Unary FA-presentable semigroups

Alan J. Cain, Nik Ruškuc, Richard M. Thomas

[AJC] Centro de Matemática, Universidade do Porto,
Rua do Campo Alegre 687, 4169–007 Porto, Portugal

Email: ajcain@fc.up.pt
Web page: www.fc.up.pt/pessoas/ajcain/

[NR] School of Mathematics and Statistics, University of St Andrews,
North Haugh, St Andrews, Fife KY16 9SS, United Kingdom

Email: nik@mcs.st-andrews.ac.uk
Web page: turnbull.mcs.st-and.ac.uk/~nik/

[RMT] Department of Computer Science, University of Leicester,
University Road, Leicester, LE1 7RH, United Kingdom

Email: rmt@mcs.le.ac.uk
Web page: www.cs.le.ac.uk/~rthomas/

ABSTRACT

Automatic presentations, also known as FA-presentations, were introduced by Khoussainov and Nerode to fulfil a need to extend finite model theory to infinite structures whilst retaining the solubility of interesting decision problems. A particular focus of research has been to attempt the classification of those structures of some species that admit automatic presentations. Whilst some successes have been obtained, this appears to be a very difficult problem in general. A restricted problem, which is also of significant interest, is to ask this question where the problem is restricted to those structures with automatic presentations over a one-letter alphabet; such structures are said to be unary FA-presentable. This paper studies unary FA-presentable semigroups.

It is proven that every unary FA-presentable structure admits an injective unary automatic presentation where the language of representatives consists of every word over a one-letter alphabet. The following results are proven: Unary FA-presentable semigroups are locally finite, but non-finitely generated unary FA-presentable semigroups may be infinite. Every unary FA-presentable semigroup satisfies some Burnside identity. In a unary FA-presentable semigroup, a $\mathcal{D}$-class cannot contain both infinitely many $\mathcal{L}$-classes and infinitely many $\mathcal{R}$-classes. The $\mathcal{H}$-classes of a unary FA-presentable semigroup are of bounded size. The class of unary FA-presentable semigroups is closed under forming the ordinal sum

Acknowledgements: The first author was supported by the project EP/C523229/1, funded by the EPSRC, and later by an FCT Ciência 2008 individual fellowship. The third author would like to thank Hilary Craig for all her help and encouragement.

of two semigroups, under taking finite Rees index subsemigroups, and under certain Rees matrix constructions. A finite Rees index extensions or an arbitrary subsemigroup of a unary FA-presentable semigroup need not be FA-presentable. A classification is given of the unary FA-presentable completely simple semigroups.

1 INTRODUCTION

Automatic presentations, also known as FA-presentations, were introduced by Khoussainov & Nerode [KN95] to fulfill a need to extend finite model theory to infinite structures while retaining the solubility of interesting decision problems, and have recently been applied to algebraic structures such as groups [OT05], rings [NT08], and semigroups [CORT08, CORT09].

One main avenue of research has been the classification of those structures of some species that admit automatic presentations. Classifications are known for finitely generated groups [OT05, Theorem 6.3] and cancellative semigroups [CORT09, Theorem 13], for integral domains (and more generally for possibly non-commutative rings with identity and no zero divisors) [NT08, Corollary 17], for Boolean algebras [KNRS04, Theorem 3.4], and for ordinals [Del04].

In several areas where general classifications remain elusive, it has been possible to classify those structures that admit unary automatic presentations (that is, automatic presentations over a one-letter alphabet), including, for example, bijective functions [Blu99, Theorem 7.12], equivalence relations [Blu99, Theorem 7.13] (see also Theorem 2.8 below), linear orders [Blu99, Theorem 7.15], graphs [Blu99, Theorem 7.16], and groups [Blu99, Theorem 7.19]. (Notice that a classification result in the non-unary case is only known for *finitely generated* groups.)

This motivates the study of semigroups admitting unary automatic presentations, which forms the subject of this paper. Whilst we do not give a complete classification of such semigroups, we do describe a number of their properties, which lead to classifications in some special cases.

First, in § 3, some very useful preliminary results are developed, two of which (Theorems 3.1 and 3.2) apply to all unary FA-presentable structures, not just to semigroups. Example 4.2 shows that infinite unary FA-presentable semigroups exist, contrasting the fact that unary FA-presentable groups are finite. However, the first main result of the paper, that unary FA-presentable semigroups are locally finite (Theorem 6.1), yields the immediate corollary that *finitely generated* unary FA-presentable semigroups are finite (Corollary 6.2). Another consequence is that for any unary FA-presentable semigroup S, there exists some $n \in \mathbb{N}$ such that $S^{n+1} = S^n$.

