Markov semigroups, monoids, and groups

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ABSTRACT

A group is Markov if it admits a prefix-closed regular language of unique representatives with respect to some generating set, and strongly Markov if it admits such a language of unique minimal-length representatives over every generating set. This paper considers the natural generalizations of these concepts to semigroups and monoids. Two distinct potential generalizations to monoids are shown to be equivalent. Various interesting examples are presented, including an example of a non-Markov monoid that nevertheless admits a regular language of unique representatives over any generating set. It is shown that all finitely generated commutative semigroups are strongly Markov, but that finitely generated subsemigroups of virtually abelian or polycyclic groups need not be. Potential connections with word-hyperbolic semigroups are investigated. A study is made of the interaction of the classes of Markov and strongly Markov semigroups with direct products, free products, and finite-index subsemigroups and extensions. Several questions are posed.

1 INTRODUCTION

The notion of Markov groups was introduced by Gromov in his seminal paper on hyperbolic groups [Gro87, § 5.2], and explored further by Ghys & de la Harpe [GdlH90a]. A group is Markov if it admits a language of unique representatives, with respect to some generating set, that can be described by a Markov grammar. In this context, a Markov grammar is essentially

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a finite state automaton with one initial state and every state being an accept state. The connection with hyperbolic groups arises because every hyperbolic group admits such a language of *minimal-length* unique representatives; such groups are said be strongly Markov [GdlH90a, Théorème 13]. Strongly Markov groups have rational growth series with respect to any generating set [GdlH90a, Corollaire 14].

The overarching aim of this paper is to begin to investigate the natural generalization to semigroups of this notion of Markov groups. A motivation for this is the fruitful generalization from groups to semigroups of concepts involving automata and languages, such as automatic structures (for groups, see [ECH⁺92], for semigroups, [CRRT01]), automatic presentations (see, for example, [OT05, CORT09]), and automaton semigroups (for groups, see the monograph [Nek05], for semigroups, see for example [Mal09, SS05]).

After recalling some necessary background definitions and results in § 2, the generalization of the definition to monoids and semigroups is given in § 3. The generalization to monoids is immediate: a Markov monoid is a monoid admitting a language of unique representatives described by a Markov grammar (again, essentially a finite state automaton with a unique initial state and every state being an accept state), which is equivalent to admitting a prefix-closed regular language of unique representatives (see Proposition 3.1 below). A monoid is strongly Markov if it admits a prefix-closed language of unique minimal-length representatives with respect to any generating set. However, since the empty word is not in general a valid representative for an element of a semigroup, generalizing the definition to semigroups entails excluding the empty word from the otherwise prefix-closed language of unique representatives. Thus there are, for monoids, distinct notions of 'Markov as a monoid' and 'Markov as a semigroup'; fortunately, the concepts turn out to be equivalent, as proved in § 4.

Some of the basic properties of Markov semigroups are explained in § 5. An example of a non-Markov monoid that nevertheless admits a regular (non-prefix-closed) language of unique representatives with respect to any generating set is given in § 6. How certain rewriting systems naturally give rise to Markov semigroups is shown in § 7. That finitely generated commutative semigroups are strongly Markov is shown in § 9. Next, § 10 shows that finitely generated subsemigroups of polycyclic or virtually abelian groups need not be Markov, and discusses the importance of these facts. § 11 exhibits some other interesting examples of Markov semigroups and some examples of non-Markov semigroups.

Given the intimate connection between hyperbolic groups and Markov groups discussed above, it is natural to look for a parallel between semigroups that are word-hyperbolic in the sense of Duncan & Gilman [DG04] and Markov semigroups. However, as discussed in § 12, a word-hyperbolic semigroup need not even admit a regular language of unique normal forms, let alone a prefix-closed one.

§§ 13–16 examine the interaction of Markov semigroups with adjoining identities and zeros, with direct products, with free products, and with finite-index subsemigroups and extensions. Finally, the class of languages that are Markov languages for semigroups is considered in § 17.

Since Markov semigroups seem to be an entirely new area, there are many possible directions for further research. Consequently, various open questions are scattered throughout the paper in the relevant contexts.

We remark that the research described in this paper has involved drawing techniques, ideas, and examples from a broad swathe of semigroup and formal language theory.

2 PRELIMINARIES

2.1 *Generators, alphabets, and words*

The notation used in this paper distinguishes a word from the element of the semigroup or monoid it represents. Let A be an alphabet representing a set of generators for a semigroup or monoid S. Formally, there is a map $\phi : A \to S$ that extends to a surjective homomorphism $\phi : A^+ \to S$ (or $\phi : A^* \to S$ if S is a monoid).

While occasionally the representation map ϕ will be explicitly mentioned, generally the following notational distinction will suffice: for a word $w \in A^*$, denote by $\overline{w}$ the element of M represented by w (so that $\overline{w} = w\phi$); for a set of words $W \subseteq A^*$, denote by $\overline{W}$ the set of all elements of S represented by at least one word in W. Notice that the emptyword ε is a valid representative word if and only if S is a monoid.

2.2 *Languages and automata*

For background information on regular and context-free languages and finite automata, see [HU79, Ch. 2–4].

Let L be a language over an alphabet A. Then L is *prefix-closed* if

$$(\forall u \in A^*, v \in A^+)(uv \in L \implies u \in L),$$

and L is *closed under taking non-empty prefixes*, or more succinctly $+$*-prefix-closed*, if

$$(\forall u \in A^+, v \in A^+)(uv \in L \implies u \in L).$$

Notice that if L is prefix-closed and non-empty, it contains the empty word ε.

2.3 *String-rewriting systems*

This subsection contains facts about string rewriting needed later in the paper. For further background information, see [BO93].

A *string rewriting system*, or simply a *rewriting system*, is a pair $(A, \mathcal{R})$, where A is a finite alphabet and $\mathcal{R}$ is a set of pairs (ℓ, r), known as *rewriting rules*, drawn from $A^* \times A^*$. The single reduction relation $\Rightarrow$ is defined as follows: $u \Rightarrow v$ (where $u, v \in A^*$) if there exists a rewriting rule $(\ell, r) \in \mathcal{R}$ and words $x, y \in A^*$ such that $u = x\ell y$ and $v = xry$. That is, $u \Rightarrow v$ if one can obtain v from u by substituting the word r for a subword ℓ of u, where (ℓ, r) is a rewriting rule. The reduction relation $\Rightarrow^*$ is the reflexive and transitive closure of $\Rightarrow$. The process of replacing a subword ℓ by a word r, where $(\ell, r) \in \mathcal{R}$, is called *reduction*, as is the iteration of this process.

A word $w \in A^*$ is *reducible* if it contains a subword ℓ that forms the left-hand side of a rewriting rule in $\mathcal{R}$; it is otherwise called *irreducible*.

The string rewriting system $(A, \mathcal{R})$ is *noetherian* if there is no infinite sequence $u_1, u_2, \ldots \in A^*$ such that $u_i \Rightarrow u_{i+1}$ for all $i \in \mathbb{N}$. That is, $(A, \mathcal{R})$ is noetherian if any process of reduction must eventually terminate with an irreducible word.

The rewriting system $(A, \mathcal{R})$ is *confluent* if, for any words $u, u', u'' \in A^*$ with $u \Rightarrow^* u'$ and $u \Rightarrow^* u''$, there exists a word $v \in A^*$ such that $u' \Rightarrow^* v$ and $u'' \Rightarrow^* v$.

The string rewriting system $(A, \mathcal{R})$ is *non-length-increasing* if $(\ell, r) \in \mathcal{R}$ implies that $|\ell| \geqslant |r|$ and is *length-reducing* if $(\ell, r) \in \mathcal{R}$ implies that $|\ell| > |r|$. Observe that any length-reducing rewriting system is necessarily noetherian.

The rewriting system $(A, \mathcal{R})$ is *monadic* if it is length-reducing and the right-hand side of each rule in $\mathcal{R}$ lies in $A \cup \{\varepsilon\}$; it is *special* if it is length-reducing and each right-hand side is the empty word ε. Observe that every special rewriting system is also monadic.

The string rewriting system $(A, \mathcal{R})$ is *finite* if the set of rules $\mathcal{R}$ is finite. A monadic rewriting system $(A, \mathcal{R})$ is *regular* (respectively, *context-free*), if, for each $a \in A \cup \{\varepsilon\}$, the set of all left-hand sides of rules in $\mathcal{R}$ with right-hand side a is regular (respectively, *context-free*).

Let $(A, \mathcal{R})$ be a confluent noetherian string rewriting system. Then for any word $u \in A^*$, there is a unique irreducible word $v \in A^*$ with $u \Rightarrow^* v$ [BO93, Theorem 1.1.12]. The irreducible words are said to be in *normal form*. The monoid presented by $\langle A \mid \mathcal{R} \rangle$ may be identified with the set of normal form words under the operation of 'concatenation plus reduction to normal form'.

3 DEFINITIONS

As defined by Ghys & de la Harpe [GdlH90a, Définition 4], a group is Markov if it admits a language of unique representatives defined by a *Markov grammar*, which is essentially a finite state automaton where every state is an accept state [GdlH90a, Définition 1]. The following result shows that the class of languages recognized by such automata are the prefix-closed regular languages. In general, arguments in this paper work with regular expressions rather than explicitly constructed automata, so this equivalences embodied in this result and in the later Proposition 3.4 are important.

PROPOSITION 3.1. *A regular language is prefix-closed if and only if it is recognized by a finite state automaton in which every state is an accept state.*

Proof of 3.1. Suppose L is prefix-closed and let $\mathcal{A}$ be a trim deterministic finite state automaton recognizing L. Let q be some state of $\mathcal{A}$. Since $\mathcal{A}$ is trim, q lies on a path from the initial state to an accept state. Let w be the label on such a path, with w' being the label before the first visit to q. Then w', being a prefix of w, also lies in L. Since $\mathcal{A}$ is deterministic, there is only one path starting at the initial state labelled by w', and this path ends at q. Since $w' \in L$, it follows that q is an accept state. Therefore, since q was arbitrary, every state of $\mathcal{A}$ is an accept state.

Suppose that L is accepted by an automaton $\mathcal{A}$ in which every state is an accept state. Let $w \in L$ and let w' be some prefix of w. Then w labels a path starting at the initial state of $\mathcal{A}$ and leading to an accept state. The prefix w' labels an initial segment of this path, ending at a state q, which, by hypothesis, is also an accept state. Thus $w' \in L$. Since $w \in L$ was arbitrary, L is prefix-closed. $\boxed{3.1}$

In light of Proposition 3.1, a group is Markov if it admits a prefix-closed regular language of unique representatives. Now, in generalizing the notion of being Markov from groups to semigroups, one must change from monoid to semigroup generating sets and modify the notion of the language of representatives appropriately. For groups, the language of representatives is taken over an alphabet representing a monoid generating set for the group, with the empty word being the representative of the identity. (Indeed, the empty word lies in any non-empty prefix-closed language.) In generalizing to arbitrary semigroups, it is necessary to use a semigroup generating set, in which case the empty word is no longer admissable as a representative, and the natural definition for the language of representatives requires not prefix-closure, but only $+$-prefix-closure.

This raises a potential problem, in that a monoid (possibly a group) could be Markov in two different ways: it could be Markov as a monoid (allowing, or rather requiring, that the identity be represented by the empty word), or Markov as a semigroup (requiring that the identity be represented by a non-empty word). It is thus conceivable that the class of monoids that are Markov as monoids and the class of monoids that are Markov as semigroups are distinct. Fortunately, however, the two notions are equivalent, as will be shown in § 4.

The definition of 'Markov as a monoid' is given first, since it is the more direct generalization from the group case:

DEFINITION 3.2. Let M be a monoid and let A be a finite alphabet representing a monoid generating set for M. For $x \in M$, let $\lambda_A(x)$ be the length of the shortest word over A representing x; this is called the natural length of x. (Notice that $\lambda(1_M) = 0$.)

A *monoid Markov language* for M over A is a regular language L that is prefix-closed and contains a unique representative for every element of M.

A *robust monoid Markov language* for M over A is a regular language L that is prefix-closed and contains a unique representative for every element of M such that $|w| = \lambda_A(\overline{w})$ for every $w \in L$.

The monoid M is *Markov* (as a monoid) if there exists a monoid Markov language for M over an alphabet representing some monoid generating set for M.

The monoid M is *robustly Markov* (as a monoid) with respect to an alphabet A representing a generating set for M if there exists a robust monoid Markov language for M over A.

The monoid M is *strongly Markov* (as a monoid) if, for every alphabet A representing a monoid generating set for M, there exists a robust monoid Markov language for M over A.

The reason for introducing the term 'robustly Markov' is because there are many natural examples of semigroups that admit a Markov languages of minimal-length representatives while not being strongly Markov (see for example Proposition 7.1), and consequently such semigroups still enjoy certain pleasant properties.

Note that Ghys & de la Harpe [GdlH90a] use different terminology: rather than 'Markov (respectively, strongly Markov) groups', they use (terms that translate as) 'groups with the Markov (respectively, strong Markov) property'. We prefer Gromov's original terminology, since it does not clash with 'Markov property' in the sense of an undecidable semigroup-theoretic property (see [Mar51] and [BO93, Theorem 7.3.7]).

DEFINITION 3.3. Let S be a semigroup and let A be a finite alphabet representing a generating set for S. For $x \in S$, let $\lambda_A(x)$ be the length of the shortest non-empty word over A representing x; this is called the natural length of x. (Notice that if S is a monoid, $\lambda_A(1_S)$ is not zero.)

A *semigroup Markov language* for S over A is a regular language L that does not contain the empty word, is $+$-prefix-closed, and contains a unique representative for every element of S.

A *robust semigroup Markov language* for S over A is a regular language L that does not contain the empty word, is $+$-prefix-closed, and contains a unique representative for every element of S such that $|w| = \lambda_A(\overline{w})$.

The semigroup S is *Markov* (as a semigroup) if there exists a semigroup Markov language for S over an alphabet representing some generating set for S.

The semigroup S is *robustly Markov* (as a semigroup) with respect to an alphabet A representing a generating set for S if there exists a robust semigroup Markov language for S over A.

The semigroup S is *strongly Markov* (as a semigroup) if, for every alphabet A representing a generating set for S, there exists a robust semigroup Markov language for S over A.

The following result is the parallel of Proposition 3.4 that applies to $+$-prefix-closed languages:

PROPOSITION 3.4. *A regular language that does not contain the empty word is $+$-prefix-closed if and only if it is recognized by a finite state automaton in which every state except the initial state is an accept state, and in which there are no incoming edges to the initial state.*

Proof of 3.4. Suppose L is $+$-prefix-closed and does not contain the empty word. Let $\mathcal{A}$ be a trim deterministic finite state automaton recognizing L. Since L does not contain the empty word, the initial state q_0 is not an accept state. Let q be some other state of $\mathcal{A}$. Since $\mathcal{A}$ is trim, q lies on a path from the initial state to an accept state. Let w be the label on such a path, with $w' \neq \varepsilon$ being the label before the first visit to q. Then w', being a non-empty prefix of w, also lies in L. Since $\mathcal{A}$ is deterministic, there is only one path starting at the initial state labelled by w', and this path ends at q. Since $w' \in L$, it follows that q is an accept state. Therefore, since q was arbitrary, every state of $\mathcal{A}$ is an accept state. Finally, suppose, with the aim of obtaining a contradiction, that there is an incoming edge from a state p to the initial state q_0. Then, since $\mathcal{A}$ is trim, there is a word w labelling a path from q_0 to an accept state, including this edge from p to q_0. Let w' be the prefix of w labelling the non-empty initial segment of the path from q_0 back to q_0. Then, since q_0 is not an accept state and $\mathcal{A}$ is deterministic, $w' \notin L$, contradicting the fact that L is $+$-prefix-closed. Hence there are no edges ending at q_0.

