On the convergence of Schröder iteration functions for pth roots of complex numbers

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Abstract

In this work a condition on the starting values that guarantees the convergence of the Schröder iteration functions of any order to a pth root of a complex number is given. Convergence results are derived from the properties of the Taylor series coefficients of a certain function. The theory is illustrated by some computer generated plots of the basins of attraction.

Key-words: basins of attraction, Bell polynomials, Faà di Bruno's formula, iteration function, order of convergence, pth root, residuals, Taylor expansions.

1 Introduction

Throughout the paper, we will assume p and j to be two integers greater or equal than 2 and w to be a given complex number not belonging to the closed negative real axis. The pth roots of w are the p solutions of the polynomial equation

$$z^p - w = 0. \tag{1.1}$$

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Let $\theta = \arg(w) \in]-\pi, \pi[$ denote the argument of w. It is well-known that for $n = 0, 1, \ldots, p-1$ each wedge of the complex plane defined by

$$\mathscr{W}_n = \left\{ z \in \mathbb{C} : \frac{(2n-1)\pi + \theta}{p} < \arg(z) < \frac{(2n+1)\pi + \theta}{p} \right\} \tag{1.2}$$

contains exactly one pth root of w.

Our interest in studying iterative methods for pth roots comes from the problem of computing matrix pth roots. This is currently an important focus for research [1,7,8,9,10,11,12] mainly because of its applications in control and finance. Since the eigenvalues of a matrix are complex (even when the matrix has only real entries), the iteration functions for pth roots of complex scalars can be extended to the matrix case.

Consider the complex function f defined by $f(z) = (1-z)^{1/p}$ and let $T_j(z)$ denote the Taylor polynomial of degree j of $f(z)$ at zero. For each $j = 2, 3, \ldots$, the pth roots of w are fixed points of

$$N_j(z) := z T_{j-1}\left(1 - wz^{-p}\right), \tag{1.3}$$

which is an iteration function with order of convergence at least j (see [4]). This means that there is an initial guess z_0 such that the sequence defined by

$$z_{k+1} = N_j(z_k) \tag{1.4}$$

converges to a pth root of w with order of convergence at least j.

It was shown in [4, Lemma 3.1] that N_j coincide with the Schröder iteration functions associated to the polynomial equation (1.1), which compels us to refer to N_j as the *Schröder iteration functions* for the pth roots of w. We refer the reader to [14,15] for more details about Schröder iteration functions.

The Taylor polynomials T_j in (1.3) are given by

$$T_j(z) := \sum_{n=0}^{j} \left(-\frac{1}{p}\right)_n \frac{z^n}{n!}, \tag{1.5}$$

where $(a)_k := a(a+1)\ldots(a+k-1)$ and $(a)_0 = 1$ represent the rising factorial of the complex number a (Pochhammer symbol). Recall that the particular case $j = 2$ is nothing more than the Newton's method for finding the zeros of the function $z^p - w$:

$$N_2(z) = z\left(1 - \frac{1}{p}(1 - wz^{-p})\right).$$

We note that the function $f(z) = (1-z)^{1/p}$ has a formal (binomial) series representation [5, p.37]:

$$f(z) = \sum_{n \geqslant 0} \left(-\frac{1}{p}\right)_n \frac{z^n}{n!}, \tag{1.6}$$

and it is absolutely convergent inside the unit circle.

A complex function that is involved in the expression of N_j is the so-called residual function

$$R(z) := 1 - wz^{-p}. \tag{1.7}$$

The successive terms of the sequence (1.4) can be related by means of the residual function (1.7):

$$R(z_{k+1}) = 1 - (1 - z_k)\Big(T_j(z_k)\Big)^{-p} \tag{1.8}$$

(see [4, Sec. 3]). Let us consider the function that corresponds the right hand side of (1.8) by

$$\widetilde{R}_j(z) = 1 - (1 - z)\Big(T_j(z)\Big)^{-p}. \tag{1.9}$$

This function will play an important role in our work.

In Section 2 we will prove our main result which is Theorem 2.1. It states that $\widetilde{R}_j(z)$ admits a representation by a power series at $z = 0$ that is convergent for any complex number z inside the unit circle and whose first j coefficients are null while the remaining ones are positive. In order to accomplish this we will need to ensure the analyticity of $\widetilde{R}_j(z)$ inside the unit circle as well as to recall some other known results. The aforementioned theorem will enable us to derive in Section 3 some convergence results on Schröder iteration functions for pth roots. In particular, we show that if the initial guess z_0 satisfies the condition $|R(z_0)| < 1$, then for any j the sequence (1.4) converges to a pth root of w with order of convergence j. We recall that the case $j = 2$ has already been proved by Guo [8] and the case $j = 3$ by the present authors [4]. Our theoretical results will be illustrated by some examples of basins of attraction generated in Matlab.