Next, every unary FA-presentable semigroup is shown to satisfy some Burnside identity $x^k = x^{k+m}$ (Theorem 7.3), and therefore to be periodic. Consequently, the Green's relations $\mathcal{D}$ and $\mathcal{J}$ coincide in such semigroups. In § 8, which focusses on the study of Green's relations for unary FA-presentable semigroups, it is proven that in such semigroups, $\mathcal{D}$-classes cannot contain both infinitely many $\mathcal{L}$-classes and infinitely many $\mathcal{R}$-classes. Furthermore, in a unary FA-presentable semigroup, there is a bound on the order of its $\mathcal{H}$-classes (Proposition 8.5).

Finally, § 9 examines the interaction of the class of unary FA-presentable semigroups with extensions and subsemigroups, the Rees matrix construction, direct products, and free products. In particular, the results on Rees matrix semigroups yield a classification of unary FA-presentable completely simple semigroups (Theorem 9.9).

2 PRELIMINARIES

This section gathers the definitions and basic results needed elsewhere in the paper.

First of all, a terminological convention: throughout the paper, 'countable' means 'countably infinite'.

The reader is assumed to be familiar with the theory of finite automata and regular languages; see [HU79, Chs 2–3] for background reading. The empty word (over any alphabet) is denoted ε.

DEFINITION 2.1. Let L be a regular language over a finite alphabet A. Define, for $n \in \mathbb{N}$,

$$L^n = \{(w_1, \ldots, w_n) : w_i \in L \text{ for } i = 1, \ldots, n\}.$$

Let \$ be a new symbol not in A. The mapping $\operatorname{conv} : (A^*)^n \to ((A \cup \{\$\})^n)^*$ is defined as follows. Suppose

$$\begin{aligned} w_1 &= w_{1,1}w_{1,2}\cdots w_{1,m_1}, \\ w_2 &= w_{2,1}w_{2,2}\cdots w_{2,m_2}, \\ &\vdots \\ w_n &= w_{n,1}w_{n,2}\cdots w_{n,m_n}, \end{aligned}$$

where $w_{i,j} \in A$. Then $\operatorname{conv}(w_1, \ldots, w_n)$ is defined to be

$$(w_{1,1}, w_{2,1}, \ldots, w_{n,1})(w_{1,2}, w_{2,2}, \ldots, w_{n,2})\cdots(w_{1,m}, w_{2,m}, \ldots, w_{n,m}),$$

where $m = \max\{m_i : i = 1, \ldots, n\}$ and with $w_{i,j} = \$$ whenever $j > m_i$.

Observe that the mapping conv maps an n-tuple of words to a word of n-tuples.

DEFINITION 2.2. Let A be a finite alphabet, and let $R \subseteq (A^*)^n$ be a relation on A^*. Then the relation R is said to be *regular* if

$$\operatorname{conv} R = \{\operatorname{conv}(w_1, \ldots, w_n) : (w_1, \ldots, w_n) \in R\}$$

is a regular language over $(A \cup \{\$\})^n$.

DEFINITION 2.3. Let $\mathcal{S} = (S, R_1, \ldots, R_n)$ be a relational structure. Let L be a regular language over a finite alphabet A, and let $\phi : L \to S$ be a surjective mapping. Then (L, ϕ) is an *automatic presentation* or an *FA-presentation* for $\mathcal{S}$ if:

1. the relation $\Lambda(=, \phi) = \{(w_1, w_2) \in L^2 : w_1\phi = w_2\phi\}$ is regular, and
2. for each relation R_i of arity r_i, the relation

$$\Lambda(R_i, \phi) = \{(w_1, w_2, \ldots, w_{r_i}) \in L^{r_i} : R(w_1\phi, \ldots, w_{r_i}\phi)\}$$

   is regular.

If $\mathcal{S}$ admits an automatic presentation, it is said to be *FA-presentable*.

If (L, ϕ) is an automatic presentation for $\mathcal{S}$ and the mapping ϕ is injective (so that every element of the structure has exactly one representative in L), then (L, ϕ) is said to be *injective*.

If (L, ϕ) is an automatic presentation for $\mathcal{S}$ and L is a language over a one-letter alphabet, then (L, ϕ) is a *unary* automatic presentation for $\mathcal{S}$, and $\mathcal{S}$ is said to be *unary FA-presentable*.

A semigroup can be viewed as a relational structure where the binary operation $\circ$ is interpreted as a ternary relation. The following definition simply restates that of a unary automatic presentation in the special case when the structure is a semigroup:

DEFINITION 2.4. Let S be a semigroup. Let L be a regular language over the alphabet $\{a\}$, and let $\phi : L \to S$ be a surjective mapping. Then (L, ϕ) is a *unary automatic presentation* for S if the relations

$$\Lambda(=, \phi) = \{(w_1, w_2) \in L^2 : w_1\phi = w_2\phi\}$$

and

$$\Lambda(\circ, \phi) = \{(w_1, w_2, w_3) \in L^3 : (w_1\phi) \circ (w_2\phi) = w_3\phi\}$$

are regular.

