-t}_{x_1}|_{E^u_{x_1}}\|}{\|DX^{-t}_{x_1}|_{E^u_{x_2}}\|} \leq K$.*

To prove the claim we use a standard application of the bounded distortion properties. The sub-bundle E^u is ν-Hölder so we can define a (C,ν)-Hölder function on Λ as $\varphi(x) = \log\|DX_x^{-1}|_{E_x^u}\|$. Since $\max\limits_{x\in\Lambda}\|DX_x^{-r}|E_x^u\|$ is bounded we consider time-1 maps,

$$\begin{aligned}
\log\frac{\|DX_{x_1}^{-n}|_{E_{x_1}^u}\|}{\|DX_{x_2}^{-n}|_{E_{x_2}^u}\|} &\leq \log\prod_{i=0}^{n-1}\|DX_{X^{-i}(x_1)}^{-1}|_{E_{X^{-i}(x_1)}^u}\| - \log\prod_{i=0}^{n-1}\|DX_{X^{-i}(x_2)}^{-1}|_{E_{X^{-i}(x_2)}^u}\| = \\
&= \sum_{i=0}^{n-1}(\varphi(X^{-i}(x_1)) - \varphi(X^{-i}(x_2))) \leq \\
&\leq \sum_{i=0}^{n-1} C d_u(X^{-i}(x_1), X^{-i}(x_2))^{\nu} \leq \sum_{i=0}^{n-1} C\sigma^{i\nu} d_u(x_1, x_2)^{\nu} \leq \\
&\leq CD^{\nu}\sum_{i=0}^{n-1}\sigma^{i\nu} \leq CD^{\nu}\sum_{i=0}^{\infty}\sigma^{i\nu} \leq CD^{\nu}S,
\end{aligned}$$

where S is the sum of the geometric series. We take $K := e^{CD^{\nu}S}$ and the claim is proved. Now we have:

$$\frac{\mu_u(W_\epsilon^u(x_t) - \Lambda)}{\mu_u(W_\epsilon^u(x_t))} \leq K\frac{\mu_u(X^{-t}(W_\epsilon^u(x_t) - \Lambda))}{\mu_u(W_\epsilon^u(x_t))} \underset{t\to\infty}{\to} 0,$$

therefore $\mu_u(W_\epsilon^u(x_t) - \Lambda) \underset{t\to\infty}{\to} 0$. $\square$

Claim 2.2 *There exists $x_0 \in \Lambda$ such that $W_\epsilon^u(x_0) \subseteq \Lambda$.*

Let $(x_t)_{t>0}$ be the orbit given by Lemma 2.6. Since Λ is closed and $x_t \in \Lambda$ we take $x_0 \in \Lambda$ an accumulation point of $(x_{t_n})_{n\in\mathbb{N}}$. By continuity of the unstable manifolds we get $W_\epsilon^u(x_t) \to W_\epsilon^u(x_0)$ and by Lemma 2.6, $\mu_u(W_\epsilon^u(x_t) - \Lambda) \underset{t\to\infty}{\to} 0$ so we conclude that $\mu_u(W_\epsilon^u(x_0) - \Lambda) = 0$. But Λ is closed and $W_\epsilon^u(x_0)$ is open on $W^u(x_0)$ so $W_\epsilon^u(x_0) - \Lambda$ is an open set on $W^u(x_0)$ with zero measure, therefore empty and $W_\epsilon^u(x_0) \subseteq \Lambda$. The claim is proved.

Lemma 2.7 *There exists a hyperbolic periodic orbit $X^t(p) \in \Lambda$ such that $W^u(p) \subseteq \Lambda$.*

Proof: By continuation of hyperbolic sets we may define the maximal invariant set $\tilde{\Lambda} = \bigcap\limits_{t\in\mathbb{R}} X^t(U_\beta)$ for any neighborhood U_β of Λ such that $d_H(\Lambda, U_\beta) < \beta$. Consider a point x_0 given by Claim 2.2 and a small transversal section Σ_0 to $\{X^t(x_0)\}_{t\in\mathbb{R}}$ at $t = 0$. Since we can always suppose that the measure is supported on Λ, the induced measure, $\tilde{\mu}$, defined on transversal sections (see section 3.1.2) verifies $\tilde{\mu}(\Sigma_0 \cap \Lambda) > 0$ so by Poincaré recurrence we have for $\tilde{\mu}$-a.e. $x \in \Sigma_0 \cap \Lambda$ a time s such that $X^s(x) \in \Sigma_0$. If all points in Σ_0 are α-close to x_0 we get a *α-pseudo periodic orbit*. By the shadowing lemma we know that given any $\beta > 0$, there exists $\alpha > 0$ such that any α-pseudo orbit in Λ is β-shadowed by an orbit in M. Take an adequate β so we obtain an orbit, $\bigcup\limits_{t\in\mathbb{R}} X^t(p)$, in $\tilde{\Lambda}$, that shadows the α-pseudo periodic orbit. Since, by expansiveness, this orbit is unique and $X^s(p)$ also shadows, we get $X^s(p) = p$ and this orbit is periodic. $p \in \tilde{\Lambda}$ is hyperbolic because it belong to $\tilde{\Lambda}$, therefore have stable/unstable manifolds that are close to the stable/unstable manifolds of x_0, we may suppose transversality between $W_\epsilon^u(x_0)$ and $W_\epsilon^s(p)$. Claim 2.2

guarantees that $W^u_\epsilon(x_0) \subseteq \Lambda$ and by Palis λ-lemma it converges to $W^u(p)$. Since Λ is closed we obtain $W^u(p) \subseteq \Lambda$. $\square$

Abbreviate $\Lambda_p = \overline{W^u(p)}$ and define $W^s_\epsilon(\Lambda_p) := \bigcup_{z\in\Lambda_p} W^s_\epsilon(z)$.

Lemma 2.8 *$W^s_\epsilon(\Lambda_p)$ is X^t-invariant and it is an open neighborhood of Λ_p.*

Proof: For $t > 0$ take $\delta \in (\sigma^t\epsilon, \epsilon)$, so:

$$X^t(W^s_\delta(\Lambda_p)) \subseteq X^t(\overline{W^s_\delta(\Lambda_p)}) \subseteq X^t(W^s_\epsilon(\Lambda_p)) \subseteq W^s_{\sigma^t\epsilon}(\Lambda_p)) \subseteq W^s_\delta(\Lambda_p)).$$

By the volume preserving property we get $\mu[W^s_\delta(\Lambda_p) - X^t(W^s_\delta(\Lambda_p))] = 0$. Since $\mu[W^s_\delta(\Lambda_p) - X^t(W^s_\delta(\Lambda_p))] \geq \mu[W^s_\delta(\Lambda_p) - X^t(\overline{W^s_\delta(\Lambda_p)})]$ we conclude that $\mu[W^s_\delta(\Lambda_p) - X^t(\overline{W^s_\delta(\Lambda_p)})] = 0$. Since $W^s_\delta(\Lambda_p)$ is open, $X^t(\overline{W^s_\delta(\Lambda_p)})$ is close and μ is Lebesgue we get that the open set $W^s_\delta(\Lambda_p) - X^t(\overline{W^s_\delta(\Lambda_p)})$ is an empty set. Hence $W^s_\delta(\Lambda_p) - X^t(\overline{W^s_\delta(\Lambda_p)})$ contains $W^s_\delta(\Lambda_p) - X^t(W^s_\epsilon(\Lambda_p))$ and so we get $W^s_\delta(\Lambda_p) - X^t(W^s_\epsilon(\Lambda_p)) = \emptyset$.

Since this is true for all $\delta < \epsilon$ we conclude $W^s_\epsilon(\Lambda_p) - X^t(W^s_\epsilon(\Lambda_p)) = \emptyset$ so:

$$W^s_\epsilon(\Lambda_p) = X^t(W^s_\epsilon(\Lambda_p)). \tag{3}$$

And we have the X^t invariance.

For the second part of the lemma, we prove that $\Lambda_p \cap W^u(z) = W^u(z)$ for any $z \in \Lambda_p$. We note that $\Lambda_p \cap W^u(z)$ is closed on $W^u(z)$, let us see that it is also open; Take $z \in \Lambda_p$, by definition of Λ_p there exists $\{z_n\}_{n\in\mathbb{N}} \in W^u(p)$ such that $z_n \underset{n\to\infty}{\to} z$. $W^u_\epsilon(z_n) \subseteq W^u(p) \subseteq \Lambda_p$ and these local unstable manifolds verifies $W^u_\epsilon(z_n) \underset{n\to\infty}{\to} W^u_\epsilon(z)$. Now, since $W^u_\epsilon(z_n) \subseteq \Lambda_p$ and Λ_p is a close set, this imply that z belongs to the interior of $\Lambda_p \cap W^u(z)$. Therefore $\Lambda_p \cap W^u(z) = W^u(z)$ so the union of all unstable manifolds of points of Λ_p is Λ_p itself. Since the local stable manifolds vary continuously with the point we get that $W^s_\epsilon(\Lambda_p)$ has an open neighborhood of Λ_p. $\square$

Proof: (Proposition 2.4) Consider a set Λ_m with m-dominated splitting, by Lemma 2.5 we get $\overline{\Lambda_m}$ hyperbolic. We take $\Lambda = \overline{\Lambda_m}$ and we follow previous lemmas assuming $\mu(\Lambda) > 0$. By (3) we get $W^s_\epsilon(\Lambda_p) = \bigcap_{t>0} X^t(W^s_\epsilon(\Lambda_p))$ but $\bigcap_{t>0} X^t(W^s_\epsilon(\Lambda_p)) = \Lambda_p$, therefore $W^s_\epsilon(\Lambda_p) = \Lambda_p$. Again by Lemma 2.8, we have that $W^s_\epsilon(\Lambda_p)$ is open so Λ_p is open. Λ_p is also closed, therefore $\Lambda_p = M$, but $\Lambda_p \subseteq \Lambda$, so $\Lambda = M$ and X is Anosov. $\square$

The conservative flow X^t is called *aperiodic* if $\mu(Per(X^t)) = 0$.

Lemma 2.9 *There exists $D \subseteq \mathfrak{X}^1_\mu(M)^*$ such that D is C^1-dense and if $X \in D$, X^t is aperiodic, X is of class C^s ($s \geq 2$) and all its sets with dominated splitting for the linear Poincaré flow have zero or full measure.*

Proof: We take the C^s-residual given by Robinson version of Kupka-Smale theorem, see [16]. This residual set of vector fields is of class C^s and the associated flows have countable periodic points. Since $\mathfrak{X}^s_\mu(M)$ is a Baire space (with respect to C^s topology), it follows that we have a C^s-dense set D, therefore a C^1-dense set, of vector fields with countable periodic orbits on $\mathfrak{X}^s_\mu(M)$. We know that $\mathfrak{X}^s_\mu(M)$ is C^1-dense on $\mathfrak{X}^1_\mu(M)$, see [17, 2], so D is C^1-dense on $\mathfrak{X}^1_\mu(M)$ and all vector fields in D are C^s. Since, by Proposition 2.4, hyperbolic sets have zero or full measure and X^t is aperiodic the lemma is proved. $\square$

2.6 Strategy for the proof of Theorem 1

Given $X \in \mathfrak{X}^1_\mu(M)$, let $\lambda^+(X,p) := \lim\limits_{t\to+\infty} \frac{1}{t}\log\|P^t_X(p)\|$ be the upper Lyapunov exponent which exists μ-a.e. $p \in M$ by Oseledets theorem. When there is no ambiguity we denote $\lambda^+(X,p)$ by $\lambda^+(p)$.

We define the "entropy function" by the integration over any X^t-invariant set $\Gamma \subseteq M$ of the upper Lyapunov exponent:

$$\begin{array}{rccc} LE(.,\Gamma): & \mathfrak{X}^1_\mu(M) & \longrightarrow & [0,+\infty) \\ & X & \longmapsto & \int_\Gamma \lambda^+(X,p)d\mu(p) \end{array}$$

Remark 2.1 *Let $f: W \longrightarrow \mathbb{R}$ where W is a topological space. f is upper semicontinuous iff for every δ the set $\{x : f(x) < \delta\}$ is open. Moreover, the infimum of continuous functions is an upper semicontinuous function.*

Lemma 2.10 *$LE(X,\Gamma) = \inf\limits_{n\geq 1} \frac{1}{n}\int_\Gamma \log\|P^n_X(p)\|d\mu(p)$, therefore it is upper semi-continuous.*

Proof: $LE(X,\Gamma) = \int_\Gamma \lambda^+(p)d\mu(p) = \lim\limits_{n\to+\infty} \frac{1}{n}\int_\Gamma \log\|P^n_X(p)\|d\mu(p)$. The sequence $x_n(X) := \int_\Gamma \log\|P^n_X(x)\|d\mu(p)$ is sub-additive, therefore satisfies,

$$\lim_{n\to+\infty} \frac{x_n(X)}{n} = \inf_{n\geq 1} \frac{x_n(X)}{n}.$$

Thus $LE(X,\Gamma) = \inf\limits_{n\geq 1} \frac{x_n(X)}{n}$, and since each $x_n(X)$ is a continuous function, by remark 2.1, $LE(X,\Gamma)$ is upper semicontinuous. $\square$

$LE(X,M)$ will be denoted by $LE(X)$. Next proposition will be crucial to prove Theorem 1.

Proposition 2.11 *Let $X \in \mathfrak{X}^2_\mu(M)^*$, with X^t aperiodic and with hyperbolic sets of zero measure. Let $\epsilon, \delta > 0$ be given. Then there exists a C^1 zero divergence vector field Y ϵ-C^1-close to X, such that $LE(Y) < \delta$.*

We assume Proposition 2.11 and prove Theorem 1. By Lemma 2.9 we have a dense set such that every X is C^2, aperiodic and with hyperbolic sets having full or zero measure. The set of conservative Anosov vector fields, denoted by $\mathbf{A}$, is open. For all $k \in \mathbb{N}$ the set $\mathbf{A}_k = \{X \in \mathfrak{X}^1_\mu(M)^* : LE(X) < k^{-1}\}$ is open by Lemma 2.10 and Remark 2.1. By Proposition 2.11 with $\delta = k^{-1}$ we get $\mathbf{A}_k$ dense in $\mathbf{A}^c$, so the set:

$$\mathfrak{R} = \bigcap_{k\in\mathbb{N}} \mathbf{A} \cup \mathbf{A}_k$$

is a C^1-residual set. But $\mathfrak{R} = \mathbf{A} \cup \bigcap\limits_{k\in\mathbb{N}} \mathbf{A}_k = \mathbf{A} \cup \{X \in \mathfrak{X}^1_\mu(M)^* : LE(X) = 0\}$, therefore for $X \in \mathfrak{R}$ we have either that X is an Anosov vector field or $LE(X) = \int_M \lambda^+ d\mu(p) = 0$. This last equality implies that Lebesgue a.e. $p \in M$ has zero Lyapunov exponents and Theorem 1 is proved.

3 Perturbation of vector fields

3.1 Auxiliary lemmas

3.1.1 A conservative straightening-out lemma

The following theorem due to Dacorogna and Moser [7], will be used to obtain a conservative local change of coordinates which trivialize a vector field.

Theorem 3.1 *(Dacorogna-Moser) Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^5 boundary $\partial\Omega$ and $g, f : \overline{\Omega} \to \mathbb{R}$ positive functions of class C^s $(s \geq 2)$. Then there exists a diffeomorphism $\varphi : \Omega \to \varphi(\Omega) \subseteq \mathbb{R}^n$ with φ, φ^{-1} of class C^s and verifying the partial differential equation:*

$$detD\varphi_q g(\varphi(q)) = \lambda f(q), \tag{4}$$

for all $q \in \Omega$ where $\lambda = \int g / \int f$. We also have $\varphi = Id$ at $\partial\Omega$.

Denote by $X^{[0,t]}(p) = \{X^s(p) : s \in [0,t]\}$. We say that the a segment of an orbit $X^{[0,m]}(p)$ is *straightened-out* if $X^{[0,m]}(p) \subseteq \{(x,0,0) : x \in \mathbb{R}\}$. Denote by $\mathfrak{N}_{X^t(p)}$ the normal plane at $X(X^t(p))$. Denote by $B(X^t(p), r)$ the ball with radius r centered at $X^t(p)$ inside $X(X^t(p))^\perp = \mathfrak{N}_{X^t(p)}$. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the constant vector field defined by $T(x,y,z) = (c,0,0)$ for some $c > 0$ and $\mathfrak{F}$, $\mathfrak{C}$ be the flowboxes $\mathfrak{F} := X^{[0,1]}(B(p,r))$ and $\mathfrak{C} := T^{[0,1]}(B(p,r))$.