Suppose that L is accepted by an automaton $\mathcal{A}$ in which every state except the initial state is an accept state, and in which the initial state has no incoming edges. Let $w \in L$ and let w' be some prefix of w. Then w labels a path starting at the initial state of $\mathcal{A}$ and leading to an accept state. The prefix w' labels an initial segment of this path, ending at a state q, which cannot be the initial state, since it has no incoming edges, and must therefore, by hypothesis, be an accept state. Thus $w' \in L$. Since $w \in L$ was arbitrary, L is prefix-closed. $\boxed{3.4}$

As remarked in § 3, it is conceivable that the class of monoids that are Markov as monoids and the class of monoids that are Markov as semigroups are distinct, and the same issue arises for being robustly Markov and strongly Markov. Fortunately, for monoids the monoid and semigroup notions are equivalent, as the following three results show:

PROPOSITION 4.1. *A monoid is Markov as a semigroup if and only if it is Markov as a monoid.*

Proof of 4.1. Let M be a monoid.

Suppose that M is Markov as a monoid. Let A be an alphabet representing a monoid generating set for M such that there is a monoid Markov language L for M over A. Then L is prefix-closed, regular, and contains a unique representative for each element of M. In particular, the identity of M is represented by $\varepsilon \in L$. Let 1 be a new symbol representing the identity for M. Then $K = (L - \{\varepsilon\}) \cup \{1\}$ is $+$-prefix-closed, regular, and contains a unique representative for every element of M. Hence K is a semigroup Markov language for M and thus M is Markov as a semigroup.

Suppose now that M is Markov as a semigroup. Let A be an alphabet representing a semigroup generating set for M such that there is a semigroup Markov language L for M over A. Then L is $+$-prefix-closed, regular, and contains a unique representative for every element of M. Let w be the unique word in L representing the identity of M. Let

$$K = \big(L - wA^*\big) \cup \{u \in A^* : wu \in L\}.$$

Since L is $+$-prefix-closed and wA^* is closed under concatenation on the right, $L - wA^*$ is also $+$-prefix closed. Furthermore, $\{u \in A^* : wu \in L\}$ is prefix-closed. (Notice that this set contains ε since w lies in L.) So K is prefix-closed. Moreover, wu and u represent the same element of M for any $u \in A^*$, so $\{u \in A^* : wu \in L\}$ consists of unique representatives for exactly those elements of M whose representatives in L have w as a prefix. Hence every element of M has a unique representative in K. Finally, notice that K is regular. Thus K is a monoid Markov language for M and so M is Markov as a monoid. $\boxed{4.1}$

PROPOSITION 4.2. *1. If a monoid is robustly Markov as a monoid with respect to some alphabet A representing a semigroup generating set, it is also robustly Markov as a semigroup with respect to A. Furthermore, if a monoid is robustly Markov as a monoid with respect to an alphabet B representing a monoid generating set that is not also a semigroup generating set, then it is robustly Markov as a semigroup with respect to $B \cup \{1\}$, where 1 represents the identity.*

2. If a monoid is robustly Markov as a semigroup with respect to some alphabet A representing a (semigroup) generating set, then it is robustly Markov as a monoid with respect to A. Furthermore, if a monoid is robustly Markov as a semigroup with respect to $B \cup \{1\}$, where B represents a monoid generating set and 1 represents the identity, then it is robustly Markov as a monoid with respect to B.

Proof of 4.2. Let M be a monoid.

1. Suppose that M admits a robust monoid Markov language L over A. Since $\overline{A}$ generates M as a semigroup, one can choose a shortest non-empty word w over A representating the identity of M. Let $w = w_1 \cdots w_n$, with $w_i \in A$.

For each non-empty prefix $w_1 \cdots w_i$ of w, let p_i be the unique element of L representing the same element of M as this prefix. Notice that if an element of L has a prefix representing $\overline{w_1 \cdots w_i}$, that prefix must be p_i by the prefix-closure of L and the fact that it maps bijectively onto M. Moreover, the length of p_i must be the same as the length of $w_1 \cdots w_i$. To find a robust semigroup Markov language for M over A, it is necessary to replace the prefixes p_i by $w_1 \cdots w_i$ and the empty word ε by w. More formally, let

$$K = \Bigl(\bigl(L - \{\varepsilon\}\bigr) - \bigcup_{i=1}^{n} p_i A^*\Bigr) \cup \{w\} \cup \bigcup_{i=1}^{n} \{w_1 \cdots w_i u : p_i u \in L\}.$$

Now, $L - \{\varepsilon\}$ is $+$-prefix-closed. Since each language $p_i A^*$ is closed under concatenation on the right,

$$\bigl(L - \{\varepsilon\}\bigr) - \bigcup_{i=1}^{n} p_i A^*$$

is $+$-prefix-closed. Furthermore,

$$\{w\} \cup \bigcup_{i=1}^{n} \{w_1 \cdots w_i u : p_i u \in L\}$$

is $+$-prefix-closed since L is and since every prefix of w is in this set. Therefore K is $+$-prefix-closed. Furthermore, K is regular and, by definition, maps bijectively onto M. Finally, since $|p_i| = |w_1 \cdots w_i|$, it follows that the representative in K of an element of M is the same length as its representative in L, excepting that the identity is represented by the non-empty word w in K. So K is a robust semigroup Markov language over A for M.

For the final claim, let L be a robust monoid Markov language for M over B. Then 1 is a shortest non-empty representative of 1_M over the alphabet $B \cup \{1\}$. Then $K = (L - \{\varepsilon\}) \cup \{1\}$ is a regular, $+$-prefix-closed, and consists of minimal-length unique representatives for M. So K is a robust semigroup Markov language for M.

2. Suppose that M admits a robust semigroup Markov language L over an alphabet A representing a semigroup generating set for M.

   Let $w \in L$ be the representative of the identity of M. Since L does not contain the empty word, $|w| \geqslant 1$. Suppose that some word $u \in L$ contains w as a proper subword, with $u = u'wu''$. Then $\overline{u'u''} = \overline{u}$ and $|u'u''| < |u|$, which contradicts the fact that representatives in L are supposed to be length-minimal. So w is not a proper subword of any word in L. In particular, $L' = L - \{w\}$ is $+$-prefix-closed.

   Notice that L' is $+$-prefix-closed, regular, and consists of unique representatives having minimal length (over A) for non-identity elements of M. Thus $K = L' \cup \{\varepsilon\}$ is prefix-closed, regular, and consists of unique representatives for all elements of M. So K is a robust monoid Markov language over A for M.

   For the final claim, let $A = B \cup \{1\}$ and follow the same reasoning. In this case, 1 is the minimal-length representative for 1_M and does not occur as a subword of any other element of L. So $L' \subseteq B^+$ and so K is a robust monoid Markov language over B for M. $\boxed{4.2}$

The following result is a consequence of Proposition 4.2:

PROPOSITION 4.3. *A monoid is strongly Markov as a semigroup if and only if it is strongly Markov as a monoid.*

Proof of 4.3. Let M be a monoid.

Suppose M is strongly Markov as a monoid. Let A be an alphabet representing a semigroup generating set for M. Then M is robustly Markov as a monoid with respect to A. By the first part of Proposition 4.2, M is robustly Markov as a semigroup with respect to A. Since A was an arbitrary alphabet representing a semigroup generating set for M, by definition M is strongly Markov as a semigroup.

Suppose M is strongly Markov as a semigroup. Let B be an alphabet representing a monoid generating set for M. Then M is robustly Markov as a semigroup with respect to $B \cup \{1\}$, where $\overline{1} = 1_M$. By the second part of Proposition 4.2, M is robustly Markov as a monoid with respect to B. Since B was an arbitrary alphabet representing a monoid generating set for M, by definition M is strongly Markov as a monoid. $\boxed{4.3}$

In light of Propositions 4.1, 4.2, and 4.3, there is no need for a terminological distinction between the conditions 'Markov as a semigroup' and 'Markov as a monoid', between 'robustly Markov as a semigroup' and 'robustly Markov as a monoid', and between 'strongly Markov as a semigroup' and 'strongly Markov as a monoid': the terms 'Markov', 'robustly Markov', and 'strongly Markov' alone will suffice.

The results in this section parallel the situation for automatic monoids: a monoid is automatic as a semigroup if and only if it is automatic as a monoid [DRR99, §5].

5 BASIC PROPERTIES

It is important to note that a Markov language does not define a group or semigroup up to isomorphism, unlike an automatic structure [KO06, Proposition 2.3]. To see this, notice that if A is a finite alphabet of size n, then A (qua language of one-letter words) is a semigroup Markov language for any semigroup of size n, and $A \cup \{\varepsilon\}$ is a monoid Markov language for any monoid or group of size $n+1$. The language $(a^* \cup (a^{-1})^*)(b^* \cup (b^{-1})^*)(c^* \cup (c^{-1})^*)$ is a Markov language for both $\mathbb{Z}^3$ and the Heisenberg group [Ghy90, § 5.2].

The *growth series* of a semigroup S with respect to a finite alphabet A representing a generating set for S is

$$\Sigma(S, A) = \sum_{s \in S} x^{\lambda_A(x)},$$

or equivalently

$$\Sigma(S, A) = \sum_{n=0}^{\infty} \sigma_A(n) x^n,$$

where $\sigma_A(n) = |\{s \in S : \lambda_A(s) = n\}|$. A growth series $\Sigma(S, A)$ is said to be *rational* if it is a power series expansion of a rational function.

THEOREM 5.1. *If a semigroup admits a robust Markov language with respect to a particular generating set, then its growth series with respect to that generating set is a rational function. A strongly Markov semigroup has rational growth series with respect to any generating set.*

Proof of 5.1. The proof for groups generalizes directly [GdlH90a, Corollaire 14]. $\boxed{5.1}$

The independent importance of semigroup growth series (see, for example, [GdlH97, § 4]) means that, as a consequence of Theorem 5.1, robust Markov semigroups are of considerably greater interest than Markov semigroups generally.

REMARK 5.2. It is worth observing that the growth rate of a Markov language need not mirror the growth of the semigroup or monoid. For example, all finitely generated polycyclic groups are Markov [GdlH90a, Corollaire 11]. Furthermore, the language of collected words for a finitely generated polycyclic group forms a Markov language [Sim94, p. 395] and is easily seen to have polynomial growth. However, a polycyclic group that is not virtually nilpotent contains a free subsemigroup of rank 2 [Ros74, Theorem 4.12] and hence has exponential growth.

Being Markov implies the existence of a regular language of unique normal forms over any finite generating set:

PROPOSITION 5.3. *Let* S *be a semigroup that admits a regular language of unique normal forms over some generating set (such as a Markov semigroup), and let* A *be a finite alphabet representing a generating set for* S*. Then there is a regular language* L *over* A *such that every element of* S *has a unique representative in* L*.*

[Notice that even if S is a Markov semigroup, the language L need not be prefix-closed.]

Proof of 5.3. Let K be a regular language of unique normal forms for S over some finite alphabet B. For each $b \in B$, let $u_b \in A^+$ be such that u_b represents $\overline{b}$. Let $R \subseteq B^+ \times A^+$ be the rational relation:

$$R = \{(b_1, u_{b_1})(b_2, u_{b_2}) \cdots (b_n, u_{b_n}) : b \in B, n \in \mathbb{N}\}$$

Notice that if $(v, w) \in R$, then $\overline{v} = \overline{w}$.

Let

$$L = K \circ R = \big\{w \in A^* : (\exists v \in K)((v, w) \in R)\big\};$$

observe that L is a regular language. Notice that, by the definition of R, for each word v in K there is exactly one word $w \in L$ with $(v, w) \in R$. Since for each $x \in S$ there is exactly one word v in K with $\overline{v} = x$, it follows that there is exactly one word $w \in L$ with $\overline{w} = x$. That is, the language L maps bijectively onto S. $\boxed{5.3}$

6 A NON-MARKOV MONOID WITH A REGULAR SET OF UNIQUE REPRESENTATIVES

This section exhibits a non-Markov monoid that nevertheless admits a regular language of unique representatives over any alphabet representing a finite generating set. (That is, regularity and uniqueness of representatives is achievable over any alphabet representing a generating set, but prefix-closure is never achievable.) This is important because it shows that the classes of Markov semigroups and monoids are properly contained in the classes of semigroups and monoids admitting regular languages of unique normal forms:

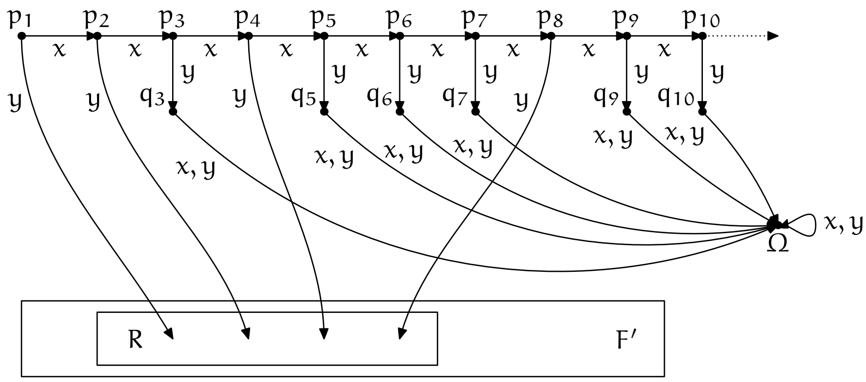

FIGURE 1. An outline of the graph of the action of X on T.

the requirement of prefix-closure properly restricts the classes under consideration.

The example depends on the following construction from [MR, § 5].

DEFINITION 6.1. For any action of a semigroup S on a set T, define a new semigroup $S[T]$ as follows. The carrier set is $S \cup T$; multiplication in S remains the same, and for $s \in S$ and $x, y \in T$,

$$sx = x, \qquad xs = x \cdot s, \qquad xy = y.$$

It is straightforward to check that this multiplication is associative.

To construct the example, proceed as follows. Let F and F' be free monoids with bases $X = \{x, y\}$ and $X' = \{x', y'\}$ respectively and let

$$R = \{w \in F' : |w|_{y'} \text{ is even}\}.$$

Let $w_0, w_1, w_2, \ldots$ be the elements of R enumerated in length-plus-lexicographic order. Define $\psi : \mathbb{N} \cup \{0\} \to R$ by $j \mapsto w_j$, so that ψ is a bijection between $\mathbb{N} \cup \{0\}$ and R. Notice that $|j\psi| < 2^j$ for all $j \in \mathbb{N} \cup \{0\}$. Let

$$\begin{aligned} P &= \{p_i : i \in \mathbb{N}\}, \\ Q &= \{q_i : i \in \mathbb{N} \wedge \neg(\exists j \in \mathbb{N} \cup \{0\})(i = 2^j)\}, \\ T &= P \cup Q \cup F' \cup \{\Omega\}. \end{aligned}$$

Define an action of the generators x and y on the set T as follows:

$$\begin{aligned} & p_i \cdot x = p_{i+1}, \\ & p_i \cdot y = \begin{cases} q_i & \text{if } i \neq 2^j \text{ for any } j \in \mathbb{N} \cup \{0\}, \\ j\psi & \text{if } i = 2^j, \end{cases} \\ & q_i \cdot x = \Omega, \qquad w \cdot x = wx' \ \ (\text{for } w \in F'), \qquad \Omega \cdot x = \Omega, \\ & q_i \cdot y = \Omega, \qquad w \cdot y = wy' \ \ (\text{for } w \in F'), \qquad \Omega \cdot y = \Omega. \end{aligned}$$

Figure 1 illustrates the graph of the action of X on T. Since F is free on X, this action extends to a unique action of F on T.

The aim is to show that $F[T]$ is not Markov but nevertheless admits a regular language of unique representatives over any finite alphabet representing a generating set.