Often, the semigroup operation $\circ$ will be denoted simply by concatenation.

PROPOSITION 2.5 ([KN95, Corollary 4.3]). *Any structure that admits an automatic presentation* (L, ϕ) *admits an injective automatic presentation* $(K, \phi|_K)$, *where* $K \subseteq L$.

DEFINITION 2.6. If (L, ϕ), where $L \subseteq a^*$, is an injective unary automatic presentation for a structure $\mathcal{S}$, and s is an element of $\mathcal{S}$, then $\ell(s)$ is the length of the unique word $w \in L$ with $w\phi = s$. [Notice that $a^{\ell(s)} = s\phi^{-1}$ for all elements s of $\mathcal{S}$.]

The fact that a tuple of elements $(s_1, \ldots, s_n)$ of a structure $\mathcal{S}$ satisfies a first-order formula $\theta(x_1, \ldots, x_n)$ is denoted $\mathcal{S} \models \theta(s_1, \ldots, s_n)$.

PROPOSITION 2.7 ([KN95]). *Let* $\mathcal{S}$ *be a structure with an automatic presentation* (L, ϕ). *For every first-order formula* $\theta(x_1, \ldots, x_n)$ *over the structure, the relation*

$$\Lambda(\theta, \phi) = \big\{(w_1, \ldots, w_n) \in L^n : \mathcal{S} \models \theta(w_1\phi, \ldots, w_n\phi)\big\}$$

is regular.

Proposition 2.7 is fundamental to the theory of automatic presentations and will be used without explicit reference throughout the paper.

The following characterization of unary FA-presentable equivalence relations will be needed later:

THEOREM 2.8 ([Blu99, Theorem 7.13]). *An equivalence relation* $\sim$ *is unary FA-presentable if and only if*

1. *the cardinality of the finite* $\sim$*-classes is bounded, and*
2. *there are only finitely many countable* $\sim$*-classes.*

For any subset X of a semigroup S, denote by X^n the set of all elements of S that can be expressed as products of elements of X of length exactly n: that is, $X^n = \{x_1x_2 \cdots x_n : x_i \in X\}$. Notice that in general $X^n \not\subseteq X^{n+1}$.

The following result shows that a unary FA-presentable structure admits an injective unary FA-presentation where the language of representatives is the language of *all* words over a one-letter alphabet. Observe that this result holds for all unary FA-presentable structures, not just for semigroups.

THEOREM 3.1. *Let $\mathcal{S}$ be an infinite relational structure that admits a unary automatic presentation. Then $\mathcal{S}$ has an injective unary automatic presentation (a^*, ψ).*

Proof of 3.1. By Proposition 2.5, assume without loss of generality, that (L, ϕ) is an injective unary automatic presentation for $\mathcal{S}$, where $L \subseteq b^*$.

Let $\mathfrak{B}$ be a deterministic complete finite automaton recognizing L. Suppose $\mathfrak{B}$ has state set Q, set of accept states Y, initial state q_0, and transition function $\delta : Q \times \{b\} \to Q$. Since the input alphabet has only one letter, each state has exactly one edge leaving it. Let $y_1, y_2, \ldots \in Y$ be the accept states in the order in which they are encountered when $\mathfrak{B}$ reads an arbitrarily long word over $\{b\}$. (The sequence of states y_i is infinite because the language L is infinite.) Let β_0 be the smallest non-negative integer such that $(q_0, b^{\beta_0})\delta \in Y$, and for each $i \in \mathbb{N}$, let β_i be the smallest positive integer such that $(y_i, b^{\beta_i})\delta \in Y$. Notice that $(q_0, b^{\beta_0})\delta = y_1$ and that $(y_i, b^{\beta_i})\delta = y_{i+1}$. For $k \in \mathbb{N} \cup \{0\}$, let $B_k = \sum_{i=0}^{k} \beta_i$; notice that since $\beta_i > 0$ for every $i \in \mathbb{N}$, the map $k \mapsto B_k$ is injective. Note that $(q_0, b^{B_k})\delta = y_{k+1}$. Therefore the map ψ from a^* to the domain of $\mathcal{S}$ defined by $a^k\psi = b^{B_k}\phi$ is a bijection

Let $\mathcal{R}$ be some relation of $\mathcal{S}$ of arity n. (Possibly, $\mathcal{R}$ is the equality relation.) Let $\mathfrak{A}$ be an n-tape synchronous automaton recognizing $\operatorname{conv}(\Lambda(\mathcal{R}, \phi))$. Suppose that $\mathfrak{A}$ has state set P, initial state p_0, transition function $\zeta : P \times \{b, \$\}^n \to P$, and set of accept states Z.