Lemma 3.2 *(Conservative flowbox theorem) Given a vector field $X \in \mathfrak{X}^s_\mu(M)$ (for $s \geq 2$) and a non-singular point $p \in M$ (eventually periodic with period $\tau > 1$), there exists a conservative C^s diffeomorphism $\Psi : \mathfrak{C} \to \mathfrak{F}$ such that $X = \Psi_* T$.*

Proof: Assume that $p = (0,0,0)$ and $X(p) \subseteq \{(x,0,0) : x \in \mathbb{R}\}$. Let $X_1(x,y,z)$ be the projection into the first coordinate of $X(x,y,z)$. We define a function $g : B(p,r) \to \mathbb{R}$ such that $g(y,z) := X_1(0,y,z)$ for $(0,y,z) \in B(p,r)$ (see Figure 1). Since $g \in C^s$ we apply Theorem 3.1 to $\Omega = B(p,r) \subseteq \mathbb{R}^2$ so there exists a diffeomorphism $\varphi : \Omega \to \varphi(\Omega) \subseteq \mathbb{R}^2$ with φ, φ^{-1} of class C^s and verifying the partial differential equation $g(\varphi(\overline{y},\overline{z}))\det D\varphi_{\overline{y},\overline{z}} = \lambda$, for all $(\overline{y},\overline{z}) \in \Omega$ where $\lambda = \int g / \int 1$, and $\varphi|_{\partial\Omega} = Id$. Now we define the C^s change of coordinates by:

$$\begin{array}{rccc} \hat{\Psi}: & \mathbb{R} \times \Omega & \longrightarrow & M \\ & (\overline{x},\overline{y},\overline{z}) & \longmapsto & X^{\lambda^{-1}\overline{x}}((0,\varphi(\overline{y},\overline{z})) \end{array}$$

First we claim that:

$$\det D\hat{\Psi}_{(0,\overline{y},\overline{z})} = 1 \text{ for all } (0,\overline{y},\overline{z}) \in \mathbb{R} \times \Omega. \tag{5}$$

Let Π_i denote the projection into the i^{th}-coordinate, for $i = 1,2,3$.
Note that, $\frac{\partial X_1^0}{\partial \overline{y}}(0,\varphi(\overline{y},\overline{z})) = \frac{\partial}{\partial \overline{y}}\Pi_1(0,\varphi(\overline{y},\overline{z})) = 0$ and for $i = 2,3$ we have that, $\frac{\partial X_i^0}{\partial \overline{y}}(0,\varphi(\overline{y},\overline{z})) = \frac{\partial}{\partial \overline{y}}\Pi_i(0,\varphi(\overline{y},\overline{z})) = \frac{\partial}{\partial \overline{y}}(\varphi_i(\overline{y},\overline{z})))$. For $\overline{z}$ we proceed analogously. Now we use these computations to derive,

$$D\hat{\Psi}_{(0,\overline{y},\overline{z})} = \begin{pmatrix} \frac{1}{\lambda}X_1(X^0(0,y,z)) & 0 & 0 \\ \frac{1}{\lambda}X_2(X^0(0,y,z)) & \frac{\partial \varphi_1}{\partial \overline{y}}|_{(\overline{y},\overline{z})} & \frac{\partial \varphi_1}{\partial \overline{z}}|_{(\overline{y},\overline{z})} \\ \frac{1}{\lambda}X_3(X^0(0,y,z)) & \frac{\partial \varphi_2}{\partial \overline{y}}|_{(\overline{y},\overline{z})} & \frac{\partial \varphi_2}{\partial \overline{z}}|_{(\overline{y},\overline{z})} \end{pmatrix}$$

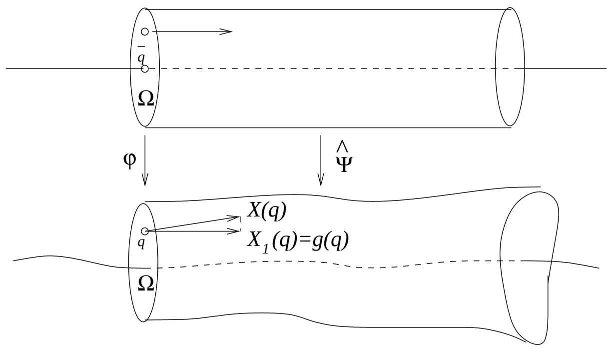

Figure 1: Construction of a conservative change of coordinates straightening-out *all* orbits.

and we get $\det D\hat{\Psi}_{(0,\overline{y},\overline{z})} = \frac{1}{\lambda}X_1((0,y,z))\det D\varphi_{(\overline{y},\overline{z})} = \frac{g(y,z)}{\lambda}\det D\varphi_{(\overline{y},\overline{z})} = 1$ by using (4) of Theorem 3.1. Therefore (5) is proved. Now we will see that:

$$\det D\hat{\Psi}_{(\overline{x}_0,\overline{y}_0,\overline{z}_0)} = 1 \text{ for all } (\overline{x}_0,\overline{y}_0,\overline{z}_0) \in \mathbb{R}\times\Omega.$$

We have:

$$\hat{\Psi}(\overline{x},\overline{y},\overline{z}) = X^{\lambda^{-1}\overline{x}_0}[X^{\lambda^{-1}(\overline{x}-\overline{x}_0)}((0,\varphi(\overline{y},\overline{z})))] = X^{\lambda^{-1}\overline{x}_0}[\hat{\Psi}(\overline{x}-\overline{x}_0,\overline{y},\overline{z})],$$

so

$$D\hat{\Psi}_{(\overline{x},\overline{y},\overline{z})} = DX^{\lambda^{-1}\overline{x}_0}_{\Psi(\overline{x}-\overline{x}_0,\overline{y},\overline{z})}D\hat{\Psi}_{(\overline{x}-\overline{x}_0,\overline{y},\overline{z})}.$$

Evaluated at $\overline{x} = \overline{x}_0$ we get:

$$D\hat{\Psi}_{(\overline{x}_0,\overline{y},\overline{z})} = DX^{\lambda^{-1}\overline{x}_0}_{\Psi(0,\overline{y},\overline{z})}D\hat{\Psi}_{(0,\overline{y},\overline{z})}.$$

We use (5) and the fact that the flow X^t is volume preserving to conclude that:

$$\det D\hat{\Psi}_{(\overline{x}_0,\overline{y}_0,\overline{z}_0)} = 1.$$

Finally take $c := \lambda$ and consider the constant vector field $T := (\lambda,0,0)$. Let $(x,y,z) = \hat{\Psi}(\overline{x},\overline{y},\overline{z})$.
We have:

$$\begin{aligned}
\hat{\Psi}_*T(x,y,z) &= D\hat{\Psi}_{(\overline{x},\overline{y},\overline{z})}T(\overline{x},\overline{y},\overline{z})) = \\
&= \begin{pmatrix} \frac{1}{\lambda}X_1(X^{\lambda^{-1}\overline{x}}(0,y,z)) & \frac{\partial X_1^{\lambda^{-1}\overline{x}}}{\partial \overline{y}}(0,\varphi(\overline{y},\overline{z})) & \frac{\partial X_1^{\lambda^{-1}\overline{x}}}{\partial \overline{z}}(0,\varphi(\overline{y},\overline{z})) \\ \frac{1}{\lambda}X_2(X^{\lambda^{-1}\overline{x}}(0,y,z)) & \frac{\partial X_2^{\lambda^{-1}\overline{x}}}{\partial \overline{y}}(0,\varphi(\overline{y},\overline{z})) & \frac{\partial X_2^{\lambda^{-1}\overline{x}}}{\partial \overline{z}}(0,\varphi(\overline{y},\overline{z})) \\ \frac{1}{\lambda}X_3(X^{\lambda^{-1}\overline{x}}(0,y,z)) & \frac{\partial X_3^{\lambda^{-1}\overline{x}}}{\partial \overline{y}}(0,\varphi(\overline{y},\overline{z})) & \frac{\partial X_3^{\lambda^{-1}\overline{x}}}{\partial \overline{z}}(0,\varphi(\overline{y},\overline{z})) \end{pmatrix} \begin{pmatrix} \lambda \\ 0 \\ 0 \end{pmatrix} = \\
&= (X_1(X^{\lambda^{-1}\overline{x}}(0,y,z)), X_2(X^{\lambda^{-1}\overline{x}}(0,y,z)), X_3(X^{\lambda^{-1}\overline{x}}(0,y,z))) = \\
&= X(\hat{\Psi}(\overline{x},\overline{y},\overline{z}))
\end{aligned}$$

therefore $X = \hat{\Psi}_*T$. $\square$

3.1.2 More notation, definitions and lemmas

Coordinates: For technical reasons, given by previous section, it is usefull to take X of class C^2, therefore we consider the vector field X in Proposition 2.11 of class C^2. Given $p \in M$ and a small $r > 0$ let $q \in X^{[0,m]}(B(p,r))$. We will use the conservative flowbox theorem to get a C^2 change of coordinates $\Psi := \hat{\Psi}^{-1}$, hence, a C^1 vector field which has all orbits straight-out, i.e., $X^{[0,1]}(B(q,r'))$ is send into $T^{[0,1]}(B(q,r'))$ by Ψ. Since for these change of coordinates we fix $t \in [0,1]$, M is compact and $\Psi \in C^2$ we conclude that exists $\Theta_1 := \max\{\|D\Psi_p\|, \|D\Psi_p^{-1}\| : p \in M\}$. We take $\Theta_2 := \max\{\|D^2\Psi_p^{-1}\| : p \in M\}$ and also $\Theta := \max\{\Theta_1, \Theta_2\}$.
Perturbations and metrics: All the perturbations in this paper will be developed using the trivial coordinates given by Ψ. Θ will be usefull to control the size of the perturbation. Therefore if $\epsilon > 0$, $X := \Psi_* T$ and $Z := T + P$ is a perturbation such that $\|P\|_{C^1} \leq \frac{\epsilon}{\Theta}$, then $\|Y - X\|_{C^1} = \|\Psi_* P\|_{C^1} \leq \epsilon$. According to Moser's Theorem, (see [14] Lemma 2), given a volume form ω there exists an atlas $\mathfrak{A} = \{\alpha_i : U_i \to \mathbb{R}^3\}$, such that $(\alpha_i)_*\omega = dx \wedge dy \wedge dz$, moreover by compactness of M we can take $\mathfrak{A}$ finite. The Riemannian norm at T_xM will not be used, instead we consider the norm $\|v\|_x := \|(D\alpha_i)_x.v\|$. Given two linear maps $A(t) : T_pM \to T_{X^t(p)}M$ and $B(t) : T_qM \to T_{X^t(q)}M$ we estimate the distance between them by using the atlas $\mathfrak{A}$ and translating the base points to $(0,0,0) \in \mathbb{R}^3$. Therefore

$$\|A(t) - B(t)\| = \|(D\alpha_{X^t(p)})_{X^t(p)} A(t) (D\alpha_p)_p^{-1} - (D\alpha_{X^t(q)})_{X^t(q)} B(t) (D\alpha_q)_q^{-1}\|.$$

Analogously we estimate the distance between the linear Poincaré flows based at different points.
Holonomy of linear flows: Assume that $p = (0,0,0)$ and also that the segment of orbit $X^{[0,T]}(p)$ is straight-out. In this case the tangent flow at p has the following simple form:

$$DX_p^t = \begin{pmatrix} \frac{\partial X_1^t}{\partial x} & \frac{\partial X_1^t}{\partial y} & \frac{\partial X_1^t}{\partial z} \\ 0 & \frac{\partial X_2^t}{\partial y} & \frac{\partial X_2^t}{\partial z} \\ 0 & \frac{\partial X_3^t}{\partial y} & \frac{\partial X_3^t}{\partial z} \end{pmatrix}_p = \begin{pmatrix} x(t) & y(t) & z(t) \\ 0 & a(t) & b(t) \\ 0 & c(t) & d(t) \end{pmatrix}, \tag{6}$$

where $x(t) = \|X(X^t(p))\| \|X(p)\|^{-1}$. Hence we have the following action,

$$\begin{array}{rccc} P_X^t : & N_p & \longrightarrow & N_{X^t(p)} \\ & (y,z) & \longmapsto & \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}. \end{array}$$

Let $q = (0,y,z) \in \mathfrak{N}_p$ and $t \in [0,T]$, then:

$$\begin{aligned} X^t(q) &= X^t(p) + DX_p^t.q + \mathfrak{O}^{\geq 2}(q) = \\ &= X^t(p) + (y(t)y + z(t)z, 0, 0) + (0, P_X^t(p)\begin{pmatrix} y \\ z \end{pmatrix}) + \mathfrak{O}^{\geq 2}(q), \end{aligned}$$

where $\mathfrak{O}^{\geq 2}(q)$ is the remainder of the Taylor expansion. So for $|y|, |z|$ small $X^t(0,y,z)$ is approximately $X^t(0,0,0) + (0, P_X^t(0,0,0).\begin{pmatrix} y \\ z \end{pmatrix})$.
Measures at transversal sections: In this context we may consider the time

arrival function $\tau(p,t) : \mathfrak{N}_p \to \mathfrak{N}_{X^t(p)}$ which is a well defined continuous function, due to the *implicit function theorem.* For $B \subseteq \mathfrak{N}_p$ denote by $X^{\tau(p,t)(B)}(B)$ the set,

$$\{X^{\tau(p,t)(q)}(q) : q \in B\} \subseteq \mathfrak{N}_{X^t(p)}.$$

Given $\delta > 0$, there exists B sufficiently small such that $X^{\tau(p,t)(B)}(B)$ is the intersection of the self-disjoint flowbox $X^{[0,t+\delta]}(B)$ with $\mathfrak{N}_{X^t(p)}$.
Let $X^{\tau(p,t)} : \mathfrak{N}_p \to \mathfrak{N}_{X^t(p)}$ be the Poincaré map between two sections. Given $n_1, n_2 \in \mathfrak{N}_p$ and using the volume form ω we can define a pair of 2-forms by $\hat{\omega}_p(n_1, n_2) := \omega_p(X(p), n_1, n_2)$ and $\overline{\omega}_p(n_1, n_2) = \omega_p(X(p)\|X(p)\|^{-1}, n_1, n_2)$. The 2-form $\hat{\omega}$ is the interior product of the volume form ω by the vector field, i.e., $\hat{\omega}_p := (i_X\omega)_p$. Denoting $P_t(.) = X^{\tau(p,t)(.)}(.)$ we have $P_t^*\hat{\omega}_p = \hat{\omega}_p$. The measure $\overline{\mu}$ induced by the 2-form $\overline{\omega}$ is not necessarily P_t-invariant, however both measures are equivalent. We call $\overline{\mu}$ the Lebesgue measure at normal sections or *modified area*. In fact given $n_1, n_2 \in N_p$ we have that $P_t^*\overline{\omega}_p(n_1, n_2) = x(t)^{-1}\overline{\omega}_p(n_1, n_2)$. By conservativeness of the flow we have $|\det P_X^t(p)|.\|X(X^t(p))\|\|X(p)\|^{-1} = 1$, so $|\det P_X^t(p)| = x(t)^{-1}$. Therefore we can give an explicit expression for the infinitesimal distortion area factor of the linear Poincaré flow, which is expressed by the following lemma (for a proof see [3]).

Lemma 3.3 *Given $\nu > 0$ and $T > 0$, there exists $r > 0$ such that for any measurable set $K \subseteq B(p,r) \subseteq \mathfrak{N}_p$ we have $|\overline{\mu}(K) - x(t).\overline{\mu}(X^{\tau(p,t)(K)}(K))| < \nu$ for all $t \in [0,T]$.*

3.2 Realizable linear flows

The next definition adapts the definition of realizable sequence given by Bochi in [4] and will also be central in the proof of our theorem, in broad terms we consider *modified* area-preserving linear flows acting in the normal bundle at p, $L^t(p) : N_p \to N_{X^t(p)}$ that do exactly what we want, and ask whether they are (γ-almost C^1) realizable as the linear Poincaré flow of Y, ϵ-C^1-close to X, computed on small transversal neighborhoods of one point.

Definition 3.1 *Given $X \in \mathfrak{X}^1_\mu(M)$, $\epsilon > 0$, $0 < \kappa < 1$ and a non-periodic point p, we say that the modified area-preserving sequence of linear flows:*

$$N_p \xrightarrow{L_0} N_{X^1(p)} \xrightarrow{L_1} N_{X^2(p)} \xrightarrow{L_2} ... \xrightarrow{L_{n-2}} N_{X^{n-1}(p)} \xrightarrow{L_{n-1}} N_{X^n(p)}$$

*is a (ϵ,κ)-**realizable linear flow of length** n **at** p if:*
$\forall\gamma > 0$, $\exists r > 0$ *such that for any open set $\emptyset \neq U \subseteq B(p,r) \subseteq \mathfrak{N}_p$, there exists:*

(a) A measurable set $K \subseteq U$,

(b) A zero divergence vector field Y, ϵ-C^1-close to X, such that:

(i) $Y^t = X^t$ outside the self-disjoint flowbox $\underset{t\in[0,n]}{\cup} X^{\tau(p,n)(U)}(U)$ and $DX_q = DY_q$ for every $q \in U, X^{\tau(p,n)(U)}(U)$;

(ii) $\overline{\mu}(K) > (1-\kappa)\overline{\mu}(U)$;

(iii) If $q \in K$, then $\|P_Y^1(Y^j(q)) - L_j\| < \gamma$ for $j \in \{0, 1, ..., n-1\}$.

Remark 3.1 *In the definition of realizable flow we consider integer iterates, but there is no restriction if we consider any intermediate linear flow, like* $L_j : N_{X^{t_j}(p)} \longrightarrow N_{X^{t_{j+1}}(p)}$ *with* $t_j < t_{j+1}$ *and* $\sum_{j=0}^{n-1} t_j = n$. *The point* p *may also be periodic, but with period larger than* n. *The realizability is with respect to the* C^1 *topology.*

Next we exhibit how to produce some elementary realizable linear flows, the linear Poincaré flow himself and also juxtaposition of realizable linear flows are realizable linear flows.

Lemma 3.4 *Let* $X \in \mathfrak{X}^1_\mu(M)$ *and* $p \in M$ *be a non-periodic point.*

(1) Given any $t \in \mathbb{R}$, $P^t_X(p)$ *(trivial linear flow) is* (ϵ, κ)*-realizable of length* t *for every* ϵ *and* κ.

