On the spectrum of generic random product of compact operators

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Abstract

In this note we consider a separable Hilbert space and family of infinite dimensional compact and invertible cocycles over a dynamical system defined in a compact and Hausdorff space X and which preserves a Borel regular measure. We prove that C^0-generically we have, for almost every $x \in X$, a trivial spectrum or else an uniformly hyperbolicity in the projective space.

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1 Introduction, basic definitions and statement of the results

In this paper we establish the typical spectrum of random products of linear operators defined in Hilbert spaces. For this purpose we use the Oseledets-like theory for the separable Hilbert spaces setting which was constructed in the early eighties by Ruelle [5]; and also a recent result of Bochi and Viana [2]. In this last-mentioned result it is proved that for a C^0-residual subset inside a very general family of finite dimensional linear cocycles based in a homeomorphism which preserves a Borel regular invariant measure, we have, for almost every point, uniform hyperbolicity in the projective space

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or else one-point spectrum. Here we show that the *Bochi-Viana* theorem is true also for infinite dimensional cocycles in Hilbert spaces. We hope that our results may be applied to the theory of functional differential equations (see [4]).

Let $\mathcal{H}$ be a separable Hilbert space, $\mathcal{C}(\mathcal{H})$ the set of linear continuous operators, compact and invertible, in $\mathcal{H}$, $f : X \to X$ a homeomorphism in the compact Hausdorff space X and μ an f-invariant Borel regular measure. Given $\mathcal{E} := X \times \mathcal{H}$, let $\pi : \mathcal{E} \to X$ be a continuous vector bundle over X. Consider a continuous map $A : X \to \mathcal{C}(\mathcal{H})$ and define

$$A^n(x) = A(f^{n-1}(x)) \cdot ... \cdot A(f(x)) \cdot A(x),$$

with $A^0(x) = Id$. In this context we define the *associated cocycle over* f: it is a continuous transformation $F(A) : \mathcal{E} \to \mathcal{E}$ such that $\pi \circ F = f \circ \pi$ and $F_x(A) : \mathcal{H}_x \to \mathcal{H}_{f(x)}$ is linear on the fiber $\mathcal{H}_x := \pi^{-1}(x)$. We then have a morphism of vector bundles covering f given by

$$\begin{array}{rccc} F(A): & X \times \mathcal{H} & \longrightarrow & X \times \mathcal{H} \\ & (x, v) & \longmapsto & (f(x), A(x) \cdot v). \end{array}$$

For simplicity of notation we call the map A a cocycle. Given a nonperiodic orbit $\{f^n(x)\}_{n \in \mathbb{N}}$ the product $A^n(x)$ for $n \in \mathbb{N}$ is what we call a *random product of operators.*

We denote by $C^0_I(X, \mathcal{C}(\mathcal{H}))$ the set of all continuous infinite dimensional cocycles such that any element $A \in C^0_I(X, \mathcal{C}(\mathcal{H}))$ satisfies the integrability condition $\int_X \log^+ \|A(x)\| d\mu(x) < \infty$, where $\log^+(y) = \max\{0, \log(y)\}$.

Given an f-invariant set Λ we say it has an *m-dominated splitting* if there exists a decomposition of the fiber $\mathcal{H}_x = E^1_x \oplus E^2_x$ (for $x \in \Lambda$) varying continuously with the point x and A-invariant and such that we may find an *uniform* $m \in \mathbb{N}$ such that the following inequality holds:

$$\Delta(p, m) = \frac{\|A^m(x)|E^1_x\|}{\mathfrak{m}(A^m(x)|E^2_x)}\| \leq \frac{1}{2}. \tag{1}$$

Where $\mathfrak{m}(L) = \|L^{-1}\|^{-1}$ is the *co-norm* of the operator L. We say that the spectrum Σ is *trivial* or a *one-point spectrum* if $\Sigma = \{s\}$ for a unique $s \in \mathbb{R}$.