Construct an n-tape synchronous automaton $\mathfrak{A}'$ as follows. The state set is $P \times Y$, the inital state is (p_0, y_0), the set of accept states is $Z \times Y$. The transition function $\kappa : (P \times Y) \times \{a, \$\}^n \to (P \times Y)$ is defined as follows:

$$\big((p, y_i), (a_1, \ldots, a_n)\big)\kappa = \big((p, (v_1, \ldots, v_n))\zeta, y_{i+1}\big),$$

where

$$v_j = \begin{cases} b^{\beta_i} & \text{if } a_j = a, \\ \$^{\beta_i} & \text{if } a_j = \$. \end{cases}$$

By construction, the new automaton $\mathfrak{A}'$ accepts $\operatorname{conv}(a^{k_1}, \ldots, a^{k_n})$ if and only if the original automaton $\mathfrak{A}$ accepts $\operatorname{conv}(b^{B_{k_1}}, \ldots, b^{B_{k_n}})$. [In particular, note that $\mathfrak{A}$ can only accept $\operatorname{conv}(b^{h_1}, \ldots, b^{h_n})$ if every h_i is B_{j_i} for some j_i, since b^{h_i} must be accepted by $\mathfrak{B}$.] Thus, since $L(\mathfrak{A}) = \operatorname{conv}(\Lambda(\mathcal{R}, \phi))$, it follows from the definition of ψ that $L(\mathfrak{A}') = \operatorname{conv}(\Lambda(\mathcal{R}, \psi))$.

Since $\mathcal{R}$ was an arbitrary relation of $\mathcal{S}$, it follows that (a^*, ψ) is a unary automatic presentation for $\mathcal{S}$. 3.1

The following result also applies to all unary FA-presentable structures.

THEOREM 3.2. *In a unary FA-presentable structure, there does not exist an infinite family of disjoint first-order definable infinite sets.*

Proof of 3.2. Let $\mathcal{S}$ be a unary FA-presentable relational structure. Suppose for *reductio ad absurdum* that there exists a countable collection $P_1, P_2, \ldots$ of disjoint first-order definable infinite sets in $\mathcal{S}$. Then $\mathcal{S}$ must be infinite and so by Theorem

3.1 admits a unary automatic presentation (a^*, ϕ). For each $i \in \mathbb{N}$, let $L_i = P_i\phi^{-1}$. Since each P_i is first-order definable and infinite each language $L_i \subseteq a^*$ is regular and infinite. Since the P_i are disjoint, so are the L_i.

Now, because the L_i are infinite regular languages over a one-letter alphabet, there exist constants b_i and p_i such that $a^{p_i k+b_i} \in L_i$ for all $k \in \mathbb{N}$. Now, it is well-known that there does not exist infinite collection of disjoint arithmetic progressions of natural numbers. So there exist $i, j \in \mathbb{N}$ such that $p_i k + b_i = p_j k' + b_j$ for some $k, k' \in \mathbb{N}$. That is, the word $a^{p_i k+b_i}$ (which is equal to $a^{p_j k'+b_j}$) lies in both L_i and L_j. This contradicts the disjointness of the languages L_i. 3.2

4 FINITE GROUPS AND INFINITE SEMIGROUPS

The following result was first observed for groups Blumensath [Blu99, Theorem 7.19]. Blumensath's proof generalizes immediately to cancellative semigroups [CORT09, Theorem 12.1], although Theorem 3.1 could be used to give a more conceptually economical proof. In particular, Theorem 3.1 makes Blumensath's notion of 'loop constants' [Blu99, § 7.1] unnecessary for the proof.

PROPOSITION 4.1. *Unary FA-presentable cancellative semigroups are finite. In particular, unary FA-presentable groups are finite.*

The following example shows that Proposition 4.1 does not extend to general semigroups, because infinite unary FA-presentable semigroups exist:

EXAMPLE 4.2. Any countable right zero semigroup or left zero semigroup is unary FA-presentable. To see this, let $S = \{z_i : i \in \mathbb{N} \cup \{0\}\}$ be a countable right zero semigroup. (The reasoning for left zero semigroups is similar.)