Notice that in $F[T]$, elements of F multiply as in the free monoid and act on T. Elements of F' are members of the set T and thus multiply like right zeroes.

PROPOSITION 6.2. *The monoid* $F[T]$ *admits a regular language of unique representatives over any finite alphabet representing a generating set.*

Proof of 6.2. By Proposition 5.3, it suffices to prove that $F[T]$ admits a regular language of unique representatives over some particular finite alphabet representing a generating set.

Let $A = \{a, b, c, d, e, f\}$, where $\overline{a} = x$, $\overline{b} = y$, $\overline{c} = x'$, $\overline{d} = y'$, $\overline{e} = p_1$, and $\overline{f} = \Omega$. Let $\rho : F' \to A^+$ be the bijection extending $x' \mapsto c$ and $y' \mapsto d$. Let

$$L = \{a, b\}^* \cup ea^* \cup ea^*b \cup (\{c, d\}^+ - R\rho) \cup \{f\}.$$

Then L maps bijectively onto $F[T]$. In particular, the subset $\{a, b\}^*$ maps bijectively onto F, the subset ea^* maps bijectively onto $\{p_i : i \in \mathbb{N}\}$, the subset ea^*b maps bijectively onto $\{q_i : i \in \mathbb{N}\} \cup R$, and the subset $\{c, d\}^+ - R\rho$ maps bijectively onto $F' - R$. So $L \subseteq A^*$ is a regular language of unique representatives for $F[T]$. [Note that L is not prefix-closed, since it does not contain words from $R\rho$ but does contain all words in $(Ry')\rho = (R\rho)d$.] $\boxed{6.2}$

PROPOSITION 6.3. *The monoid* $F[T]$ *is not Markov.*

Proof of 6.3. Suppose, with the aim of obtaining a contradiction, that $F[T]$ admits a Markov language L over some alphabet A.

Informally, the strategy is to reach a contradiction by proving the following:

1. Sufficiently long elements of R must have representatives in L that label paths that run through P for most of their length (excepting a short prefix) and enter $R \subseteq F'$ on their last letter. (Lemma 6.5.)
2. Sufficiently long elements of $F' - R$ have representatives in L that label paths that run through F' for most of their length (excepting a short prefix). (Lemma 6.6.)
3. Taking a suitable prefix of a representative of an element of $F' - R$ yields a representative of an element of R that is not of the form described in step 1. (Conclusion of proof.)

As as preliminary, define several subalphabets of A and several constants that will be used later to clarify what 'sufficiently long' means in the plan above. Let

$$\begin{aligned} A_P &= \{a \in A : \overline{a} \in P\}, \\ A_Q &= \{a \in A : \overline{a} \in Q\}, \\ A_{F'} &= \{a \in A : \overline{a} \in F'\}, \\ A_F &= \{a \in A : \overline{a} \in F\}, \\ A_x &= \{a \in A : \overline{a} \in x^+\}, \\ A_\Omega &= \{a \in A : \overline{a} = \Omega\}; \end{aligned}$$

notice that A is the disjoint union of A_P, A_Q, $A_{F'}$, A_F, and A_Ω, and that $A_x \subseteq A_F$. Let

$$\begin{aligned} m_1 &= |u|, \text{where } u \text{ is the unique representative in } L \text{ of } \Omega, \\ m_2 &= \max\{i : p_i \in \overline{A_P}\}, \\ m_3 &= \max\{i : q_i \in \overline{A_Q}\}, \\ m_4 &= \max\{|\overline{a}| : a \in A_{F'}\}, \\ m &= \max\{m_1, m_2, m_3, m_4\}. \end{aligned}$$

Let $k = \max\{|\overline{a}| : a \in A_F\}$.

Let $\mathcal{A}$ be a deterministic finite automaton recognizing L. Consider the set of labels on simple loops in $\mathcal{A}$. Let V be the set of such labels that lie in A_x^*. Let n be a constant that is a multiple of all of the lengths of the elements of V and that also exceeds the number of states in $\mathcal{A}$.

LEMMA 6.4. *Let $uav \in L$, where $a \in A - A_F$. Then $|u| < n$. That is, any letter from $A_P \cup A_Q \cup A_{F'} \cup A_\Omega$ in a word in L must lie in the first n letters, and hence $L \subseteq A^{\leqslant n} A_F^*$.*

Proof of 6.4. Suppose for *reductio ad absurdum* that $uav \in L$ is as in the hypothesis but that $|u| > n$. Then by the pumping lemma, u factorizes as $u'u''u'''$ such that $u'(u'')^\alpha u''' av \in L$ for all $\alpha \in \mathbb{N} \cup \{0\}$. Since $\overline{a} \in T$, it follows from the definition of multiplication in $F[T]$ that

$$\overline{u'(u'')^\alpha u''' av} = \overline{av},$$

for every $\alpha \in \mathbb{N} \cup \{0\}$, which contradicts the uniqueness of representatives in L. Hence $|u| \leqslant n$. $\boxed{6.4}$

LEMMA 6.5. *The representative in L of every $w \in R \subseteq F'$ with $|w| > m+n+k+kn$ has the form vc, where $v \in A^*$, $c \in A_F - A_x$, $\overline{v} \in P$ and $\overline{c} = x^\beta y$ for some $\beta < k$.*

Proof of 6.5. Let j be such that $j\psi = w$. Since $|w| > m+n+k+kn$, it follows that $2^j > |w| > m+n$ and hence $2^j - n > m$. It also follows that $2^j > n + kn \geqslant 2n$, and so $n < 2^{j-1}$. Hence $2^j - n > 2^{j-1}$. Thus $2^j - n$ is not a power of 2 and so there is an element $q_{2^j - n} \in Q$.

Let t be the representative in L of q_{2^j-n}. Since $2^j - n > m$, the rightmost letter a from $A - A_F$ in the word t cannot be such that $\overline{a} = q_{2^j-n}$ by the definition of m; therefore a must lie in A_P. By Lemma 6.4, t factorizes as uas, where $|u| < n$ and $s \in A_F^*$. Let $s = s'cs''$, where $s' \in A_x^*$ and $c \in A_F - A_x$. (Such a letter c must exist, otherwise $\overline{ubs} \in P$.) Now, $\overline{uas'c} \in Q$. Since the action of F on any element of Q leads to the sink element Ω, it follows that s'' is the empty word. Hence $t = uas'c$.

Let $\overline{c} = x^\beta yz$, where $z \in \{x,y\}^*$. Then $\overline{uas'}x^\beta y \in Q$, and so $z = \varepsilon$ since otherwise $\overline{uas'}x^\beta yz = \Omega$. Since $|\overline{c}| \leqslant k$, it follows *a fortiori* that $\beta < k$.

Furthermore, since $\overline{uas'c} = q_{2^j-n}$, it follows that $\overline{uas'} = p_{2^j-n-\beta}$. Hence, since $\overline{ua} = \overline{a} = p_{m'}$ for some $m' \leqslant m$, it follows that

$$|\overline{s'}| = 2^j - n - \beta - m' \geqslant 2^j - n - k - m > kn.$$

Thus $|s'| > n$ since each letter of s' represents an element of F whose length is at most k.

Thus by the pumping lemma s' factorizes as $v'v''v'''$, where $|\overline{v''}|$ divides n (by the definition of n) and $v'(v'')^\alpha v''' \in L$ for all $\alpha \in \mathbb{N} \cup \{0\}$. Set $\alpha = n/|\overline{v''}|+1$. Then $\overline{uav'(v'')^\alpha v'''} = p_{2^j-\beta}$. Thus

$$\overline{uav'(v'')^\alpha v'''c} = p_{2^j-\beta} x^\beta y = p_{2^j} y = p_{2^j}\psi = w.$$

Set $v = uav'(v'')^\alpha v'''$ to see that the representative t of w has the form vc. $\boxed{6.5}$

LEMMA 6.6. *Let $w \in F' - R$. Then the representative in L of w factorizes as uv where $\overline{u} \in F'$ with $|\overline{u}| < m + k + kn$ and $v \in A_F^*$.*

Proof of 6.6. Let w be in the hypothesis and let t be its representative in L. Since t cannot lie in A_F^*, it contains some letter from $A_P \cup A_Q \cup A_{F'} \cup A_\Omega$. The rightmost such letter cannot lie in $A_Q \cup A_\Omega$, since this would force $\overline{t}$ to lie in $Q \cup \{\Omega\}$. So the rightmost such letter is either from A_P or $A_{F'}$.

If the rightmost such letter is from $A_{F'}$, then by Lemma 6.4, $t = u'av$, where $a \in A_{F'}$, $|u'| < n$, $v \in A_F^*$. Set $u = u'a$. Then $\overline{u} = \overline{a}$ and so $|\overline{u}| < m < m + nk$ and there is nothing more to prove.

So suppose the rightmost such letter is from A_P. Then by Lemma 6.4, $t = t'bt''$, where $b \in A_P$, $|t'| < n$, $t'' \in A_F^*$. Then $w = \overline{t} = \overline{bt''}$. Now, if $t'' \in A_x^*$, then $\overline{bt''} \in P$ by the definition of the action. So t'' contains some letter from $A_F - A_x$. Let $t'' = scv$, where this distinguished letter c is the leftmost letter of t'' that is from $A_F - A_x$, so that $s \in A_x^*$. Then $\overline{bs} \in P$ and $\overline{bsc} \in F'$ since the alternative $\overline{bsc} \in Q \cup \{\Omega\}$ cannot happen since this set is closed under the action of v.

Thus far t has been factorized as $t'bscv$. The next step is to show that $|s| < n$. Suppose for *reductio ad absurdum* that $|s| \geqslant n$. Then s factorizes as $s's''s'''$, where $t'bs'(s'')^\alpha s'''cv \in L$ for all $\alpha \in \mathbb{N} \cup \{0\}$. Now, since $s = s's''s''' \in A_x^*$, the elements $\overline{t'bs'(s'')^\alpha s'''}$ are a sequence of elements p_{i_α} whose indices i_α form a linear progression. But the indices of the elements $p_i \in P$ such that $p_i \cdot \overline{c} \in F'$ are the terms of an exponential function. So there are infinitely many $\alpha \in \mathbb{N} \cup \{0\}$ such that $\overline{t'bs'(s'')^\alpha s'''c} = q_j \in Q \cup \{\Omega\}$.

Reasoning as in the third paragraph of the proof of Lemma 6.5, $c = x^\beta y$. Now, if $v \neq \varepsilon$, then $\overline{t'bs'(s'')^\alpha s'''cv} = \Omega$ for infinitely many $\alpha \in \mathbb{N} \cup \{0\}$, which contradicts uniqueness of representatives. If, on the other hand, $v = \varepsilon$, then $w = \overline{t} = \overline{t'bsc} = j\psi$ for some j since $\overline{t'bs} \in P$, $\overline{c} = x^\beta y$, and $\overline{t'bsc} \in F'$. So $w = j\psi \in R$, which contradicts the hypothesis of the lemma. Hence $|s| < n$.

Therefore $|\overline{s}| < kn$ since each letter of s represents a word in A_x^* of length at most k.

Now, $\overline{b} = p_{m'}$, where $m' < m$ by the definition of m. Hence $\overline{bs} = p_{m'}\overline{s} = p_h$ for some $h < m + kn$ by the definition of the action of x on the p_i. Suppose $c = x^\beta yz$ for some and $z \in \{x, y\}^*$. Then $\beta + |z| < k$. Since $\overline{t'bsc} \in F'$, it follows that $h + \beta = 2^j$ for some $j \in \mathbb{N} \cup \{0\}$. Hence $\overline{t'bsc} = wz$, where $w \in R$ with $|w| < 2^j$. Now,

$$|\overline{t'bsc}| = |wz| = |w| + |z| < |j\psi| + |z| < 2^j + |z| = h + \beta + |z| < h + k < m + k + kn.$$

Let $u = t'bsc$. Then $t = uv$ with $|\overline{u}| < m + k + kn$. $\boxed{6.6}$

Choose $w \in R$ with $|w| > m + n + k + kn$. Then $|w|_{y'}$ is even and so $|w(x')^{2k}y'|_{y'}$ is odd, so that $w(x')^{2k}y' \notin R$. Let t be the representative in L of $w(x')^{2k}y'$. Then by Lemma 6.6, t factorizes as uv, where the v is the longest suffix lying in A_F^* and $\overline{u} \in F'$ with $|\overline{u}| < m + k + kn$

In particular, $|w(x')^{2k}y'| > m + 3k + kn$. Since $|\overline{u}| < m + k + kn$, it follows that $|\overline{v}| > 2k$. Since each letter of v represents an element of F of length at most k, the word v has length at least 2. So let $v = v'ab$, where $a, b \in A_F$. Since $|\overline{a}|, |\overline{b}| < k$, $\overline{t} = \overline{uv} = w(x')^{2k}y'$ and $\overline{u} \in F'$, it follows from the action of F on $F' \subseteq T$ that $\overline{uv'} = w(x')^\alpha$ for some $\alpha \in \{1, \ldots, 2k\}$, $\overline{a} = x^\beta$ (so that $a \in A_x$), and $b \in A_F - A_x$.

Let $t' = uv'a$. Then $\overline{t'} = w(x')^{\alpha+\beta}$. By prefix-closure, $t' \in L$. Observe that t' ends with $a \in A_x$.

Now, the word $w(x')^{\alpha+\beta}$ lies in R since $|w|_{y'} = |w(x')^{\alpha+\beta}|_{y'}$ is even. So by Lemma 6.5, its unique representative t' must factorize as sc, where $\overline{c} = x^\beta y$, so that $c \in A_F - A_x$. This contradicts the fact that $\overline{t'}$ ends with a letter from A_x.

Thus $F[T]$ does not admit a Markov language. $\boxed{6.3}$

7 REWRITING SYSTEMS

Confluent noetherian rewriting systems form a natural source of examples of Markov semigroups. The following result is easily noticed, but will prove very useful:

PROPOSITION 7.1. *Let $(A, \mathcal{R})$ be a confluent noetherian rewriting system with the set of left-hand sides of rewriting rules in $\mathcal{R}$ being regular. Then the monoid presented by $\langle A \mid \mathcal{R} \rangle$ is Markov, and its language of normal forms is a Markov language. Furthermore, if $(A, \mathcal{R})$ is non-length-increasing, then the language of normal forms is a robust Markov language for the monoid.*

Proof of 7.1. The language $L = A^* - \{\ell : (\ell, r) \in \mathcal{R}\}$, which is the language of normal forms of $(A, \mathcal{R})$, is regular, prefix-closed, and maps bijectively onto the monoid presented by $\langle A \mid \mathcal{R} \rangle$. For the final observation, notice that if $(A, \mathcal{R})$ is non-length-increasing, then the language of normal forms consists of minimal-length representatives. $\boxed{7.1}$

It is worth emphasizing that Proposition 7.1 says that being Markov is a necessary condition for a semigroup to be presented by a confluent noetherian rewriting system, although it is probably not as useful as other necessary conditions such as finite derivation type [SOK94], which are independent of the choice of generating set.

However, the following example shows that a semigroup presented by a finite confluent noetherian non-length-increasing rewriting system can admit a robust Markov language that looks very different from its language of normal forms:

EXAMPLE 7.2. Let $A = \{a, b\}$ and $\mathcal{R} = \{(a^2, ba), (b^2, ab)\}$. Then $(A, \mathcal{R})$ is confluent and noetherian. Let L be its language of normal forms; this is a robust Markov language by Proposition 7.1. Then L is the language of words over A that do contain neither two consecutive letters a nor two consecutive letters b; thus L is the language of alternating products of letters a and b:

$$\begin{aligned} L &= (A^* - A^*aaA^*) - A^*bbA^* \\ &= (ab)^* \cup (ab)^*a \cup (ba)^* \cup (ba)^*b. \end{aligned}$$

Let M be the monoid presented by $\langle A \mid \mathcal{R} \rangle$. Let

$$K = ab^* \cup ba^*.$$

The aim is to show that K is also a Markov language for M. Notice first that K is prefix-closed and regular and so it remains to show that it consists of unique minimal-length representatives for M.