(2) Let $\{L_0, ..., L_{n-1}\}$ *be* (ϵ, κ_1)*-realizable of length* n *at* p *and* $\{L_n, ..., L_{n+m-1}\}$ *be* (ϵ, κ_2)*-realizable of length* m *at* $X^n(p)$. *For* κ *such that* $\kappa < \kappa_1 + \kappa_2 < 1$ *the linear flow* $\{L_0, ..., L_{n+m-1}\}$ *is* (ϵ, κ)*-realizable.*

Proof:

(1) Given $\gamma > 0$, by continuity of the linear Poincaré flow, there exists a sufficiently small $r > 0$ such that for all $q \in B(p, r)$ we have the inequality $\|P^1_X(X^j(q)) - P^1_X(X^j(p))\| < \gamma$ for $j = \{0, ..., n-1\}$. Let $r > 0$ be also sufficiently small such that $X^{[0,t]}(B(p, r))$ is a self-disjoint flowbox. For any open set $U \subseteq B(p, r)$, we choose $K \subseteq U$ verifying (ii) of Definition 3.1 and Y equal to X. So (i) of Definition 3.1 follows by choice of Y and r, (ii) follows by choice of K and (iii) is clearly true.

(2) Let r_1, r_2 be the radius according to Definition 3.1 related to realizable linear flows $\{L_0, ..., L_{n-1}\}$ and $\{L_n, ..., L_{n+m-1}\}$ respectively. We take any non-empty open set $U \subseteq B(p, r_1)$, if we have $X^{\tau(B(p,r_1),n)}(B(p, r_1)) \subseteq B(X^n(p), r_2)$ fine, otherwise we choose a smaller $r < r_1$. Given $\nu > 0$, decrease $r > 0$ if necessary, by using Lemma 3.3, to get $|\overline{\mu}(K) - x(t)\overline{\mu}(X^{\tau(p,t)(K)})(K)| < \nu$ for all $t \in [0, n]$ and any measurable set $K \subseteq B(p, r)$. By definition and choice of the radius $r > 0$, we have that the flowbox $\{X^{\tau(p,t)(U)}(U) : t \in [0, n+m]\}$ is self-disjoint, again by definition, given any $U \subseteq B(p, r)$ we get a measurable $K_1 \subseteq U$ and a vector field Y_1 verifying (i), (ii) and (iii) of Definition 3.1. Also for any non-empty open subset of $B(X^n(p), r_2)$, in particular for $X^{\tau(p,n)(U)}(U)$, we get a measurable $\hat{K}_2 \subseteq X^{\tau(p,n)(U)}(U) =: \hat{U}$ and a vector field Y_2 verifying (i), (ii) and (iii) of Definition 3.1.
Now define the vector field $Y = Y_1$ in the flowbox $\{X^{\tau(p,t)(U)}(U) : t \in [0, n]\}$, $Y = Y_2$ in the flowbox $\{X^{\tau(X^n(p),t+n)(U)}(U) : t \in [0, m]\}$ and $Y = X$ elsewhere. Y is C^1 because by definition $(DY_1)_q = DX_q = (DY_2)_q$ for any $q \in X^{\tau(p,n)(U)}(U)$, so Y and U verifies (i). To check (ii) we define $K := K_1 \cap K_2$ where K_2 is such that $X^{\tau(p,n)(K_2)}(K_2) = \hat{K}_2$. By Lemma 3.3 we get $x(n)\overline{\mu}(\hat{U}) < \overline{\mu}(U) + \nu$ and also $\overline{\mu}(U - K_2) < x(n)\overline{\mu}(\hat{U} - \hat{K}_2) + \nu$. So we get:

$$\begin{aligned}
\overline{\mu}(U - K) &= \overline{\mu}(U - (K_1 \cap K_2)) \leq \overline{\mu}(U - K_1) + \overline{\mu}(U - K_2) < \\
&< \kappa_1\overline{\mu}(U) + x(n)\overline{\mu}(\hat{U} - \hat{K}_2) + \nu < \\
&< \kappa_1\overline{\mu}(U) + x(n)\kappa_2\overline{\mu}(\hat{U}) + \nu < \\
&< \kappa_1\overline{\mu}(U) + \kappa_2\overline{\mu}(U) + \kappa_2\nu + \nu = \\
&= \kappa\overline{\mu}(U) + (1 + \kappa_2)\nu.
\end{aligned}$$

Therefore $\overline{\mu}(K) = \overline{\mu}(U) - \overline{\mu}(U - K) > (1-\kappa)\overline{\mu}(U) - (1+\kappa_2)\nu$ and the result follows considering a sufficient small ν. Finally (iii) follows by definition. $\square$

Next lemma says that we only have to prove realizability on balls.

Lemma 3.5 *We only have to prove the realizability of the sequence of linear flows* $\{L_0, ..., L_{n-1}\}$ *for* $U = B(p', r')$ *where* $B(p', r') \subseteq B(p, r)$.

Proof: We use Vitali's covering lemma and we cover the open set U with a finite number of balls $\{B(p_i, r_i)\}_{i=1,...,m}$ such that $\overline{\mu}(U - \cup_{i=1}^m B(p_i, r_i))$ is as small as we want. By hypotheses, for each $B(p_i, r_i)$ there exists a measurable set $K_i \subseteq B(p_i, r_i)$ and a zero divergence vector field Y_i, ϵ-C^1-close to X, such that:

(a) $Y_i^t = X^t$ outside the flowbox $\{X^{\tau(p_i,t)(B(p_i,r_i))}(B(p_i, r_i)) : t \in [0, n]\}$ and $DX_q = (DY_i)_q$ for every $q \in B(p_i, r_i), X^{\tau(p,n)(B(p_i,r_i))}(B(p_i, r_i))$;

(b) $\overline{\mu}(K_i) > (1-\kappa)\overline{\mu}(B(p_i, r_i))$;

(c) For every $q \in K_i$ we have $\|P^1_{Y_i}(Y_i^j) - L_j\| < \gamma$ for $j \in \{0, 1, ..., n-1\}$.

Then we may define the same objects in U by taking $K = \underset{i=1,..,n}{\cup} K_i$; and $Y = X$ outside the self-disjoint flowbox $\{X^{\tau(p,t)(B(p_i,r_i))}(B(p_i, r_i)) : t \in [0, s]\}$ and $Y = Y_i$ inside it and the lemma is proved. $\square$

3.2.1 Small rotations

Next we will construct realizable linear flows of time-1 length at p, which rotate by a small angle ξ the action of the linear Poincaré flow, i.e., $L_0 := P_X^1(p) \circ R_\xi$ where R_ξ is a rotation of angle ξ. We may expect that increasing the length will allow us to rotate by a larger angle but unfortunately this is not possible, because the size of the perturbation depends on the dynamics, on the angles between the bundles and also on the change of coordinates given by Lemma 3.2.

Denote the rotation matrix by:

$$R_\theta := \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sin(\theta) & cos(\theta) \end{pmatrix}.$$

Given a 3×3 matrix A denote by $\hat{A}$ the 2×2 matrix obtained after removing the first row and first line from $\hat{A}$ so:

$$\hat{R}_\theta := \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}.$$

Lemma 3.6 *Given* $X \in \mathfrak{X}^2_\mu(M)$, *a non-periodic point* $p \in M$, $\epsilon > 0$, $0 < \kappa < 1$, $\gamma > 0$ *and a fixed time* $T = 1$. *There exists* $r > 0$ *(depending on* p*), an angle* ξ *(not depending on* p*) and a zero divergence vector field* Y, ϵ-C^1*-close to* X *such that:*

(a) $Y - X$ *is supported in the flowbox* $\{X^{\tau(p,t)(B(p,r))}(p)(B(p, r)) : t \in [0, 1]\}$,

(b) For every $q \in B(p, r\sqrt{1-\kappa})$ *we have* $\|P^1_Y(q) - P^1_X(p) \circ \hat{R}_\xi\| < \gamma$.

Proof: We take $p \in M$ a non-periodic point. Let $C := \max\{\|DX^1_p\| : p \in M\}$ and suppose that this constant is valid for any vector field ϵ-C^1-close to X. Using Lemma 3.2 we obtain a C^2 conservative diffeomorphism $\Psi : \mathfrak{F} \to \mathfrak{C}$. Consider the constant Θ defined at section 3.1.2 and suppose that this constant is also valid for any vector field ϵ-C^1- close to X.
Next we define two C^∞ *bump-functions* g and G. Note that for $q \in B(p,r)$ the orbit $X^{[0,1]}(q)$ in general crosses $\mathfrak{N}_{X^1(p)}$, nevertheless for all $\eta > 0$ exists a small $r > 0$ such that we have $X^{[0,1-\eta/2]}(q) \cap \mathfrak{N}_{X^1(p)} = \emptyset$, for all $q \in B(p,r)$.
Let $\eta > 0$ be sufficiently small to get:

(i) $\|R_{\xi\alpha} - R_\xi\| < \gamma/8C$ for $\alpha \in [1-\eta, 1-\eta/2]$ and also for all $q \in B(p,r)$ we have $X^{[0,\alpha]}(q) \cap \mathfrak{N}_{X^1(p)} = \emptyset$.

Now let $g : \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that $g(t) = 0$ for $t < 0$, $g(t) = t$ for $t \in [\eta, 1-\eta]$ and $g(t) = \alpha$ for $t \geq 1-\eta/2$ where $\alpha \in]1-\eta, 1-\eta/2[$ is fixed. We take a sufficiently small $r > 0$ such that for all $q \in B(p,r)$ we have:

(ii) $\|\Pi_{Y^1(q)} - \Pi_{X^1(p)}\| < \frac{\gamma}{2C}$ for any vector field $Y = X$ outside $\mathfrak{F}$ and ϵ-C^1-close to X;

(iii) $\|D\Psi_p - Id\| < \frac{\gamma}{16\Theta}$;

(iv) For $q \in B(p,r)$ and any vector field $Z = T$ outside $\mathfrak{C}$ and ϵ-C^1-close to X we have;

1. $\|D\Psi^{-1}_{Z^1(\Psi(q))} - D\Psi^{-1}_{Z^1(\Psi(p))}\| < \frac{\gamma}{24C\Theta}$;
2. $\|DZ^1_{\Psi(q)} - DZ^1_{\Psi(p)}\| < \frac{\gamma}{24\Theta^2}$ and
3. $\|D\Psi_q - D\Psi_p\| < \frac{\gamma}{24C\Theta}$.

(v) $|y|, |z| < \frac{c\epsilon}{\Theta^2 \ddot{g}(x)\xi}$ for any ξ such that $0 < \xi < 1$;

(vi) $|y|, |z| < \frac{\epsilon}{\Theta}$.

We take the angle ξ such that,

$$\xi < \frac{\epsilon(1 - \sqrt{1 - \frac{\kappa}{2}})}{2\Theta^2}.$$

For $r > 0$ verifying the properties above let $G : \mathbb{R} \to [0,1]$ be a C^∞ function such that $G(\rho) = 1$ for $\rho \leq r\sqrt{1 - \frac{\kappa}{2}}$ and $G(\rho) = 0$ for $\rho \geq r$. Note that $\max|\dot{G}| \leq \frac{2}{(1-\sqrt{1-\frac{\kappa}{2}})r}$. Let $\rho = \sqrt{y^2 + z^2}$ and consider the rotation flow acting on $\mathfrak{N}_p$ defined by:

$$R_{\xi g(t) G(\rho)}(0,y,z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(\xi g(t)G(\rho)) & -sin(\xi g(t)G(\rho)) \\ 0 & sin(\xi g(t)G(\rho)) & cos(\xi g(t)G(\rho)) \end{pmatrix} \begin{pmatrix} 0 \\ y \\ z \end{pmatrix}$$

Denote the time derivative by $\dot{R}_{\xi g(t)G(\rho)} = \frac{d}{dt}(R_{\xi g(t)G(\rho)})$. A simple computation shows that,

$$\dot{R}_{\xi g(t)G(\rho)}.R^{-1}_{\xi g(t)G(\rho)} = \xi \dot{g}(t) G(\rho) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \qquad (7)$$

Inducing the adequate vector field:

We consider the flow T^t associated to the vector field T and we define for $q=(0,y,z)\in B(p,r)$, $\Upsilon(t,q):=T^t(R_{\xi g(t)G(\rho)}(q))$. We denote $R_{\xi g(t)G(\rho)}(q)$ by $R_t(q)$. Let $H(t,q):=(t,R_t(q))$ and $F(t,R_t(q)):=T^t(R_t(q))$ so we obtain $\Upsilon(t,q)=F\circ H(t,q)$. We take time derivatives at $t=s$:

$$\begin{aligned}\frac{d}{dt}\Upsilon(t,q)|_{t=s} &= \frac{d}{dt}T^t(R_t(q))|_{t=s}=DF_{H(s,q)}.DH_s=\\ &= \begin{pmatrix}\partial_1 F & \partial_2 F\end{pmatrix}_{H(s,q)}\begin{pmatrix}\partial_1 H\\ \partial_2 H\end{pmatrix}_s=\\ &= \begin{pmatrix}T(T^s(R_s)) & DT^s_{R_s(q)}\end{pmatrix}\begin{pmatrix}1\\ \dot{R}_s(q)\end{pmatrix},\end{aligned}$$

and we get

$$\frac{d}{dt}\Upsilon(t,q)|_{t=s}=T(T^s\circ R_s(q))+DT^s_{R_s(q)}\circ\dot{R}_s(q).$$

So the C^1 vector field Z is defined in flowbox coordinates by:

$$Z(.)=T(.)+DT^s_{R_s(q)}\circ\dot{R}_s(R_{-s}\circ T^{-s})(.)$$

Let $T^s(R_s(q))=(cs,y,z)$. Since $T^{-s}(cs,y,z)=(0,y,z)$ and $DT^s_{R_s(q)}=Id$ by (7) we obtain that the C^1-perturbation is defined by $Z=T+P$ where:

$$P(x,y,z)=\xi\dot{g}(x)G(\sqrt{y^2+z^2})(0,-z,y). \tag{8}$$

Properties of $Z=T+P$:

Z^t is volume preserving;

$$\begin{aligned}\text{div}Z(x,y,z) &= \text{div}T(x,y,z)-\frac{\partial G}{\partial y}\xi\dot{g}(x)+\frac{\partial G}{\partial z}\xi\dot{g}(x)=\\ &= \xi\dot{g}(x)[-\dot{G}\frac{y}{\sqrt{y^2+z^2}}z+\dot{G}\frac{z}{\sqrt{y^2+z^2}}y]=0.\end{aligned}$$

We also have that the support of the perturbation P is $B(p,r)\times[0,c\alpha]$.

Estimation of the C^1 norm of the perturbation P:

By (vi) $|y|,|z|<\epsilon/\Theta$ so $\|P\|_{C^0}\leq\epsilon/\Theta$. To compute the C^1 norm we take derivatives:

$$DP_{(x,y,z)}=\begin{pmatrix}0 & 0 & 0\\ -\xi\ddot{g}(x)G(\rho)zc^{-1} & -\xi\dot{g}(x)\frac{\partial G}{\partial y}z & -\xi\dot{g}(x)[\frac{\partial G}{\partial z}z+G(\rho)]\\ \xi\ddot{g}(x)G(\rho)yc^{-1} & \xi\dot{g}(x)[\frac{\partial G}{\partial y}y+G(\rho)] & \xi\dot{g}(x)\frac{\partial G}{\partial z}y\end{pmatrix} \tag{9}$$

For the first column we use (v), but we must verify also that the other terms are unaffected by the choice of $r>0$ small. We take, for example, $\frac{\partial G}{\partial y}z$ and polar coordinates $(y,z)=(\rho\cos(\beta),\rho\sin(\beta))$ then we have,

$$\begin{aligned}\frac{\partial G}{\partial y}z &= \frac{\partial G(\sqrt{y^2+z^2})}{\partial y}z=\dot{G}\frac{yz}{\sqrt{y^2+z^2}}\leq\dot{G}\frac{2\rho^2}{\rho}\leq\\ &\leq \frac{2\rho}{(1-\sqrt{1-\frac{\kappa}{2}})r}\leq\frac{2}{(1-\sqrt{1-\frac{\kappa}{2}})}.\end{aligned}$$

For the other three terms, $\frac{\partial G}{\partial y}y$, $\frac{\partial G}{\partial z}y$ and $\frac{\partial G}{\partial z}z$ we proceed analogously. Since G and $\dot{g}$ are bounded and $\xi < \frac{\epsilon(1-\sqrt{1-\frac{\kappa}{2}})}{2\Theta^2}$ we get $\|DP\|_{C^0} < \epsilon/\Theta^2$. Note that we are allowed to take y, z close to zero without interfering with the size of the perturbation. This is a key property of the C^1 topology. Next we make use of this fact to get properties (a) and (b). The perturbation is defined, in the original coordinates, by $P_1(.) = D\Psi^{-1}_{\Psi(.)}P(\Psi(.))$. We have $\dot{g}(x) = 0$ for $x \geq 1-\eta/2$ and $x \leq 0$. For $t \in [0, 1-\eta/2]$ and $q \in B(p,r)$ we guarantee that $X^t(q)$ does not intersect $\mathfrak{N}_{X^1(p)}$, so by (i) above we get that $P_1 = Y - X$ is supported inside the flowbox $\underset{t\in[0,1]}{\cup} X^{\tau(p,t)(B(p,r))}(p)(B(p,r))$ and (a) follows. Now we are interested in κ's close to zero and since $\Psi|_{\partial B(p,r)} = Id$ we have $\Psi(B(p, r\sqrt{1-\kappa})) \subseteq B((0,0,0), r\sqrt{1-\frac{\kappa}{2}})$. Therefore for $q \in B(p, r\sqrt{1-\kappa})$ we have $Z^1(\Psi(q)) = (c, R_{\xi\alpha}(\Psi(q)))$, so:

$$DZ^1_{\Psi(q)} = R_{\xi\alpha}. \tag{10}$$

We use (iii) and $X = \Psi_* T$ to get:

$$\begin{aligned}
& \|DX^1_p R_{\xi\alpha} - D\Psi^{-1}_{Z^1(\Psi(p))} DZ^1_{\Psi(p)} D\Psi_p\| = \\
= \;& \|D\Psi^{-1}_{T^1(\Psi(p))} DT^1_{\Psi(p)} D\Psi_p R_{\xi\alpha} - D\Psi^{-1}_{T^1(\Psi(p))} DT^1_{\Psi(p)} R_{\xi\alpha} D\Psi_p\| \leq \\
\leq \;& \|D\Psi^{-1}_{T^1(\Psi(p))}\| \|DT^1_{\Psi(p)}\| \|D\Psi_p R_{\xi\alpha} - R_{\xi\alpha} D\Psi_p\| \leq \\
\leq \;& \Theta \|(D\Psi_p - Id)R_{\xi\alpha} + R_{\xi\alpha}(Id - D\Psi_p)\| \leq \gamma/8.
\end{aligned}$$