2 Series representation of $\widetilde{R}_j(z)$

Lemma 2.1 *The roots of the Taylor polynomial $T_j(z)$ given by (1.5) lie outside the unit circle and consequently $\widetilde{R}_j(z)$ is analytic for any z such that $|z| < 1$.*

PROOF. For any complex number z such that $|z| < 1$, we successively have

$$\begin{aligned} |T_j(z)| &= \left|1 + \sum_{\nu=1}^{j} \frac{(-1/p)_\nu}{\nu!}\, z^\nu\right| = \left|1 - \frac{1}{p}\sum_{\nu=1}^{j} \frac{(1-1/p)_{\nu-1}}{\nu!}\, z^\nu\right| \\ &\geqslant 1 - \frac{1}{p}\sum_{\nu=1}^{j} \frac{(1-1/p)_{\nu-1}}{\nu!}\, |z|^\nu > 1 - \frac{1}{p}\sum_{\nu=1}^{j} \frac{(1-1/p)_{\nu-1}}{\nu!} \\ &\geqslant 1 - 1 + \frac{(1-1/p)_j}{j!}, \end{aligned}$$

whence $|T_j(z)| > 0$ which implies $T_j(z) \neq 0$. Inasmuch as $\widetilde{R}_j(z)$ is a rational function whose poles lie outside the unit circle, its analyticity inside this domain is guaranteed.

Based on the *Faà di Bruno's formula* [5,6,13] it is possible to derive the expression (although a tricky one) of the nth derivative of the function $\left(T_j(z)\right)^{-p}$ by means of the (partial) *Bell polynomials* [3], or, more precisely through the so called *potential polynomials*. We recall the result:

Proposition 2.1 *[5, p.141] Consider the function* $G(z) = 1 + \sum_{n\geqslant 1} g_n \frac{z^n}{n!}$ *where* $g_n = \frac{d^n}{dz^n} G(z)\Big|_{z=a}$, $n \geqslant 1$. *For any integer number r, the nth order derivative of the power function* $(G(z))^r$ *at the point* $z = a$ *is given by*

$$\begin{aligned}\frac{d^n}{dz^n}\left(G(z)\right)^r\Big|_{z=a} &= P_n^{(r)}(g_1, g_2, g_3, \ldots, g_n) \\ &:= \sum_{k=1}^{n} (-1)^k (-r)_k B_{n,k}(g_1, g_2, \ldots, g_{n-k+1}), \quad n \geqslant 0,\end{aligned}$$

with $B_{n,k}$ *representing the partial Bell polynomials:*

$$B_{n,k}(x_1, \ldots, x_{n-k+1}) = \sum \frac{1}{c_1! c_2! \ldots c_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{c_1} \left(\frac{x_2}{2!}\right)^{c_2} \ldots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{c_{n-k+1}}$$

where the summation is taken over all partitions of n into exactly k non-negative parts, i.e., over all solutions in non-negative integers c_i *such that* $c_1 + 2c_2 + 3c_3 + \ldots + (n-k+1)c_{n-k+1} = n$ *and* $c_1 + c_2 + c_3 + \ldots + c_{n-k+1} = k$.

Corollary 2.1 *For any integer number r, whenever* $g_i = \left(-\frac{1}{p}\right)_i$ *for any* $i = 1, 2, 3, \ldots$, *the following equality holds:*

$$P_n^{(r)}(g_1, g_2, \ldots, g_n) = \left(-\frac{r}{p}\right)_n, \quad n \geqslant 0. \tag{2.1}$$

PROOF. The function $(g(z))^r = (1-z)^{r/p}$ admits the series representation

$$(g(z))^r = \sum_{n\geqslant 0} \left(-\tfrac{r}{p}\right)_n \frac{z^n}{n!}$$

insofar as

$$\frac{d^n}{dz^n}\left(g(z)\right)^r\Big|_{z=0} = \left(-\tfrac{r}{p}\right)_n, \quad n \geqslant 0.$$

On the other hand, from Proposition 2.1, it follows

$$\frac{d^n}{dz^n}\left(g(z)\right)^r\Big|_{z=0} = P_n^{(r)}(g_1, g_2, \ldots, g_n), \quad n \geqslant 0,$$

whence the result.