Notice that for any $\alpha \in \mathbb{N} \cup \{0\}$,

$$\overline{(ab)^\alpha} = \overline{ab(ab)^{\alpha-1}} = \overline{ab(b^2)^{\alpha-1}} = \overline{ab^{2\alpha-1}}$$

and

$$\overline{(ab)^\alpha a} = \overline{ab(ab)^{\alpha-1}a} = \overline{bb(ab)^{\alpha-1}a} = \overline{b(ba)^\alpha} = \overline{b(a^2)^\alpha} = \overline{ba^{2\alpha}}.$$

Parallel reasoning shows that $\overline{(ba)^\alpha} = \overline{ba^{2\alpha-1}}$ and $\overline{(ba)^\alpha b} = \overline{ab^{2\alpha}}$. Thus every word in L represents the same element as exactly one element of K and vice versa. Furthermore, the lengths of the corresponding words in L and K are the same. Hence, since L is a robust Markov language for M by Proposition 7.1, K is also a robust Markov language for M.

QUESTION 7.3. Is every Markov semigroup presented by a confluent noetherian rewriting system where the language of left-hand sides of rewriting rules is regular? (That is, where the language of *all* left-hand sides is regular: Example 11.9 below shows that the language of left-hand sides of rules with a particular right-hand side may be irregular.)

8 MARKOV, ROBUSTLY MARKOV, AND STRONGLY MARKOV SEMIGROUPS

The example in § 6 consists of a non-Markov monoid that admitted a regular language of unique representatives over any alphabet representing a generating set. The present section gives an example of a monoid that is Markov but not robustly Markov (Example 8.1) and an example of a monoid that is robustly Markov but not strongly Markov (Example 8.4). These three examples together show that the classes of Markov, robustly Markov, and strongly Markov semigroups are distinct.

EXAMPLE 8.1. Let

$$\begin{aligned}
P &= \{p_i : i \in \mathbb{N}\},\\
Q &= \{q_i : i \in \mathbb{N} \wedge \neg(\exists j \in \mathbb{N})(i = 2^j)\},\\
R &= \{r_i : i \in \mathbb{N}\},\\
S &= \{s_i : i \in \mathbb{N}\},\\
T &= P \cup Q \cup R \cup S \cup \{\Omega\}.
\end{aligned}$$

Let F be a free monoid with basis $X = \{x, y\}$. Define an action of X on T as follows

$$\begin{aligned}
p_i \cdot x &= p_{i+1}, & p_i \cdot y &= \begin{cases} q_i & \text{if } i \neq 2^j \text{ for any } j \in \mathbb{N} \cup \{0\},\\ s_j & \text{if } i = 2^j \text{ for some } j \in \mathbb{N} \cup \{0\},\end{cases}\\
q_i \cdot x &= \Omega, & q_i \cdot y &= \Omega,\\
r_i \cdot x &= r_{i+1}, & r_i \cdot y &= s_i,\\
s_i \cdot x &= \Omega, & s_i \cdot y &= \Omega,\\
\Omega \cdot x &= \Omega, & \Omega \cdot y &= \Omega.
\end{aligned}$$

Since F is free on X, this action extends to a unique action of F on T. Figure 2 shows the graph of the action of X on T. Propositions 8.2 and 8.3 below show that $F[T]$ is strongly Markov but not robustly Markov.

PROPOSITION 8.2. *The monoid* $F[T]$ *is Markov.*

Proof of 8.2. Let $A = \{a, b, c, d, e\}$ be an alphabet representing elements of $F[T]$ as follows:

$$\overline{a} = x, \quad \overline{b} = y, \quad \overline{c} = p_1, \quad \overline{d} = r_1, \quad \overline{e} = \Omega.$$

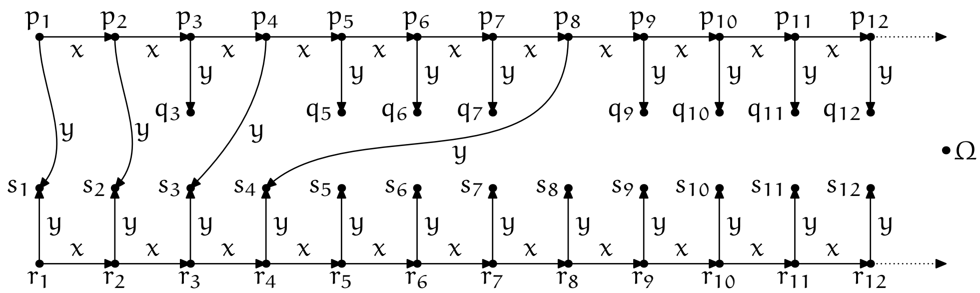

FIGURE 2. Part of the graph of the action of X on T. Edges which lead to Ω are not shown

Let $K = \{a, b\}^* \cup ca^* \cup ca^*b \cup da^* \cup \{e\}$. Then K is prefix-closed, regular, and maps bijectively onto $F[T]$. In particular, the subset $\{a, b\}^*$ maps bijectively onto F, the subset ca^* maps bijectively onto P, the subset ca^*b maps bijectively onto $Q \cup S$, and the subset da^* maps bijectively onto R. Thus K is a Markov language for $F[T]$. $\boxed{8.2}$

PROPOSITION 8.3. *The monoid* $F[T]$ *is not robustly Markov.*

Proof of 8.3. Suppose, with the aim of obtaining a contradiction, that $F[T]$ admits a robust Markov language L over some alphabet A.

Define the following subalphabets of A:

$$\begin{aligned}
A_P &= \{a \in A : \overline{a} \in P\},\\
A_Q &= \{a \in A : \overline{a} \in Q\},\\
A_R &= \{a \in A : \overline{a} \in R\},\\
A_S &= \{a \in A : \overline{a} \in S\},\\
A_F &= \{a \in A : \overline{a} \in F\},\\
A_x &= \{a \in A : \overline{a} \in x^+\},\\
A_\Omega &= \{a \in A : \overline{a} = \Omega\};
\end{aligned}$$

notice that A is the disjoint union of A_P, A_Q, A_R, A_S, A_F, and A_Ω. Let

$$\begin{aligned}
m_1 &= \max\{i : p_i \in \overline{A_P}\},\\
m_2 &= \max\{i : q_i \in \overline{A_Q}\},\\
m_3 &= \max\{i : r_i \in \overline{A_R}\},\\
m_4 &= \max\{i : s_i \in \overline{A_S}\},\\
m &= \max\{m_1, m_2, m_3, m_4\}.
\end{aligned}$$

Let $k = \max\{|\overline{a}| : a \in A_F\}$.

Reasoning as in the proof of Lemma 6.5, one sees that for i sufficiently large, s_i is represented by a word of the form vc, where $v \in A^*$, $c \in A_F - A_x$, $\overline{v} \in P$, and $\overline{c} = x^\beta y$ for some $\beta < k$.

Let $v = v'bv''$, where $v'' \in A_F$. Then $b \in A_P$ and so $\overline{v'b} = \overline{b} = p_{m'}$ for some $m' < m$. Now, $s_i = \overline{vc} = \overline{v'bv''c} = p_{m'}\overline{v''}x^\beta y$, and so by the definition of the action, $p_{2^i} = p_{m'}\overline{v''}x^\beta$. Thus $\overline{v''} = s^{2^i - m' - \beta}$. So each letter of v'' lies in A_x. Furthermore, since each such letter represents an element of length at most k, it follows that $|v''| > (2^i - m - \beta)/k$ and further that $|v| > (2^i - m - \beta)/k + 2$.

Since $v'' \in A_x^*$, the subalphabet A_x must be non-empty. Let $a \in A_x$, with $\overline{a} = x^\gamma$. Since $(T - Q) \cdot F$ does not contain any element of Q, the subalphabet A_Q is non-empty and contains some letter b with $\overline{b} = q_\alpha$.

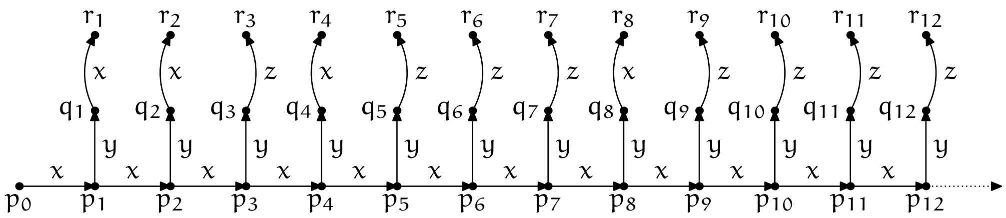

FIGURE 3. Part of the graph of the action of X on T. Edges corresponding to actions which fix elements of T are not shown

Then $\overline{ba^h c} = q_\alpha x^{\gamma h+\beta} y = q_{\alpha+\gamma h+\beta} y = s_{\alpha+\gamma h+\beta}$. By choosing h large enough, $s_{\alpha+\gamma h+\beta}$ is represented in L by a word v of length greater than $(2^{\alpha+\gamma h+\beta} - m - \beta)/k + 2$. Again choosing h large enough, so that

$$(2^{\alpha+\gamma h+\beta} - m - \beta)/k + 2 > h + 2.$$

one obtains $|v| > |ba^h c|$. Thus v is not a minimal-length representative of $s_{\alpha+\gamma h+\beta}$, which contradicts L being a robust Markov language for $F[T]$. $\boxed{8.3}$

EXAMPLE 8.4. Let

$$\begin{aligned} P &= \{p_i : i \in \mathbb{N} \cup \{0\}\}, \\ Q &= \{q_i : i \in \mathbb{N}\}, \\ R &= \{r_i : i \in \mathbb{N}\}, \\ T &= P \cup Q \cup R. \end{aligned}$$

Let F be a free monoid with basis $X = \{x, y, z\}$. Define an action of X on T as follows

$$\begin{aligned} &p_i \cdot x = p_{i+1}, \quad && q_i \cdot x = \begin{cases} q_i & \text{if } i \neq 2^j \text{ for any } j \in \mathbb{N} \cup \{0\}, \\ r_i & \text{if } i = 2^j \text{ for some } j \in \mathbb{N} \cup \{0\}, \end{cases} \quad && r_i \cdot x = r_i, \\ &p_i \cdot y = q_i, && q_i \cdot y = q_i, && r_i \cdot y = r_i, \\ &p_i \cdot z = p_i, && q_i \cdot z = \begin{cases} q_i & \text{if } i = 2^j \text{ for some } j \in \mathbb{N} \cup \{0\}, \\ r_i & \text{if } i \neq 2^j \text{ for any } j \in \mathbb{N} \cup \{0\}, \end{cases} && r_i \cdot z = r_i. \end{aligned}$$

(Notice that q_i is fixed by one of x or z and sent to r_i by the other, and that which letter fixes q_i and which sends it to r_i depends on whether i is a power of 2.) Since F is free on X, this action extends to a unique action of F on T. Figure 3 shows the graph of the action of X on T. Propositions 8.5 and 8.6 below show that $F[T]$ is robustly Markov but not strongly Markov.

PROPOSITION 8.5. *The monoid* $F[T]$ *is robustly Markov.*

Proof of 8.5. Let $A = \{a, b, c, d, e, f\}$ be an alphabet representing elements of $F[T]$ as follows:

$$\overline{a} = x, \quad \overline{b} = y, \quad \overline{c} = z, \quad \overline{d} = yx, \quad \overline{e} = yz, \quad \overline{f} = p_0.$$

Let $A' = A - \{f\}$. Then $(A', \{(ba, d), (bc, e)\})$ is a confluent noetherian rewriting system presenting the subsemigroup F of $F[T]$. Hence its language of normal forms $K_1 = A^* - A^*(ba \cup bc)A^*$ is a robust Markov language for the subsemigroup F of $F[T]$ by Proposition 7.1.

Let $K_2 = fa^* \cup fa^+d \cup fa^+e$. Then K_2 is $+$-prefix-closed and regular. The subset fa^* maps bijectively onto P. The subsets fa^+d and fa^+e map bijectively onto $Q \cup R$, since for each $i \in \mathbb{N}$, exactly one of the following cases holds:

- $\overline{fa^id} = p_0x^iyx = p_iyx = q_ix = r_i$ and $\overline{fa^ie} = p_0x^iyz = p_iyz = q_iz = q_i$ (this holds if $i = 2^j$ for some $j \in \mathbb{N}$);
- $\overline{fa^id} = p_0x^iyx = p_iyx = q_ix = q_i$ and $\overline{fa^ie} = p_0x^iyz = p_iyz = q_iz = r_i$ (this holds if $i \neq 2^j$ for any $j \in \mathbb{N}$).

Thus K_2 maps bijectively onto T.

It remains to show that every word in K_2 is a minimal length representative. Let $u \in A^*$ represent p_i. Then u must contain f, since all other letters in A represent elements of F. So let $u = u'fu''$, where $u'' \in (A - \{f\})^*$, so that this distinguished letter f is the rightmost such letter in u. Each symbol in $A - \{f\}$ represents an element of F that contains at most one letter x. So, by the definition of the action on the p_i, it follows that u'' must contain at least i letters. Hence $|u| \geqslant i+1$. Any word over A representing q_i or r_i must therefore have length at least $i+2$. By the observations in the preceding paragraph, the representative in K_2 of p_i has length $i+1$, and those of q_i and r_i both have length $i+2$.

Therefore the language $K_1 \cup K_2$ is prefix-closed, regular, and consists of minimal-length representatives for $F[T]$. So $K_1 \cup K_2$ is a robust Markov language for $F[T]$. $\boxed{8.5}$

PROPOSITION 8.6. *The monoid* $F[T]$ *is not strongly Markov.*

Proof of 8.6. Suppose, with the aim of obtaining a contradiction, that $F[T]$ is strongly Markov. Let $A = \{a, b, c, f\}$ represent elements of $F[T]$ as follows:

$$\overline{a} = x, \quad \overline{b} = y, \quad \overline{c} = z, \quad \overline{f} = p_0.$$

Since $F[T]$ is strongly Markov, it admits a robust Markov language L over the alphabet A. Let n be greater than the number of states in an automaton recognizing L. Choose k such that $2^k > n$.

It is easy to see that the unique shortest word over A representing r_{2^k} is $fa^{2^k}ba$. Therefore this word lies in L. By the pumping lemma, a^{2^k} factorizes as $v'v''v'''$, where $v', v'', v''' \in a^*$ and $fv'(v'')^\alpha v'''ba \in L$ for every $\alpha \in \mathbb{N} \cup \{0\}$. Choose α so that $m = |v'(v'')^\alpha v'''|$ is not a power of 2. Then $\overline{fv'(v'')^\alpha v'''b} = q_m$, and $\overline{fv'(v'')^\alpha v'''ba} = q_mx = q_m$. Hence $fv'(v'')^\alpha v'''b$ and $fv'(v'')^\alpha v'''ba$ represent the same element of $F[T]$. Since both these words lie in L by prefix-closure, this contradicts the uniqueness of representatives in L. $\boxed{8.6}$

9 COMMUTATIVE SEMIGROUPS

That finitely generated commutative semigroups are Markov could be deduced from Proposition 7.1, and the fact that finitely generated commutative monoids have presentations via finite confluent noetherian rewriting systems [Die86], and the closure of the class of Markov semigroups under adjoining and removing an identity (Proposition 13.1 below). However, a stronger result holds:

PROPOSITION 9.1. *Finitely generated commutative semigroups are strongly Markov.*

[The first part of the following proof parallels the proof that all commutative cancellative semigroups are automatic; see [Cai05, Theorem 5.4.2].]