Therefore:

$$\|DX^1_p R_{\xi\alpha} - D\Psi^{-1}_{Z^1(\Psi(p))} DZ^1_{\Psi(p)} D\Psi_p\| \leq \frac{\gamma}{8}. \tag{11}$$

Since $Y = \Psi_* Z$ we get $DY^1_q = D\Psi^{-1}_{Z^1(\Psi(q))} DZ^1_{\Psi(q)} D\Psi_q$ so using (iv) we get:

$$\begin{aligned}
& \|DY^1_q - D\Psi^{-1}_{Z^1(\Psi(p))} DZ^1_{\Psi(p)} D\Psi_p\| = \\
= \;& \|D\Psi^{-1}_{Z^1(\Psi(q))} DZ^1_{\Psi(q)} D\Psi_q - D\Psi^{-1}_{Z^1(\Psi(p))} DZ^1_{\Psi(p)} D\Psi_p\| \leq \\
\leq \;& \|D\Psi^{-1}_{Z^1(\Psi(q))} DZ^1_{\Psi(q)} D\Psi_q - D\Psi^{-1}_{Z^1(\Psi(p))} DZ^1_{\Psi(q)} D\Psi_q\| + \\
+ \;& \|D\Psi^{-1}_{Z^1(\Psi(p))} DZ^1_{\Psi(q)} D\Psi_q - D\Psi^{-1}_{Z^1(\Psi(p))} DZ^1_{\Psi(p)} D\Psi_q\| + \\
+ \;& \|D\Psi^{-1}_{Z^1(\Psi(p))} DZ^1_{\Psi(p)} D\Psi_q - D\Psi^{-1}_{Z^1(\Psi(p))} DZ^1_{\Psi(p)} D\Psi_p\| = \\
= \;& \|(D\Psi^{-1}_{Z^1(\Psi(q))} - D\Psi^{-1}_{Z^1(\Psi(p))}) DZ^1_{\Psi(q)} D\Psi_q\| + \\
+ \;& \|D\Psi^{-1}_{Z^1(\Psi(p))} (DZ^1_{\Psi(q)} - DZ^1_{\Psi(p)}) D\Psi_q\| + \\
+ \;& \|D\Psi^{-1}_{Z^1(\Psi(q))} DZ^1_{\Psi(p)} (D\Psi_q - D\Psi_p)\| \leq \\
\leq \;& \|D\Psi^{-1}_{Z^1(\Psi(q))} - D\Psi^{-1}_{Z^1(\Psi(p))}\| \|DZ^1_{\Psi(q)}\| \|D\Psi_q\| + \\
+ \;& \|D\Psi^{-1}_{Z^1(\Psi(p))}\| \|DZ^1_{\Psi(q)} - DZ^1_{\Psi(p)}\| \|D\Psi_q\| + \\
+ \;& \|D\Psi^{-1}_{Z^1(\Psi(q))}\| \|DZ^1_{\Psi(p)}\| \|D\Psi_q - D\Psi_p\| \leq \\
\leq \;& \|DZ^1\| \|D\Psi\| \frac{\gamma}{24C\Theta} + \|D\Psi\| \|D\Psi^{-1}\| \frac{\gamma}{24\Theta^2} + \|D\Psi^{-1}\| \|DZ^1\| \frac{\gamma}{24C\Theta} \leq \\
\leq \;& \frac{\gamma}{8}.
\end{aligned}$$

Therefore:

$$\|DY^1_q - D\Psi^{-1}_{Z^1(\Psi(p))} DZ^1_{\Psi(p)} D\Psi_p\| \leq \frac{\gamma}{8} \tag{12}$$

and (11) and (12) imply $\|DY_q^1 - DX_p^1 R_{\xi\alpha}\| \leq \frac{\gamma}{4}$. Jointly with (i) above we get:

$$\begin{aligned}\|DY_q^1 - DX_p^1 R_\xi\| &\leq \|DY_q^1 - DX_p^1 R_{\xi\alpha}\| + \|DX_p^1 R_{\xi\alpha} - DX_p^1 R_\xi\| \leq \\ &\leq \gamma/4 + \|DX_p^1\|\|R_{\xi\alpha} - R_\xi\| \leq \\ &\leq \gamma/2.\end{aligned}$$

Finally we use (ii) and we obtain:

$$\begin{aligned}\|P_Y^1(q) - P_X^1(p)\circ \hat{R}_\xi\| &= \|\Pi_{Y^1(q)}\circ DY_q^1 - \Pi_{X^1(p)}\circ DX_p^1 R_\xi\| = \\ &= \|\Pi_{Y^1(q)}\circ DY_q^1 - \Pi_{Y^1(q)}\circ DX_p^1 R_\xi\| + \\ &+ \|\Pi_{Y^1(q)}\circ DX_p^1 R_\xi - \Pi_{X^1(p)}\circ DX_p^1 R_\xi\| \leq \\ &\leq \|\Pi_{Y^1(q)}\|\|DY_q^1 - DX_p^1 R_\xi\| + \\ &+ \|\Pi_{Y^1(q)} - \Pi_{X^1(p)}\|\|DX_p^1 R_\xi\| \leq \\ &\leq \gamma.\end{aligned}$$

Estimation of the C^1 norm of P_1:

Above in (vi) we consider $|y|,|z| < \epsilon/\Theta$ and we choose $\xi < \frac{\epsilon(1-\sqrt{1-\frac{\kappa}{2}})}{2\Theta^2}$ and we obtain $\|DP\|_{C^0} < \epsilon/\Theta^2$. Now since $P_1(q) = D\Psi^{-1}_{\Psi(q)}P(\Psi(q))$ and we have $(DP_1)_q = D^2\Psi^{-1}_{\Psi(q)}DP_{\Psi(q)}D\Psi_q$ we obtain that $\|P_1\|_{C^0} \leq \Theta\|P\|_{C^0} \leq \epsilon$ and also that $\|DP_1\|_{C^0} \leq \|D^2\Psi^{-1}\|\|DP\|\|D\Psi\| \leq \Theta^2\|DP\| \leq \epsilon$. We conclude that,

$$\|Y - X\|_{C^1} = \|P_1\|_{C^1} \leq \epsilon,$$

and the lemma is proved. $\square$

Lemma 3.7 *Given $X \in \mathfrak{X}^2_\mu(M)$, a non-periodic point $p \in M$, $\epsilon > 0$, $0 < \kappa < 1$, $\gamma > 0$ and a fixed time $T = 1$. There exists $r > 0$ (depending on p), an angle ξ (not depending on p) and a zero divergence vector field Y, ϵ-C^1-close to X such that:*

(a) $Y - X$ is supported in the flowbox $\bigcup_{t\in[0,1]} X^{\tau(p,-t)(B(p,r))}(B(p,r))$,

(b) For every $q \in X^{\tau(p,-1)(B(p,\sqrt{1-\kappa}r))}(B(p,(1-\kappa)r))$ we have the following inequality $\|P_Y^1(q) - \hat{R}_\xi \circ P_X^1(X^{-1}(p))\| < \gamma$.

Proof: We proceed like in Lemma 3.6, this time for X^{-t} finding a change of coordinates $\hat{\Psi}(x,y,z) = X^{-\lambda^{-1}x}(0,\varphi(y,z))$. Then we consider $R^{-1}_{\xi g(t)G(\rho)}$, $t > 0$ and we find Z. We define $Y = \Psi_* Z$ and we get:

$$\begin{aligned}P_Y^1(q) &= [P_Y^{-1}(Y^1(q))]^{-1} \approx [P_X^{-1}(p)\circ \hat{R}_\xi^{-1}]^{-1} = \\ &= \hat{R}_\xi \circ [P_X^{-1}(p)]^{-1} = \hat{R}_\xi \circ P_X^1(X^1(p)),\end{aligned}$$

and the arguments are equal to the ones used in the proof of Lemma 3.6. $\square$

Now we use the two previous lemmas to build some realizable linear flows.

Lemma 3.8 *Given $X \in \mathfrak{X}^2_\mu(M)$, $\epsilon > 0$, $0 < \kappa < 1$, a non-periodic point p and a fixed time $T = 1$. Then there exists an angle ξ (not depending on p) such that $L_0 = P_X^1(p)\circ \hat{R}_\xi$ and $L_0 = \hat{R}_\xi \circ P_X^1(p)$ are (ϵ,κ)-realizable linear flows of length 1 at p.*

Proof: We prove that $L_0 = P^1_X(p) \circ \hat{R}_\xi$ is (ϵ,κ)-realizable. Let $\gamma > 0$. By Lemma 3.5 we may choose the open set U to be a ball, say $B(p',r') \subseteq B(p,r)$. Now we apply Lemma 3.6 and we get a zero divergence vector field Y, ϵ-C^1-close to X such that $Y - X$ is supported inside the flowbox defined by $\{X^{\tau(p',t)(B(p',r'))}(B(p',r')) : t \in [0,1]\}$ and for every $q \in B(p', r'\sqrt{1-\kappa})$ we have $\|P^1_Y(q) - P^1_X(p') \circ \hat{R}_\xi\| < \gamma$. Note that since $r > 0$ is small the arrival time for points at $B(p,r)$ is almost 1.
The support of the perturbation is contained in the flowbox. For the perturbation P defined in Lemma 3.6 we have $DP_{\Psi(q)} = [0]$ for $\Psi(q) \in B(0,r')$, $T^t(B(0,r'))$ and for $t \geq 1 - \eta/2$ so $DX_q = DY_q$ for any q belonging to $B(p',r')$ and also to $X^{\tau(p,1)(B(p',r'))}(B(p',r'))$. Therefore (i) on Definition 3.1 is true. We take $K \subseteq U$ equal to $\overline{B}(p', r'\sqrt{1-\kappa})$ and we get $\frac{\overline{\mu}(K)}{\overline{\mu}(U)} = \frac{\pi(1-\kappa)r'^2}{\pi.r'^2} = 1 - \kappa$ and (ii) follows. Finally (iii) follows from (b) of Lemma 3.6 and the continuity of the linear Poincaré flow.
For $L_0 = \hat{R}_\xi \circ P^1_X(p)$ we proceed analogously now using Lemma 3.7. Given any open set $\hat{U} \subseteq \mathfrak{N}_{X^{-1}(p)}$ we take $U = X^{\tau(X^{-1}(p),1)(\hat{U})}(\hat{U}) \subseteq \mathfrak{N}_p$, then we measure theoretically fill up this open set U by taking a finite union of balls $\{B_i\}_{i=1,\dots,m}$. We denoted this covering by $\mathfrak{C}$. Let $\hat{\mathfrak{C}} \subseteq \hat{U}$ be such that $X^{\tau(X^{-1}(p),1)(\hat{\mathfrak{C}})}(\hat{\mathfrak{C}}) = \mathfrak{C}$. Of course $\overline{\mu}(\hat{U} - \hat{\mathfrak{C}})$ can be made as small as we want, and the realizability follows. □

We continue to produce realizable linear flows.

Lemma 3.9 *Given $X \in \mathfrak{X}^2_\mu(M)$, $\epsilon > 0$, $0 < \kappa < 1$ and a non-periodic point p, there exists an angle ξ such that for $|\xi_i| < \xi$, $i = 1, 2$;*

$$N_p \overset{P^1_X(p)\circ\hat{R}_{\xi_1}}{\longrightarrow} N_{X^1(p)} \overset{P^r_X(p)}{\longrightarrow} N_{X^{1+r}(p)} \overset{\hat{R}_{\xi_2}\circ P^1_X(X^{r+2}(p))}{\longrightarrow} N_{X^{r+2}(p)}$$

is a (ϵ,κ)-realizable linear flow of length $r+2$ at p.

Proof: Take $\gamma > 0$. By Lemma 3.8 for $\kappa_1 < \kappa$ we get ξ such that $P^1_X(p) \circ \hat{R}_{\xi_1}$ and $\hat{R}_{\xi_2} \circ P^1_X(X^{r+2}(p))$ are (ϵ,κ_1)-realizable. By Lemma 3.4 (1) the trivial flow P^r_X is (ϵ,κ_1)-realizable. Now if $\kappa_1 < \kappa/3$ then we use Lemma 3.4 (2) and obtain the (ϵ,κ)-realizability. □

3.2.2 Large rotations

Now we find conditions under which we can rotate by large angles. In the previous section we were able to rotate by time-1, so what happens if we increase time? We want to rotate by an angle 2π, thus we take a time m such that $\xi m = 2\pi$. But ξ is in general very small, so m must be very large. Note that the choice of m may affect the norm of the perturbation because $\|\Psi\|$, for Ψ given by Lemma 3.2, depends on m and in general increases with m. Furthermore the dynamics along the orbit may also obstruct the construction of a small norm perturbation. Let us consider a situation for which this last problem is minimized, say when we have simultaneously:

(a) No domination conditions for each bundle, i.e. $P^t_X(p)$ is "almost conformal" for all $t \in [0,m]$.

(b) The angle between $N^u_{X^t}(p)$ and $N^s_{X^t}(p)$ is larger than a fixed ξ for all $t \in [0,m]$.

Even if we have properties (a) and (b) our perturbations may not have a small C^1-norm, because as we already said, the entries $y(t)$ and $z(t)$ of DX^t_p (see (6)) may obstruct the construction of a vector field $Y - X$ with small norm. So we will concatenate several small rotations, however this concatenation worsens κ. In [4] this problem is bypassed by using a nested rotation lemma (Lemma 3.7 in [4]) and here we will adapt this procedure. Note that if we had $y(t), z(t)$ bounded, then under conditions (a) and (b) we could perform large rotations with just one single perturbation. In fact this is what we did when we carry out the development of perturbations for linear differential systems, see [3].

Since we will juxtapose several rotations beginning with a ball it turns out that after the first time-1 iteration, by the linear Poincaré flow, it will became an ellipse. We consider vector fields which induce elliptical rotations over normal sections, so take the elliptical rotation flow defined by:

$$E_{\xi g(t)G(\rho)} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & a\cos(\xi g(t)G(\rho)) & -a\sin(\xi g(t)G(\rho)) \\ 0 & d\sin(\xi g(t)G(\rho)) & d\cos(\xi g(t)G(\rho)) \end{pmatrix},$$

where $d \geq a$ are the axis of the ellipse. Let $E = \sqrt{\frac{d}{a}}$. As in [4] we call E the eccentricity of the ellipse, so eccentricity close to one is equivalent to be almost conformal. A simple computation shows that:

$$\dot{E}_{\xi g(t)G(\rho)}.E^{-1}_{\xi g(t)G(\rho)} = \xi \dot{g}(t)G(\rho) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -E^{-2} \\ 0 & E^2 & 0 \end{pmatrix}. \tag{13}$$

The number E measures how large is the norm of a modified area-preserving linear map that sends a ball into the ellipse. The ball $B(p,1)$ is mapped into an ellipse denoted by $\mathfrak{B}(p) \subseteq \mathfrak{N}_p$. Let $0 < \zeta < 1$ and we denote by $\mathfrak{B}(p,\zeta)$ the ellipse $\mathfrak{B}(p)$ after a shrunk by a factor of ζ.

Lemma 3.10 *Given $X \in \mathfrak{X}^2_\mu(M)$, a non-periodic point $p \in M$, $\epsilon > 0$, $0 < \kappa < 1$, $\gamma > 0$ a fixed time $T = 1$ and $E \geq 1$, then there exists $r > 0$ and $\hat{\epsilon} > 0$ such that if $\mathfrak{B}(p,r)$ is an ellipse with eccentricity less than E, $diam(\mathfrak{B}(p,r)) < \epsilon$ and $\|P^1_X(p)\hat{E}_\xi - P^1_X(p)\| < \hat{\epsilon}$ for $\xi > 0$, we may find a C^1 zero divergence vector field Y, ϵ-C^1-close to X such that:*

(a) $Y - X$ is supported in the flowbox $\underset{t\in[0,1]}{\cup} X^{\tau(p,t)(\mathfrak{B}(p,r))}(\mathfrak{B}(p,r))$,

(b) For every $q \in \mathfrak{B}(p, r\sqrt{1-\kappa})$ we have $\|P^1_Y(q) - P^1_X(p) \circ \hat{E}_\xi\| < \gamma$.