Define $\phi : a^* \to S$ by $a^n \mapsto z_n$ for all $n \in \mathbb{N} \cup \{0\}$. Then

$$\Lambda(=, \phi) = \{(a^p, a^p) : p \in \mathbb{N} \cup \{0\}\}$$

and

$$\begin{aligned} \Lambda(\circ, \phi) &= \{(a^p, a^q, a^r) : a^p\phi \circ a^q\phi = a^r\phi,\ p, q, r \in \mathbb{N} \cup \{0\}\} \\ &= \{(a^p, a^q, a^r) : z_p \circ z_q = z_r,\ p, q, r \in \mathbb{N} \cup \{0\}\} \\ &= \{(a^p, a^q, a^r) : z_q = z_r,\ p, q, r \in \mathbb{N} \cup \{0\}\} \\ &= \{(a^p, a^q, a^q) : p, q \in \mathbb{N} \cup \{0\}\}, \end{aligned}$$

and so $\Lambda(=, \phi)$ and $\Lambda(\circ, \phi)$ are regular. Thus (a^*, ϕ) is a unary automatic presentation for S.

Note in passing that any finite semigroup — indeed, any finite structure — admits a unary automatic presentation.

5 ADJOINING AN IDENTITY

Although the natural place for the following result would be in the discussion of semigroup constructions in § 9, it is required in § 6 and so is proved here instead:

PROPOSITION 5.1. *Let S be a semigroup. Then S is unary FA-presentable if and only if S^1 is unary FA-presentable.*

Proof of 5.1. Suppose that S admits an injective unary FA-presentation (a^*, ϕ). Define a new map $\psi : a^* \to S^1$ by $\varepsilon\psi = 1$ and $(aw)\psi = w\phi$. [The idea is to lengthen all representatives by a single letter a and use the empty word ε to represent the adjoined identity.] Then ψ is injective and

$$\begin{aligned}\Lambda(\circ, \psi) &= \{(u, v, w) : u, v, w \in a^*, (u\psi)(v\psi) = w\psi\}\\ &= \{(u, v, w) : u, v, w \in a^+, (u\psi)(v\psi) = w\psi\}\\ &\qquad \cup \{(u, \varepsilon, u), (\varepsilon, u, u) : u \in a^*\}\\ &= \{(au', av', aw') : u', v', w' \in a^+, (u'\phi)(v'\phi) = w'\phi\}\\ &\qquad \cup \{(u, \varepsilon, u), (\varepsilon, u, u) : u \in a^*\}\\ &= (a, a, a)\Lambda(\circ, \phi)\\ &\qquad \cup \{(u, \varepsilon, u), (\varepsilon, u, u) : u \in a^*\},\end{aligned}$$

which is regular. So (a^*, ψ) is an injective unary FA-presentation for S^1.

Suppose now that (a^*, ϕ) is an injective unary FA-presentation for S^1. Let $u \in a^*$ be the unique word representing the adjoined identity. Then $a^* - \{u\}$ maps injectively onto S and

$$\Lambda(\circ, \phi|_{a^*-\{u\}}) = \Lambda(\circ, \phi) \cap \big((a^* - \{u\}) \times (a^* - \{u\}) \times (a^* - \{u\})\big)$$

is regular; hence $(a^* - \{u\}, \phi|_{a^*-\{u\}})$ is a unary FA-presentation for S. 5.1

6 FINITELY GENERATED UNARY FA-PRESENTABLE SEMIGROUPS

While unary FA-presentable groups are finite by Proposition 4.1, Example 4.2 shows that unary FA-presentable semigroups may be infinite. However, with the extra condition of finite generation, finitude is guaranteed:

THEOREM 6.1. *Unary FA-presentable semigroups are locally finite.*

Proof of 6.1. Let S be a unary FA-presentable semigroup. Let Y be a finite subset of S. The aim is to show that the subsemigroup T generated by Y is finite.

By Proposition 5.1 and Theorem 3.1, S^1 admits an injective unary FA-presentation (a^*, ϕ). Let $X = Y \cup \{1\}$. Then X generates the subsemigroup T^1 of S^1. Let $R = \max\{l(a) : a \in X\}$. By [CORT10, Lemma 7.5], there is a constant N such that, for all $m \in \mathbb{N}$,

$$\max\{\ell(a_1 \cdots a_m) : a_i \in X\} \leqslant R + \lceil \log_2 m \rceil N. \tag{6.1}$$

In a language over a one-letter alphabet, words are uniquely determined by their lengths. It thus follows from (6.1) that for all $m \in \mathbb{N}$,

$$|X^m| \leqslant R + \lceil \log_2 m \rceil N. \tag{6.2}$$

Since X contains the identity 1, it follows that $X^m \subseteq X^{m+1}$. So $|X^m| \leqslant |X^{m+1}|$.

Suppose that $|X^m| < |X^{m+1}|$ for all $m \in \mathbb{N}$. Then since $|X^m|$ must be an integer, $|X^m| \geqslant m$ for all $m \in \mathbb{N}$. Hence $m \leqslant R + \lceil \log_2 m \rceil N$ for all $m \in \mathbb{N}$ by (6.2), which is a contradiction, for this inequality is false for sufficiently large m. Therefore there is some $m \in \mathbb{N}$ such that $|X^m| = |X^{m+1}|$.