Now we are able to prove our main result.

Theorem 2.1 *Consider the functions* $f(z) = \sqrt[p]{1-z}$ *and* $\widetilde{R}_j(z)$ *defined in* (1.9)*. If*

$$c_{n,j}^{(p)} := c_n = \frac{1}{n!}\,\frac{d^n}{dz^n}\Big(T_j(z)\Big)^{-p}\bigg|_{z=0}\,, \quad n \geqslant 0, \tag{2.2}$$

$$d_{n,j}^{(p)} := d_n = \frac{1}{n!}\,\frac{d^n}{dz^n}\widetilde{R}_j(z)\bigg|_{z=0}\,, \quad n \geqslant 0, \tag{2.3}$$

then

$$d_n = 0\,, \quad n = 1, 2, \ldots, j \tag{2.4}$$

$$d_n > 0\,, \quad n \geqslant j+1 \tag{2.5}$$

and both series $\sum_{k\geqslant 0} c_k z^k$ *and* $\sum_{k\geqslant 0} d_k z^k$ *are absolutely convergent for any* $z \in \mathbb{C}$ *such that* $|z| < 1$*. Moreover, within the same domain we are able to write*

$$\widetilde{R}_j(z) = \sum_{n\geq j+1} d_n z^n.$$

PROOF. We begin by showing that (2.4) holds to afterward prove (2.5) and, at the end, we will prove the convergence of the aforementioned series.

Let us consider the formal power series expansion at the point $z = 0$ of the function $q(z)$:

$$q(z) := \Big(T_j(z)\Big)^{-p} = \sum_{n\geqslant 0} c_n\, z^n \tag{2.6}$$

where

$$c_n = \frac{1}{n!}\,\frac{d^n}{dz^n} q(z)\bigg|_{z=0}\,, \quad n \geqslant 0.$$

Naturally the coefficients c_n and d_n are connected by:

$$d_n = c_{n-1} - c_n \quad , \quad n \geqslant 1. \tag{2.7}$$

From Proposition 2.1, it follows that

$$\frac{d^n}{dz^n}\Big(T_j(z)\Big)^{-p}\bigg|_{z=0} = P_n^{(-p)}\Big(t_1, t_2, \ldots, t_n\Big)$$

where

$$t_n = \begin{cases} (-1/p)_n \,, & n = 1, 2, \ldots, j \\ 0 \,, & n \geqslant j+1 \end{cases}. \tag{2.8}$$

Corollary 2.1 ensures $\frac{d^n}{dz^n}\Big(T_j(z)\Big)^{-p}\Big|_{z=0} = (1)_n = n!$ for $n = 1,2,\ldots,j$, providing $c_n = 1$ for $n = 1,2,\ldots,j$, whence (2.4).

For the sake of simplicity, from this point forward we will use the notation $g^{(i)}(a) := \frac{d^i}{dz^i} g(z)\Big|_{z=a}$.

After a single differentiation of $q(z)$ defined in (2.6), we have

$$T_j(z)\; q'(z) = -p\; T_j'(z)\; q(z)\;.$$

In order to infer about the kth-order derivative of $q(z)$ for any $k \geqslant j+1$, we consider the k-times differentiation of this latter relation, which according to the Leibniz rule yields

$$\sum_{\sigma=0}^{j} \binom{k}{\sigma} T_j^{(\sigma)}(z)\; q^{(k-\sigma+1)}(z) = -p \sum_{\sigma=0}^{j-1} \binom{k}{\sigma} T_j^{(\sigma+1)}(z)\; q^{(k-\sigma)}(z) \tag{2.9}$$

because $T_j(z)$ is a polynomial of degree j (which implies $\frac{d^\sigma}{dz^\sigma} T_j(z) = 0$ for any $\sigma \geqslant j+1$). The relation (2.9) may be equivalently written

$$T_j(z) q^{(k+1)}(z) = \sum_{\sigma=0}^{j-1} \binom{k}{\sigma} \frac{-(k-\sigma+p(\sigma+1))}{\sigma+1}\; T_j^{(\sigma+1)}(z)\; q^{(k-\sigma)}(z)\;.$$

Now, by taking $z \to 0$, we have

$$q^{(k+1)}(0) = \sum_{\sigma=0}^{j-1} \binom{k}{\sigma} \frac{-(k-\sigma+p(\sigma+1))}{\sigma+1}\; (-1/p)_{\sigma+1}\; q^{(k-\sigma)}(0)\;.$$