Proof: The proof is the same of Lemma 3.6, but the angle ξ depends also on E, because rotations of ellipses with large E imply large perturbations. By (13) we get $\|\dot{E}_{\xi g(t)G(\rho)}.E^{-1}_{\xi g(t)G(\rho)}\| \leq E^2\xi$. Let $\hat{\epsilon} > 0$ be sufficiently small. So we have $\|P^1_X(p)\hat{E}_\xi - P^1_X(p)\| < \hat{\epsilon}$, therefore $a, d \approx 1$. Consequently $E^2 \approx 1$ and we rotate approximately ξ. □

Next lemma says that if we fix a small ellipse in a small ball $B(p,r) \subseteq \mathfrak{N}_p$ and consider its arrival into $\mathfrak{N}_{X^1(p)}$, then this set is almost the image by the

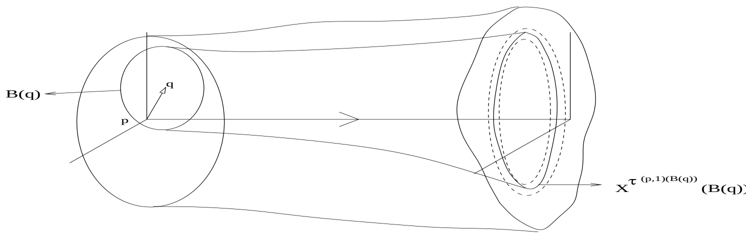

Figure 2: For small $r > 0$, $X^{\tau(p,1)(\mathfrak{B}(q))}(\mathfrak{B}(q))$ is almost an ellipse.

linear Poincaré flow at p of the same ellipse modulo translations. A similar statement is proved in Lemma 3.6 of [4] (see Figure 2).

Lemma 3.11 *Let $X^t : M \longrightarrow M$ be a C^1-flow, $\zeta \in]0,1[$ (near 1), $E \geq 1$. There exists $r > 0$ such that, for all ellipsis $\mathfrak{B}(q) \subseteq B(p,r) \subseteq \mathfrak{N}_p$ with eccentricity less or equal than E we have:*

(A) $P^1_X(p)(\mathfrak{B}(q,\zeta) - q) + X^{\tau(p,1)(q)}(q) \subseteq X^{\tau(p,1)(\mathfrak{B}(q))}(\mathfrak{B}(q))$

(B) $X^{\tau(p,1)(\mathfrak{B}(q))}(\mathfrak{B}(q)) \subseteq P^1_X(p)(\mathfrak{B}(q, 2-\zeta) - q) + X^{\tau(p,1)(q)}(q)$.

Since $P^t_X(p)$ is modified area-preserving, we measure the non-conformality using its norm $\|P^t_X(p)\|$ in the following way. Suppose that $P^t_X(p)$ has a matrix representation $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ with $d \geq a$. Then the eccentricity of $P^t_X(p)(B(p,1))$ is $E = \sqrt{\frac{d}{a}}$. Since $\|P^t_X(p)\| = d$, and by volume-preserving we have $a^{-1} = d.x(t)$ we conclude that $E = \sqrt{\frac{d}{a}} = \sqrt{d^2 x(t)} = d\sqrt{x(t)} = \|P^t_X(p)\|\sqrt{x(t)}$.

In next lemma we adapt Lemma 3.3 of [4].

Lemma 3.12 *Given $X \in \mathfrak{X}^2_\mu(M)$, a non-periodic point $p \in M$, $\epsilon > 0$, $0 < \kappa < 1$, $\gamma > 0$ and $E \geq 1$, suppose that for a fixed $n \in \mathbb{N}$ we have $\|P^j_X(p)\| \leq E\sqrt{x^{-1}(j)}$ for $j = 0, 1, ..., n-1$. Then there exists $\hat{\epsilon} > 0$ such that if:*

$$N_p \xrightarrow{L_0} N_{X^1(p)} \xrightarrow{L_1} ... \xrightarrow{L_{n-1}} N_{X^n(p)}$$

is a sequence of linear flows verifying:

(a) $L_{j-1} \circ ... \circ L_0(B(p,1)) = P^j_X(p)(B(p,1))$ *for* $j = 0, 1, ..., n-1$;

(b) $\|P^1_X(X^j(p)) - L_j\| < \hat{\epsilon}$ *for* $j = 0, 1, ..., n-1$,

then $\{L_0, L_1, ..., L_{n-1}\}$ is a (ϵ, κ)-realizable linear flow.

Proof: Let $\gamma > 0$ be given. Take $\hat{\epsilon}$ given by Lemma 3.10 and depending on X, ϵ, κ, E and $\gamma/3$.

Choice of r:

We choose $r_0 > 0$ such that:

(1) $\|P_X^j(q)\| < 2E\sqrt{x^{-1}(j)}$ for all $q \in B(p, r_0\sqrt{1-\kappa})$ and for $j = 0, 1, ..., n-1$;

(2) For all $q \in X^{\tau(p,j)(B(p,r_0))}(B(p, r_0))$ we have $\|P_X^1(q) - P_X^1(X^j(p))\| < \frac{\gamma}{3E}$;

(3) Any vector field Y ϵ-C^1-close to X and also such that $X = Y$ outside $X^{[0,n]}(B(p, r_0))$ verify: $\|P_Y^1(Y^j(q)) - P_Y^1(x_j)\| < \frac{\gamma}{3}$, for any $q \in B(p, r_0)$ and $x_j \in \mathfrak{N}_{X^j(p)}$.

By hypothesis $\|P_X^j(p)\| \leq E\sqrt{x^{-1}(j)}$ and also we have by *(a)* $L_{j-1} \circ ... \circ L_0(B(p, 1)) = P_X^j(p)(B(p, 1))$ for $j = 0, 1, ..., n-1$, then we take E_j the elliptical rotation with eccentricity less or equal than E and define,

$$L_j := P_X^1(X^j(p)) \circ \hat{E}_j.$$

Now we choose $\kappa_0 < \kappa$ by taking $\lambda \in]0, 1[$ near 1 verifying $\lambda^{4n}(1 - \kappa_0) > 1 - \kappa$. We take $\zeta \in]0, 1[$, such that $\zeta \in]\lambda, 1[$ and $2 - \zeta \in]1, \lambda^{-1}[$. By Lemma 3.11 there exists $r_1 > 0$ such that for all ellipsis $\mathfrak{B}(q) \subseteq B(p, r_1) \subseteq \mathfrak{N}_p$ with eccentricity less or equal than E we have:

(A) $P_X^1(p)(\mathfrak{B}(q, \zeta) - q) + X^{\tau(p,1)(q)}(q) \subseteq X^{\tau(p,1)(\mathfrak{B}(q))}(\mathfrak{B}(q))$

(B) $X^{\tau(p,1)(\mathfrak{B}(q))}(\mathfrak{B}(q)) \subseteq P_X^1(p)(\mathfrak{B}(q, 2-\zeta) - q) + X^{\tau(p,1)(q)}(q)$.

Again by Lemma 3.11 we have for all $j \in \{1, ..., n-1\}$ that there exists $r_{j+1} > 0$ such that for any ellipse $\mathfrak{B}(q) \subseteq B(X^j(p), r_{j+1}) \subseteq \mathfrak{N}_{X^j(p)}$ with eccentricity less or equal than E we have:

$$X^{\tau(X^j(p),1)(\mathfrak{B}(q))}(\mathfrak{B}(q)) \supseteq P_X^1(X^j(p))(\mathfrak{B}(q, \zeta) - q) + X^{\tau(X^j(p),1)(q)}(q).$$

After applying Lemma 3.11 n times, we choose $r_{n+1} > 0$ such that:

$$X^{\tau(p,j)(B(p,r_{n+1}))(B(p,r_{n+1}))} \subseteq B(X^j(p), r_{j+1}) \text{ for } j \in \{1, ..., n-1\}.$$

We define the value of $r > 0$ in Definition 3.1 by:

$$r := \frac{\min\{r_i\sqrt{x(j)}\}_{i=0}^{n+1}}{3E}.$$

Defining Y and K:

By Lemma 3.5 we consider $U = B(p', r') \subseteq B(p, r)$. We define a sequence of ellipsis $\mathfrak{B}_s^j \subseteq \mathfrak{N}_{X^j(p)}$ for $j \in \{0, 1, ..., n-1\}$ all of eccentricity $\leq E$ by:

$\mathfrak{B}_s^0 = B(p', sr')$ for $s \in]0, 1]$;

$\mathfrak{B}_s^j = P_X^j(p')(B(p', sr') - p') + X^{\tau(p,j)(p')}(p')$ for $j \in \{1, ..., n-1\}$.

Denote $X^{\tau(p,j)(p')}(p') = p'_j$. It follows from $P_X^j(B(p, r)) \subseteq B(X^j(p), E\sqrt{x^{-1}(j)}r)$, $X^{\tau(p,j)(B(p,r))}(B(p, r)) \subseteq B(X^j(p), 2E\sqrt{x^{-1}(j)}r)$ and from the choice of r that,

$$\mathfrak{B}_s^j \subseteq B(X^j(p), 3E\sqrt{x^{-1}(j)}r) \subseteq B(X^j(p), r_{j+1}) \text{ for all } j = \{0, ..., n-1\}.$$

These ellipsis are in the conditions of Lemma 3.11 so for all $j \in \{0, 1, ..., n-1\}$ we have:

$$\mathfrak{B}_{s\lambda^{-1}}^{j+1} \supseteq \mathfrak{B}_{s(2-\zeta)}^{j+1} \supseteq X^{\tau(X^j(p),1)(\mathfrak{B}_s^j)}(\mathfrak{B}_s^j) \supseteq \mathfrak{B}_{s\zeta}^{j+1} \supseteq \mathfrak{B}_{s\lambda}^{j+1}.$$

We apply Lemma 3.10 to p'_j, κ_0, $\mathfrak{B}_s^j$ and E_j, with $s = \lambda^n$. So there exists an angle $\xi(\hat{\epsilon})$ and a vector field $Y_j \in \mathfrak{U}$ such that:

(a) $Y_j - X$ is supported in the flowbox $\underset{t\in[0,1]}{\cup} X^{\tau(p'_j,t)(\mathfrak{B}^j_{\lambda^n})}(\mathfrak{B}^j_{\lambda^n})$;

(b) For every $q_j \in \mathfrak{B}^j_{\lambda^n\sqrt{1-\kappa_0}}$ we have $\|P^1_Y(q_j) - P^1_X(p'_j)\circ \hat{E}_\xi\| < \gamma/3$.

We get Y_j for $j = 0, ..., n-1$ with disjoint supports and define $Y := \sum_{j=0}^{n-1} Y_j$. Defining $K := \overline{\mathfrak{B}}^0_{\lambda^{2n}\sqrt{1-\kappa_0}} = \overline{B}(p', \lambda^{2n}\sqrt{1-\kappa_0})$ we get,

$$\frac{\overline{\mu}(K)}{\overline{\mu}(U)} = \frac{\pi(\lambda^{2n}\sqrt{1-\kappa_0}r')^2}{\pi r'^2} = \lambda^{4n}(1-\kappa_0) > 1-\kappa.$$

Let us see that when we iterate we have a nested sequence, i.e., for all $q \in K$, we have $Y^j(q) \in \mathfrak{B}^j_{\lambda^n\sqrt{1-\kappa_0}}$ for all $j \in \{0, 1, ..., n-1\}$.
We have $Y^{\tau(p',1)(\overline{\mathfrak{B}}^0_s)}(\overline{\mathfrak{B}}^0_s) \subseteq \overline{\mathfrak{B}}^1_{s(2-\zeta)} \subseteq \overline{\mathfrak{B}}^1_{s\lambda^{-1}}$, so for every $j = \{1, 2, ..., n-1\}$ we obtain $Y^{\tau(p',j)(\overline{\mathfrak{B}}^0_s)}(\overline{\mathfrak{B}}^0_s) \subseteq \overline{\mathfrak{B}}^j_{s\lambda^{-j}} \subseteq \overline{\mathfrak{B}}^j_{s\lambda^{-n}}$. Hence for $s = \lambda^{2n}\sqrt{1-\kappa_0}$ we get $Y^{\tau(p',1)(K)}(K) \subseteq \overline{\mathfrak{B}}^j_{\lambda^n\sqrt{1-\kappa_0}}$, and the orbit of q will be always inside the domain of the rotations.
Finally we prove that for all $q \in K$ we have $\|P^1_Y(Y^j(q)) - L_j\| < \gamma$ by using (3), (b) and (2),

$$\begin{aligned}
\|P^1_Y(Y^j(q)) - L_j\| &= \|P^1_Y(Y^j(q)) - P^1_X(X^j(p))\circ\hat{E}_j\| \leq \\
&\leq \|P^1_Y(Y^j(q)) - P^1_Y(q_j)\| + \|P^1_Y(q_j) - P^1_X(p'_j)\circ\hat{E}_j\| + \\
&+ \|P^1_X(p'_j)\circ\hat{E}_j - P^1_X(X^j(p))\circ\hat{E}_j\| \leq \\
&\leq \gamma/3 + \gamma/3 + \|P^1_X(p'_j) - P^1_X(X^j(p))\|\|\hat{E}_j\| \leq \gamma
\end{aligned}$$

and the lemma is proved. □

4 Exchange of the Oseledets directions along an orbit segment

When we have an orbit without dominated splitting, along an orbit segment $X^{[0,m]}(p)$ of this orbit it may occur an "exchange on the dominance" during a period of time r, i.e. for $c > 1$,

$$\Delta(X^t(p), r) = \frac{\|P^r_X(X^t(p))|N^s_{X^t(p)}\|}{\|P^r_X(X^t(p))|N^u_{X^t(p)}\|} \geq c.$$

Therefore the dynamics sends vectors near $N^u_{X^r(p)}$ into vectors near $N^s_{X^{t+r}(p)}$ during that period. Next simple lemma, whoose prove may be found in [4], explicit this behavior. Denote by $n^\sigma_t \in N^\sigma_{X^t(p)}$ the unitary vectors, for $\sigma = u, s$.

Lemma 4.1 *Given an angle ξ, there exists $c > 1$, such that if we have $\Delta(X^t(p), r) > c$ then there exists a non-zero vector $v \in N_{X^t(p)}$ such that $\angle(v, n^u_t) < \xi$ and $\angle(P^r_X(X^t(p)).v, n^s_{t+r}) < \xi$.*

Now we give sufficient conditions under which we may apply Lemma 3.12:

Lemma 4.2 *Let $\xi > 0$ and $d > 1$ be given, there exists $E > 1$ such that: If for all $t \in [0,m]$: $\angle(N^u_{X^t(p)}, N^s_{X^t(p)}) > \xi$ and $d^{-1} \leq \frac{\|P^t_X(p)|_{N^s_p}\|}{\|P^t_X(p)|_{N^u_p}\|} \leq d$ then $\|P^t_X(p)\| \leq E\sqrt{x(t)^{-1}}$ for all $t \in [0,m]$.*

Proof: We define $\angle(N^u_{X^t(p)}, N^s_{X^t(p)}) =: \xi_t > \xi$ for all $t \in [0,m]$. By volume preserving we get $\|P^t_X(p)|_{N^s_p}\| = \sin^{-1}\xi_t x(t)^{-1}\|P^t_X(p)|_{N^u_p}\|^{-1}\sin\xi_0$ therefore,

$$\|P^t_X(p)|_{N^s_p}\|^2 = x(t)^{-1}\frac{\|P^t_X(p)|_{N^s_p}\|}{\|P^t_X(p)|_{N^u_p}\|}\sin^{-1}\xi_t \leq x(t)^{-1}d.\sin^{-1}\xi$$

and we obtain $\|P^t_X(p)|_{N^s_p}\| \leq \sqrt{x(t)^{-1}d.\sin^{-1}\xi}$. Analogously:

$$\|P^t_X(p)|_{N^u_p}\|^2 \leq x(t)^{-1}\frac{\|P^t_X(p)|_{N^u_p}\|}{\|P^t_X(p)|_{N^s_p}\|}\sin^{-1}\xi_t \leq x(t)^{-1}d.\sin^{-1}\xi$$

and we also obtain $\|P^t_X(p)|_{N^u_p}\| \leq \sqrt{x(t)^{-1}d.\sin^{-1}\xi}$. We conclude that $\|P^t_X(p)\| \leq \sqrt{2x(t)^{-1}d.\sin^{-1}\xi}$, for all $t \in [0,m]$, so the statement holds by taking $E = \sqrt{2d.\sin^{-1}\xi}$. □

Now we are able to mix the Oseledets subspaces by small perturbations along orbits with lack of hyperbolicity.