In Sections 4 and 5 we will prove that:

Theorem 1.1 *There exists a C^0-residual subset $\mathcal{R} \subset C^0_I(X, \mathcal{C}(\mathcal{H}))$ such that, if $A \in \mathcal{R}$, then for μ-a.e. $x \in X$ we have trivial spectrum or else m-dominated splitting for some $m \in \mathbb{N}$.*

2 A dichotomy for finite dimensional cocycles

Let S denotes any subgroup of finite d-dimensional cocycles which acts transitively in the projective space $\mathbb{R}P^{d-1}$(see [2] Example 4).

Theorem 2.1 *(Bochi-Viana [2]) There exists a residual $\mathcal{R} \subset S$ (in the C^0-topology) such that for any $A \in \mathcal{R}$ we have for μ-a.e. point $x \in X$ that the Oseledets's splitting is m-dominated or else is trivial.*

We note that the linear groups of matrices $GL(d,\mathbb{R})$ and $SL(d,\mathbb{R})$ act transitively in $\mathbb{R}P^{d-1}$. Our interest is centered on the correspondent infinite dimensional set of invertible operators. As a first approach we may consider a C^0-approximation of our original system by other with finite rank. Then applying Theorem 2.1 we obtain a system satisfying the stated dichotomy. However, for this new system the dichotomy would not be so precise: for example, we could have for μ-generic orbits the existence of a two-point spectrum with $-\infty$ as another element of the spectrum; moreover the dominated splitting might not be extended to $\{\mathcal{H}_x\}_{x\in\Lambda}$; and finally this result only gives a C^0-dense instead of a C^0-residual.

As we will see in the next section it is essential to consider compactness and one gives an integrability condition to guarantee Ruelle's version of the multiplicative ergodic theorem. The m-dominated splitting for some $m \in \mathbb{N}$ allows us to conclude that we have uniform hyperbolic structure in the projective space. The notion of hyperbolicity in our setting will be over the infinite real projective space $\mathbb{R}P^\infty$, which is built by applying the telescoping construction, where through the natural inclusions we glue together the real projective spaces of finite dimension (see [3]).

3 The multiplicative ergodic theorem for compact operators

The following result gives us a spectral decomposition for random products of compact operators.

Theorem 3.1 *(Ruelle [5]) Let μ and $f : X \to X$ be as above. If $A \in C^0_I(X,\mathcal{C}(\mathcal{H}))$, then for μ-a.e $x \in X$ we have the following properties:*
(a) The limit $\lim\limits_{n\to\infty}(A(x)^{n*}A(x)^n)^{\frac{1}{2n}}$ exists and is a compact operator $\mathcal{L}(x)$.
(b) Let $e^{\lambda_1(x)} > e^{\lambda_2(x)} > \ldots$ be the nonzero eigenvalues of $\mathcal{L}(x)$ and $U_1(x)$, $U_2(x)$,... be the associated eigenspaces whose dimensions are denoted by

$n_i(x)$. The real functions $\lambda_i(x)$ are called ***Lyapunov exponents****. The sequence possibly terminates at $\lambda_j(x)$, otherwise we write $j = \infty$. Then $j = j(x)$, $\lambda_i(x)$ and $n_i(x)$ are f-invariant functions and depend in a measurable way on x.*
(c) Let $V_i(x)$ be the orthogonal complement of $U_1(x) \oplus U_2(x) \oplus ... \oplus U_{i-1}(x)$ for $i < j(x) + 1$. Let $V_{j(x)+1}(x) = Ker(\mathcal{L}(x))$. Then

(i) $\lim_{n\to\infty} \frac{1}{n} \log \|A^n(x) \cdot u\| = \lambda_i(x)$ *if* $u \in V_i(x) \setminus V_{i+1}(x)$ *for* $i < j(x) + 1$*;*

(ii) $\lim_{n\to\infty} \frac{1}{n} \log \|A^n(x) \cdot u\| = -\infty$ *if* $u \in V_{j(x)+1}(x)$*.*

We denote by $\mathcal{O}(A)$ the full measure set of points given by Theorem 3.1.