Since $q^{(\tau)}(0) = \tau!\; c_\tau$ for any $\tau \geqslant 1$, we finally find a recursive relation of order j for the coefficients c_τ:

$$c_{k+1} = \frac{1}{k+1} \sum_{\sigma=0}^{j-1} \frac{-(k-\sigma+p(\sigma+1))}{(\sigma+1)!}\; (-1/p)_{\sigma+1}\; c_{k-\sigma}. \tag{2.10}$$

In particular,

$$c_{j+1} = \frac{1}{k+1} \sum_{\sigma=0}^{j-1} \frac{-(k-\sigma+p(\sigma+1))\,(-1/p)_{\sigma+1}}{(\sigma+1)!} = 1 - \frac{(1-1/p)_j}{(j+1)!}$$

and we have $0 < c_{j+1} < 1$. Proceeding by finite induction, we observe that

$$0 < c_k \leqslant 1\;,\; \text{for any}\;\; k \geqslant 0. \tag{2.11}$$

Now, based on (2.7) and (2.10), we have

$$\begin{aligned} d_{k+1} &= -c_{k+1}+c_k \\ &= \frac{1}{k+1}\sum_{\sigma=0}^{j-1}\frac{-(k-\sigma+p(\sigma+1))}{(\sigma+1)!}\,(-1/p)_{\sigma+1}\,(-c_{k-\sigma}+c_{k-\sigma-1}) \\ &\quad+\sum_{\sigma=0}^{j-1}\frac{-(-1/p)_{\sigma+1}}{(\sigma+1)!}\left(\frac{k-1-\sigma+p(\sigma+1)}{k}-\frac{k-\sigma+p(\sigma+1)}{k+1}\right)c_{k-\sigma-1} \end{aligned}$$

therefore

$$\begin{aligned} d_{k+1} &= \frac{1}{k+1}\sum_{\sigma=0}^{j-1}\frac{-(-1/p)_{\sigma+1}\,(k-\sigma+p(\sigma+1))}{(\sigma+1)!}\,d_{k-\sigma} \\ &\quad+\sum_{\sigma=0}^{j-1}\frac{-(-1/p)_{\sigma+1}\,(p-1)}{\sigma!\,k\,(k+1)}\,c_{k-\sigma-1}. \end{aligned} \tag{2.12}$$

In particular,

$$d_{j+1}=1-c_{j+1}=\frac{(1-1/p)_j}{(j+1)!}>0\,.$$

Insofar as $c_k>0$ for any $k\geqslant 0$, then from (2.12) we obtain

$$d_{k+1}>\frac{1}{k+1}\sum_{\sigma=0}^{j-1}\frac{-(-1/p)_{\sigma+1}\,(k-\sigma+p(\sigma+1))}{(\sigma+1)!}\,d_{k-\sigma}+0\geqslant 0\quad,\quad k\geqslant 0.$$

Now, on account of (2.4), the assumption over the positiveness of d_k for any $k\geqslant j+1$ permits to infer from this latter inequality that (2.5) holds.

Finally, we show the convergence of the sequence of the partial sums $\left\{S_n=\sum_{k=0}^{n}d_k\right\}_{n\geqslant 0}$. The equality (2.7) compels $S_n=c_0-c_n=1-c_n$. Thus $\{S_n\}_{n\geqslant 0}$ is a convergent sequence as long as $\{c_n\}_{n\geqslant 0}$ is. Following (2.5) and (2.11), $\{c_n\}_{n\geqslant 0}$ is a decreasing and bounded sequence and therefore converges. By virtue of (2.10), we have $\lim_k c_k=0$ and consequently

$$\lim_n\sum_{k=0}^{n}d_k=1\,.$$

On the basis of Abel's lemma, (2.11) implies the absolute convergence of $\sum_{k\geqslant 0}c_kz^k$ for any z such that $|z|<1$. Likewise, the convergence of $\sum_{k\geqslant 0}d_k$ implies the absolute convergence of the series $\sum_{k\geqslant 0}d_kz^k$ for any $z\in\mathbb{C}$ for which $|z|<1$.