Lemma 4.3 *Let $X \in \mathfrak{X}^2_\mu(M)$, $\epsilon > 0$ and $\kappa \in (0,1)$. There exists $m \in \mathbb{N}$, such that for every $p \in \Delta_m(X)$ there exists a (ϵ,κ)-realizable linear flow such that:*

$$L^m(N^u_p) = N^s_{X^m(p)}.$$

Proof: First we set up the constants. Take $\xi > 0$ the minimum of the angles verifying simultaneously Lemma 3.8 and Lemma 3.9 and depending on X, ϵ and $\kappa/2$. Take $C := \max\{\|DX^{\pm 1}\| : p \in M\}$ and c given by Lemma 4.1 depending on the angle ξ. It also will be usefull to take $c > C^2$. Lemma 4.2 gives us $E > 1$ depending on ξ and $d = 2c^2$. Let $\hat{\epsilon} > 0$ depending on X, ϵ, κ and E given by Lemma 3.12. Let $\beta > 0$ be such that for $\xi_0 < \beta$, $\|R_{\xi_0} - Id\| \leq C^{-1}E^{-2}\hat{\epsilon}$. Finally we take a very large $m \in \mathbb{N}$ verifying $m \geq \frac{2\pi}{\beta}$.
I - Angle between Oseledets subspaces small:
If along the orbit segment there is a time r such that the angle between $N^u_{X^r(p)}$ and $N^s_{X^r(p)}$ is less than ξ say:

$$\text{For some } r \in [0,T] \text{ we have } \angle(N^u_{X^r(p)}, N^s_{X^r(p)}) < \xi. \tag{14}$$

We take advantage of this fact and define a realizable linear flow of length 1 in the following way; If $r < m-1$ the linear flow is based at $X^r(p)$ and defined by $L_0 := P^1_X(X^r(p)) \circ R_\xi$ and if $r > m-1$ the linear flow is based at $X^{r-1}(p)$ and defined by $L_0 := R_\xi \circ P^1_X(X^{r-1}(p))$. Now we use Lemma 3.8 and concatenate from right and left, if necessary, with trivial realizable linear flows by using (1) of Lemma 3.4. We obtain $L^m(N^u_p) = N^s_{X^m(p)}$.
II - Locally N^s dominates N^u:
Now we suppose that:

$$\text{For some } 0 \leq r+t \leq m \text{ we have } \Delta(X^t, r) > c. \tag{15}$$

We use Lemma 4.1 and there exists a vector $v \in N_{X^t(p)}$ such that $\angle(v, n_t^u) < \xi$ and $\angle(P_X^r(X^t(p)).v, n_{t+r}^s) < \xi$. Since ξ is small we apply Lemma 3.8 to both extremes at $X^t(p)$ and at $X^{t+r}(p)$. So by choice of c we get $r > 2$ we have disjoint perturbations. Therefore our first rotation allow us to send $N^u_{X^t(p)}$ onto $v.\mathbb{R}$, the dynamics help us and maps this direction into $P_X^r(X^t(p)).v$ in time r, finally and another rotation sends $P_X^r(X^t(p)).v.\mathbb{R}$ onto $N^s_{X^{t+r}(p)}$. Now we use Lemma 3.4 and concatenate the three realizable linear flows, say rotation-trivial-rotation, by using Lemma 3.9 and we get $L^m(N_p^u) = N^s_{X^m(p)}$.

III - Lack of dominance behavior:
We suppose that we do not have both (14) and (15). We set up the conditions of Lemma 4.2. Since $\Delta(p,m) \geq \frac{1}{2}$ and (15) is false we conclude that,

$$\Delta(X^r(p), t) = \Delta(X^{t+r}(p), m-t-r)^{-1}\Delta(p,m)\Delta(p,r)^{-1} \geq (2c^2)^{-1},$$

therefore since $d = 2c^2$ and we get:

$$d^{-1} \leq \frac{\|P_X^t(p)|_{N^s_{X^r(p)}}\|}{\|P_X^t(p)|_{N^u_{X^r(p)}}\|} \leq d \text{ for all } r, t \text{ such that } 0 \leq r+t \leq m.$$

In particular for $r = 0$ we have $\forall t \in [0,m]$: $\angle(N^u_{X^t(p)}, N^s_{X^t(p)}) > \xi$ we use Lemma 4.2 and conclude that $\|P_X^t(p)\| \leq E\sqrt{x(t)^{-1}}$ for all $t \in [0,m]$.

Take $\xi_1, \xi_2, ..., \xi_{m-1}$ such that each ξ_j is less than β and also take $\sum_{j=0}^{m-1}\xi_j = \angle(N_p^u, N_p^s)$. We define:

$$\begin{array}{rccc} L_j: & N_{X^j(p)} & \longrightarrow & N_{X^{j+1}(p)} \\ & v & \longmapsto & P_X^{j+1}(p) \circ R_{\xi_j} \circ [P_X^j(p)]^{-1}.v \end{array}$$

Let us see that we are in the hypotheses of Lemma 3.12:
Since by definition $L_{j-1} \circ ... \circ L_0 = P_X^j(p) \circ R_{\sum_{j=0}^{m-1}\xi_j}$ we have

$$L_{j-1} \circ ... \circ L_0(B(p,1)) = P_X^j(p)(B(p,1))$$

and it verifies (a). Now we have:

$$\begin{aligned} \|P_X^1(X^j(p)) - L_j\| &\leq \|P_X^1(X^j(p)) - P_X^{j+1}(p) \circ R_{\xi_j} \circ [P_X^j(p)]^{-1}\| = \\ &= \|P_X^1(X^j(p))[Id - P_X^j(p) \circ R_{\xi_j} \circ [P_X^j(p)]^{-1}]\| \leq \\ &\leq \|P_X^1(X^j(p))\|\|P_X^j(p)[Id - R_{\xi_j}][P_X^j(p)]^{-1}]\| \leq \\ &\leq \|P_X^1(X^j(p))\|\|P_X^j(p)\|\|[P_X^j(p)]^{-1}\|\|Id - R_{\xi_j}\| \leq \\ &\leq CE\sqrt{x^{-1}(j)}E\sqrt{x(j)}\|Id - R_{\xi_j}\|. \end{aligned}$$

In last inequality we use $\|P_X^{-t}\| \leq E\sqrt{x(t)}$. Therefore we have:

$$\|P_X^1(X^j(p)) - L_j\| \leq CE^2\|Id - R_{\xi_j}\| \leq \hat{\epsilon}$$

and (b) is true, so by Lemma 3.12 we have the realizability of our linear flow therefore:

$$L^m(N_p^u) = L_{m-1} \circ ... \circ L_0(N_p^u) = P_X^m(p) \circ R_{\sum_{j=0}^{m-1}\xi_j}(N_p^u) = P_X^m(p).N_p^s = N^s_{X^m(p)},$$

which proves the lemma. □

5 Lowering the norm - Local procedure

Now we consider two lemmas, the first one which we adapt from [4](see Lemma 3.12), gives information about *when* we have a recurrence to a positive measure set. The second lemma is an elementary result, see [5], and it relates the original norm with a new norm which is better when we do the computations.

Lemma 5.1 *Let $X^t : M \to M$ be a measurable μ-invariant flow, $\Delta \subseteq M$ a positive measure set, its saturate $\Gamma = \underset{t\in\mathbb{R}}{\cup} X^t(\Delta)$ and $\gamma > 0$. There exists a measurable function $T : \Gamma \to \mathbb{R}$ such that for μ-a.e. $p \in \Gamma$, all $t \geq T(p)$ and every $\tau \in [0,1]$ there exists some $s \in [0,t]$ verifying $|\frac{s}{t} - \tau| < \gamma$ and $X^s(p) \in \Delta$.*

Proof: See [4]. □

Consider $p, q := X^t(p) \in \Gamma$ and the map $P : N_p \to N_q$ with matrix relatively to *Oseledets basis* (given by $\{n^u_p, n^u_p\}$ and $\{n^u_q, n^u_q\}$):

$$P = \begin{pmatrix} a^{uu} & a^{us} \\ a^{su} & a^{ss} \end{pmatrix}.$$

Let $\|P\|_{\max} = \max\{|a^{uu}|, |a^{us}|, |a^{su}|, |a^{ss}|\}$.

Lemma 5.2 *(a) $\|P\| \leq 4\frac{1}{sin\angle(N^u_p, N^s_p)}\|P\|_{max}$, (b) $\|P\|_{max} \leq \frac{1}{sin\angle(N^u_q, N^s_q)}\|P\|$*

Proof: See [5], Lemma 4.5. □

Lemma 5.3 *Let $X \in \mathfrak{X}^3_\mu(M)$, with X^t aperiodic and the measure of hyperbolic sets zero. Let $\epsilon, \delta > 0$, $0 < \kappa < 1$. There exists a measurable function $T : M \to \mathbb{R}$ such that for μ-a.e. $p \in M$ and every $t \geq T(p)$, there exists a (ϵ, κ)-realizable linear flow at p with length t such that $\|L^t(p)\| \leq e^{t\delta}$.*

Proof: First we take $m \in \mathbb{R}$ large enough given by Lemma 4.3 depending on $X, \epsilon, \kappa/2$, abbreviate $\Gamma = \Gamma^+_m(X)$ and $\Delta = \Delta_m(X)$. The union of Γ with the Oseledets points with zero Lyapunov exponents is a full measure set, otherwise we could get a positive measure set with m-dominated splitting and by Proposition 2.4 X is Anosov and this contradicts the hypothesis of hyperbolic sets have zero measure. We suppose $\mu(\Gamma) > 0$ because if $\mu(\Gamma) = 0$ then for μ-a.e. point $p \in M$ we have $\lambda^+(p) = 0$ and the proof is over because a trivial linear flow do the work. Remember that $\Gamma = \underset{t\in\mathbb{R}}{\cup} X^t(\Delta)$. For μ-a.e. $p \in \Gamma$ the Oseledets theorem give us $Q(p)$ such that $\forall t \geq Q(p)$ we have:

(1) $\frac{1}{t}\log\|P^t_X(p).n^u\| < \lambda^+(p) + \delta$ for all $n^u \in N^u_p - \{0\}$;

(2) $\frac{1}{t}\log\|P^t_X(p).n^s\| < -\lambda^+(p) + \delta$ for all $n^u \in N^s_p - \{0\}$;

(3) $\log\frac{1}{sin\angle(N^u_{X^t(p)}, N^s_{X^t(p)})} < t\delta$.

By using Lemma 5.1 with $\tau = 1/2$ we get recurrence to Δ approximately in the middle of the orbit segment, but to get good estimates to the norm of the linear flow L^t, points in the orbit after this time, must also verify (1) and (2), that is why we consider the following sets; We define $B_n := \{p \in \Gamma : Q(p) \leq n\}$ for

$n \in \mathbb{N}$, of course that $B_n \subseteq B_{n+1}$ and $\mu(\Gamma - B_n) \underset{n\to\infty}{\to} 0$. Consider a family of sets defined by:

$$C_0 := \emptyset, C_1 := \underset{t\in\mathbb{R}}{\cup} X^t(\Delta \cap X^{-m}(B_1)), ..., C_n := \underset{t\in\mathbb{R}}{\cup} X^t(\Delta \cap X^{-m}(B_n)), ...$$

Clearly $C_n \underset{n\to\infty}{\to} \Gamma$, so the measurable function $T : \Gamma \to \mathbb{R}$ will be μ-a.e. defined on each $C_n - C_{n-1}$ for $n \in \mathbb{N}$. Taking $c > 2\max\{\|DX^1_p\| : p \in M\}$ yields the Lyapunov exponents of any $p \in \mathfrak{O}$ less than c. For p we have non-null Lyapunov exponents so we have the Oseledets 1-dimensional subspaces N^u_p and N^s_p. Let $\gamma = \min\{1/6, \delta/c\}$. Now we use Lemma 5.1 substituting Δ by $\Delta \cap X^{-m}(B_n)$ and Γ by $\underset{t\in\mathbb{R}}{\cup} X^t(\Delta \cap X^{-m}(B_n))$, so by this lemma for each n there exists a measurable function $T_n : C_n \to \mathbb{R}$ such that for μ-a.e $p \in C_n$, and for all $t \geq T_n(p)$, there exists some $s \in [0, t]$ verifying $X^s(p) \in \Delta \cap X^{-m}(B_n)$ and $|\frac{s}{t} - \frac{1}{2}| < \gamma$. Now we define a sufficiently large $T(p)$ for $p \in C_n - C_{n-1}$;

$$T(p) \geq \max\{T_n(p), \frac{m}{\gamma}, 6Q(p), \frac{1}{\delta}\log\frac{4}{sin\angle(N^u_p, N^s_p)}\} \tag{16}$$

Let $p \in C_n - C_{n-1}$ and $t \geq T(p)$, since $t \geq T(p) \geq T_n(p)$ by (a) above $X^s(p) \in \Delta$, so by Lemma 4.3 we may define a $(\epsilon, \kappa/2)$-realizable linear flow $L_1 : N_{X^s(p)} \longrightarrow N_{X^{s+m}(p)}$, sending $N^u_{X^s(p)}$ into $N^s_{X^{s+m}(p)}$, now we concatenate from right and left with trivial linear flows so by Lemma 3.4 we obtain a (ϵ, κ)-realizable linear flow defined by:

$$N_p \xrightarrow{L_0} N_{X^s(p)} \xrightarrow{L_1} N_{X^{s+m}(p)} \xrightarrow{L_2} N_{X^t(p)}$$

with $L_0 = P^s_X(p)$ and $L_2 = P^{t-m-s}_X(X^{s+m}(p))$. Now we estimate $\|L^t(p)\|$, and for that we consider the linear maps relatively to a suitable unitary basis $\{n^u_{X^r(p)}, n^s_{X^r(p)}\}$ for $r \in [0, t]$ that is invariant for the linear Poincaré flow, so they have the form:

$$L_2 = \begin{pmatrix} c^{uu} & 0 \\ 0 & c^{ss} \end{pmatrix}, L_1 = \begin{pmatrix} b^{uu} & b^{us} \\ b^{su} & b^{ss} \end{pmatrix}, L_0 = \begin{pmatrix} a^{uu} & 0 \\ 0 & a^{ss} \end{pmatrix}$$

The key observation is that $b^{uu} = 0$, and this is the reason why we send $N^u_{X^s(p)}$ into $N^s_{X^{s+m}(p)}$. Hence we will be able to get all entries of the product matrix small, whereas if $b^{uu} \neq 0$ this could not be done. So consider the product matrix:

$$L^t(p) = \begin{pmatrix} 0 & a^{uu}b^{us}c^{ss} \\ a^{ss}b^{su}c^{uu} & a^{ss}b^{ss}c^{ss} \end{pmatrix}$$

Claim 5.1 *For $p \in C_n - C_{n-1}$ and $t \geq T(p)$ we have:*

(a) $log|a^{uu}| < \frac{1}{2}t(\lambda^+(p) + 4\delta)$;

(b) $log|a^{ss}| < \frac{1}{2}t(-\lambda^+(p) + 4\delta)$;

(c) $log|c^{uu}| < \frac{1}{2}t(\lambda^+(p) + 4\delta)$;

(d) $log|c^{ss}| < \frac{1}{2}t(-\lambda^+(p) + 4\delta)$.

Proof: (of the claim) For (a) we have $s > t(1/2-\gamma) > t/3 > T(p)/3 \geq Q(p)$ so by Oseledets theorem we have $log|a^{uu}| = log|P_X^s(p).n_p^u| < s(\lambda^+(p)+\delta)$ and also $log|a^{ss}| = log|P_X^s(p).n_p^s| < s(-\lambda^+(p)+\delta)$. Since $\gamma\lambda^+(p) < \gamma c < \delta$ and $\gamma < 1/2$ we obtain;

$$s(\lambda^+(p)+\delta) < t(1/2+\gamma)(\lambda^+(p)+\delta) < t(\lambda^+(p)/2+\delta/2+\lambda^+(p)\gamma+\gamma\delta) <$$

$$< t(\lambda^+(p)/2+\delta/2+\delta+\delta/2) < \frac{1}{2}t(\lambda^+(p)+4\delta)$$

and (a) follows. We note that (b) is analog to (a) by taking $-\lambda^+(p)$ instead. For (c) we make use of the fact that $X^s(p) \in X^{-m}(B_n)$, therefore $X^{s+m}(p) \in B_n$ and by definition of B_n, $Q(X^{s+m}(p)) \leq n$, so we will have the approximation rate given by Oseledets theorem if $t-m-s > n$. By (16) for $t \geq T(p)$ we have $-m/t \geq -\gamma$, $-s/t > -\frac{1}{2}-\gamma$ and we know that $-\gamma \geq -1/6$ so:

$$t-m-s = t(1-\frac{m}{t}-\frac{s}{t}) > t(\frac{1}{2}-2\gamma) > \frac{t}{6} > Q(p) \geq n.$$

Now:

$$\begin{aligned} log|c^{uu}| &= log|P_X^{t-m-s}(X^{s+m}(p)).n^u_{X^{s+m}(p)}| < (t-m-s)(\lambda^+(p)+\delta) < \\ &< t(1-m/t-s/t)(\lambda^+(p)+\delta) < t(\gamma+1/2)(\lambda^+(p)+\delta) = \\ &= t(\gamma\lambda^+(p)+\gamma\delta+\lambda^+(p)/2+\delta/2) < \\ &< t(\delta+\delta/2+\lambda^+(p)/2+\delta/2) = \frac{1}{2}t(\lambda^+(p)+4\delta). \end{aligned}$$

Again (d) is analog to (c) by taking $-\lambda^+(p)$ instead and the claim is proved. □
Now we estimate $\|L_1\|_{\max}$. First note that;

$$s+m > t(1/2-\gamma+m/t) > t(1/2-\gamma) > t/6 > Q(p) \geq n,$$

so again by Oseledets theorem (3) we have an estimate for the angle, i.e.,

$$sin^{-1}\angle(N^u_{X^{s+m}(p)}, N^s_{X^{s+m}(p)}) < e^{(s+m)\delta} < e^{t\delta}.$$

Since L_1 is (ϵ,κ)-realizable we conclude that $\|L_1 - P_X^m(X^s(p))\|$ is small, therefore since $t > T(p) \geq m/\gamma$ and $\gamma c < \delta$ we have $\|L_1\| < e^{mc} < e^{t\gamma c} < e^{t\delta}$. By Lemma 5.2 (b) we get:

$$\|L_1\|_{\max} \leq sin^{-1}\angle(N^u_{X^{s+m}(p)}, N^s_{X^{s+m}(p)})\|L_1\| \leq e^{2t\delta}.$$