4 Upper semi-continuity of the entropy function

Given a bounded linear operator $A : \mathcal{H} \to \mathcal{H}$ we can induce an operator on the space $\wedge^p(\mathcal{H})$ of the p^{th} exterior power of $\mathcal{H}$ by

$$\wedge^p(A)(e_1 \wedge e_2 \wedge ... \wedge e_p) = A(e_1) \wedge A(e_2) \wedge ... \wedge A(e_p).$$

For details see [6], chapter V.

Lemma 4.1 *If A is compact, then $\wedge^p(A)$ is also compact.*

Proof: Given any bounded sequence $(y_n)_n$ of elements of $\wedge^p\mathcal{H}$ we will prove that there exists a subsequence $(y_{n_k})_k$ such that $(\wedge^p(A)(y_{n_k}))_k$ converges. Let $y_n = v_{a_1^n} \wedge ... \wedge v_{a_p^n}$ for each $n \in \mathbb{N}$ where $v_{a_j^n} \in \mathcal{H}$ for $j = 1, ..., p$. Hence $\wedge^p(A)(y_n) = A(v_{a_1^n}) \wedge ... \wedge A(v_{a_p^n})$ for each $n \in \mathbb{N}$. So, by applying the hypothesis p-times we conclude that $v_{a_j^n}$ admits a subsequence $v_{a_j^{n_k}}$ converging to $u_j \in \mathcal{H}$, for $j = 1, ..., p$. Therefore there exists a subsequence $(y_{n_k})_k$ such that $(\wedge^p(A)(y_{n_k}))_k$ converges and so the operator $\wedge^p(A)$ is compact. □

Since $\wedge^p(A)$ is compact we can apply Theorem 3.1 and conclude that for μ-a.e. x the following limit exists:

$$\lim_{n\to+\infty} \frac{1}{n} \log \| \wedge^p A(x)^n \| = \lambda_1^{\wedge p}(x).$$

This limit is the largest Lyapunov exponent given by the dynamics of the operator $\wedge^p(A)$ at x. Moreover for μ-a.e. x we have $\sum_{i=1}^{p} \lambda_i(x) = \lambda_1^{\wedge p}(x)$ (see [1]).

Given any $p \in \mathbb{N}$ and $\Gamma \subseteq X$ an f-invariant set, we define the *p-entropy function* at Γ by:

$$\begin{array}{rccc} LE_p(\Gamma): & C^0_I(X,\mathcal{C}(\mathcal{H})) & \longrightarrow & \mathbb{R}^+_0 \\ & A & \longmapsto & \int_\Gamma \sum_{i=1}^p \lambda_i(A,x)d\mu(x). \end{array}$$

Note that as the Lyapunov exponents vary in a measurable fashion, there is no reason to believe that the function LE_p is continuous. Therefore, it is interesting to see that previous observations allows us to conclude that the function LE_p is at least upper-semicontinuous.

If we denote $a_n(A) = \int_\Gamma \log \| \wedge^p A(x)^n \| d\mu(x)$, then

$$LE_p(\Gamma)(A) = \int_\Gamma \sum_{i=1}^p \lambda_i(A,x)d\mu(x) = \int_\Gamma \lambda_1^{\wedge p}(x) = \lim_{n\to+\infty} \frac{a_n}{n}.$$

Since a_n is sub-additive (see [2] Section 2.1.3.) we obtain $LE_p(\Gamma)(A) = \inf_{n\in\mathbb{N}} \frac{a_n}{n}$. Finally, since $\wedge^p(A)$ is a continuous operator, we have that the function defined by $\frac{1}{n}\int_\Gamma \log \| \wedge^p A(x)^n \| d\mu(x)$ is continuous; therefore $LE_p(\Gamma)(A)$ is the infimum of a sequence of continuous functions, hence upper-semicontinuous. The next proposition will be crucial to prove Theorem 1.1.

Proposition 4.2 *If A is a point of continuity of LE_p for all $p \in \mathbb{N}$, then for μ-a.e. $x \in X$ we have that the splitting given by Theorem 3.1 is trivial or else is m-dominated.*

5 Proof of Proposition 4.2

Let us consider the following simple lemma which allows us to perform useful measurable C^0-perturbations of our original system.