Proof of 9.1. Let A be a finite alphabet representing an arbitrary generating set for some commutative semigroup S. Suppose $A = \{a_1, \ldots, a_n\}$. Consider elements of S using tuples: identify the tuple $(\alpha_1, \dots, \alpha_n)$ with the element $\overline{a_1^{\alpha_1} \cdots a_n^{\alpha_n}}$. Define the ShortLex ordering $\prec_{\text{SLex}}$ of these tuples by

$$
\begin{aligned}
(\alpha_1, \dots, \alpha_n) \prec_{\text{SLex}} (\beta_1, \dots, \beta_n) \iff & \sum_{i=1}^{n} \alpha_i < \sum_{i=1}^{n} \beta_i, \text{ or} \\
& \Bigg[\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i \\
& \qquad \text{and } (\alpha_1, \dots, \alpha_n) \sqsubset_{\text{Lex}} (\beta_1, \dots, \beta_n) \Bigg],
\end{aligned}
$$

where $\sqsubset_{\text{Lex}}$ is the lexicographical order of tuples: $(\alpha_1, \dots, \alpha_n) \sqsubset_{\text{Lex}} (\beta_1, \dots, \beta_n)$ if the leftmost non-zero coördinate of $(\beta_1 - \alpha_1, \dots, \beta_n - \alpha_n)$ is positive.

Rédei's Theorem [Réd63] asserts that S is finitely presented. An approach to this theorem found in [RGS99, Chapter 5] (which is a modification of the proof in [Gri93]) shows that the semigroup S is isomorphic to

$$[(\mathbb{N} \cup \{0\})^n - \{(0, \dots, 0)\}] / \{(u_1, v_1), \dots, (u_n, v_n)\}^\#,$$

where $u_i \prec_{\text{SLex}} v_i$, and such that the ShortLex-minimal representative of $\overline{w} \in (\mathbb{N} \cup \{0\})^n - \{(0, \dots, 0)\}$ can be found by repeatedly replacing w by $w - v_i + u_i$ whenever every coördinate of $w - v_i$ is non-negative. (Addition is performed componentwise on tuples.)

Since the ShortLex order is compatible with the operation (that is, for all $x \in S$, $u \prec_{\text{SLex}} v \implies u + x \prec_{\text{SLex}} v + x$), the set of ShortLex-minimal elements is simply

$$M = \{w \in (\mathbb{N} \cup \{0\})^n - \{(0, \dots, 0)\} : w - v_i \text{ is not in } (\mathbb{N} \cup \{0\})^n \text{ for any } i\}.$$

Let

$$K = \{a_1^{\alpha_1} \cdots a_n^{\alpha_n} : (\alpha_1, \dots, \alpha_n) \in M\}.$$

Since the number of v_i is finite, a finite state automaton can check whether a word $a_1^{\alpha_1} \cdots a_n^{\alpha_n}$ lies in K. Therefore K is regular.

Finally, notice that if a word $a_1^{\alpha_1} \cdots a_n^{\alpha_n}$ lies in K, then one obtains its longest proper prefix by decreasing by 1 the right-most non-zero exponent α_i. (Recall that some of the α_i, but not all, can be 0.) Thus if w is the tuple in M corresponding to a word in K, then the tuple w' corresponding to its longest proper prefix is obtained by decreasing the right-most non-zero coördinate by 1. Hence if $w - v_i \notin (\mathbb{N} \cup \{0\})^n$ then $w' - v_i \notin (\mathbb{N} \cup \{0\})^n$. Consequently K is closed under taking longest proper non-empty prefixes, and so, by iteration, is $+$-prefix-closed. By the definition of the ShortLex ordering, the language K consists of minimal-length representatives. So K is a robust Markov language for S. Since the generating set represented by A was arbitrary, S is strongly Markov. $\boxed{9.1}$

Finitely generated abelian groups are Markov, as a consequence of the more general result that finitely generated polycyclic groups are Markov [GdlH90a, Corollaire 11]. However, that finitely generated abelian groups are strongly Markov (an immediate corollary of Proposition 9.1) does not seem to have been explicitly noted anywhere, although it is implicit in [ECH+92, Chs 3–4].

It is known that nilpotent groups need not be strongly Markov, since they may have irrational (indeed, transcendental) growth functions with respect to some generating sets [Sto96, Theorem B]. Furthermore, there exist virtually abelian groups that do not admit any regular language of minimal length representatives over some generating set (that is, even without requiring uniqueness) [NS97]. Thus virtually abelian groups are not in general strongly Markov.

QUESTION 10.1. Are finitely generated semigroups that are nilpotent (in the sense of Malcev [Mal53]) Markov? In particular, are all finitely generated subsemigroups of nilpotent groups are Markov?

This section exhibits two examples to show that finitely generated subsemigroups of virtually abelian groups and of polycyclic groups need not be Markov. All finitely generated sub*groups* of such groups are Markov, since these classes of groups are closed under taking subgroups.

The example of a non-Markov subsemigroup of a virtually abelian group (Example 10.4) is particularly important: First, it shows that the class of groups all of whose finitely generated subsemigroups are Markov is not closed under forming finite extensions. Second, virtually abelian groups satisfy a non-trivial semigroup identity and thus have the following property: if S is a subsemigroup and H the subgroup it generates, then H is [isomorphic to] the universal group of S. In general groups, this is not true: H is in general a homomorphic image of the universal group of S. (The universal group of S is the group obtained by taking a presentation for S and considering it as a group presentation; see [CP67, Ch. 12] or the discussion in [Cai05, § 5.2.1] for background information.) Thus the example is a non-Markov semigroup with a Markov universal group.

The following technical result will be used in proving both examples non-Markov:

LEMMA 10.2. *Let S be a semigroup and $A = \{a, b, c, d, e, f, g, h, i, j\}$ an alphabet representing a finite generating set for S. Suppose that for $\alpha, \beta \in \mathbb{N} \cup \{0\}$ with $\alpha \neq \beta$ the following conditions hold:*

1. *The element represented by $ab^\alpha cd^\beta e$ is represented by no other word over A.*
2. *The element represented by $fg^\alpha hi^\beta j$ is represented by no other word over A.*
3. *The equality $\overline{ab^\alpha cd^\alpha e} = \overline{fg^\alpha hi^\alpha j}$ holds, and the only words representing this element are $ab^\alpha cd^\alpha e$ and $fg^\alpha hi^\alpha j$.*

Then S is not Markov.

Proof of 10.2. Suppose for *reductio ad absurdum* that S is Markov. Then it admits a regular language of unique representatives L over A by Proposition 5.3. So $K = L \cap (ab^*cd^*e \cup fg^*hi^*j)$ is regular. By assumption, when $\alpha \neq \beta$, the element represented by $ab^\alpha cd^\beta e$ is represented by no other word over A. Thus $K \supseteq \{ab^\alpha cd^\beta e : \alpha \neq \beta\}$ and similarly $K \supseteq \{fg^\alpha hi^\beta j : \alpha \neq \beta\}$. Thus $ab^*cd^*e - K \subseteq \{ab^\alpha cd^\alpha e : \alpha \in \mathbb{N} \cup \{0\}\}$ and $fg^*hi^*j - K \subseteq \{fg^\alpha hi^\alpha e : \alpha \in \mathbb{N} \cup \{0\}\}$.

Furthermore, the only representatives over A of the element $\overline{ab^\alpha cd^\alpha e}$ are $ab^\alpha cd^\alpha e$ and $fg^\alpha hi^\alpha j$. So at least one of the regular languages $ab^*cd^*e - K$ and $fg^*hi^*j - K$ is infinite. Assume the former; the latter case is similar. Since $ab^*cd^*e - K \subseteq \{ab^\alpha cd^\alpha e : \alpha \in \mathbb{N} \cup \{0\}\}$ is infinite, it contains arbitrarily long words $ab^\alpha cd^\alpha e$. So a string of symbols b can be pumped, which contradicts the fact that every word in this language is of the form $ab^\alpha cd^\alpha e$. Thus S is not Markov. $\boxed{10.2}$

EXAMPLE 10.3. The semigroup presented by

$$\langle a, b, c, d, e, f, g, h, i, j \mid ab^\alpha cd^\alpha e = fg^\alpha hi^\alpha j, \alpha \in \mathbb{N} \cup 0 \rangle ,$$

which is isomorphic to a subsemigroup of a polycyclic group [Cai09, § 3], is not Markov by Lemma 10.2 above.

EXAMPLE 10.4. Let $\mathcal{S}_{11}$ be the symmetric group on eleven elements. Let $\mathbb{Z}^{11}$ be the direct product of eleven copies of the integers under addition. View elements of $\mathbb{Z}^{11}$ as 11-tuples of integers. Let $G = \mathcal{S}_{11} \ltimes \mathbb{Z}^{11}$, where $\mathcal{S}_{11}$ acts (on the right) by permuting the components of elements of $\mathbb{Z}^{11}$. (The $\mathbb{Z}$-components are indexed from 1 at the left to 11 at the right.) The abelian subgroup $\mathbb{Z}^{11}$ of G has index $11!$, so G is a virtually abelian group.

Let $A = \{a, b, c, d, e, f, g, h, i, j\}$ be an alphabet representing elements of G in the following way:

$$\begin{aligned}
\overline{a} &= [(1\,3),\ (0,1,1,0,0,0,1,0,0,0,0)], & \overline{f} &= [(1\,5),\ (0,1,0,0,1,0,1,0,0,0,0)],\\
\overline{b} &= [\mathrm{id},\ (0,0,1,0,0,0,0,0,0,1,0)], & \overline{g} &= [\mathrm{id},\ (0,0,0,0,1,0,0,0,0,0,1)],\\
\overline{c} &= [(1\,3)(2\,4),\ (1,0,0,0,0,0,0,1,0,0,0)], & \overline{h} &= [(1\,5)(2\,6),\ (1,0,0,0,0,0,0,1,0,0,0)],\\
\overline{d} &= [\mathrm{id},\ (0,0,0,1,0,0,0,0,0,-1,0)], & \overline{i} &= [\mathrm{id},\ (0,0,0,0,0,1,0,0,0,0,-1)],\\
\overline{e} &= [(2\,4),\ (0,1,0,0,0,0,0,0,1,0,0)], & \overline{j} &= [(2\,6),\ (0,1,0,0,0,0,0,0,1,0,0)].
\end{aligned}$$

Let S be the subsemigroup of G generated by $\overline{A}$. The aim is show that S is not Markov. [We admit that the generators in $\overline{A}$ may look intimidating. However, they interact in a fairly nice way, and the method in their madness will become apparent.]

First of all, some preliminaries are necessary. For any $\alpha, \beta \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}
&\overline{ab^\alpha cd^\beta e}\\
&= [(1\,3),\ (0,1,1,0,0,0,1,0,0,0,0)][\mathrm{id},\ (0,0,\alpha,0,0,0,0,0,0,\alpha,0)]\overline{cd^\beta e}\\
&= [(1\,3),\ (0,1,\alpha+1,0,0,0,1,0,0,\alpha,0)][(1\,3)(2\,4),\ (1,0,0,0,0,0,0,1,0,0,0)]\overline{d^\beta e}\\
&= [(2\,4),\ (\alpha+2,0,0,1,0,0,1,1,0,\alpha,0)][\mathrm{id},\ (0,0,0,\beta,0,0,0,0,0,-\beta,0)]\overline{e}\\
&= [(2\,4),\ (\alpha+2,0,0,\beta+1,0,0,1,1,0,\alpha-\beta,0)][(2\,4),\ (0,1,0,0,0,0,0,0,1,0,0)]\\
&= [\mathrm{id},\ (\alpha+2,\beta+2,0,0,0,0,1,1,1,\alpha-\beta,0)],
\end{aligned}$$

and

$$\begin{aligned}
&\overline{fg^\alpha hi^\beta j}\\
&= [(1\,5),\ (0,1,0,0,1,0,1,0,0,0,0)][\mathrm{id},\ (0,0,0,0,\alpha,0,0,0,0,0,\alpha)]\overline{hi^\beta j}\\
&= [(1\,5),\ (0,1,0,0,\alpha+1,0,1,0,0,0,\alpha)][(1\,5)(2\,6),\ (1,0,0,0,0,0,0,1,0,0,0)]\overline{i^\beta j}\\
&= [(2\,6),\ (\alpha+2,0,0,0,0,1,1,1,0,0,\alpha)][\mathrm{id},\ (0,0,0,0,0,\beta,0,0,0,0,-\beta)]\overline{j}\\
&= [(2\,6),\ (\alpha+2,0,0,0,0,\beta+1,1,1,0,0,\alpha-\beta)][(2\,6),\ (0,1,0,0,0,0,0,0,1,0,0)]\\
&= [\mathrm{id},\ (\alpha+2,\beta+2,0,0,0,0,1,1,1,0,\alpha-\beta)].
\end{aligned}$$

In particular, $\overline{ab^\alpha cd^\alpha e} = \overline{fg^\alpha hi^\alpha j}$.

LEMMA 10.5. *Let* $\alpha, \beta \in \mathbb{N} \cup \{0\}$ *with* $\alpha \neq \beta$.

1. *The only word over* A *representing* $\overline{\mathtt{ab}^\alpha\mathtt{cd}^\beta\mathtt{e}}$ *is* $\mathtt{ab}^\alpha\mathtt{cd}^\beta\mathtt{e}$, *and the only word over* A *representing* $\overline{\mathtt{fg}^\alpha\mathtt{hi}^\beta\mathtt{j}}$ *is* $\mathtt{fg}^\alpha\mathtt{hi}^\beta\mathtt{j}$.
2. *The only words over* A *representing* $\overline{\mathtt{ab}^\alpha\mathtt{cd}^\alpha\mathtt{e}} = \overline{\mathtt{fg}^\alpha\mathtt{hi}^\alpha\mathtt{j}}$ *are* $\mathtt{ab}^\alpha\mathtt{cd}^\alpha\mathtt{e}$ *and* $\mathtt{fg}^\alpha\mathtt{hi}^\alpha\mathtt{j}$.

Proof of 10.5. Let $\alpha, \beta \in \mathbb{N} \cup \{0\}$. For the present, allow the possibility that α and β are equal.

Let $w \in A^+$ be some word representing

$$s = \overline{\mathtt{ab}^\alpha\mathtt{cd}^\beta\mathtt{e}} = [\mathrm{id},\ (\alpha+2, \beta+2, 0, 0, 0, 0, 1, 1, 1, \alpha-\beta, 0)]. \tag{10.1}$$

Let $A' = \{\mathtt{a}, \mathtt{c}, \mathtt{e}, \mathtt{f}, \mathtt{h}, \mathtt{j}\}$; observe that A' consists of exactly those elements of A representing elements with non-zero seventh, eighth, and ninth $\mathbb{Z}$-components, which are also exactly those that have non-identity $\mathcal{S}_{11}$-components. Let $A'' = A - A' = \{\mathtt{b}, \mathtt{d}, \mathtt{g}, \mathtt{i}\}$ observe that A'' consists of exactly those elements of A representing elements with non-zero tenth and eleventh $\mathbb{Z}$-components, which are also exactly those that have identity $\mathcal{S}_{11}$-components.

First, consider which letters from A' can appear in w. Examining the seventh, eighth, and ninth $\mathbb{Z}$-components (which are unaffected by the actions of any of the $\mathcal{S}_{11}$-components), shows that w contains one letter $\mathtt{a}$ or letter $\mathtt{e}$, one letter $\mathtt{c}$ or letter $\mathtt{h}$, and one letter $\mathtt{e}$ or letter $\mathtt{j}$, and no other letter from A'. For the product of the $\mathcal{S}_{11}$-components to be id, the letters from A' in w must then be $\mathtt{a}, \mathtt{c}, \mathtt{e}$ or $\mathtt{f}, \mathtt{h}, \mathtt{j}$ (in some order).

