On the theory of convolution integral equations related to Lebedev's type operators

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Abstract

We draw a parallel with the Gakhov- Cherskii method to investigate a class of convolution integral equations related to the Kontorovich -Lebedev and Lebedev's type transformations. A relationship with the Cauchy type integral is obtained. General convolution equation is solved being reduced to the Riemann boundary value problem by means of the Kontorovich-Lebedev transform.

Keywords: *Convolution integral equations, singular equations, Kontorovich-Lebedev transform, modified Bessel function, Fourier transform, Riemann boundary value problem, Cauchy's kernel, Banach ring*

AMS subject classification: 45A05, 45E05, 44A15, 33C10, 30E20, 30E25

1 Introduction

In this paper we will deal with an integral equation, which contains two convolution type operators, namely

$$f(t) + \lambda_1 (f * m_1)_1(t) + \lambda_2 (f * m_2)_2(t) = g(t), \quad t > 0, \tag{1.1}$$

where $\lambda_1, \ \lambda_2 \in \mathbb{C}$ are parameters, $m_1(t), m_2(t), g(t)$ are given functions, $f(t)$ is to be determined and

$$(f * m_1)_1(t) = \frac{1}{2t} \int_0^\infty \int_0^\infty e^{-\frac{1}{2}\left(t\frac{x^2+y^2}{xy} + \frac{yx}{t}\right)} f(x) m_1(y) dx dy, \tag{1.2}$$

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$$(f * m_2)_2(t) = \int_0^\infty \int_0^\infty \mathcal{K}(x, y, t) f(x) m_2(y) dx dy \tag{1.3}$$

with

$$\mathcal{K}(x, y, t) = y \int_1^\infty e^{-vx} \frac{K_1(\sqrt{t^2 + y^2 + 2tyv})}{\sqrt{t^2 + y^2 + 2tyv}} dv, \tag{1.4}$$

where $K_\nu(z)$ the modified Bessel function [1]. As we will see below operator (1.2) can be factorized by the Kontorovich - Lebedev transform (see [10], [11], [12])

$$K_{ix}[f] = \int_0^\infty K_{ix}(t) f(t) dt, \ x \in \mathbb{R}, \tag{1.5}$$

with the kernel

$$K_{ix}(t) = \frac{1}{2} \int_{-\infty}^\infty e^{-t \cosh u} e^{iux} du, \ t > 0, \tag{1.6}$$

and operator (1.3) will be treated with the aid of the auxiliary transformation $M_{ix}[f]$ (see [9], [16])

$$M_{ix}[f] = \int_0^\infty M_{ix}(t) f(t) dt, \ x \in \mathbb{R}, \tag{1.7}$$

which involves the kernel $M_{ix}(t)$ related to the Bessel functions (cf. [6])

$$M_{ix}(t) = \int_0^\infty e^{-t \cosh u} \sin xu \ du, \ t > 0. \tag{1.8}$$

Equation (1.1) will be investigated in a certain class of functions, which is related to mapping properties of transformations (1.5), (1.7). We will apply the so-called Gakhov-Cherskii method (see [4]) reducing this equation to the Riemann boundary value problem [3] for the half-plane. Such an approach was used in [4] to investigate a class of the Fourier type convolution integral equations. This scheme has been also considered formally in [5] for a similar equation to (1.1) from the intersection of various weighted L_2-spaces. Concerning convolution integral equations of the first kind, which are associated with the Kontorovich-Lebedev transform, see [13], [15].

As it is known, the modified Bessel function $K_\nu(z)$ satisfies the differential equation

$$z^2 \frac{d^2u}{dz^2} + z \frac{du}{dz} - (z^2 + \nu^2) u = 0, \tag{1.9}$$

for which it is the solution that remains bounded as z tends to infinity on the real line. It has the asymptotic behaviour (see [1], relations (9.6.8), (9.6.9), (9.7.2))

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} [1 + O(1/z)], \qquad z \to \infty, \tag{1.10}$$

and near the origin

$$K_\nu(z) = O\left(z^{-|\mathrm{Re}\nu|}\right),\ z \to 0, \tag{1.11}$$

$$K_0(z) = -\log z + O(1),\ z \to 0. \tag{1.12}$$

When $x \in \mathbb{R}$, then (see (1.6)) $K_{ix}(t)$ is real-valued and even with respect to the pure imaginary index ix. Furthermore, this integral can be extended to the strip $\delta \in [0, \pi/2)$ in the upper half-plane, i.e.

$$K_{ix}(t) = \frac{1}{2}\int_{i\delta-\infty}^{i\delta+\infty} e^{-t\cosh z + ixz} dz,$$

and leads for each $t > 0$ to a uniform estimate

$$|K_{ix}(t)| \le e^{-|x|\arccos\beta} K_0(\beta t), \quad 0 < \beta \le 1. \tag{1.13}$$