Now we give estimates for each of the entries of the product matrix:

$$\begin{aligned} |a^{uu}b^{us}c^{ss}| &\leq e^{\frac{1}{2}t(\lambda^+(p)+4\delta)+2t\delta+\frac{1}{2}t(-\lambda^+(p)+4\delta)} = e^{6t\delta}. \\ |a^{ss}b^{su}c^{uu}| &\leq e^{\frac{1}{2}t(-\lambda^+(p)+4\delta)+2t\delta+\frac{1}{2}t(\lambda^+(p)+4\delta)} = e^{6t\delta}. \\ |a^{ss}b^{ss}c^{ss}| &\leq e^{\frac{1}{2}t(-\lambda^+(p)+4\delta)+2t\delta+\frac{1}{2}t(-\lambda^+(p)+4\delta)} \leq e^{-t\lambda^+(p)+6t\delta} \leq e^{6t\delta}. \end{aligned}$$

This imply the inequality $\|L^t(p)\|_{\max} < e^{6t\delta}$. Again by Lemma 5.2 (a) we have:

$$\|L^t(p)\| \leq 4\frac{1}{sin\angle(N_p^u, N_p^s)}\|L^t(p)\|_{\max}.$$

But $t \geq T(p) \geq \frac{1}{\delta}\log\frac{4}{sin\angle(N_p^u,N_p^s)}$ so $\frac{4}{sin\angle(N_p^u,N_p^s)} \leq e^{t\delta}$ and we get $\|L^t(p)\| \leq e^{7t\delta}$. Replacing δ by $\delta/7$ we conclude that $\|L^t(p)\| \leq e^{t\delta}$ and the lemma is proved. □

5.1 Realizing vector fields

Let $X \in \mathfrak{X}^2_\mu(M)^*$, with X^t aperiodic and also with hyperbolic with zero measure. Given $\epsilon, \delta > 0$ and $0 < \kappa < 1$, we suppose that m is large enough to verify Lemma 4.3. By Lemma 5.3 there exists a measurable function $T : M \to \mathbb{R}$ such that for μ-a.e. $p \in M$, for every $t \geq T(p)$, there exists a (ϵ, κ)-realizable linear flow at p with length t such that $\|L^t(p)\| \leq e^{t\delta}$. By Definition 3.1 $\forall \gamma > 0$,

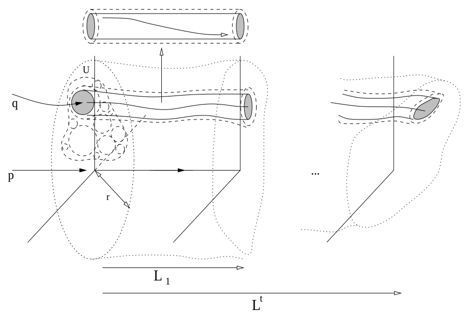

Figure 3: Realizing vector fields given a linear flow L^t.

$\exists r(p,t) > 0$ such that for all open set $\emptyset \neq U \subseteq B(p,r)$, there exists:

(a) A measurable set $K \subseteq U$,

(b) A zero divergence C^1 vector field Y, ϵ-C^1-close to X, such that:

(i) $Y = X$ outside the self-disjoint flowbox $\{X^{\tau(p,s)(U)}(U) : s \in [0,t]\}$ and for every $q \in U, X^{\tau(p,t)(U)}(U)$ we have $DX_q = DY_q$;

(ii) $\overline{\mu}(K) > (1-\kappa)\overline{\mu}(U)$;

(iii) If $q \in K$, then $\|P^t_Y(q) - L^t\| < \gamma$.

By (iii) and $\|L^t(p)\| \leq e^{t\delta}$ we conclude that $\|P^t_Y(q)\| \leq e^{\delta t} + \gamma$ for all $q \in K$, and we note that γ is very small. So the vector field Y is the one who *realizes* the property of having small norm for the orbit of p, and this property is shared by large percentage of points inside any open set inside $\mathfrak{N}_p$ near p (see Figure 3). This property is crucial because after we perturb X the point p may no longer be in $\mathfrak{O}(Y)$, however most points (relatively to Lebesgue measure) near p have norm close to the norm of p, therefore small norm.

6 Lowering the norm - Global procedure

6.1 Sections of flows and special flows

Now we use the local construction of realizable linear flows with small norm to get a conservative vector field Y near X with $LE(Y)$ small. First we define a special flow build under a ceiling function h. Consider a measure space Σ, a map $R : \Sigma \to \Sigma$, a measure $\tilde{\mu}$ defined in Σ and an integrable function $h : \Sigma \to \mathbb{R}^+$, with $h(x) \geq \alpha > 0$ for all $x \in \Sigma$ and $\int_\Sigma h(x) d\tilde{\mu}(x) = 1$. Consider the flow on the product space $M_h \subseteq \Sigma \times \mathbb{R}$ where M_h is the set bellow the graphic of $h(x)$, on which the dynamics is defined by:

$$\begin{array}{rccc} S^s: & \Sigma \times \mathbb{R} & \longrightarrow & \Sigma \times \mathbb{R} \\ & (x,r) & \longmapsto & (R^n(x), r+s-\sum_{i=0}^{n-1} h(R^i(x))) \end{array}$$

and $n \in \mathbb{Z}$ is uniquely defined by $\sum_{i=0}^{n-1} h(R^i(x)) \leq r+s < \sum_{i=0}^{n} h(R^i(x))$. Informally speaking this flow S^t moves any point $(y,r) \in M_h$ to $(y, r+s)$ at time-one speed until hits the graphic of h after that the point returns to the base Σ and proceed its journey. The following lemma, see [1], gives a representation of an aperiodic flow by a flow build from a ceiling function h.

Lemma 6.1 *(Ambrose-Kakutani) Any aperiodic flow $X^t : M \to M$ is isomorphic to some special flow.*

This Lemma is aplicable to vector fields of Proposition 2.11, because these vector fields are all aperiodic. The isomorphism is given by $W : M \to M_h$. The measure $\mu^* = W^*\mu$ is decomposed into the product of lebesgue measure in $\mathbb{R}$ and an R-invariant measure $\tilde{\mu}$ in Σ, i.e. $\int_{M_h} f(x,s)d\mu^* = \int_\Sigma \int_0^{h(x)} f(x,s)dsd\tilde{\mu}(x)$. So we have a simplified representation of our flow $X^s(p)$. In what follows we consider that our flow have this representation. Given a special flow over a section Σ the set $Q = \bigcup_{t\in\mathbb{R}} X^t(\Sigma)$ is called the *Kakutani castle* and the *tower of height i*, which is denoted by T_i, is the set bellow the graphic of $h(B_i)$ where $B_i = \{x \in \Sigma : h(x) = i\}$ so $T_i = \cup X^{[0,i]}(B_i)$. Next we consider a lemma which is a continuous-time version of Lemma 4.1 of [4]:

Lemma 6.2 *Let $X^t : M \to M$ be μ-preserving aperiodic flow. For every positive measure set $U \subseteq M$ and every $h \in \mathbb{R}$, there exists a $\tilde{\mu}$-positive measure section $B \subseteq U$ such that $X^{[0,h]}(B)$ is a self-disjoint flowbox and B is maximal (i.e. no set containing B and with larger measure has the same properties as B).*

Proof: Suppose that for all $B_1 \subseteq U$ with $\tilde{\mu}$ positive measure, we have $\tilde{\mu}(X^{[0,n[}(B_1)\Delta X^{]0,n]}(B_1)) = 0$, therefore $\tilde{\mu}$-a.e. $x \in B_1$ is fixed for X^t or periodic with period less then n but X^t is aperiodic, so there exists $B_1 \subseteq U$ with $\tilde{\mu}(B_1) > 0$, such that $X^{[0,n]}(B_1)$ is a self-disjoint flowbox.

If $\mu(U - X^{[-n,n]}(B_1)) = 0$ there is no chance of getting a set B_1 with larger measure. If $\mu(U - X^{[-n,n]}(B_1)) > 0$, then we extract $B_2 \subseteq \{U - X^{[-n,n]}(B_1)\}$ such that $X^{[0,n]}(B_2)$ is a self-disjoint flowbox and repeat by method of exhaustion. □

For μ-generic point p, Lemma 5.3 gives us $T(p)$, which, in general is very large. Hence Lemma 6.2 will be very usefull to avoid overlapping of perturbations.

6.2 The construction of an adequate section

Now we prove Proposition 2.11 and for that purpose the next step is the construction of a special flow over a section. Consider $X \in \mathfrak{X}^1_\mu(M)$, of class C^2, aperiodic, with hyperbolic sets measuring zero, ϵ and δ like in Proposition 2.11.

For all Y $\epsilon - C^1$ close to X we define $C := \max\{\|P_Y^1(p)\| : p \in M\}$. We take $\kappa = \delta^2$. Using the measurable function given by Lemma 5.3 we define:

$$Z_h = \{p \in M : T(p) \leq h\}. \tag{17}$$

Of course that $\mu(M - Z_h) \underset{h\to\infty}{\to} 0$ so taking h sufficiently large guarantees

$$\mu(M - Z_h) < \delta^2. \tag{18}$$

We intend to build a special flow with ceiling function with height not less than h and section inside Z_h. Since Z_h has almost full measure by Lemma 6.2 we get a $\tilde{\mu}$-positive measure set $B \subseteq Z_h$. $h(x)$ verify $h(x) \geq h$ and since $x \in B \subseteq Z_h$ we have $h \geq T(x)$ so we are in the conditions of Lemma 5.3. Let $\hat{Q}$ be the castle with base B, i.e. $\hat{Q} = \underset{t\in\mathbb{R}}{\cup} X^t(B)$. We have $\hat{Q} \supseteq Z_h$ in measure theoretical sense, this follows because if by contradiction there exists $U_1 \subseteq Z_h$ with $\mu(U_1) > 0$ and $U_1 \cap \hat{Q} = \emptyset$, then by Lemma 6.2 we could extract a section $B_1 \subseteq U_1$ with $\tilde{\mu}(B_1) > 0$ and $X^{[0,h]}(B_1)$ would be a self-disjoint flowbox. But since $\hat{Q} = \underset{t\in\mathbb{R}}{\cup} X^t(B)$ and $\underset{t\in\mathbb{R}}{\cup} X^t(B) \cap U_1 = \emptyset$ we contradict the maximality of B. So by (18) we get the inequality:

$$\mu(\hat{Q}^c) \leq \delta^2. \tag{19}$$

Define the subcastle $Q \subseteq \hat{Q}$ by excluding the towers of $\hat{Q}$ with height bigger than $3h$ and we (like in [4] Lemma 4.2) obtain:

Lemma 6.3 $\mu(\hat{Q} - Q) < 3\delta^2$.

Proof: Let $B_i = \{x \in B : h(x) = i\}$, $T_i = \underset{t\in[0,i]}{\cup} X^t(B_i)$ and $\hat{Q} = \underset{i\geq h}{\cup} T_i$. Take $i, j \in \mathbb{R}$ with $i \geq 2h$ and $j \in [h, i-h]$. We have, by definition of tower, that $\underset{j\in[h,i-h]}{\cup} X^j(B_i)$ is self-disjoint, furthermore is disjoint from $\underset{t\in[0,h[}{\cup} X^t(B)$, by choice of i and j. $\underset{j\in[h,i-h]}{\cup} X^j(B_i) \subseteq Z_h^c$ in measure theoretical sense, otherwise since the set $\underset{j\in[h,i-h]}{\cup} X^j(B_i)$ is disjoint from $\underset{t\in[0,h]}{\cup} X^t(B)$ we extend B with more elements and obtain $\mu(\underset{j\in[h,i-h]}{\cup} X^j(B_i) \cap Z_h) \neq 0$. Each T_i for $i \geq 2h$ decomposes into three floors T_i^1, T_i^2 and T_i^3 where:

$T_i^1 \subseteq \underset{t\in[0,h[}{\cup} X^t(B)$ with length h;

$T_i^2 \subseteq \underset{t\in[h,i-h[}{\cup} X^t(B)$ with length $\geq i - 2h$;

$T_i^3 \subseteq \underset{t\in[i-h,i]}{\cup} X^t(B)$ with length h.

So if $i \geq 3h$, then the length of T_i^2 is bigger or equal than h, hence by (18) we have, $\mu(\underset{i\geq 3h}{\cup} T_i) \leq 3\mu(\underset{i\geq 3h}{\cup} T_i^2) \leq 3\mu(Z_h^c) \leq 3\delta^2$. $\square$

6.3 The zero divergence vector field Y ϵ-C^1-close to X

Now we make use of the realizability of vector fields and the properties of special flows to construct a conservative vector field Y inside the subcastle Q by gluing a finite number of local perturbations supported on self-disjoint flowboxes. We note that the measures $\tilde{\mu}$ and $\overline{\mu}$ are equivalent. Next we follow Lemma 4.14 of [5].

Lemma 6.4 *Given $\gamma > 0$, there exists Y, ϵ-C^1-close to X, a castle U for Y^t and a subcastle K for Y^t such that:*

(a) The castle U is open;

(b) $\mu(U - Q) < \gamma$ and $\mu(Q - U) < 2\gamma$;

(c) $\mu(U - K) < \kappa(1 + \gamma)$;

(d) $Y^t(U) = X^t(U)$ and $Y^t = X^t$ outside the castle U;

(e) If q is in the base of K and $h(q)$ is the height of the tower of K that contains q, then

$$\|P_Y^{h(q)}(q)\| \leq e^{\delta h(q)} + \gamma.$$

Proof: The castle Q is a measurable set and since μ is Borel regular, there exists a compact $J \subseteq Q$ with:

$$\mu(Q - J) < \gamma\mu(\hat{Q}). \tag{20}$$

We choose this compact such that it is a X^t-castle with the same structure as Q (i.e. preserving the same dynamics of bases and towers as the castle Q do). Now we choose an open castle V such that $J \subseteq V$ with:

$$\mu(V - J) < \gamma\mu(\hat{Q}), \tag{21}$$

and also with the same structure of Q and J.

For every point p_1 in $J \cap B$, we have $p_1 \in B$ and so it follows that $h(p_1) \geq h$. Since $(J \cap B) \subseteq Z_h$ we have $T(p_1) \leq h$, therefore $T(p_1) \leq h \leq h(p_1)$. So we are able to contruct a conservative vector field which realizes a linear flow who has the property of having small norm, i.e., for all $t_1 \geq T(p_1)$, and for γ fixed, there exists a radius $r_1(p_1, t_1)$ (take a smaller one if we leave the open castle V) such that for almost (related with $\kappa = \delta^2$) all point in $U_1 = B(p_1, r_1) \subseteq \mathfrak{N}_{p_1}$, more precisely for all point $q \in K_1 \subseteq U_1$, we have a vector field Y_1 supported in a small tubular neighborhood of the orbit segment $X^{[0,t_1]}(p_1)$ such that:

$$\|P_{Y_1}^{t_1}(q)\| \leq e^{\delta t_1} + \gamma.$$

Now we continue by choosing p_i's and by Vitali's arguments we fill up the base of J, denoted by B_J, by finite pairwise self-disjoint balls U_i's verifying $\tilde{\mu}(B_J - U_i) \leq \gamma\tilde{\mu}(B_J)$ so,

$$\mu(J - U) \leq \gamma\mu(J). \tag{22}$$

U is a X^t-castle with section the union of the U_i's. So we get a vector field Y_i supported in a small tubular neighborhood of the orbit segment $X^{[0,t_i]}(p_i)$, ϵ-C^1-close to X and such that for all $q \in K_i \subseteq U_i$:

$$\|P^{t_i}_{Y_i}(q)\| \leq e^{\delta t_i} + \gamma.$$

Now we define $Y = Y_i$ inside the flowbox $\underset{t\in[0,t_i]}{\cup} X^t(U_i)$ and $Y = X$ outside. Since these flowboxes are pairwise disjoint, the vector field is well defined and it is ϵ-C^1-close to X. Note that V is also a castle for Y^t, U is also a Y^t-subcastle of the Y^t-castle V and have for base the union of all $U_i's$. We take K the Y^t-subcastle with section (base of the castle) the union of $K_i's$. By construction of U we get (a), (d) and (e).

Now we prove (b). By (22) and since by (ii) of Definition 3.1 $\overline{\mu}(K_i) > (1-\kappa)\overline{\mu}(U_i)$, we obtain $\mu(U-K) < \kappa\mu(U)$. By (21) and recalling that $V \supseteq U$ and $J \subseteq K$ we get $\mu(U-Q) < \mu(V-J) < \gamma\mu(\hat{Q}) \leq \gamma$.
To prove that $\mu(Q-U) < 2\gamma$ we use (20) and (22) and we conclude that $\mu(Q-U) \leq \mu(Q-J) + \mu(J-U) < 2\gamma\mu(\hat{Q}) < 2\gamma$. Finally for (c) since (b) implies that $\mu(U) < \mu(Q) + \mu(U-Q) < 1+\gamma$, we use the inequality $\mu(U-K) < \kappa\mu(U)$ and get $\mu(U-K) < \kappa(1+\gamma)$. □

6.4 Computing $LE(Y)$

By Lemma 2.10 we have $LE(Y) = \inf_{n\geq 1} \int_M \frac{1}{n}\log\|P^n_Y(p)\|d\mu(p)$. The next inequality is valid for any positive integer in particular for $t = h\delta^{-1}$ (we may assume that this is an integer), $LE(Y) \leq \int_M \frac{1}{t}\log\|P^t_Y(p)\|d\mu(p)$. By what we did above for orbit segments inside the castle K and starting in the base, we guarantee small upper Lyapunov exponent, so we define the set of points whose orbit stay for a long time in K by, $G := \{p \in M : Y^s(p) \in K\ \forall s \in [0,t]\}$, its complementary set is $G^c := \{p \in M : \exists s \in [0,t] : p \in Y^{-s}(K^c)\}$.