Consider these two cases separately:

1. Suppose first that the letters from A' in w are $\mathtt{a}, \mathtt{c}, \mathtt{e}$. Since the $\mathcal{S}_{11}$-components of $\overline{\mathtt{a}}, \overline{\mathtt{c}}, \overline{\mathtt{e}}$ do not affect the fifth and sixth $\mathbb{Z}$-components, and since these are both 0 in s, w cannot contain letters $\mathtt{g}$ or $\mathtt{h}$. So w is a rearrangement of $\mathtt{aceb}^\gamma\mathtt{d}^\delta$ for some $\gamma, \delta \in \mathbb{N} \cup \{0\}$. Now, in w the letter $\mathtt{a}$ must precede the letter $\mathtt{c}$, for otherwise the third $\mathbb{Z}$-component of $\overline{w}$ would be non-zero. Similarly, $\mathtt{c}$ must precede $\mathtt{e}$, for otherwise the fourth $\mathbb{Z}$-component of $\overline{w}$ would be non-zero. The letters $\mathtt{b}$ must all lie between $\mathtt{a}$ and $\mathtt{c}$, for otherwise the third $\mathbb{Z}$-component of $\overline{w}$ would be non-zero, and similarly the letters $\mathtt{d}$ must all lie between $\mathtt{c}$ and $\mathtt{e}$, for otherwise the fourth $\mathbb{Z}$-component of $\overline{w}$ would be non-zero. So $w = \mathtt{ab}^\gamma\mathtt{cd}^\delta\mathtt{e}$. Examining the first and second $\mathbb{Z}$-components forces $\gamma = \alpha$ and $\delta = \beta$. So if the letters from A' in w are $\mathtt{a}, \mathtt{c}, \mathtt{e}$, then $w = \mathtt{ab}^\alpha\mathtt{cd}^\beta\mathtt{e}$.
2. Suppose now that the letters from A' in w are $\mathtt{f}, \mathtt{h}, \mathtt{j}$. Since the $\mathcal{S}_{11}$-components of $\overline{\mathtt{f}}, \overline{\mathtt{h}}, \overline{\mathtt{j}}$ do not affect the third and fourth $\mathbb{Z}$-components, and since these are both 0 in s, w cannot contain letters $\mathtt{b}$ or $\mathtt{c}$. So w is a rearrangement of $\mathtt{fhjg}^\gamma\mathtt{i}^\delta$ for some $\gamma, \delta \in \mathbb{N} \cup \{0\}$. Now, in w, the letter $\mathtt{f}$ must precede the letter $\mathtt{h}$, for otherwise the fifth $\mathbb{Z}$-component of $\overline{w}$ would be non-zero. Similarly, $\mathtt{h}$ must precede $\mathtt{j}$, for otherwise the sixth $\mathbb{Z}$-component of $\overline{w}$ would be non-zero. The letters $\mathtt{g}$ must all lie between $\mathtt{f}$ and $\mathtt{h}$, for otherwise the fifth $\mathbb{Z}$-component of $\overline{w}$ would be non-zero, and similarly the letters $\mathtt{i}$ must all lie between $\mathtt{h}$ and $\mathtt{j}$, for otherwise the sixth $\mathbb{Z}$-component of $\overline{w}$ would be non-zero. So $w = \mathtt{fg}^\gamma\mathtt{hi}^\delta\mathtt{j}$. Examining the first and second $\mathbb{Z}$-components forces $\gamma = \alpha$ and $\delta = \beta$. So if the letters from A' in w are $\mathtt{f}, \mathtt{h}, \mathtt{j}$, then $w = \mathtt{fg}^\alpha\mathtt{hi}^\beta\mathtt{j}$. In this case,

$$\overline{w} = [\mathrm{id},\ (\alpha+2, \beta+2, 0, 0, 0, 0, 1, 1, 1, 0, \alpha-\beta)].$$

By (10.1), this forces $\alpha = \beta$, and $w = \mathtt{fg}^\alpha\mathtt{hi}^\alpha\mathtt{j}$.

So if $\alpha \neq \beta$, only the first case holds and $w = ab^{\alpha}cd^{\beta}e$. If, on the other hand, $\alpha = \beta$, then both cases can hold and w is either $ab^{\alpha}cd^{\alpha}e$ or $fg^{\alpha}hi^{\alpha}j$.

Parallel reasoning shows that if $\alpha \neq \beta$, the element represented by $fg^{\alpha}hi^{\beta}j$ is represented by no other word over A. $\boxed{10.5}$

By Lemma 10.5, the semigroup S satisfies the hypotheses of Lemma 10.2 and so is not Markov.

11 MISCELLANEOUS EXAMPLES OF MARKOV AND NON-MARKOV SEMIGROUPS

This section gathers miscellaneous examples to illustrate particular aspects of the class of Markov semigroups.

First, here is an example of a non-Markov semigroup:

EXAMPLE 11.1. Let $A = \{a, b, c, d\}$ and let

$$\mathcal{S} = \{(ba, ab), (bc, aca), (acc, d)\} \cup \{(dx, d), (xd, d) : x \in A\}.$$

The monoid presented by $\langle A \mid \mathcal{S} \rangle$ does not admit a regular language of unique representatives by [OKK98, Example 4.6], and thus is not Markov.

Since free groups of finite rank are Markov (either by Proposition 7.1 or as a corollary of [GdlH90a, Proposition 9]) and indeed strongly Markov (since they are hyperbolic; see [GdlH90a, Théorème 13]), the following example is worth noting:

EXAMPLE 11.2. The free inverse monoid of rank 1 is not Markov, because it admits no regular language of unique normal forms over the generating set [CS01, Proof of Theorem 2.7].

The Baumslag–Solitar groups play their customary rôle of being pleasant and easy to understand but slightly eccentric. This is a consequence of the following theorem of Groves:

THEOREM 11.3 ([Gro96, Corollary in § 1]). *There is no regular language of minimal-length representatives for the Baumslag–Solitar groups*

$$\left\langle a, t \mid (t^{-1}at, a^{p}) \right\rangle,$$

where $p > 1$ *with respect to the alphabet* $\{a, a^{-1}, t, t^{-1}\}$.

[The original statement of this result by Groves is phrases in terms of minimal-length (unique) normal forms. However, the property of uniqueness is not used anywhere in the proof. Groves states the result in these terms because he places the result in the context of calculating growth series.]

EXAMPLE 11.4. The Baumslag-Solitar group $\left\langle a, t \mid (t^{-1}at, a^{2}) \right\rangle$ is presented by the a confluent noetherian rewriting system [ECH+92, p. 156], and is therefore Markov by Proposition 7.1. However, since it admits no regular language of minimal-length representatives by Theorem 11.3, it is not strongly Markov. (However, it does admit a one-counter language of minimal-length normal forms [Eld05, §§ 4–5].)

This example leads on to the following question:

QUESTION 11.5. Is every one-relation semigroup Markov?

If every one-relation semigroup can be presented by a confluent noetherian rewriting system (an open question, since it would imply a solution to the world problem), this question would have a positive answer by Proposition 7.1.

A robustly Markov monoid may not be residually finite:

EXAMPLE 11.6. Let $A = \{a, b\}$ and let $\mathcal{R} = \{(ab^2, b)\}$. Then $(A, \mathcal{R})$ is a confluent noetherian rewriting system and so the monoid M presented by $\langle A \mid \mathcal{R}\rangle$ is Markov by Proposition 7.1. This monoid M is known to be non-residually finite [Lal74].

A strongly Markov monoid may not be finitely presented:

EXAMPLE 11.7. Let $A = \{a, b, c, d, e, f\}$ and $\mathcal{R} = \{(ab^n c, de^n f) : n \in \mathbb{N}\}$. Let M be the monoid presented by $\langle A \mid \mathcal{R}\rangle$. Then M is not finitely presented since no relation in $\mathcal{R}$ can be deduced from the others. But M is strongly Markov: since every generators in $\overline{A}$ is indecomposable, any alphabet representing a generating set for M must contain a subalphabet representing $\overline{A}$; thus $A^* - A^*ab^*cA^*$ is a robust Markov language for M over any alphabet representing a generating set.

This example suggests the following question:

QUESTION 11.8. Does there exist a strongly Markov group that is not finitely presented? If not, does there exists a non-finitely presented Markov or robustly Markov group? [The authors conjecture that the answers to these questions are both yes, for intuition suggests that a Markov or robust Markov language does not impose enough structure on a group to guarantee finite presentability.]

The following easy example shows that it is possible for a robustly Markov monoid to have unsolvable word problem:

EXAMPLE 11.9. Let I be a non-recursive subset of $\mathbb{N}$. Let $A = \{a, b, c, x, y\}$ and

$$\mathcal{R} = \{(ab^\alpha c, x) : \alpha \in I\} \cup \{(ab^\alpha c, y) : \alpha \notin I\}.$$

The rewriting system $(A, \mathcal{R})$ is confluent because left-hand sides of rules in $\mathcal{R}$ overlap only when they are identical. It is noetherian because it is length-reducing. The language of left-hand sides of rules in $\mathcal{R}$ is

$$\{ab^\alpha c : \alpha \in I\} \cup \{ab^\alpha c : \alpha \notin I\} = ab^*c$$

and so is regular. By Proposition 7.1, L is a robust Markov language for the monoid presented by $\langle A \mid \mathcal{R}\rangle$.

However, this monoid does not have solvable word problem, since $ab^\alpha c$ and x represent the same element of the semigroup if and only if $\alpha \in I$. But membership of I is undecidable since I is non-recursive.

However, a *finitely presented* Markov semigroup will have soluble word problem, as does any finitely presented semigroup that admits a recursively enumerable language of normal forms [CS01, Theorem 1.5].

Ghys et al. proved that hyperbolic groups are Markov using a direct approach [GdlH90a, §3]. It also follows using the machinery of automatic groups: over any generating set, the language of geodesics is regular and forms part of a prefix-closed automatic structure [ECH+92, Theorem 3.4.5], and the construction of an automatic structure with uniqueness [ECH+92, Theorem 2.5.1] preserves prefix-closure when applied in this particular case (although *not* in the general case).

Hyperbolicity can be generalized from groups to semigroups in either a geometric or linguistic sense. The latter generalization, which is termed *word-hyperbolicity*, is due to Duncan & Gilman [DG04]. It informally says that a semigroup is word-hyperbolic if it admits a regular language of representatives such that the multiplication table in terms of these representatives is a context-free language.

DEFINITION 12.1. A *word-hyperbolic structure* for a semigroup S is a pair (A, L), where A is a finite alphabet representing a generating set for S and L is a regular language over A such that $\overline{L} = S$ and the language

$$M(L) = \{u\#_1 v\#_2 w^{\mathrm{rev}} : u, v, w \in L \wedge \overline{uv} = \overline{w}\}$$

(where $\#_1$ and $\#_2$ are new symbols not in A) is context-free.

A semigroup is *word-hyperbolic* if it admits a word-hyperbolic structure.

A group is word-hyperbolic in the sense of Definition 12.1 if and only if it is hyperbolic in the sense of Gromov [DG04, Corollary 4.3]. For further background information on word-hyperbolic semigroups, see [DG04, HKOT02].

The following example is taken from [CM11, Example 4.2]:

EXAMPLE 12.2. Let $A = \{a, b, c, d\}$ and let $\mathcal{R} = \{(ab^\alpha c^\alpha d, \varepsilon) : \alpha \in \mathbb{N}\}$. Let M be the monoid presented by $\langle A \mid \mathcal{R}\rangle$. Since the rewriting system $(A, \mathcal{R})$ is context-free, M is word-hyperbolic by [CM11, Theorem 3.1]. The reasoning in [CM11, Example 4.2] shows that it does not admit a regular language of unique normal forms over any generating set, and so in particular cannot be Markov by Proposition 5.3.

Thus word-hyperbolic monoids are not in general Markov. Moreover if the regularity condition on the left-hand sides of rewriting rules in Proposition 7.1 is weakened to being context-free (or even just to being one-counter), then the semigroups or monoids thus presented are not Markov in general.

Example 12.2 is not finitely presented, and it does not admit a word-hyperbolic structure with uniqueness [CM11, Example 4.2]. This provokes the following questions:

QUESTION 12.3. Does there exist a non-Markov finitely presented word-hyperbolic monoid?

QUESTION 12.4. Does there exist a non-Markov monoid that admits a word-hyperbolic structure with uniqueness?

Since satisfying a linear isoperimetric inequality is one of several equivalent characterizations of hyperbolic groups (see, for example, [ABC+91, Ch. 1]), the following question is of interest:

QUESTION 12.5. Does there exist a non-Markov semigroup with linear isoperimetric inequality?

Markov groups are not in general automatic, since all polycyclic groups are Markov [GdlH90a, Corollaire 11], but a nilpotent group that is not virtually abelian cannot be automatic [ECH+92, Theorem 8.2.8].

QUESTION 12.6. Are automatic semigroups Markov? (Note that, unlike the situation for groups, an automatic semigroup need not be word-hyperbolic.) This question relates to the long-standing open question of whether an automatic semigroup or group admits a prefix-closed automatic structure with uniqueness [ECH+92, Open Question 2.5.10]. Admitting such an automatic structure entails being Markov.

13 ADJOINING AN IDENTITY OR ZERO

This section and those that follow examines the interaction of the classes of Markov, robustly Markov, and strongly Markov semigroups with various semigroup constructions. The main questions are whether these classes of semigroups are closed under a particular construction, and whether the semigroup resulting from such a construction being Markov, robustly Markov, or strongly Markov implies that the original semigroup is (or the original semigroups are) Markov, robustly Markov, or strongly Markov.

Arguably the simplest semigroup construction are the adjoining of an identity or zero, and it is reassuring that both questions have positive answers for these constructions:

PROPOSITION 13.1. *Let S be a semigroup. Then:*

1. *S is Markov if and only if S^1 is Markov.*
2. *S is robustly Markov if and only if S^1 is robustly Markov.*
3. *S is strongly Markov if and only if S^1 is strongly Markov.*

Proof of 13.1. Let A be a finite alphabet representing a semigroup generating set for S. Let 1 be a new symbol not in A representing the adjoined identity of S^1.

Let L be a semigroup Markov language for S with respect to A. Then L is regular, $+$-prefix-closed, and maps bijectively onto S. Let $K = L \cup \{1\}$. Then K is regular, $+$-prefix-closed, and maps bijectively onto S^1. Thus K is a semigroup Markov language for S^1.

Furthermore, if L is a robust semigroup Markov language, then so is K, since 1 is the unique shortest word representing the adjoined identity, and the natural lengths of elements in S over A and over $A \cup \{1\}$ are equal.

Now let L be a semigroup Markov language for S^1 over an alphabet B representing some generating set for S^1. Now, B must be of the form $A \cup \{1\}$, where 1 represents the adjoined identity and A represents a generating set for S, since no product of elements of S equals the adjoined identity.

Suppose some $w \in L$ contains the symbol 1. Then $w = w'1w''$ and so w' and $w'1$ represent the same element of S^1, unless w' is the empty word, which is not a member of the semigroup Markov language L. So such a word w can only contain a single instance of the symbol 1, and it must be the first symbol of w. (If L is a robust semigroup Markov language, the only such word is $w = 1$, since otherwise $w'w''$ would be a shorter word representing $\overline{w}$, as in the proof of Proposition 4.3.)

Let

$$K = \Big(\big(L - \{1\}\big) - 1A^*\Big) \cup \{u \in A^+ : 1u \in L\}.$$

Arguing as in the proof of Proposition 4.1, it follows that K is +-prefix-closed, is regular, and contains a unique representative for each element of S. Thus K is a semigroup Markov language for S over the alphabet A.