For a product of the modified Bessel functions of different arguments the Macdonald formula is true [5, Vol. II, relation (2.16.9.1)]

$$K_\nu(x)K_\nu(y) = \frac{1}{2}\int_0^\infty e^{-\frac{1}{2}\left(t\frac{x^2+y^2}{xy} + \frac{yx}{t}\right)} K_\nu(t)\frac{dt}{t}. \tag{1.14}$$

Meanwhile, kernel (1.8) has a relationship with the modified Bessel function by means of the following integral representation [9]

$$\frac{\pi}{\sinh \pi\tau} M_{i\tau}(t) = \int_0^\infty \frac{e^{-t-y}}{t+y} K_{i\tau}(y)dy,\ \tau \in \mathbb{R}\backslash\{0\}, \tag{1.15}$$

which will be used in the sequel. In particular, substituting in the left-hand side of (1.15) the value of $M_{i\tau}(t)$ in terms of the integral (1.8), we let $\tau \to 0$ through (1.15) via the absolute and uniform convergence to find the result

$$\lim_{\tau\to 0}\frac{\pi M_{i\tau}(t)}{\sinh \pi\tau} = \int_0^\infty \frac{e^{-t-y}}{t+y} K_0(y)dy = \int_0^\infty e^{-t\cosh u} u du,\ t > 0, \tag{1.16}$$

which defines (1.15) for all $\tau \in \mathbb{R}$. Finally in this section, putting $\nu = i\tau$ in (1.14) we multiply both sides of this equality on $\frac{e^{-t-y}}{t+y}$ and we integrate with respect to y. Changing the order of integration by Fubini's theorem and employing (1.15) we derive

$$\frac{\pi}{\sinh \pi\tau} M_{i\tau}(t)K_{i\tau}(x) = \frac{1}{2}\int_0^\infty\int_0^\infty e^{-\frac{1}{2}\left(u\frac{x^2+y^2}{xy} + \frac{yx}{u}\right) - t - y} K_{i\tau}(u)\frac{du dy}{u(t+y)}$$

$$= \frac{1}{2}\int_0^\infty\int_0^\infty\int_1^\infty e^{-\frac{1}{2}\left(y\frac{x^2+u^2}{xu} + \frac{ux}{y}\right) - v(t+y)} \frac{K_{i\tau}(u)}{u} du dy dv$$

$$= \frac{1}{2}\int_0^\infty \frac{K_{i\tau}(u)}{u}\int_1^\infty e^{-vt}\int_0^\infty e^{-\frac{1}{2}\left(y\frac{x^2+u^2+2vxu}{xu}+\frac{ux}{y}\right)}dydvdu. \tag{1.17}$$

The inner integral with respect to y is calculated by relation (2.3.16.1) in [7, Vol. I]. Therefore we come out with the following integral representation

$$\frac{\pi}{\sinh \pi\tau}M_{i\tau}(t)K_{i\tau}(x) = x\int_0^\infty K_{i\tau}(u)\int_1^\infty e^{-vt}\frac{K_1\left(\sqrt{x^2+u^2+2vxu}\right)}{\sqrt{x^2+u^2+2vxu}}dvdu$$

$$= \int_0^\infty K_{i\tau}(u)\mathcal{K}(t,x,u)du, \ \tau \in \mathbb{R}, \tag{1.18}$$

where $\mathcal{K}(t,x,u)$ is defined by (1.4). This kernel can be written in terms of the inversion formula for the Kontorovich-Lebedev transformation (1.5) (see [11, Chapter 2]). Thus we obtain

$$\mathcal{K}(t,x,u) = \frac{1}{\pi u}\int_{-\infty}^\infty \tau M_{i\tau}(t)K_{i\tau}(x)K_{i\tau}(u)d\tau, \ t,x,u > 0. \tag{1.19}$$

2 Key properties of the Kontorovich-Lebedev transform. A relationship with the Cauchy type integral

In this section we will give necessary mapping properties of the Kontorovich-Lebedev transform (1.5) in the Lebesgue spaces, which we will use to establish a solvability theory for integral equation (1.1). In particular, operator (1.5) is well defined in the Banach ring $L^\alpha \equiv L_1(\mathbb{R}_+; K_\alpha(t)\ dt), \alpha \in \mathbb{R}$ (see [10], [11], [14]), i.e. the space of all summable functions $f : \mathbb{R}_+ \to \mathbb{C}$ with respect to the measure $K_\alpha(t)\ dt$ for which

$$||f||_{L^\alpha} = \int_0^\infty |f(t)|K_\alpha(t)dt \tag{2.1}$$

is finite. It is shown (see [11, Chapter 4]) that the operation of multiplication for two elements f, g of the ring L^α is the convolution (1.2) $(f * g)_1$. Moreover, the Macdonald formula (1.14) is used to prove the factorization property for the convolution (1.2) in terms of the Kontorovich-Lebedev transform (1.5) in the space L^α, namely