Lemma 6.5 *For $p \in G$ we have $\|P^t_Y(p)\| < e^{t(1+6logC)\delta}$.*

Proof: Let $p \in G$. We split the orbit segment $X^{[0,t]}(p)$ by return-times at B_K (the section of the castle K), say $t = b + r_n + \dots + r_2 + r_1 + a$ where all $X^a(p), X^{r_1+a}(p), X^{r_2+r_1+a}(p), \dots, X^{\sum_{i=1}^n r_i+a}(p)$ are in the base B_K. By restriction of height $a, b, r_i\ \in]0, 3h]$ except when $p \in B_K$, where $a = 0$, and $X^t(p) \in B_K$, where $b = 0$. Note that,

$$\begin{aligned}
\|P^t_Y(p)\| &= \|P^{b+\sum_{i=1}^n r_i+a}_Y(p)\| \leq \\
&\leq \|P^b_Y(X^{\sum_{i=1}^n r_i+a}(p)))\| \times \|P^{r_n}_Y(X^{\sum_{i=1}^{n-1} r_i+a}(p)))\| \times \dots \\
&\dots \times\|P^{r_1}_Y(X^a(p))\| \times \|P^a_Y(p)\|.
\end{aligned}$$

But these maps are based at points in B_K so we recall Lemma 5.3 and get:

$$\begin{aligned}
\|P^t_Y(p)\| &\leq C^{3h}e^{\sum_{i=1}^n r_i\delta}C^{3h} \leq e^{(b+\sum_{i=1}^n r_i+a)\delta}C^{6h} \leq \\
&\leq e^{t\delta}C^{6h} \leq e^{t\delta}C^{6\delta t} \leq e^{t(1+6\log C)\delta}.
\end{aligned}$$

□

So if $p \in G$, then $\|P^t_Y(p)\|$ is small, but we still do not know what happens outside G, however next lemma says that G^c has small measure.

Lemma 6.6 *Let $\gamma < \delta^2 h^{-1}$ then $\mu(U \cup \Gamma^*_m(X) - G) < 12\delta$.*

Proof: See [5] Lemma 4.16. □

By Lemma 6.6 we obtain $\mu(G^c) < 12\delta$. Now we finish the proof of Proposition 2.11, therefore Theorem 1:

$$\begin{aligned} LE(Y) &= \inf_{n\geq 1}\int_M \frac{1}{n}\log\|P_Y^n(p)\|d\mu(p) \leq \int_M \frac{1}{t}\log\|P_Y^t(p)\|d\mu(p) \leq \\ &\leq \int_G \frac{1}{t}\log\|P_Y^t(p)\|d\mu(p) + \int_{G^c}\frac{1}{t}\log\|P_Y^t(p)\|d\mu(p) \leq \\ &\leq (1+6\log C)\delta\mu(G) + \log C\mu(G^c) = (1+18\log C)\delta. \end{aligned}$$

And the Proposition 2.11 is proved by substitution of δ by $\frac{\delta}{(1+18\log C)}$ along the proof.

7 Dichotomy for vector fields with singularities

7.1 Global dichotomy under additional hypotheses

Consider the following hypotheses again in the 3-dimensional context:

Hypotheses 7.1 *Let $\mathfrak{X}^2_\mu(M)$ and Λ_m be a X^t-invariant set with m-dominated splitting for the linear Poincaré flow. Then $\mu(\overline{\Lambda_m}) = 0$ or X is Anosov.*

Under this hypotheses we prove an analog to Proposition 2.11, this time over $\mathfrak{X}^2_\mu(M)$. Since X^t is aperiodic, the measure of all singularities are zero, so we do not need to make any perturbations on singularities. Moreover, when we estimate the C^1 norm of the perturbation P, defined in (8), the first column of DP, see (9), is given by, $(0, -\xi\ddot{g}(x)G(\sqrt{y^2+z^2})c^{-1}z, \xi\ddot{g}(x)G(\sqrt{y^2+z^2})c^{-1}y)$. Note that near singularities c^{-1} is very large, however since the radius depends on the point p we can decrease the radius $r(p)$ and control the C^1 norm. So the perturbations we developed work equally on this setting.

Theorem 7.2 *Under Hypotheses 7.1 there exists a residual $\mathfrak{R} \subseteq \mathfrak{X}^1_\mu(M)$ such that if $X \in \mathfrak{R}$ then we have that X is Anosov or for μ-a.e. $p \in M$ we have zero Lyapunov exponents.*

7.2 Proof of Theorem 2

7.2.1 Adapting the proof of Theorem 1

If in Theorem 3.1 we take Ω with C^∞ boundary and g, f also C^∞, the diffeomorphism φ, provided by Dacorogna-Moser, is also C^∞. So our conservative flowbox theorem guarantee a conservative change of coordinates $\Psi \in C^\infty$. Note that the perturbation P, defined in (8), is also C^∞, moreover we know by [17] (for other proof see Theorem 3.1 of [2]) that $\mathfrak{X}^\infty_\mu(M)$ is C^1-dense in $\mathfrak{X}^1_\mu(M)$. The following Proposition is similar to Proposition 2.11. The main diference is *where* the computation of the entropy function is done.

Proposition 7.3 *Let $X \in \mathfrak{X}^\infty_\mu(M)$ and $\epsilon, \delta > 0$. There exists $m \in \mathbb{N}$ and a zero divergence C^∞ vector field Y, ϵ-C^1-close to X that equals X outside the open set $\Gamma_m(X)$ and such that $LE(Y, \Gamma_m(X)) < \delta$.*

Proof: We note that for a fixed $m \in \mathbb{N}$ we have $p \in \Gamma_m^+(X) - \Gamma_m^*(X)$ if p is periodic, has positive Lyapunov exponent and belong to $\Gamma_m(X)$. We consider the following simple claim, for a proof see [3].

Claim 7.1 *For any $\delta > 0$, there exists $m \in \mathbb{N}$ such that we have $\mu(\Gamma_m^+(X) - \Gamma_m^*(X)) < \delta$.*

To find $m \in \mathbb{N}$ we first proceed like in Lemma 4.3, then we take $m \in \mathbb{N}$ sufficiently large to verify also Claim 7.1. Now we consider the measurable function $T : \Gamma_m^*(X) \to \mathbb{R}$ similar to the function of Lemma 5.3. We define $Z_h = \{p \in \Gamma_m^*(X) : T(p) \leq h\}$. Of course that $\mu(\Gamma_m^*(X) - Z_h) \underset{h\to\infty}{\to} 0$ so we take h sufficiently large to verify $\mu(\Gamma_m^*(X) - Z_h) < \delta^2\mu(\Gamma_m^*(X))$. Now we increase h, if necessary, and use Oseledets theorem, which is an asymptotic result, to get for $p \in \mathfrak{O}^0(X)$ the inequality:

$$\|P_X^t(p)\| < e^{t\delta} \text{ for all } t \geq h. \tag{23}$$

Clearly $X^t : \Gamma_m^*(X) \to \Gamma_m^*(X)$ is an aperiodic flow. Now we follow the construction of section 6.3 and finally we compute $LE(Y, \Gamma_m^*(X))$. We define again the set of "good" points, $G := \{p \in \Gamma_m^*(X) : Y^s(p) \in K, \forall s \in [0,t]\}$. By Lemma 6.6 $\mu(U \cup \Gamma_m^*(X) - G) < 12\delta$. Define $A = A(p,t,Y) := \frac{1}{t}\log\|P_Y^t(p)\|$.

$$\begin{aligned} LE(Y,\Gamma_m(X)) &\leq \int_{\Gamma_m(X)} A d\mu(p) \leq \\ &\leq \int_{\Gamma_m(X)-(U\cup\Gamma_m^+(X))} A d\mu(p) + \int_{U\cup\Gamma_m^+(X)-G} A d\mu(p) + \int_G A d\mu(p). \end{aligned}$$

By (23) and since $Y = X$ outside U we obtain,

$$\int_{\Gamma_m(X)-(U\cup\Gamma_m^+(X))} A(p,t,Y) d\mu(p) \leq \int_{\Gamma_m(X)-\Gamma_m^+(X)} A(p,t,X) d\mu(p) \leq \delta.$$

Since $C := \max\{\|P_X^1(p)\| : p \in M\}$ we use Claim 7.1 and Lemma 6.6 to conclude that, $\int_{U\cup\Gamma_m^+(X)-G} A d\mu(p) \leq 13\delta$. Finally at G our construction allow us to obtain $\int_G A d\mu(p) \leq \delta$ and the proposition is proved. $\square$

7.2.2 End of the proof of Theorem 2

Let $\tilde{X} \in \mathfrak{X}_\mu^1(M)$ and $\tilde{\epsilon} > 0$ be given. We will prove that exists $Y \in \mathfrak{X}_\mu^1(M)$, $\tilde{\epsilon}$-C^1-close to X verifying the conclusions of Theorem 2. For $\epsilon = \tilde{\epsilon}/2$, there exists $X \in \mathfrak{X}_\mu^\infty(M)$ ϵ-C^1-close to $\tilde{X}$. It suffices to prove Theorem 2 for the vector field X and $\epsilon > 0$.

Proof: (of Theorem 2)
Let $X \in \mathfrak{X}_\mu^\infty(M)$ and $\epsilon > 0$. We will find Y ϵ-C^1-close to X and a partition $M = D \cup O$ into Y^t-invariant sets such that:
(a) For $p \in O$ we have zero Lyapunov exponents.
(b) D is a countable increasing union of compact invariant sets Λ_{m_n} admitting a m_n-dominated splitting for the Linear Poincaré flow. We define the sequence $\{X_n\}_{n\geq 0} \in \mathfrak{X}_\mu^\infty(M)$, $m_n \in \mathbb{N}$ and eventually $\epsilon_n > 0$ for $n \geq 0$.
Take $X_0 = X$, $\theta > 1$ (near 1) and $\delta_n \underset{n\to 0}{\to} 0$.
If $\int_{\Gamma_m(X)} \lambda^+(X) d\mu = 0$ for some $m \in \mathbb{N}$, then we are finished by taking $Y = X$,

$D = \Lambda_m(X)$ and O a full measure subset of $\Gamma_m(X)$. Otherwise for some $m = m_0$ we have $\int_{\Gamma_{m_0}(X)} \lambda^+(X)d\mu > 0$. Let $\epsilon_0 \in (0, \epsilon/2)$ be sufficiently small such that:

$$\int_{\Gamma_{m_0}(X_0)} \lambda^+(Z)d\mu \leq \theta \int_{\Gamma_{m_0}(X_0)} \lambda^+(X_0)d\mu,$$

for all Z $2\epsilon_0$-C^1-close of X_0 and $Z = X_0$ outside $\Gamma_{m_0}(X_0)$. ϵ_0 always exists because $LE(., \Gamma_{m_0}(X_0))$ is upper semicontinuous and $\Gamma_{m_0}(X_0)$ is simultaneous invariant for X_0^t and Z^t.

Knowing X_0, m_0 and ϵ_0 we are going to define $X_1 \in \mathfrak{X}^\infty_\mu(M)$, $m_1 \in \mathbb{N}$ and eventually $\epsilon_1 > 0$.

By Proposition 7.3, there exists $m_1 \in \mathbb{N}$ and $X_1 \in \mathfrak{X}^\infty_\mu(M)$ a perturbation of X_0 ϵ_0-C^1-close, with $X_1 = X_0$ outside $\Gamma_{m_1}(X_0)$ and such that:

$$\int_{\Gamma_{m_1}(X_0)} \lambda^+(X_1) < \delta_1.$$

Suppose that $m_1 \geq m_0$. Note that $\Gamma_{m_1}(X_1) \subseteq \Gamma_{m_0}(X_1) \subseteq \Gamma_{m_0}(X_0)$. If $\int_{\Gamma_{m_1}(X_1)} \lambda^+(X_1) = 0$, then we are finished by taking $Y = X_1$, $D = \Lambda_{m_1}(X_1)$ and O a full measure subset of $\Gamma_{m_1}(X_1)$. Otherwise if $\int_{\Gamma_{m_1}(X_1)} \lambda^+(X_1) > 0$ we choose $\epsilon_1 \in (0, \epsilon_0/2)$ such that $B(X_1, 2\epsilon_1) \subseteq B(X_0, \epsilon_0)$ and also

$$\int_{\Gamma_{m_1}(X_1)} \lambda^+(Z)d\mu \leq \theta \int_{\Gamma_{m_1}(X_1)} \lambda^+(X_1)d\mu,$$

for all Z $2\epsilon_1$-C^1-close of X_1 and $Z = X_1$ outside $\Gamma_{m_1}(X_1)$.

Recursively knowing X_{n-1}, m_{n-1} and $\epsilon_{n-1} \in (0, \epsilon 2^{-n})$ we are going to define $X_n \in \mathfrak{X}^\infty_\mu(M)$, $m_n \in \mathbb{N}$ and eventually $\epsilon_n > 0$.

Again by Proposition 7.3, there exists $m_n \in \mathbb{N}$ and $X_n \in \mathfrak{X}^\infty_\mu(M)$ a perturbation of X_{n-1} ϵ_{n-1}-C^1-close, with $X_n = X_{n-1}$ outside $\Gamma_{m_{n-1}}(X_{n-1})$ and such that:

$$\int_{\Gamma_{m_n}(X_{n-1})} \lambda^+(X_n) < \delta_n.$$

Suppose that $m_n \geq m_{n-1}$. Now $\Gamma_{m_n}(X_n) \subseteq \Gamma_{m_{n-1}}(X_n) \subseteq \Gamma_{m_{n-1}}(X_{n-1})$. If $\int_{\Gamma_{m_n}(X_n)} \lambda^+(X_n) = 0$, then we are finished by taking $Y = X_n$, $D = \Lambda_{m_n}(Y)$ and O a full measure subset of $\Gamma_{m_n}(Y)$. Otherwise if $\int_{\Gamma_{m_n}(X_n)} \lambda^+(X_n) > 0$ we choose $\epsilon_n \in (0, \epsilon_{n-1}/2)$ so that $B(X_n, 2\epsilon_n) \subseteq B(X_{n-1}, \epsilon_{n-1})$ and also

$$\int_{\Gamma_{m_n}(X_n)} \lambda^+(Z)d\mu \leq \theta \int_{\Gamma_{m_n}(X_n)} \lambda^+(X_n)d\mu,$$

for all Z $2\epsilon_n$-C^1-close of X_n and $Z = X_n$ outside $\Gamma_{m_n}(X_n)$.

We continue this procedure and if for some $n \in \mathbb{N}$ we obtain $\int_{\Gamma_{m_n}(X_n)} \lambda^+(X_n) = 0$ we are over, otherwise the sequence $\{X_n\}_{n\geq 0}$ converges C^1 to some $Y \in \mathfrak{X}^1_\mu(M)$, moreover since $\epsilon_n < \epsilon/2^n$ we have Y ϵ-C^1-close to X.

Let $D = \underset{n\in\mathbb{N}}{\cup} \Lambda_{m_n}(X_n)$. Since $\Lambda_{m_n}(X_n) \supseteq \Lambda_{m_{n-1}}(X_{n-1})$ and $Y = X_n$ at $\Lambda_{m_n}(X_n)$, Y^t has m_n-dominated splitting at $\Lambda_{m_n}(X_n)$.

Let $\Gamma := [\underset{n\in\mathbb{N}}{\cup} \Lambda_{m_n}(X_n)]^c = \underset{n\in\mathbb{N}}{\cap} \Gamma_{m_n}(X_n)$, clearly $\Gamma \subseteq \Gamma_{m_n}(X_n)$. To finish the

proof of Theorem 2 we must see if $\int_\Gamma \lambda^+(Y)d\mu = 0$.
Note that $Y \in B(X_n, 2\epsilon_n)$ for all $n \in \mathbb{N}$. So we have

$$\int_\Gamma \lambda^+(Y)d\mu < \int_{\Gamma_{m_n}(X_n)} \lambda^+(Y)d\mu \leq \theta \int_{\Gamma_{m_n}(X_n)} \lambda^+(X_n)d\mu = \theta\delta_n \underset{n\to\infty}{\to} 0.$$

We conclude that we have zero Lyapunov exponents in a full measure subset O of Γ and Theorem 2 is proved. □

Finally we consider the reason why Theorem 2 is stated for dense subset instead of residual subset? In [5], we find a strategy developed to obtain a residual subset, unfortunately this are not applied to our case, let us see why. They start with a C^1 system which is a continuity point X of the function $LE(., X)$. Then they define the "jump" of the function at X by $LE(X, \Gamma_\infty(X))$ where $\Gamma_\infty(X) := \underset{m\in\mathbb{N}}{\cap} \Gamma_m(X)$. Of course that being a continuity point implies that the "jump" is zero. So $\mu(\Gamma_\infty(X)) = 0$ or $\lambda^+(p) = 0$ for μ-a.e. point $p \in \Gamma_\infty(X)$ and the statements of Theorem 2 are verified. Note that to estimate a lower bound for the "jump" we perturb the original vector field X like we did to prove Theorem 2. But our conservative flowbox theorem may not be applied to X, unless X is of class C^2, so this argument only works for $X \in \mathfrak{X}^2_\mu(M)$. However this set equipped with C^1 topology is not a Baire space, so in general residual sets are meaningless.

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