Lemma 5.1 *Let $A \in C^0_I(X,\mathcal{C}(\mathcal{H}))$, $x \in X$ and $\epsilon > 0$. For any 2-dimensional subspace $E \subset \mathcal{H}_x$, there exists an angle ξ (not depending on x) and a measurable integrable cocycle B such that:*

(a) $B(x) \cdot u = A(x) \cdot u$, $\forall u \in E^\perp$;

(b) $B(x) \cdot u = A(x) \cdot R_\xi \cdot u$, $\forall u \in E$, *where R_ξ is the rotation of angle ξ in E;*

(c) $\|A - B\| \leq \epsilon$.

Proof: For any $v \in \mathcal{H}_x$ we can write $v = v_1 + v_2$ where $v_1 \in E^\perp$ and $v_2 \in E$. Denote by

$$R_\theta = \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix},$$

the matrix of the rotation of angle θ in an orthonormal basis of E.
Take $\xi > 0$ such that $\|Id - R_\xi\| \leq \frac{\epsilon}{\|A\|}$ (note that $\|A\| \neq 0$) and define the perturbation cocycle by:

$$B(y) = \begin{cases} A(y), & \text{if } y \neq x \\ B(x) \cdot v = A(x) \cdot v_1 + A(x) \cdot R_\xi \cdot v_2, & \text{if } y = x \end{cases}$$

Clearly this cocycle verifies the properties (a), (b) and (c) and the lemma is proved. □

The following lemma allows us to interchange directions and to prove it we use Lemma 5.1 jointly with Proposition 7.1 of [2]. The main idea is to use the absence of hyperbolic behavior to concatenate several small rotations of the form given by Lemma 5.1 in order to mix different directions.

Lemma 5.2 *Given A, $\epsilon > 0$ and $m(\epsilon) = m \in \mathbb{N}$ large enough, then the following holds: let $y \in X$ be a nonperiodic point and suppose it is endowed a nontrivial splitting $\mathcal{H}_y = E \oplus F$ such that $\frac{\|A^m(y)|_F\|}{\mathfrak{m}(A^m(y)|_E)} \geq 1/2$. Then there exist operators $L_j : \mathcal{H}_{f^j(y)} \to \mathcal{H}_{f^{j+1}(y)}$, for each $j = 0, ..., m-1$, with $\|L_j - A(f^j(y))\| < \epsilon$ such that $L_{m-1} \cdot ... \cdot L_0 \cdot v = w$ for some nonzero vectors $v \in E$ and $w \in A^m(y)(F)$.*

Let $\Lambda_p(A, m)$ be the set of points x such that there exists an m-dominated splitting (of index p) along the orbit of x and $\Gamma_p(A, m) = X \setminus \Lambda_p(A, m)$. Let us consider $\Gamma_p^*(A, m)$ defined by the points $x \in \mathcal{O}(A) \cap \Gamma_p(A, m)$, which are nonperiodic and also such that $\lambda_p(A, x) > \lambda_{p+1}(A, x)$. Then we get a local result:

Lemma 5.3 *Given A, $\epsilon, \delta > 0$ and $p \in \mathbb{N}$, if $m \in \mathbb{N}$ is large enough, then there exists a measurable function $N : \Gamma_p^*(A, m) \to \mathbb{N}$ such that, for μ-a.e. $x \in \Gamma_p^*(A, m)$ (with all $\lambda_i(A, x)'s$ for $i \leq p+1$ different from $-\infty$) and every $n \geq N(x)$, there exist operators $L_0, ..., L_{m-1}$ as in Lemma 5.2 satisfying*

$$\frac{1}{n} \log^+ \| \wedge^p (L_{n-1} \cdot ... \cdot L_0)\| \leq \sum_{i=1}^{p-1} \lambda_i(A, x) + \frac{\lambda_p(A, x) + \lambda_{p+1}(A, x)}{2} + \delta. \quad (2)$$