$$K_{ix}[(f*g)_1] = K_{ix}[f]K_{ix}[g], \quad x \in \mathbb{R}, \tag{2.2}$$

where the integral (1.5) exists as a Lebesgue integral. It is also proved, that the Kontorovich-Lebedev transformation is a bounded operator from L^α into the space $C_0(\mathbb{R})$ of bounded continuous functions on $\mathbb{R}$ vanishing at infinity, admitting the following composition representation

$$K_{ix}[f] = \sqrt{\frac{\pi}{2}}\left(Fh\right)(x), \tag{2.3}$$

as a Fourier transform [8] of the the function $h(u) = \int_0^\infty e^{-t\cosh u} f(t)dt \in L_1(\mathbb{R}_+; dt)$. The latter fact can be easily done by the estimate (see (2.1))

$$\int_0^\infty |h(u)|du \le \int_0^\infty |f(t)| \int_0^\infty e^{-t\cosh u} du dt = \int_0^\infty |f(t)| K_0(t) dt$$

$$\le \int_0^\infty |f(t)| K_\alpha(t) dt < \infty.$$

Furthermore, the convolution (1.2) of two functions $f, g \in L^\alpha$ exists as a Lebesgue integral and belongs to L^α. It satisfies the Young type inequality

$$||(f * g)_1||_{L^\alpha} \le ||f||_{L^\alpha} ||g||_{L^\alpha}. \tag{2.4}$$

Next, we will calculate the transform (1.5) of the convolution $(f * m_2)_2(t)$. Indeed, taking into account (1.3), (1.4), we change the order of integration by Fubini's theorem and by using (1.18) we obtain the equality

$$K_{ix}[(f * m_2)_2] = \frac{\pi}{\sinh \pi x} \int_0^\infty M_{ix}(t) f(t) dt \int_0^\infty K_{ix}(y) m_2(y) dy$$

$$= \frac{\pi}{\sinh \pi x} M_{ix}[f] K_{ix}[m_2], \ x \in \mathbb{R}. \tag{2.5}$$

The motivation of this interchange can be done for any $f, m_2 \in L^\alpha$, $|\alpha| \ge 1$. But first we appeal to (1.8), (1.13), (1.16), (1.17) to find the estimate

$$\sup_{x\in\mathbb{R}} \left| \frac{\pi}{\sinh \pi x} M_{ix}(t) K_{ix}(y) \right| \le \sup_{x\in\mathbb{R}} \left| \frac{\pi x}{\sinh \pi x} \right| K_0(y) \int_0^\infty e^{-t\cosh u} u \, du$$

$$\le e^{-t} K_0(y) \int_0^\infty e^{-2t\sinh^2(u/2)} u du \le e^{-t} K_0(y) \int_0^\infty e^{-tu^2/2} \, u du = \frac{e^{-t}}{t} K_0(y).$$

Therefore the iterated integral in the right-hand side of (2.5) converges absolutely and uniformly with respect to x. Precisely, we have for $x \in \mathbb{R}$ (see (1.10), (1.11))

$$\frac{\pi}{|\sinh \pi x|} \int_0^\infty |M_{ix}(t) f(t)| dt \int_0^\infty |K_{ix}(y) m_2(y)| dy \le \int_0^\infty \frac{e^{-t}}{t} |f(t)| dt \int_0^\infty K_0(y) |m_2(y)| dy$$

$$\le \sup_{t>0} \left[\frac{e^{-t}}{K_\alpha(t) \, t} \right] \, ||f||_{L^\alpha} ||m_2||_{L^\alpha} < \infty, \ |\alpha| \ge 1,$$

which motivates the proof of equality (2.5).

Considering, in turn, the Kontorovich-Lebedev integral (1.5) in the case $f(t) \in L_2(\mathbb{R}_+; tdt)$, i.e.

$$||f||_{L_2(\mathbb{R}_+;tdt)} = \left(\int_0^\infty |f(t)|^2 tdt\right)^{1/2} < \infty \tag{2.6}$$

it is not difficult to verify that it generally, does not exist in Lebesgue's sense (take, for instance

$$f(t) = \begin{cases} \frac{1}{t\log t}, & \text{if } 0 < t \leq \frac{1}{2}, \\ 0, & \text{if } t > \frac{1}{2}, \end{cases}$$

and use asymptotic formula (1.11)). Thus we define it in the form

$$K_{ix}[f] = \lim_{N\to\infty} \int_{1/N}^\infty K_{ix}(t) f(t)\ dt \tag{2.7}$$

with the necessary truncation in the origin, where the limit is taken in the mean square sense with respect to the norm of the space $L_2(\mathbb{R}; x\sinh \pi x\ dx)$. It has been proved (see [11, Chapter 2], [17]) that the range $K_{ix}(L_2(\mathbb{R}_+; tdt))$ coincides with the subspace of even functions in the weighted Hilbert space $L_2(\mathbb{R}; x\sinh \pi x\ dx)$. Operator (2.7) is bounded and its square of the norm satisfies the Parseval identity of the form

$$\int_{-\infty}^\infty x\sinh \pi x |K_{ix}[f]|^2 dx = \pi^2 \int_0^\infty t|f(t)|^2 dt. \tag{2.8}$$