Remark. It is worth to note that, according to (2.10), the nth order derivative of $(T_j(z))^{-p}$ at the point $z=0$, say $P_n^{(-p)}(t_1,t_2,\ldots,t_n)$ with $t_i=\frac{d^i}{dz^i}T_j(z)\Big|_{z=0}$ given by

(2.8), fulfill the jth order recurrence relation

$$P_n^{(-p)}(t_1,\ldots,t_n) = \begin{cases} n! & , n=1,\ldots,j \\ \sum_{\nu=0}^{j-1} \binom{n-1}{\nu} \frac{-(-1/p)_{\nu+1}(n-1-\nu+p(\nu+1))}{\nu+1} P_{n-\nu-1}^{(-p)}(t_1,\ldots,t_{n-\nu-1}) & , n \geqslant j+1 \,. \end{cases}$$

As far as we are concerned this is a new expression for the derivatives of such composite function.

3 Convergence regions of Schröder iteration functions

Due to Theorem 2.1, we are able to derive a condition on the initial guess that provide the convergence of the Schröder iteration functions (1.3) to a pth root of w.

Corollary 3.1 *Consider the residual function $R(z)$ defined in (1.7) and let $\{z_k\}_{k=0}^{\infty}$ the sequence defined by (1.4). If*
$|R(z_0)| = |1-wz_0^{-p}| < 1$, then, for any j,

(i) $|R(z_{k+1})| \le |R(z_k)|^j \le |R(z_0)|^{j^k}$, for all $k=1,2,\ldots$;
(ii) the sequence (1.4) converges to a pth root of w with order of convergence j.

PROOF.

(i) From (1.8) and Theorem 2.1 the following holds:

$$\begin{aligned} |R(z_1)| &= \left|1-(1-R(z_0))\Big(T_j(R(z_0))\Big)^{-p}\right| = \left|\sum_{n\ge 0} d_n\,(R(z_0))^n\right| \\ &\le |R(z_0)|^j \sum_{n\ge j+1} d_n\,|R(z_0)|^{n-j} \le |R(z_0)|^j \sum_{n\ge j+1} d_n = |R(z_0)|^j. \end{aligned}$$

Now the result follows by finite induction.

(ii) The result follows using statement (i) and proceeding in a similar way as in the proof of [4, Lem. 3.2].

Proposition 3.1 *For each $n=0,1,\ldots,p-1$, consider the sets*

$$\mathscr{S}_n = \left\{ z\in\mathbb{C} : |1-wz^{-p}| < 1 \wedge \frac{(2n-1)\pi+\theta}{p} < \arg(z) < \frac{(2n+1)\pi+\theta}{p} \right\}. \tag{3.1}$$

If $z_0 \in \mathscr{S}_n$ then for any j the sequence (1.4) converges with order j to the unique pth root of w which lies on the wedge $\mathscr{W}_n$ defined in (1.2).

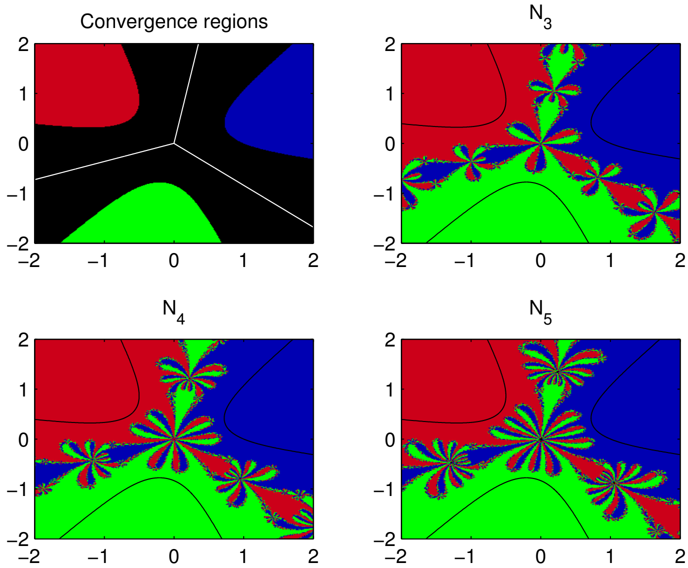

Fig. 1. Convergence regions defined by the residuals associated to N_j (top left) and basins of attraction of the 3rd roots of $w = 1/2 + i\sqrt{3}/2$ for N_3, N_4 and N_5.

PROOF. On account of Corollary 3.1 and Lemma 3.3 in [4], the proof is analogous to the one of Theorem 3.3 in [4].

The following proposition gives a condition for the convergence in the particular case when (1.3) is restricted to real numbers.

Proposition 3.2 *Assume that w is a positive real number and let z_0 be a real number such that*

$$z_0 > \left(\frac{w}{2}\right)^{1/p}.$$

Then for each j the sequence (1.4) converges with order j to the real positive pth root of w.