So $X^m = X^{m+1}$. Hence X^m contains all the elements of X and is closed under right- and left-multiplication by elements of X. So X^m must be the subsemigroup generated by X, which is T^1. Hence T^1 is finite and thus so is T.

Since X was an arbitrary finite subset of the unary FA-presentable semigroup S, it follows that S is locally finite. 6.1

COROLLARY 6.2. *A finitely generated semigroup is unary FA-presentable if and only if it is finite.*

Proof of 6.2. In one direction, the result is obvious: if a semigroup is finite it admits a unary automatic presentation. In the other, it is a consequence of Theorem 6.1. 6.2

Notice that Corollary 6.2 gives a classification of those finitely generated semigroups that admit unary automatic presentations. Finite generation seems to be a useful tool for proving classification results for general (not just unary) FA-presentable structures; witness the classifications of finitely generated FA-presentable groups [OT05, Theorem 6.3] and finitely generated FA-presentable cancellative semigroups [CORT09, Theorem 13].

THEOREM 6.3. *Let S be a unary FA-presentable semigroup. Then there exists $n \in \mathbb{N}$ such that $S^{n+1} = S^n$.*

Proof of 6.3. For each $k \in \mathbb{N}$, let $D_k = S^k - S^{k+1}$. Then each set D_k consists of those elements of S that can be written as a product of length k but not of length $k+1$. Each set S_k is first-order definable, so each set D_k is first first-order definable. Furthermore, the sets D_k are pairwise disjoint. Hence by Theorem 3.2, all but finitely many of them are finite. So suppose D_k is finite for every $k \geqslant r$.

LEMMA 6.4. *If $s_1 s_2 ... s_k \in D_k$, then every subproduct $s_i s_{i+1} \cdots s_j$ belongs to D_{j-i+1}.*

Proof of 6.4. As a product of $i-j+1$ elements of S, the product $s_{i+1} \cdots s_j$ belongs to D_h for some $h \geqslant i-j+1$. If $h > i-j+1$, then $s_{i+1} \cdots s_j = t_1 \cdots t_h$ and so

$$s_1 s_2 \cdots s_k = s_1 \cdots s_{i-1} t_1 \cdots t_k s_{j+1} \cdots s_k,$$

which is a product of more than k elements of S and so cannot lie in D_k. 6.4

It follows that every element of the set $D_r \cup D_{r+1} \cup \ldots$ can be written as a product of elements from $D_r \cup D_{r+1} \cup \ldots \cup D_{2r-1}$: just bracket the products in groups of r, each of which lies in D_r by Lemma 6.4, except for the last one which has length between r and $2r-1$, which lies in $D_r \cup D_{r+1} \cup \ldots \cup D_{2r-1}$ by Lemma 6.4.

So the set $D_r \cup D_{r+1} \cup \ldots$ lies in the subsemigroup generated by $D_r \cup D_{r+1} \cup \ldots \cup D_{2r-1}$. But $D_r \cup D_{r+1} \cup \ldots \cup D_{2r-1}$ is finite, and S is locally finite by Theorem 6.1, so the set $D_r \cup D_{2+1} \cup \ldots$ is finite as well. Hence there exists $n \in \mathbb{N}$ such that $D_n = \emptyset$, and so $S^{n+1} = S^n$. 6.3

7 BURNSIDE IDENTITIES

The present section is dedicated to proving that any unary FA-presentable semigroup satisfies some Burnside identity; that is, some semigroup identity $x^k = x^{k+m}$. (The constants $k, m \in \mathbb{N}$ are dependent on the semigroup in question.) In particular, any such semigroup is periodic and has bounded period.

First, two technical results are needed. The first restricts the length of the word representing a product of two elements in terms of the lengths of the words representing those elements themselves. In the language a^*, of course, the length of a word uniquely determines that word, so this restriction is very useful.

LEMMA 7.1. *Let S be an infinite semigroup admitting an injective unary automatic presentation (a^*, ϕ) (by Theorem 3.1). Then there is a constant $n \in \mathbb{N}$ such that, for any $x, y \in S$, one of the following conditions holds:*

1. $\ell(x) - n \leqslant \ell(xy) \leqslant \ell(x) + n$,
2. $\ell(y) - n \leqslant \ell(xy) \leqslant \ell(y) + n$,
3. $\ell(xy) \leqslant n$.