More generally, it gives

$$\int_{-\infty}^\infty x\sinh \pi x K_{ix}[f] K_{ix}[g] dx = \pi^2 \int_0^\infty tf(t)g(t)dt, \tag{2.9}$$

where $f, g \in L_2(\mathbb{R}_+; tdt)$. The two definitions (1.5) and (2.6) of the Kontorovich-Lebedev operator are equivalent, if we take $f \in L_2^\alpha \equiv L_2(\mathbb{R}_+; tdt) \cap L^\alpha$. The inverse operator in the latter case is given by the formula $f(t) = \lim_{N\to\infty} f_N(t)$, where

$$f_N(t) = \frac{1}{\pi^2} \int_{-N}^N x\sinh \pi x \frac{K_{ix}(t)}{t} K_{ix}[f] dx \tag{2.10}$$

with the necessary truncation at infinity and the convergence is in the mean square sense with respect to the norm (2.4) of $L_2(\mathbb{R}_+; tdt)$. Denoting by

$$\mathcal{K}L_{ix} \equiv \{G \in K_{ix}(L_2(\mathbb{R}_+; tdt));\ G(x) = K_{ix}[f], f \in L_2^\alpha\}$$

a set of images under the Kontorovich-Lebedev transform (1.5), which, in turn, is a subspace of $K_{ix}(L_2(\mathbb{R}_+; tdt))$, we will consider a restriction of this map to $K_{ix} : L_2^\alpha \to \mathcal{K}L_{ix}$. As we see above, for instance, it has $\mathcal{K}L_{ix} \subset C_0(\mathbb{R})$.

Let us consider operator (1.5) of the complex variable

$$K_{iz}[f] = \int_0^\infty K_{iz}(t)f(t)dt, \quad z \in \mathbb{C}. \tag{2.11}$$

It is not difficult to prove, that if $f \in L^\alpha$, then $K_{iz}[f]$ is analytic in the horizontal strip $|\mathrm{Im}\ z| \le |\alpha|$. Indeed, via (1.6) we observe, that $K_{iz}(t)$ is entire with respect to z and $|K_{iz}(t)| \le K_{\mathrm{Im}\ z}(t) \le K_\alpha(t)$. Moreover, integral (2.11) is convergent absolutely and uniformly in the strip $|\mathrm{Im}\ z| \le |\alpha|$, representing there an analytic function. In particular, when $\alpha = 0$, we find that $K_{ix}[f]$ is infinite times continuously differentiable on the real axis. Furthermore, it satisfies there the Hölder condition [3] of any $\lambda,\ 0 < \lambda \le 1$.

Let us establish a relationship of the integral (2.11) and the Cauchy type integral over real axis with the density function $K_{i\tau}[f],\ \tau \in \mathbb{R}$ [3]. Assuming that $f(t) \in L_2^\alpha$, it is easily seen from discussions above that the Kontorovich-Lebedev transform $K_{i\tau}[f] \in C_0(\mathbb{R}) \cap\ L_2(\mathbb{R}; \tau \sinh \pi\tau\ d\tau)$. Therefore it belongs to $L_2(\mathbb{R};\ d\tau)$ and via representation (2.3) and the Parseval equality for the Fourier transform [8] we obtain

$$\frac{1}{2\pi i}\int_{-\infty}^\infty \frac{K_{i\tau}[f]}{\tau - z} d\tau = \frac{1}{2}\int_{-\infty}^\infty h(u)\ \frac{1}{2\pi i}\int_{-\infty}^\infty \frac{e^{i\tau u}}{\tau - z} d\tau du$$

$$= \frac{1}{2}\left[\int_{-\infty}^0 + \int_0^\infty\right] h(u)\ \frac{1}{2\pi i}\int_{-\infty}^\infty \frac{e^{i\tau u}}{\tau - z} d\tau du.$$

Hence accounting the value of the inner integral with respect to τ (see, for instance, in [4]) we come out with the equalities

$$\frac{1}{2\pi i}\int_{-\infty}^\infty \frac{K_{i\tau}[f]}{\tau - z} d\tau = \frac{1}{2}\int_0^\infty h(u)e^{izu}du,\ \mathrm{Im} z > 0, \tag{2.12}$$

$$\frac{1}{2\pi i}\int_{-\infty}^\infty \frac{K_{i\tau}[f]}{\tau - z} d\tau = -\frac{1}{2}\int_0^\infty h(u)e^{-izu}du,\ \mathrm{Im} z < 0, \tag{2.13}$$

where

$$h(u) = \int_0^\infty e^{-t\cosh u} f(t)dt.$$

Substituting this value into (2.12), (2.13), we change the order of integration by Fubini's theorem since

$$\int_0^\infty \left|\int_0^\infty e^{-t\cosh u} f(t)\ dt\right| e^{\pm izu} du\ dt \le \int_0^\infty \int_0^\infty e^{-t\cosh u - |\mathrm{Im} z| u} |f(t)|\ dtdu$$

$$\le \int_0^\infty \int_0^\infty e^{-t\cosh u} |f(t)|\ dtdu = \int_0^\infty K_0(t)|f(t)|dt \le ||f||_{L^\alpha} < \infty.$$