PROOF. Similar to the proof of [4, Cor. 3.3].

Figure 1 displays the convergence regions defined in Proposition 3.1 together with the basins of attraction of the 3rd roots of $w = 1/2 + i\sqrt{3}/2$. We recall that the basin of attraction of a given 3rd root $\tilde{w}$ of w is the set of initial values for which the sequence (1.4) converges to $\tilde{w}$. The top-left plot displays the sets $\mathscr{S}_0, \mathscr{S}_1$ and $\mathscr{S}_2$ defined in (3.1) with different colors together with the boundaries of the sets $\mathscr{W}_n$ defined in (1.2) which are the rays

$$\mathscr{R}_n = \left\{ z \in \mathbb{C} : \ \arg(z) = \frac{(2n-1)\pi + \theta}{p} \right\},$$

($n = 0, 1, 2$). For each n, $\mathscr{S}_n$ corresponds to a specific 3rd root of w and according to the previous results they define convergence regions for the iterations N_j, i. e., they are contained in the basins of attraction. The plots on the top–right, bottom–left and bottom–right show Matlab generated plots of the basins of attraction of each 3rd root of w associated to N_3, N_4 and N_5, respectively. A point z_0 in the rectangle $[-2,2] \times [-2,2]$ is marked with the same color of the 3rd root $\tilde{w}$ of w, whenever $|z_{50} - \tilde{w}| < 10^{-4}$. We have also overlapped the boundaries of the regions of convergence $\mathscr{S}_n$ on the basins of attraction. Black color is assigned to points that are not in $\mathscr{S}_n$ (top-left) or that do not belong to the basins of attraction of the pth roots of w (remaining plots).

We can observe that the convergence regions do not intersect the boundaries of the basins of attractions, which are Julia sets. Recall that if the initial guess z_0 belongs to these boundaries then the iteration functions N_j do not converge to a pth root of w (see [15] and also [2] for more details about Julia sets).

References

[1] D. A. Bini, N. J. Higham, and B. Meini, Algorithms for the matrix pth root, *Numer. Algorithms*, 39 (2005), 349–378.

[2] P. Blanchard, Complex analytic dynamics on the Riemann sphere, *Bulletin of the A.M.S*, 11 (1984), 85–141.

[3] E. T. Bell, Partition polynomials, *Ann. Math.*, 24 (1992), 38-46.

[4] J. R. Cardoso, and A. F. Loureiro, Iteration functions for pth roots of complex numbers, to appear in *Numer. Algorithms*, `DOI:10.1007/s11075-010-9431-8`.

[5] L. Comtet, Advanced Combinatorics: the art of finite and infinite expansions, D. Reidel Publishing Co.,Dordrecht, 1974.

[6] F. Faà di Bruno, Sullo sviluppo delle Funzioni, *Annali di Scienze Matematiche e Fisiche*, 6 (1855), 479-480.

[7] C. -H. Guo, N.J. Higham, A Schur-Newton method for the matrix pth root and its inverse, *SIAM J. Matrix Anal. Appl.*, 28 (2006) 788–804.

[8] C. -H. Guo, On Newton's method and Halley's method for the principal pth root of a matrix, *Linear Algebra Appli.*, 432, 1905–1922 (2010).

[9] N. J. Higham, and L. Lin, On pth roots of stochastic matrices, *Linear Algebra Appli.*, doi:10.1016/j.laa.2010.04.007 (2010)

[10] B. Iannazzo, On the Newton method for the matrix pth root, *SIAM J. Matrix Anal. Appl.* 28 (2006) 503–523.

[11] B. Iannazzo, A family of rational iterations and its applications to the computation of the matrix pth root, *SIAM J. Matrix Anal. Appl.* 30 (2008) 1445–1462.

[12] B. Laszkiewicz and K. Ziętak, A Padé family of iterations for the matrix sector function and the matrix pth root, *Num. Lin. Algebra Appl.* 16 (2009) 951–970.

[13] J. Riordan, Combinatorial Identities, John Wiley & Sons, New York, 1968.

[14] E. Schröder, Ueber unendlich viele Algorithmen zur Auflösung der Gleichungen, *Mathematische Annalen*, 2 (1870), 317–365. English translation by G. W. Stewart: E. Schröder, On infinitely many algorithms for solving equations. 1993.

[15] E. R. Vrscay, Julia sets and Mandelbrot–like sets associated with higher order Schröder rational iterative maps: a computer assisted study, *Math. Comp.*, Vol. 46, 151–169 (1986).