Proof: We use Lemma 5.2 and follow the arguments of Proposition 4.2 of [2]. □

Remark 5.1 *Suppose that for some $p \in \mathbb{N}$ we have,*

$$\lambda_p(A,x) \neq \lambda_{p+1}(A,x) = -\infty.$$

Therefore if we swap $\lambda_{p+1}(A,x)$ by $\kappa_{p+1}(A,x) \in \mathbb{R}^-$ (with $\kappa_{p+1}(A,x) << 0$) and $\kappa_{p+1}(A,x) < \lambda_p(A,x)$, then the left hand of the inequality (2) can easily be made as small as we want. For this reason, in the sequel, we assume that $\lambda_p(A,x) \neq -\infty$, $\forall p \in \mathbb{N}$.

Finally we consider the global case.

Proposition 5.4 *Let A be a compact operator, $\epsilon, \delta > 0$ and $p \in \mathbb{N}$. Then there exists $m \in \mathbb{N}$ and a continuous operator B, with $\|B - A\|_\infty < \epsilon$, that equals A outside the open set $\Gamma_p(A,m)$ and is such that*

$$\int_{\Gamma_p(A,m)} \lambda_1^{\wedge p}(B,x)d\mu(x) < \int_{\Gamma_p(A,m)} \sum_{i=1}^{p-1} \lambda_i(A,x) + \frac{\lambda_p(A,x) + \lambda_{p+1}(A,x)}{2} d\mu(x) + \delta.$$

Proof: Like in [2] we use Lemma 5.3 and a Kakutani tower argument and prove an easier result for a measurable and essentially bounded operator; then, using Lusin's theorem we approximate the measurable operator by a continuous one. See Proposition 7.3 and Lemma 7.4 of [2] for all details. □

Once we have settled Proposition 5.4 we define for each $p \in \mathbb{N}$ and for $\Gamma_p(A,\infty) := \bigcap_m \Gamma_p(A,m)$, the *jump function* at A by

$$J_p(A) = \int_{\Gamma_p(A,\infty)} \lambda_p(A,x) - \lambda_{p+1}(A,x) d\mu(x).$$

By using Lemma 4.17 of [2] jointly with Proposition 5.4 we obtain that, given a continuous cocycle A, any $p \in \mathbb{N}$, $\epsilon > 0$ and $\delta > 0$, there exists a continuous cocycle B, ϵ-close to A and such that:

$$\int_X \lambda_1^{\wedge p}(B,x)d\mu(x) < \delta + \int_X \lambda_1^{\wedge p}(A,x)d\mu(x) - 2J_p(A).$$

Now to end the proof of Proposition 4.2 we note that if A is a point of continuity of $LE_p(X)$ for all $p \in \mathbb{N}$, then $J_p(A) = 0$ for every p. Therefore for every p and μ-a.e. $z \in \Gamma_p(A,\infty)$ we have $\lambda_p(A,z) = \lambda_{p+1}(A,z)$. Now if

x is inside the full measure set given by Theorem 3.1 and its spectrum is trivial, then our result is proved. If the spectrum of x is not trivial, hence $\lambda_p(A,x) > \lambda_{p+1}(A,x)$ for some p and so $x \notin \Gamma_p(A,m)$. Therefore, for some m, we have $x \in \Lambda_p(A,m)$. Hence we have an m-dominated splitting of index p; $\mathcal{H}_{f^n(x)} = E_n \oplus F_n$ along the orbit of x with $E_n = U_1(x) \oplus ... \oplus U_p(x)$ and $F_n = U_{p+1}(x) \oplus ....$ Moreover, we obtain that the splitting given by Theorem 3.1 is dominated and the Proposition 4.2 is proved.
By Proposition 4.2, for each p, there exists a residual $\mathcal{R}_p$ of points of continuity of LE_p. Take $\mathcal{R} = \bigcap_{p\in\mathbb{N}} \mathcal{R}_p$ and this is the residual where the statement of Theorem 1.1 holds.

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