Then taking into account relations (1.6), (1.7), (1.8), (2.11) we can write (2.12), (2.13) in the form

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{K_{i\tau}[f]}{\tau - z}d\tau = \frac{1}{2}\left[K_{iz}[f] + iM_{iz}[f]\right],\ \mathrm{Im} z > 0, \tag{2.14}$$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{K_{i\tau}[f]}{\tau - z}d\tau = -\frac{1}{2}\left[K_{iz}[f] - iM_{iz}[f]\right],\ \mathrm{Im} z < 0. \tag{2.15}$$

Consequently, since the Cauchy type integral represents a piecewise analytic function

$$G(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{K_{i\tau}[f]}{\tau - z}d\tau = \begin{cases} G^{+}(z), & \text{if } \mathrm{Im} z > 0,\\ G^{-}(z), & \text{if } \mathrm{Im} z < 0,\end{cases} \tag{2.16}$$

in the cut z- plane along the real axis, we have from (2.14), (2.15) that functions

$$G^{+}(z) = \frac{1}{2}\left[K_{iz}[f] + iM_{iz}[f]\right], \tag{2.17}$$

$$G^{-}(z) = -\frac{1}{2}\left[K_{iz}[f] - iM_{iz}[f]\right] \tag{2.18}$$

are analytic in the upper and lower half-plane, respectively. Moreover, the Sokhotski formulas take place for the limit values on the real axis

$$G^{+}(x) = \frac{1}{2}\left[K_{ix}[f] + iM_{ix}[f]\right] = \frac{1}{2}\left[K_{ix}[f] + \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{K_{i\tau}[f]}{\tau - x}d\tau\right], \quad x \in \mathbb{R}, \tag{2.19}$$

$$G^{-}(x) = -\frac{1}{2}\left[K_{ix}[f] - iM_{ix}[f]\right] = -\frac{1}{2}\left[K_{ix}[f] - \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{K_{i\tau}[f]}{\tau - x}d\tau\right], \quad x \in \mathbb{R}, \tag{2.20}$$

which are equivalent to the following relations

$$G^{+}(x) - G^{-}(x) = K_{ix}[f], \quad x \in \mathbb{R}, \tag{2.21}$$

$$G^{+}(x) + G^{-}(x) = iM_{ix}[f] = \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{K_{i\tau}[f]}{\tau - x}d\tau, \quad x \in \mathbb{R}. \tag{2.22}$$

Besides, since $K_{ix}[f] \in C_0(\mathbb{R})$, it gives the condition $\lim_{|x|\to\infty} K_{ix}[f] = 0$.

Definition [2]. *A function $G^{+}(z)$ $(G^{-}(z))$ $(z = x + iy)$ belongs to the Hardy class $\mathbb{H}_2^{+}$ $(\mathbb{H}_2^{-})$ if it is analytic in the upper (lower) half- plane $y > 0$ $(y < 0)$ and satisfies the inequality*

$$\sup_{y>0\ (y<0)}\int_{-\infty}^{\infty}\left|G^{(\pm)}(x+iy)\right|^2 dx < \infty.$$

We are ready to prove the following result.

Theorem 1. *A function $G(x) \in \mathcal{K}L_{ix}$ is a limit value of $G^+(z) \in \mathbb{H}_2^+$ $(G^-(z) \in \mathbb{H}_2^-)$ if and only if $K_{ix}[f] = (\pm)\ iM_{ix}[f]$.*

Proof. *Necessity.* Indeed, let us suppose that $G(x)$ is a limit value of $G^+(z) \in \mathbb{H}_2^+$. Since $G(x)$ is from the class $\mathcal{K}L_{ix}$, we have $G(x) = K_{ix}[f],\ f \in L_2^\alpha$. Meanwhile (see [2], [8]), $G^+(z)$ is representable in the upper half-plane in terms of the Cauchy type integral (2.16) and hence $G^+(z) = K_{iz}[f]$. Thus $G^+(x) = K_{ix}[f]$ and from (2.17) we find the condition $K_{ix}[f] = iM_{ix}[f]$. If, in turn, $G(x) = K_{ix}[f]$ is a limit value of $G^-(z)$ in the lower half-plane, then the representation by the Cauchy type integral (2.16) gives the equality $G^-(z) = -K_{iz}[f]$. Consequently, $G^-(x) = -K_{ix}[f]$ and from (2.18) we deduce $K_{ix}[f] = -iM_{ix}[f]$.