Furthermore, if L is a robust semigroup Markov language, the only word in L containing the symbol 1 is the word 1 itself, so in this case

$$K = L - \{1\}.$$

From these arguments, it follows that S is Markov if and only if S^1 is Markov and that S is robustly Markov if and only if S^1 is robustly Markov. From the arbitrary choice of generating sets, and the fact that any alphabet representing a generating set for S^1 must be of the form $A \cup \{1\}$, where 1 represents the adjoined identity and A represents a generating set for S, it follows that S is strongly Markov if and only if S^1 is strongly Markov. $\boxed{13.1}$

PROPOSITION 13.2. *Let* S *be a semigroup. Then:*

1. *S is Markov if and only if S^0 is Markov.*
2. *S is robustly Markov if and only if S^0 is robustly Markov.*
3. *S is strongly Markov if and only if S^0 is strongly Markov.*

Proof of 13.2. By reasoning parallel to the proof Proposition 13.1, substituting 0 for 1 and S^0 for S^1 as appropriate, it follows that if L is a [robust] Markov language for S, then $L \cup \{0\}$ is a [robust] Markov language for L.

Now let L be a Markov language for S^0 over an alphabet B representing some generating set for S^0. Now, B must be of the form $A \cup \{0\}$, where 0 represents the adjoined zero and A represents a generating set for S, since no product of elements of S equals the adjoined zero.

Suppose some $w \in L$ contains the symbol 0, with $w = w'0w''$. Then $w'0$ and w both represent the zero of the semigroup, which contradicts the uniqueness of representatives in L unless w'' is the empty word. So such a word w can contain only a single symbol 0, and this must be the last letter of the word. (If L is a robust Markov language, the only such word is $w = 0$ since this is the unique shortest word over $A \cup \{0\}$ representing the adjoined zero.) Notice that there can only be one such word, since any other word containing the symbol 0 would also represent the adjoined zero. So L contains a unique word $w = w'0$ containing the symbol 0, and this word is not the prefix of any other word in L.

Let $K = L - \{w'0\}$. Then K is +-prefix-closed (since $w'0$ is not a prefix of any other word in L), is regular, and contains a unique representative for each element of S. Finally, $K \subseteq A^+$ by the observation at the end of the last paragraph. Thus K is a Markov language for S over the alphabet A.

From these arguments, it follows that S is Markov if and only if S^0 is Markov and that S is robustly Markov if and only if S^0 is robustly Markov. From the arbitrary choice of generating sets, and the fact that any alphabet representing a generating set for S^0 must be of the form $A \cup \{0\}$, it follows that S is strongly Markov if and only if S^1 is strongly Markov. $\boxed{13.2}$

The class of Markov groups is closed under direct products, as a special case of the fact that an extension of one Markov group by another is also Markov [GdlH90a, Proposition 10]. For monoids, the result is also positive:

THEOREM 14.1. *1. If M and N are Markov monoids, then $M \times N$ is a Markov monoid.*

2. If M and N are robust Markov monoids, then $M \times N$ is a robust Markov monoid.

Proof of 14.1. 1. Let A and B be finite alphabets representing monoid generating sets for M and N with representation maps $\phi_A : A \to M$ and $\phi_B : B \to N$, respectively, and let K and L be monoid Markov languages over A and B for M and N, respectively. Then $H = KL$ is prefix-closed, regular, and maps bijectively onto $M \times N$ under the representation map $\phi : A \cup B \to M \times N$ defined by $a \mapsto (a\phi_A, 1_N)$ and $b \mapsto (1_M, b\phi_B)$.

2. Proceed as in the previous part, but with K and L being robust Markov languages. Then KL is a robust Markov language for $M \times N$ since (with respect to the representation map ϕ) $\lambda_{A \cup B}(\overline{uv}) = \lambda_A(\overline{u}) + \lambda_B(\overline{v})$ for all $u \in K$ and $v \in L$. $\boxed{14.1}$

However, for semigroups, the situation is obscure. First of all, a direct product of finitely generated semigroups is not necessarily finitely generated. For example, the direct product of two copies of the natural numbers $\mathbb{N}$ (excluding 0) is not finitely generated. (Notice that $\mathbb{N}$ is strongly Markov.) Even when the direct product is finitely generated, the relationship of a finite generating set to the finite generating sets of the direct factors is complex; see the discussion in [RRW98, § 2]. It is possible to prove that a direct product of a Markov semigroup and a finite semigroup is Markov if it is finitely generated (Theorem 14.2 below). The general idea of the proof is similar to that used by Campbell et al. to prove the analogous result for automatic semigroups [CRRT00, Theorem 1.1(ii)], but more sophisticated reasoning is required here to ensure that prefix-closure and uniqueness are preserved. However, the issue of prefix-closure seems to make it impossible to adapt and strengthen the idea used by Campbell et al. for direct products of infinite semigroups. An entirely new approach may be required in this case.

THEOREM 14.2. *Let S be a Markov semigroup and let T be finite. Then $S \times T$ is a Markov semigroup if and only if it is finitely generated.*

Proof of 14.2. One direction of the result is trivial: if $S \times T$ is a Markov semigroup, then by definition it is finitely generated.

Suppose that $S \times T$ is finitely generated. Then by [RRW98, Lemma 2.3], the finite semigroup T is such that $T^2 = T$.

Since S is a Markov semigroup, it admits a Markov language L over some finite alphabet A representing a generating set for S.

Let B be a finite alphabet in bijection with T. Since $T^2 = T$, it follows that, $T^n = T$ for all $n \in \mathbb{N}$ and so for any $t \in T$ and $n \in \mathbb{N}$, there is word of length n over B representing t. Let

$$R = \{(u, v) : u, v \in B^+, |u| = |v|, \overline{u} = \overline{v}\};$$

notice that R is a synchronous rational relation. Let $\sqsubset_{\mathrm{Lex}}$ be the lexicographic ordering on B^+ based on some total ordering of B. Then

$$R' = \{u : (\forall v \in A^*)((u, v) \in R \implies u \sqsubset_{\mathrm{Lex}} v)\}.$$

The language R' contains exactly one (lexicographically minimal) representative of each length for each element of T. Furthermore, the language R' is $+$-prefix-closed, for if u is not $\sqsubset_{\mathrm{Lex}}$-minimal amongst words of length $|u|$ representing $\overline{u}$, then for any $a \in A$, the word ua is not $\sqsubset_{\mathrm{Lex}}$-minimal amongst words of length $|ua|$ representing $\overline{ua}$.

Define

$$\delta : \bigcup_{n=0}^{\infty} (A^n \times B^n) \to (A \times B)^*$$

(so that $(u, v)\delta$ is defined when $u \in A^*$ and $v \in B^*$ have equal length) by

$$(a_1 a_2 \cdots a_n, b_1 b_2 \cdots b_n) \mapsto (a_1, b_1)(a_2, b_2) \cdots (a_n, b_n),$$

where $a_i \in A$, $b_i \in B$.

Let $K = \{(w, u) : w \in L, u \in R', |w| = |u|\}$. Then $K\delta$ is a regular language over $A \times B$. Since both L and R' are $+$-prefix-closed, so is $K\delta$.

Now let $(s, t) \in S \times T$. Then since L maps onto S, there is a word $w \in L$ with $\overline{w} = s$. There is a word u' of length $|w|$ over B such that $\overline{u'} = t$. Let u be the $\sqsubset_{\mathrm{Lex}}$-minimal such word. Then $|u| = |w|$ and so $(w, u) \in K$ and so $(w, u)\delta \in K\delta$ represents (s, t). So $K\delta$ maps onto $S \times T$.

Now suppose $(w, u)\delta, (w', u')\delta \in K\delta$ represent the same element of $S \times T$. Then $\overline{w} = \overline{w'}$ and $\overline{u} = \overline{u'}$. Since L is a Markov language for S, it maps bijectively onto S and so $w = w'$. In particular, $|w| = |w'|$, and so $|u| = |u'|$ by the definition of K. Since $\overline{u} = \overline{u'}$ and $|u| = |u'|$, and R' contains exactly one representative of $\overline{u}$ of length $|u|$, it follows that $u = u'$. Hence $(w, u)\delta = (w', u')$. Therefore $K\delta$ maps bijectively onto $S \times T$.

Thus $K\delta$ is a Markov language for $S \times T$ and so $S \times T$ is a Markov semigroup. $\boxed{14.2}$

THEOREM 14.3. *Let S be a robustly Markov semigroup and let T be finite. Then $S \times T$ is a robustly Markov semigroup if and only if it is finitely generated.*

Proof of 14.3. Proceed as in the proof of Theorem 14.3, with L being a robust Markov language for S. Since $\lambda_B(t) = 1$ for all $t \in T$, it follows that $\lambda_{(A\times B)\delta}(s, t) = \lambda_A(s)$. So, by its construction, $K\delta$ is a robust Markov language for $S \times T$. $\boxed{14.3}$

The corresponding result for being strongly Markov is still open:

QUESTION 14.4. Let S be strong Markov and T finite. If $S \times T$ is finitely generated, is it strongly Markov?

We conjecture that the answer to this question is 'yes', but probably requires more complex reasoning than in the proofs of Theorems 14.2 and 14.3, because the generating set for $S \times T$ may not project onto T, which complicates the relationship between minimal lengths of representatives of elements of $S \times T$ and T.

As remarked above, the following question is open:

QUESTION 14.5. Let S and T be Markov. If $S \times T$ is finitely generated, is it Markov?

The following question also arises:

QUESTION 14.6. Is it true that whenever $S \times T$ is Markov, then both factors S and T are Markov?

The answer to this question may shed light on the long-standing open question of whether direct factors of automatic groups, monoids, or semigroups must themselves be automatic (see [ECH+92, Open Question 4.1.2] and [CRRT01, Question 6.6]).

15 FREE PRODUCTS

THEOREM 15.1. *The class of Markov monoids is closed under forming (monoid) free products.*

Proof of 15.1. The proof for groups generalizes directly [GdlH90a, Proposition 9]. $\boxed{15.1}$

THEOREM 15.2. *The class of Markov semigroups, the class of robustly Markov semigroups, and the class of strongly Markov semigroups are all closed under forming (semigroup) free products.*

Proof of 15.2. Let S and T be Markov semigroups. Let $K \subseteq A^+$ and $L \subseteq B^+$ be semigroup Markov languages for S and T, respectively. Let

$$M = (KL)^+ \cup (KL)^*K \cup (LK)^+ \cup (LK)^*K.$$

Since the languages K and L are prefix-closed and regular, so is the language M. Any element of the free product $S * T$ has a unique representation as an alternating product of elements of S and T. That is $S * T$ is the disjoint union of

$$\begin{aligned}
X_1 &= \{s_1t_1 \cdots s_nt_n : s_i \in S, t_i \in T, n \in \mathbb{N}\},\\
X_2 &= \{s_1t_1 \cdots s_nt_ns_{n+1} : s_i \in S, t_i \in T, n \in \mathbb{N} \cup \{0\}\},\\
X_3 &= \{t_1s_1 \cdots t_ns_n : s_i \in S, t_i \in T, n \in \mathbb{N}\},\\
X_4 &= \{t_1s_1 \cdots t_ns_nt_{n+1} : s_i \in S, t_i \in T, n \in \mathbb{N} \cup \{0\}\}.
\end{aligned}$$

Since the languages K and L do not contain the empty word, every element of X_1 (respectively X_2, X_3, X_4) has a unique representative in $(KL)^+$ (respectively $(KL)^*K$, $(LK)^+$, (LK^*K). So every element of $S * T$ has a unique representative in M. So M is a Markov language for $S * T$.

Following the same reasoning with S and T being robustly Markov semigroups and K and L being robust Markov languages shows that M is a robust Markov language for $S * T$, since,

$$\lambda_{A\cup B}(s_1t_1 \cdots s_nt_n) = \sum_{i=1}^{n} \big(\lambda_A(s_i) + \lambda_B(t_i)\big),$$

and similarly for alternating products in $X_2 \cup X_3 \cup X_4$.

Finally, suppose that S and T are strongly Markov semigroups. Let C be a finite alphabet representing a generating set for $S * T$. Since $S * T$ is a *semigroup* free product, C contains subalphabets A and B representing generating sets for S and T respectively. Since S and T are strongly Markov semigroups, there exist robust Markov languages $K \subseteq A^+$ and $L \subseteq B^+$ for S and T respectively. Thus, by the preceding paragraph, $M \subseteq (A \cup B)^+ \subseteq C^+$ is a robust Markov language for $S * T$. Since C was arbitrary, $S * T$ is strongly Markov. $\boxed{15.2}$

Many properties of groups are known to be preserved under passing from groups to finite-index extensions and subgroups; for example, finite generation and presentability. For semigroups, the most well-known notion of index is the Rees index: if T is subsemigroup of a semigroup S, then T has finite index in S if $S - T$ is finite. Many properties of semigroups are known to be preserved on passing to finite Rees index extensions and subsemigroups; for example, finite generation [Ruš98, Theorem 1.1], finite presentability [Ruš98, Theorem 1.3], and automaticity [HTR02, Theorem 1.1]. The following result fits this pattern:

THEOREM 16.1. *The class of Markov semigroups is closed under forming finite Rees index extensions and subsemigroups.*

Proof of 16.1. Let S be a semigroup and let T be a finite Rees index subsemigroup of S.

Suppose that T is Markov and that L is a Markov language for T over some finite alphabet A representing a generating set for T. Let B be an alphabet in bijection with $S - T$; then B is finite since T has finite Rees index in S. Without loss of generality, assume that B and A are disjoint. Then $L \cup B$ is a Markov language for S.

Now suppose that S admits a Markov language L over an alphabet A.

Define

$$L(A,T) = \{w \in A^+ : \overline{w} \in T\}.$$

Let C be an alphabet of unique representatives for $S - T$. For any word $w \in A^* - L(A,T)$, let $\underline{w}$ be the unique element of $C \cup \{\varepsilon\}$ representing $\overline{w}$, or ε if $w = \varepsilon$.

Define the alphabet

$$D = \{d_{\rho,a,\sigma} : \rho, \sigma \in C \cup \{\varepsilon\}, a \in A, a\sigma \in L(A,T) \wedge \rho a\sigma \in L(A,T)\},$$

and let it represent elements of T as follows:

$$\overline{d_{\rho,a,\sigma}} = \overline{\rho a\sigma}.$$

Notice that if A is finite, D too must be finite.

Let $\mathcal{R} \subseteq A^+ \times D^+$ be the relation consisting of pairs

$$\big(w_{n+1}a_n w_n a_{n-1} w_{n-1} \cdots a_2 w_2 a_1 w_1, d_{\underline{w_{n+1}},a,\underline{w_n}} d_{\varepsilon,a,\underline{w_{n-1}}} \cdots d_{\varepsilon,a,\underline{w_2}} d_{\varepsilon,a,\underline{w_1}}\big)$$

where the left-hand side lies in $L(A,T)$ and the factorization of the left-hand side is obtained in the following way: start by letting the left-hand side be w_0'; a partial factorization

$$w_{i+1}' a_i w_i \cdots a_1 w_1$$

is complete if $w_{i+1}' \notin L(A,T)$; if on the other hand $w_{i+1}' \in L(A,T)$ set $a_{i+1}w_{i+1}$ to be the shortest suffix of w_{i+1}' lying in $L(A,T)$ and let w_{i+2}' be the remainder of w_{i+1}'.

Notice that if $(w,u) \in \mathcal{R}$ then $\overline{w} = \overline{u}$ by the definition of how the alphabet D represents element of T, and that each word w determines a unique word u such that $(w,u) \in \mathcal{R}$.

LEMMA 16.2. *The relation $\mathcal{R}$ is rational.*

Proof of 16.2. It is easier to explain a how a two-tape finite state automaton $\mathcal{A}$ can recognize $\mathcal{R}$ when reading from right-to-left; since the class of rational relations is closed under reversal, it will then follow that $\mathcal{R}$ is rational.