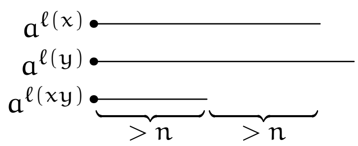
Proof of 7.1. Let $\mathfrak{A}$ be an automaton recognizing $\operatorname{conv}\Lambda(\circ, \phi)$ and let n be the number of states in $\mathfrak{A}$.

Let $x, y \in S$. If $\ell(xy) \leqslant n$, then condition 3 holds and there is nothing to prove. So suppose $\ell(xy) > n$. Assume that $\ell(x) \leqslant \ell(y)$; the other case is similar. Suppose, with the aim of obtaining a contradiction, that neither condition 1 nor condition 2 holds. Then one of the following conditions holds:

$$n < \ell(xy) < \ell(x) - n, \text{ or } \ell(x) + n < \ell(xy) < \ell(y) - n, \text{ or } \ell(xy) > \ell(y) + n.$$

Each of the possible ranges for $\ell(xy)$ leads to a contradiction:

1. $n < \ell(xy) < \ell(x) - n$. Then the following diagram describes the situation:



So the word $\operatorname{conv}(a^{\ell(x)}, a^{\ell(y)}, a^{\ell(xy)})$ can be pumped before the end of $a^{\ell(xy)}$ and between the end of $a^{\ell(xy)}$ and the end of $a^{\ell(x)}$. That is, there exist $p, q \in \mathbb{N}$ with $0 < p, q < n$ such that

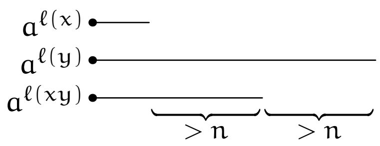
$$\left(a^{\ell(x)+ip+jq}, a^{\ell(y)+ip+jq}, a^{\ell(xy)+ip}\right) \in \Lambda(\circ, \phi)$$

for all $i, j \in \mathbb{N} \cup \{0\}$. Setting $i = q$ and $j = 0$ and then $i = 0$ and $j = p$ shows that

$$\left(a^{\ell(x)+qp}, a^{\ell(y)+qp}, a^{\ell(xy)+qp}\right), \left(a^{\ell(x)+pq}, a^{\ell(y)+pq}, a^{\ell(xy)}\right) \in \Lambda(\circ, \phi),$$

which implies that $a^{\ell(xy)+qp}\phi = a^{\ell(xy)}\phi$, contradicting the injectivity of ϕ.

2. $\ell(x) + n < \ell(xy) < \ell(y) - n$. Then the following diagram describes the situation:



So the word $\operatorname{conv}(a^{\ell(x)}, a^{\ell(y)}, a^{\ell(xy)})$ can be pumped between the end of $a^{\ell(x)}$ and the end of $a^{\ell(xy)}$ and between the end of $a^{\ell(xy)}$ and the end of $a^{\ell(y)}$. That is, there exist $p, q \in \mathbb{N}$ with $0 < p, q < n$ such that

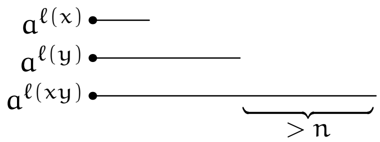
$$\left(a^{\ell(x)}, a^{\ell(y)+ip+jq}, a^{\ell(xy)+ip}\right) \in \Lambda(\circ, \phi)$$

for all $i, j \in \mathbb{N} \cup \{0\}$. Setting $i = q$ and $j = 0$ and then $i = 0$ and $j = p$ shows that

$$\left(a^{\ell(x)}, a^{\ell(y)+qp}, a^{\ell(xy)+qp}\right), \left(a^{\ell(x)}, a^{\ell(y)+pq}, a^{\ell(xy)}\right) \in \Lambda(\circ, \phi),$$

which implies that $a^{\ell(xy)+qp}\phi = a^{\ell(xy)}\phi$, contradicting the injectivity of ϕ.

3. $\ell(y) + n < \ell(xy)$. Then the following diagram describes the situation:



So the word $\mathrm{conv}(a^{\ell(x)}, a^{\ell(y)}, a^{\ell(xy)})$ can be pumped between the end of $a^{\ell(y)}$ and the end of $a^{\ell(xy)}$. That is, there exists $p \in \mathbb{N}$ with $0 < p < n$ such that

$$\left(a^{\ell(x)}, a^{\ell(y)}, a^{\ell(xy)+ip}\right) \in \Lambda(\circ, \phi)$$

for all $i \in \mathbb{N} \cup \{0\}$. Setting $i = 0$ and then $i = 1$ shows that

$$\left(a^{\ell(x)}, a^{\ell(y)}, a^{\ell(xy)}\right), \left(a^{\ell(x)}, a^{\ell(y)}, a^{\ell(xy)+p}\right) \in \Lambda(\circ, \phi),$$

which implies that $a^{\ell(xy)}\phi = a^{\ell(xy)+p}\phi$, contradicting the injectivity of ϕ.