Sufficiency. Conversely, if $K_{ix}[f] = iM_{ix}[f]$, then from Sokhotski's formulas (2.21), (2.22) we get $G^-(x) = 0$ and therefore $G^+(x) = K_{ix}[f] = G(x)$ is a limit value of $G^+(z)$. In the case of $K_{ix}[f] = -iM_{ix}[f]$ we find from the same relations that $G^+(x) = 0$ and therefore, $G^-(x) = -K_{ix}[f]$ is a limit value of $G^-(z)$. Theorem 1 is proved.

3 A solvability of convolution type equation (1.1) in the class L_2^α

We begin to consider a simple case of equation (1.1) letting $\lambda_2 = 0$. So we have a convolution type integral equation with operator (1.2)

$$f(t) + \lambda_1 (f * m_1)_1(t) = g(t), \quad t > 0, \tag{3.1}$$

where $\lambda_1 \neq 0$, $m_1(t),\ g(t)$ are given functions in the class L_2^α. We seek a solution in the same class L_2^α. Taking the operator (2.11) of the Kontorovich-Lebedev transformation from both sides of (3.1) we use the factorization property (2.2) and we come out with the algebraic equation with respect to $K_{iz}[f]$

$$K_{iz}[f]\,(1 + \lambda_1\ K_{iz}[m_1]) = K_{iz}[g],\ |\mathrm{Im} z| \leq |\alpha|. \tag{3.2}$$

Assuming the normality condition

$$1 + \lambda_1\ K_{iz}[m_1] \neq 0,\ |\mathrm{Im} z| \leq |\alpha|, \tag{3.3}$$

the unique solution of (3.2) is

$$K_{iz}[f] = \frac{K_{iz}[g]}{1 + \lambda_1\ K_{iz}[m_1]},\ |\mathrm{Im} z| \leq |\alpha|. \tag{3.4}$$

But the Wiener type theorem for the Kontorovich-Lebedev transform (see [11, Theorem 4.15]) says, that there exists a unique element $q(t)$ of the Banach ring L^α (see (2.1)) such that

$$\frac{1}{1 + \lambda_1\ K_{iz}[m_1]} = 1 + \lambda_1 K_{iz}[q],\ |\mathrm{Im} z| \leq |\alpha|. \tag{3.5}$$

Therefore (3.4) becomes

$$K_{iz}[f] = K_{iz}[g]\left(1 + \lambda_1 K_{iz}[q]\right). \tag{3.6}$$

Letting $z = x \in \mathbb{R}$ in (3.6) we observe that since $K_{ix}[g] \in KL_{ix}$ and $1 + \lambda_1 K_{ix}[q]$ is bounded, then the right-hand side of (3.6) belongs to $L_2(\mathbb{R}; x \sinh \pi x\, dx)$. Thus we have $K_{ix}[f] \in L_2(\mathbb{R}; x \sinh \pi x\, dx)$ and by virtue of inversion (2.10) it defines reciprocally a unique solution $f(t) \in L_2(\mathbb{R}_+; tdt)$ by the formula

$$f(t) = \lim_{N\to\infty} \frac{1}{\pi^2} \int_{-N}^{N} x \sinh \pi x \frac{K_{ix}(t)}{t} K_{ix}[g]\left(1 + \lambda_1 K_{ix}[q]\right) dx, \ t > 0. \tag{3.7}$$

However, our goal is to show that $f(t) \in L_2^\alpha$. In fact, relation (3.7), factorization equality (2.2) being written for $g, q \in L^\alpha$ and the boundedness of $K_{ix}[q]$ will guarantee the property $(g * q)_1(t) \in L_2^\alpha$. Moreover, from (3.7) we deduce

$$f(t) = g(t) + \lim_{N\to\infty} \frac{\lambda_1}{\pi^2} \int_{-N}^{N} x \sinh \pi x \frac{K_{ix}(t)}{t} K_{ix}[g] K_{ix}[q] dx$$

$$= g(t) + \lim_{N\to\infty} \frac{\lambda_1}{\pi^2} \int_{-N}^{N} x \sinh \pi x \frac{K_{ix}(t)}{t} K_{ix}[(g * q)_1] dx$$

$$= g(t) + \lambda_1 (g * q)_1(t), \quad t > 0.$$

Hence

$$f(t) = g(t) + \lambda_1 (g * q)_1(t), \quad q \in L^\alpha \tag{3.8}$$

is the desired unique L_2^α-solution of equation (3.1) and we have proved the following

Theorem 2. *Under normality condition* (3.3) *there exists a unique solution of the convolution integral equation* (3.1) *in the class* $L_2^\alpha,\ \alpha \in \mathbb{R}$ *given by formula* (3.8).

