ABEL-GONCHAROV'S POLYNOMIALS AND THE CASAS-ALVERO CONJECTURE

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ABSTRACT. While exploiting new bounds and genetic sum's representations for the Abel -Goncharov interpolation polynomials, we make a relationship with complex polynomials of degree $n \geq 2$, having, possibly, at least two distinct roots and sharing a root with each of its derivatives up to order $n-1$. In 2001 Casas-Alvero conjectured, that such a complex univariate polynomial must be a power of a linear polynomial. We investigate particular cases, especially for polynomials with only real roots, when the conjecture holds true.

1. INTRODUCTION AND PRELIMINARY RESULTS

Let $P_n(z)$ be a polynomial of degree $n \geq 1$ with complex coefficients

$$P_n(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n, \; a_0 \neq 0. \tag{1}$$

In 2001 Casas- Alvero [2] formulated the following

Conjecture. *Let $P_n(z)$ be a polynomial of degree $n \geq 1$ with complex coefficients. Then it is of the form $P_n(z) = a(z-b)^n, a, b \in \mathbb{C}$, if and only if P_n shares a root with each of its derivatives $P_n^{(1)}, P_n^{(2)}, \ldots, P_n^{(n-1)}$.*

We will call a polynomial P_n which has a common root with each of its derivatives up to the order $n-1$ as CA-polynomial, and by non-trivial CA-polynomial if it has at least two distinct roots. The conjecture says that there exist no non-trivial CA-polynomials. The problem is still open. However, it is proved for infinitely many degrees, for instance, for all powers n, when n is a prime (see in [3], [7], [4]).

It is easy to prove, solving the first order differential equation, that a polynomial has the form $a(z-b)^n$, i.e. it is trivial, if and only if it is divisible by its first derivative. Thus there exist no non-trivial CA-polynomials, which are divisible, correspondingly, by their first derivatives. Therefore we obtain the following equivalence of the Casas-Alvero problem.

Equivalent conjecture. *A polynomial P_n with complex coefficients is divisible by its first derivative $P_n^{(1)}$, if and only if it has a common root with each of its derivatives $P_n^{(1)}, P_n^{(2)}, \ldots, P_n^{(n-1)}$.*

Consequently, the Casas-Alvero conjecture is false, if there exists a polynomial of degree n, which is not divisible by its first derivative, however it shares a root with each of its derivatives up to the order $n-1$. Such a kind of non-trivial CA-polynomial $P_n, \; n \geq 2$ cannot have all distinct roots since at least one root is common with its first derivative. Therefore it has a multiplicity at least 2 and a maximum of possible distinct roots is $n-1$.

Remark 1. Without loss of generality one can assume that $P_n(z)$ is a monic polynomial, i.e. $a_0 = 1$ in (1). Since $\hat{P}_n(z) = a^{-n} P_n(az+b)$, where $a \neq 0, b$ are complex numbers is also a monic CA-polynomial, we may choose and put one of multiple roots to be equal to zero. If it has at least two distinct roots, by the same motivation we can assume that one root is zero and another is equal to one.

We will start to study non-trivial CA-polynomials, if any, drawing a parallel with the familiar Abel-Goncharov interpolation polynomials [5], [9], [10]. We begin with

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Definition. *A sequence of complex numbers (repeated terms are permitted) $z_0, z_1, z_2, \ldots, z_{n-1}, n \geq 1$ is called the CA-sequence, if each $z_\nu, \nu = 0, 1, \ldots, n-1$ corresponds to a common root of the ν-th derivative $P_n^{(\nu)}$ and P_n.*

Clearly, the Casas-Alvero conjecture holds true, if and only if there exist no non-stationary CA-sequences.

Let us consider a monic CA-polynomial $P_n(z),\ n \geq 2$ having n roots $w_1, w_2, \ldots, w_n$, where multiple roots are repeated according to them multiplicity. One can choose a CA-sequence $\{z_\nu\}$ such that $P_n^{(\nu)}(z_\nu) = 0,\ \nu = 0, 1, \ldots, n-1$ and clearly $P_n^{(n)}(z) = n!$ Since $P_n^{(n-1)}(z)$ has the only one root z_{n-1} a maximum possible number of CA-sequences is n^{n-1}. Then we represent $P_n(z)$ in the form

$$P_n(z) = z^n + Q_{n-1}(z), \tag{2}$$

where $Q_{n-1}(z)$ is a polynomial of degree at most $n-1$. To determine $Q_{n-1}(z)$ we differentiate the latter equality ν times, and we calculate the corresponding derivatives in z_ν to obtain

$$Q_{n-1}^{(\nu)}(z_\nu) = -\frac{n!}{(n-\nu)!} z_\nu^{n-\nu}, \quad \nu = 0, 1, \ldots, n-1. \tag{3}$$

But this is the Abel-Goncharov interpolation problem (see [5]) and the polynomial $Q_{n-1}(z)$, which corresponds to the CA-sequence $\{z_\nu\}$ can be uniquely determined via the linear system (3) of n equations with n unknowns and triangular matrix with non-zero determinant. So, following [5], we derive

$$Q_{n-1}(z) = -\sum_{k=0}^{n-1} \frac{n!}{(n-k)!} z_k^{n-k} G_k(z), \tag{4}$$

where $G_k(z), k = 0, 1, \ldots, n-1$ is the system of the Abel-Goncharov polynomials. On the other hand it is known that

$$G_n(z) = z^n - \sum_{k=0}^{n-1} \frac{n!}{(n-k)!} z_k^{n-k} G_k(z). \tag{5}$$

Thus comparing with (2), we find that $G_n(z) = P_n(z)$, and the CA-polynomial is just the Abel-Goncharov polynomial, whose coefficients in (1) are functions of $\{z_\nu\}$, i.e. $a_j \equiv a_j(z_0, z_1, \ldots, z_{n-1}),\ j = 1, 2, \ldots, n$. Meanwhile, we have

$$G_n(z) \equiv G_n(z, z_0, z_1, z_2, \ldots, z_{n-1}),$$

and satisfies additional conditions

$$G_n(w_j, z_0, z_1, z_2, \ldots, z_{n-1}) = P_n(w_j) = 0, \quad j = 1, 2, \ldots, n.$$

However, latter equalities can be written as

$$G_n(\lambda_j, z_0, z_1, z_2, \ldots, z_{n-1}) = 0, \quad j = 1, \ldots, k, \tag{6}$$

where $\lambda_j, j = 1, \ldots, k, 1 \leq k \leq n-1$ are distinct multiple roots of $P_n(z)$ with the corresponding multiplicities $r_1, r_2, \ldots, r_k,\ r_1 + r_2 + \ldots r_k = n$. Consequently, employing the known and new properties of the Abel-Goncharov polynomials we will derive important characteristics of non- trivial CA-polynomials, if any, which hopefully will lead us to an affirmative solution of the Casas- Alvero conjecture.

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