By the dual of [RT98, Theorem 4.3], S admits a left congruence Λ of finite index (that is, having finitely many equivalence classes) contained within $(T \times T) \cup \Delta_{S-T}$, where Δ_{S-T} is the diagonal relation on $S - T$ (that is, $\{(s,s) : s \in S - T\}$).

Imagine the automaton $\mathcal{A}$ reading letters from A from its left-hand input tape and outputting symbols from D on its right-hand tape. Suppose the content of its left-hand tape is w. As it reads symbols from w (moving from right to left along the tape), it keeps track of the Λ-class of the element represented by the suffix of w read so far. (This is possible because Λ is a left congruence with only finitely many equivalence classes.) In particular, $\mathcal{A}$ knows whether the element represented by the suffix read so far lies in T (or equivalently, whether the suffix read so far lies in $L(A,T)$), or, if the element so represented lies in $S - T$, which letter of $C \cup \{\varepsilon\}$ represents it. When $\mathcal{A}$ reads a symbol a such that the suffix read so far — say aw' — lies in $L(A,T)$, it non-deterministically chooses one of two actions:

1. It outputs $d_{\varepsilon,a,\underline{w'}}$, resets its store of the suffix read so far to ε, and continues to read from its left-hand tape.
2. It outputs $d_{c,a,\underline{w'}}$, where c is a non-deterministically chosen element of $C \cup \{\varepsilon\}$, then reads the remainder v of its left-hand tape and accepts if and only if $\underline{v} = c$. (Notice that this is the only way that $\mathcal{A}$ can accept.)

By induction on the subscripts of the letters a_i, the automaton $\mathcal{A}$ can accept only by outputting letters $d_{\varepsilon,a,\underline{w_i}}$ immediately after reading the suffix $a_i w_i \cdots a_1 w_1$ and the letter $d_{\underline{w_{n+1}},a_n,\underline{w_n}}$ immediately after reading $a_n w_n \cdots a_1 w_1$, and can accept only when $w_{n+1} \notin L(A,T)$. So $\mathcal{A}$ recognizes $\mathcal{R}$, reading from left-to-right. $\boxed{16.2}$

By Lemma 16.2,

$$K = L \circ \mathcal{R} = \big\{u \in D^* : (\exists v \in L)\big((u,v) \in \mathcal{R}\big)\big\}.$$

is regular. Since the set of left-hand sides of elements of $\mathcal{R}$ is $L(A,T)$, the language K maps onto T.

Suppose $u_1, u_2 \in K$ are such that $\overline{u_1} = \overline{u_2}$. Let $w_1, w_2 \in L$ be such that $(w_1,u_1),(w_2,u_2) \in \mathcal{R}$. Since L maps bijectively onto S and $\overline{w_1} = \overline{u_1} = \overline{u_2} = \overline{w_2}$, the words w_1 and w_2 must be identical. Since every $w \in L(A,T)$ determines a unique $u \in D^+$ with $(w,u) \in \mathcal{R}$, it follows that u_1 and u_2 are identical. So K maps bijectively onto T.

Finally, let $u \in K$ with $|u| \geqslant 2$. Then $u = d_{c_{n+1},a_n,c_n} \cdots d_{\varepsilon,a_2,c_2} d_{\varepsilon,a_1,c_1}$, with $n \geqslant 2$. Then there is some word $w \in L$ with $(w,u) \in \mathcal{R}$. By the definition of $\mathcal{R}$, the word w factorizes as $w_{n+1}a_n w_n \cdots a_2 w_2 a_1 w_1 \in L$ with $\underline{w_i} = c_i$, and $a_1 w_1, a_2 w_2, \ldots, w_{n+1} a_n w_n \in L(A,T)$.

Since L is prefix-closed, $w_{n+1} a_n w_n \cdots a_2 w_2 \in L$. Since $a_2 w_2, \ldots, w_{n+1} a_n w_n \in L(A,T)$, it follows that $w_{n+1} a_n w_n \cdots a_2 w_2 \in L(A,T)$. So, by the definition of $\mathcal{R}$, it follows that $d_{c_{n+1},a_n,c_n} \cdots d_{\varepsilon,a_2,c_2} \in K$.

This shows that K is closed under taking longest proper non-empty prefixes. By induction, K is $+$-prefix-closed. Hence K is a Markov language for T. $\boxed{16.1}$

However, the Rees index has the disadvantage that is does not generalize the group index. This motivated Gray & Ruškuc [GR08] to develop the notion of Green index, which does generalize the group index. The definition and only the necessary properties of the Green index and related topics are given here; the reader is referred to [GR08, § 1] for further details.

DEFINITION 16.3. Let S be a semigroup and let T be a subsemigroup of S. The T-relative Green's relations $\mathcal{R}^T$, $\mathcal{L}^T$, and $\mathcal{H}^T$ are defined on S as follows: for $x, y \in S$,

$$\begin{aligned} x \mathrel{\mathcal{R}^T} y &\iff xT^1 = yT^1 \\ x \mathrel{\mathcal{L}^T} y &\iff T^1x = T^1y \\ x \mathrel{\mathcal{H}^T} y &\iff x \mathrel{\mathcal{R}^T} y \wedge x \mathrel{\mathcal{L}^T} y; \end{aligned}$$

these are equivalence relations [GR08, § 1]. The T-relative $\mathcal{R}^T$-, $\mathcal{L}^T$-, and $\mathcal{H}^T$-classes (that is, the equivalence classes of these relations) respect T, in the sense that each such class lies either wholly in T or wholly in $S - T$.

The *Green index* of T in S is defined to be one more than the number of $\mathcal{H}^T$-classes in $S - T$.

Several properties are known to be preserved under passing to finite Green index extensions and subsemigroups, such as finite generation [CGR09, Theorems 4.1 & 4.3], others are known to hold on passing to finite Green index subsemigroups and not on passing to finite Green index extensions, such as automaticity [CGR09, Theorem 10.1 & Example 10.3]. The following example shows that neither the class of Markov semigroups nor the class of strongly Markov semigroups is not closed under finite Green index extensions. Indeed, a finite Green index extension of a strongly Markov semigroup need not be Markov:

EXAMPLE 16.4. Let G a finitely generated infinite torsion group. Let B be an alphabet representing a generating set for G. Let A be a finite alphabet in bijection with B. Let F be the free group with basis $\overline{A}$. The bijection from $\overline{A}$ to $\overline{B}$ naturally extends to a surjective homomorphism $\phi : F \to G$. Let S be the strong semilattice of groups $\mathcal{S}(F, G, \phi)$. (See [How95, §§ 4.1–4.2] for background on strong semilattices of groups.)

The free group is hyperbolic and therefore strongly Markov. Moreover, F is a finite Green index subsemigroup of S, with $S - F$ consisting of the single $\mathcal{H}_F$-class G.

Suppose that S is Markov. Then by Proposition 5.3, S admits a regular language of unique normal forms L over the alphabet $A \cup B$. By the definition of multiplication in a strong semilattice of monoids, the words in L representing elements of G are precisely those that include at least one letter B. That is, the language of words in L representing elements of G is $K = L - A^*$. Since L is regular, K is also. Since L maps bijectively onto S and $K \subseteq L$, it follows that K maps bijectively onto G. So if each letter $a \in A$ is interpreted as representing the element $\overline{a}\phi$ of G, then K is a regular language of unique normal forms for G. However, G, as a finitely generated infinite torsion group, does not admit a regular language of unique normal forms by the reasoning in [ECH+92, Example 2.5.12]. This is a contradiction, and so S cannot be Markov.

This example is similar in spirit to examples showing that neither the class of finitely presented semigroups nor the class of automatic semigroups is not closed under forming finite Green index extensions [CGR09, Examples 6.5 & 10.3]. However, with an extra condition on the Schützenberger groups of the T-relative $\mathcal{H}$-classes in the complement, a positive result does hold. First of all, recall the definitions of Schützenberger groups:

DEFINITION 16.5. Retain notation from Definition 16.3. Let H be an $\mathcal{H}_T$. Let $\mathrm{Stab}(H) = \{t \in T^1 : Ht = H\}$ (the *stabilizer* of H in T), and define an equivalence $\sigma(H)$ on $\mathrm{Stab}(H)$ by $(x, y) \in \sigma(H)$ if and only if $hx = hy$ for all $h \in H$. Then $\sigma(H)$ is a congruence on $\mathrm{Stab}(H)$ and $\mathrm{Stab}(H)/\sigma(H)$ is a group, called the *Schützenberger group* of the $\mathcal{H}_T$-class H and denoted $\Gamma(H)$.

PROPOSITION 16.6. *Let S be a semigroup and T a subsemigroup of S of finite Green index. Suppose that T is Markov and that the Schützenberger group of every T-relative $\mathcal{H}$-class in $S - T$ is Markov. Then S is Markov.*

Proof of 16.6. Let L be a semigroup Markov language for T over some finite alphabet A representing a generating set for T under the map $\phi : A \to T$. Since T has finite Green index in S, there are finitely many T-relative H-classes $H_1, \ldots, H_n$ in $S - T$. By hypothesis, every Schützenberger group $\Gamma(H_i)$ admits a semigroup Markov language L_i over some finite alphabet A_i representing a generating set for $\Gamma(H_i)$ under the map $\phi_i : A_i \to \Gamma(H_i)$. For brevity, let $\sigma_i = \sigma(H_i)$.

For each $i = 1, \ldots, n$, fix an element $h_i \in H_i$. For each $i = 1, \ldots, n$ and $a \in A_i$, fix elements $s_{i,a} \in \mathrm{Stab}(H_i)$ such that $a\phi_i = [s_{i,a}]_{\sigma_i}$.

Let A_i' be a new alphabet in bijection with A_i under the map $\alpha_i : A_i \to A_i'$. (Without loss of generality, assume that the alphabet A and the various alphabets A_i and A_i' are pairwise disjoint.) Define a map $\psi_i : A_i \cup A_i' \to S$ as follows:

$$a\psi_i = \begin{cases} s_{i,a} & \text{if } a \in A_i, \\ h_i s_{a,i} & \text{if } a \in A_i'. \end{cases} \tag{16.1}$$

Let

$$L_i' = \{(a\alpha_i)u \in A_i' A_i^* : au \in L_i, a \in A_i\}.$$

(So L_i' is the language obtained from L_i by taking each word in $L_i \subseteq A_i^+$ and replacing its first letter with the corresponding letter from A_i'.) Notice that since L_i is regular and $+$-prefix-closed, so is L_i'.

Since $\Gamma(H_i)$ acts regularly on H_i via

$$x \cdot [s]_{\sigma_i} = xs,$$

it follows that for every $y \in H_i$ there is a unique element $[s]_{\sigma_i} \in \Gamma(H_i)$ such that $h_i \cdot [s]_{\sigma_i} = y$. Thus it follows from (16.1) and the fact that L_i is a Markov language for $\Gamma(H)$ that for every $y \in H_i$ there is a unique $w \in L_i$ such that $h_i(w\phi_i) = y$. Hence, by (16.1) and the definition of L_i', for every $y \in H_i$ there is a unique word $v \in L_i'$ with $v\psi_i = y$. Thus L_i' maps bijectively onto H_i.

Finally, let

$$K = L \cup \bigcup_{i=1}^{n} L_i'.$$

Then K is $+$-prefix-closed and regular. Define

$$\psi : A \cup \bigcup_{i=1}^{n} (A_i \cup A_i') \to S, \qquad a\psi = \begin{cases} a\phi & \text{if } w \in A, \\ a\psi_i & \text{if } w \in A_i \cup A_i'. \end{cases}$$

Then ϕ maps K bijectively onto S. Hence K is a semigroup Markov language for L . $\boxed{16.6}$

Proposition 16.6 parallels [CGR09, Theorem 6.1], which shows that if T is a finite Green index subsemigroup of S, and T *and* all the Schützenberger groups of the T-relative $\mathcal{H}$-classes in $S - T$ are finitely presented, then S is finitely presented. (As remarked above, without the condition on the finite presentability, this result does not hold.) This is in marked contrast to the situation for automatic groups: even if T and all the Schützenberger groups are automatic, S may not be automatic; see [CGR09, Example 10.3].

QUESTION 16.7. Let T be a subsemigroup of finite Green index in a semigroup S. Let also S be Markov. Is T Markov?

QUESTION 16.8. Is the property of being Markov preserved under passing to subsemigroups and extensions of finite Grigorchuk index for finitely generated cancellative semigroups (so that both of the semigroups are finitely generated)?

17 THE CLASS OF MARKOV LANGUAGES

This final section examines the class of languages that are Markov languages for some semigroup or monoid. First, notice that not every regular language is a Markov language:

EXAMPLE 17.1. Let $L = a^+ \cup a^+b$. Suppose L is a Markov language for a semigroup S. Then $\overline{b}$ lies in S and so must be represented by an element of L. If $\overline{b} = \overline{a^k}$ for some k then $\overline{ab} = \overline{aa^k} = \overline{a^{k+1}}$. Since both ab and a^{k+1} lie in L, this contradicts the uniqueness of representives in L. If, on the other hand, $\overline{b} = \overline{a^kb}$ for some k, then $\overline{ab} = \overline{aa^kb} = \overline{a^{k+1}b}$, again contradicting the uniqueness of representives in L. So L is not a semigroup Markov language.

Indeed, if instead $L' = L \cup \{\varepsilon\} = a^* \cup a^+b$, then the same contradictions show that L' is not a monoid Markov language.

Starting from a Markov language and adding or removing a finite number of words can yield a prefix-closed regular language that is not a Markov language, as the following two examples show:

EXAMPLE 17.2. Let $K = L' \cup \{b\} = a^* \cup a^*b$, where L' is the language from Example 17.1. Then K is a Markov language for the semigroup presented by $\langle a, b \mid (b^2, b), (ba, b) \rangle$. To see this, notice that $(\{a, b\}, \{(b^2, b), (ba, b)\})$ is a confluent noetherian rewriting system and its language of normal forms is K, and apply Proposition 7.1. Thus removing the single word b from the Markov language K yields the non-Markov language L'.

EXAMPLE 17.3. Let $L = a^* \cup \{a^2c, a^4c\}$. Suppose L is a Markov language for a semigroup S. Then $\overline{ac}$ lies in S and so must be represented by an element of L. Now, if $\overline{ac} = \overline{a^\alpha}$, then $\overline{a^2c} = \overline{a^{\alpha+1}}$, contradicting the uniqueness of representatives in L. If $\overline{ac} = \overline{a^2c}$, then $\overline{ac} = \overline{a^2c} = \overline{a^3c} = \overline{a^4c}$, again contradicting the uniqueness of representatives in L. So $\overline{ac} = \overline{a^4c}$.

Now, $\overline{a^3c}$ must also be represented by an element of L. If $\overline{a^3c} = \overline{a^\alpha}$, then $\overline{a^4c} = \overline{a^{\alpha+1}}$, contradicting the uniqueness of representatives in L. If $\overline{a^3c} = \overline{a^2c}$, then $\overline{a^2c} = \overline{a^3c} = \overline{a^4c}$, again contradicting the uniqueness of representatives in L. So $\overline{a^3c} = \overline{a^4c}$, which, by the preceding paragraph, implies $\overline{ac} = \overline{a^3c}$, which in turn implies $\overline{a^2c} = \overline{a^4c}$. This contradicts the uniqueness of representatives in L, and so L cannot be a Markov language.

Thus adding the two words a^2c and a^4c to the Markov language a^* yields the non-Markov language L.

There are two main questions about the class of Markov languages:

QUESTION 17.4. Is there an algorithm that takes a regular language that is prefix-closed or $+$-prefix-closed and decides whether it is a Markov language for some monoid or semigroup?

QUESTION 17.5. Is every finite language that is prefix-closed or $+$-prefix-closed a Markov language for a (necessarily finite) monoid or semigroup?

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