Let us consider convolution integral equation (1.1), where $\lambda_1, \lambda_2 \in \mathbb{R}\backslash\{0\}, m_1(t), m_2(t)$ and $g(t)$ are given functions in the class $L_2^\alpha,\ |\alpha| \geq 1$, assuming that $m_1(t), m_2(t)$ are real-valued. We will seek a solution in the same class. In fact, taking the Kontorovich-Lebedev transform (1.5) from both sides of (3.9) and invoking relations (2.2), (2.5) we obtain

$$K_{ix}[f]\left(1 + \lambda_1 K_{ix}[m_1]\right) + \frac{\lambda_2 \pi}{\sinh \pi x} M_{ix}[f] K_{ix}[m_2] = K_{ix}[g], \ x \in \mathbb{R}. \tag{3.9}$$

But relations (2.21), (2.22) yield the equation

$$(G^+(x) - G^-(x))\left(1 + \lambda_1 K_{ix}[m_1]\right) - i(G^+(x) + G^-(x)) \frac{\lambda_2 \pi K_{ix}[m_2]}{\sinh \pi x} = K_{ix}[g], \ x \in \mathbb{R}, \tag{3.10}$$

which can be rewritten as

$$G^+(x) = D(x) G^-(x) + H(x), \quad x \in \mathbb{R}, \tag{3.11}$$

where

$$H(x) = \frac{\sinh \pi x K_{ix}[g]}{\sinh \pi x \,(1 + \lambda_1 K_{ix}[m_1]) - i\lambda_2 \pi K_{ix}[m_2]},$$

$$D(x) = \frac{\sinh \pi x \,(1 + \lambda_1 K_{ix}[m_1]) + i\lambda_2 \pi K_{ix}[m_2]}{\sinh \pi x \,(1 + \lambda_1 K_{ix}[m_1]) - i\lambda_2 \pi K_{ix}[m_2]}, \tag{3.12}$$

and

$$\sinh^2 \pi x \,(1 + \lambda_1 K_{ix}[m_1])^2 + \lambda_2^2 \pi^2 K_{ix}^2[m_2] \neq 0, \; x \in \mathbb{R} \tag{3.13}$$

via the normality condition. Consequently, we have arrived at the Riemann boundary value problem (3.11) for the half-plane. Namely, the problem is to find a piecewise bounded analytic function $G(z)$ in the cut plane along the real axis whose limit values satisfy the boundary condition (3.11). Moreover, we seek solutions in the class of functions vanishing at infinity due to an asymptotic behavior of the Kontorovich-Lebedev transformation (1.5). This problem is solved in detail in [3], and we will appeal to the necessary formulas for the solution. Indeed, denoting by κ the index of the problem (3.11) $\kappa = \mathrm{Ind} D(x)$ we have accordingly:

1. If $\kappa > 0$ then the problem (3.11) is solvable and its solution can be written in the form

$$G(z) = X(z) \left[\Psi(z) + \frac{P_{\kappa-1}(z)}{(z+i)^{\kappa}} \right], \tag{3.14}$$

where $P_{\kappa-1}(z)$ is an arbitrary polynomial of degree $\kappa - 1$. The so-called canonic function in (3.14) $X(z)$ by definition represents a piecewise analytic function satisfying the boundary condition $X^+(t) = D(t) X^-(t), \; t \in \mathbb{R}$, where

$$X^+(z) = e^{\Gamma^+(z)}, \quad X^-(z) = \left(\frac{z-i}{z+i} \right)^{-\kappa} e^{\Gamma^-(z)}, \tag{3.15}$$

and

$$\Gamma(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \log \left[\left(\frac{\tau - i}{\tau + i} \right)^{-\kappa} D(\tau) \right] \frac{d\tau}{\tau - z} \tag{3.16}$$

Meanwhile, with the Sokhotzki formulas [3] we find from (3.16)

$$\Gamma^+(t) = \frac{1}{2} \log \left[\left(\frac{t-i}{t+i} \right)^{-\kappa} D(t) \right] + \Gamma(t),$$

$$\Gamma^-(t) = -\frac{1}{2} \log \left[\left(\frac{t-i}{t+i} \right)^{-\kappa} D(t) \right] + \Gamma(t).$$

Therefore, relations (3.15) yield

$$X^+(t) = e^{\Gamma(t)} \left[\left(\frac{t-i}{t+i} \right)^{-\kappa} D(t) \right]^{1/2}, \ t \in \mathbb{R}$$

$$X^-(t) = e^{\Gamma(t)} \left[\left(\frac{t-i}{t+i} \right)^{-\kappa} D(t) \right]^{-1/2}, \ t \in \mathbb{R}.$$

Now we observe from (3.16) taking the definition of $D(x)$ that $\Gamma(z) = \Gamma(-z)$. Hence we see that $X^+(-t) = X^-(t)$. An analytic function $\Psi(z)$ in (3.14) is defined by the Cauchy type integral as follows (see (3.11))

$$\Psi(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{H(\tau)}{X^+(\tau)} \frac{d\tau}{\tau - z}, \tag{3.17}$$

and again invoking (3.12), the property $D(-t) = 1/D(t)$ and the definitions of $H(t),\ X^+(t)$ we derive the relation $\Psi(z) = -\Psi(-z)$. So returning to (3.14) we write solutions in the form

$$G^+(t) = X^+(t) \left[\Psi^+(t) + \frac{P_{\kappa-1}(t)}{(t+i)^\kappa} \right],$$

$$G^-(t) = X^-(t) \left[\Psi^-(t) + \frac{P_{\kappa-1}(t)}{(t+i)^\kappa} \right],$$

where accordingly from (3.17) it has

$$\Psi^+(t) = \frac{1}{2} \frac{H(t)}{X^+(t)} + \Psi(t),$$

$$\Psi^-(t) = -\frac{1}{2} \frac{H(t)}{X^+(t)} + \Psi(t).$$

Since $G^+(t) - G^-(t) = K_{it}[f]$ is even we should get

$$X^+(-t) \left[\Psi^+(-t) + \frac{P_{\kappa-1}(-t)}{(i-t)^\kappa} \right] - X^-(-t) \left[\Psi^-(-t) + \frac{P_{\kappa-1}(-t)}{(i-t)^\kappa} \right]$$

$$= X^+(t) \left[\Psi^+(t) + \frac{P_{\kappa-1}(t)}{(i+t)^\kappa} \right] - X^-(t) \left[\Psi^-(t) + \frac{P_{\kappa-1}(t)}{(i+t)^\kappa} \right] \tag{3.18}$$

and taking into account our discussions above the latter equality will be true if and only if

$$\left(X^+(-t) - X^-(-t) \right) \frac{P_{\kappa-1}(-t)}{(i-t)^\kappa} = \left(X^+(t) - X^-(t) \right) \frac{P_{\kappa-1}(t)}{(i+t)^\kappa}, \ t \in \mathbb{R},$$

i.e.

$$\frac{P_{\kappa-1}(-t)}{(i-t)^{\kappa}} = -\frac{P_{\kappa-1}(t)}{(i+t)^{\kappa}}, \ t \in \mathbb{R}, \tag{3.19}$$

because $X^+(t) \neq X^-(t),\ t \in \mathbb{R}$. Consequently, we will consider for our solutions only the polynomials $P_{\kappa-1}(z)$, which satisfy the boundary condition (3.19). However, the Liouville theorem immediately concludes that the only solution of (3.19) is $P_{\kappa-1}(z) \equiv 0$. Therefore the Kontorovich-Lebedev transform $K_{it}[f] = G^+(t) - G^-(t) \in \mathcal{K}L_{it}$ can be written as the right-hand side of the equation (3.18)

$$K_{it}[f] = \frac{1}{2}\ H(t)\left(1 + \frac{1}{D(t)}\right) + \left(X^+(t) - X^-(t)\right)\Psi(t),\ t \in \mathbb{R}, \tag{3.20}$$

and the L_2^{α}- solution f of the convolution integral equation (1.1) will be found by the inversion formula (2.10)

$$f(x) = \frac{1}{\pi^2}\int_{-\infty}^{\infty} t \sinh \pi t \frac{K_{it}(x)}{x}\left[\frac{1}{2}\ H(t)\left(1 + \frac{1}{D(t)}\right)\right.$$
$$\left. + \left(X^+(t) - X^-(t)\right)\Psi(t)\right] dt,\ x > 0, \tag{3.21}$$

where the convergence of the integral is in L_2 -sense.

2. When $\kappa \leq 0$, formula (3.14) simply becomes $G(z) = X(z)\Psi(z)$. However, this unique solution is zero when $\kappa = 0$ and when $\kappa < 0$ for its existence it is necessary and sufficient the fulfilment of $-\kappa$ solvability conditions

$$\int_{-\infty}^{\infty} \frac{H(\tau)}{X^+(\tau)}\ \frac{d\tau}{(\tau+i)^k}, \quad k = 1, 2, \ldots, -\kappa. \tag{3.22}$$

In this case as above, the solution of the convolution integral equation (1.1) is given by (3.21). Thus we have proved the final

Theorem 3. *Let $\lambda_1, \lambda_2 \in \mathbb{R}\backslash\{0\}, m_1(t), m_2(t)$ and $g(t)$ be given functions in the class $L_2^{\alpha},\ |\alpha| \geq 1$, assuming that $m_1(t), m_2(t)$ are real-valued. Let also the normality condition* (3.13) *be true. Denoting by $\kappa = Ind\ D(x),\ x \in \mathbb{R}$, where $D(x)$ is defined by* (3.12)*, the solution of equation* (1.1) *is given by formula* (3.21) *for $\kappa > 0$. When $\kappa = 0$ the solution is trivial. Finally, for $\kappa < 0$ it is represented by* (3.21) *under the existence conditions* (3.22)*.*

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