Each case leads to a contradiction; this completes the proof. 7.1

The second technical result relates the lengths of representatives for an element and for powers of that element:

LEMMA 7.2. *Let S be a semigroup admitting an injective unary automatic presentation (a^*, ϕ). For all $x \in S$ and $k \in \mathbb{N}$, one of the following conditions holds:*

1. $\ell(x^k) \leqslant n\lceil \log_2 k \rceil$,
2. $|\ell(x^k) - \ell(x)| \leqslant n\lceil \log_2 k \rceil$,

where n is the constant of Lemma 7.1.

Proof of 7.2. Proceed by strong induction on k. For $k = 1$, the values of $|\ell(x^k) - \ell(x)|$ and $\log_2 k$ are both 0, so condition 2 holds for $k = 1$.

For the induction step, suppose that $k > 1$ and that for every $h < k$ one of the following conditions holds:

1. $\ell(x^h) \leqslant n\lceil \log_2 h \rceil$,
2. $|\ell(x^h) - \ell(x)| \leqslant n\lceil \log_2 h \rceil$.

The aim is to show that one of these two conditions holds for $h = k$. Now, $x^k = x^{\lfloor k/2 \rfloor} x^{\lceil k/2 \rceil}$ and both $\lfloor k/2 \rfloor$ and $\lceil k/2 \rceil$ are strictly less than k since $k > 1$. Thus, by Lemma 7.1, one of the following holds:

$$\ell(x^k) \leqslant n, \tag{7.1}$$
$$\left|\ell(x^k) - \ell(x^{\lfloor k/2 \rfloor})\right| \leqslant n, \tag{7.2}$$
$$\left|\ell(x^k) - l(x^{\lceil k/2 \rceil})\right| \leqslant n. \tag{7.3}$$

Consider each case in turn:

1. Suppose that (7.1) holds: $\ell(x^k) \leqslant n$. Then $\lceil \log_2 k \rceil \geqslant 1$ since $k \geqslant 2$, and so $\ell(x^k) \leqslant n\lceil \log_2 k \rceil$. Thus condition 1 holds.
2. Suppose that (7.2) holds: $|\ell(x^k) - \ell(x^{\lfloor k/2 \rfloor})| \leqslant n$. By the induction hypothesis with $h = \lfloor k/2 \rfloor$, one of the following holds:

$$\ell(x^{\lfloor k/2 \rfloor}) \leqslant n\lceil \log_2 \lfloor k/2 \rfloor \rceil, \tag{7.4}$$
$$\left|\ell(x^{\lfloor k/2 \rfloor}) - \ell(x)\right| \leqslant n\lceil \log_2 \lfloor k/2 \rfloor \rceil. \tag{7.5}$$

So there are two sub-cases:

(a) Suppose (7.4) holds. Then:

$$\begin{aligned}
&\ell(x^k)\\
&= \left|\ell(x^k) - \ell(x^{\lfloor k/2 \rfloor}) + \ell(x^{\lfloor k/2 \rfloor})\right|\\
&\leqslant \left|\ell(x^k) - \ell(x^{\lfloor k/2 \rfloor})\right| + \ell(x^{\lfloor k/2 \rfloor}) && \text{(by the triangle inequality)}\\
&\leqslant n + n\lceil \log_2 \lfloor k/2 \rfloor \rceil && \text{(by (7.2) and (7.4))}\\
&\leqslant n\lceil \log_2 \lfloor k/2 \rfloor + 1\rceil\\
&\leqslant n\lceil \log_2 \lfloor k \rfloor \rceil,
\end{aligned}$$

and so condition 1 holds.

(b) Suppose (7.5) holds. Then:

$$\begin{aligned}
&\left|\ell(x^k) - \ell(x)\right|\\
&= \left|\ell(x^k) - \ell(x^{\lfloor k/2 \rfloor}) + \ell(x^{\lfloor k/2 \rfloor}) - \ell(x)\right|\\
&\leqslant \left|\ell(x^k) - \ell(x^{\lfloor k/2 \rfloor})\right| + \left|\ell(x^{\lfloor k/2 \rfloor}) - \ell(x)\right| && \text{(by the triangle inequality)}\\
&\leqslant n + n\lceil \log_2 \lfloor k/2 \rfloor \rceil && \text{(by (7.2) and (7.5))}\\
&\leqslant n\lceil \log_2 \lfloor k/2 \rfloor + 1\rceil\\
&\leqslant n\lceil \log_2 \lfloor k \rfloor \rceil,
\end{aligned}$$

and so condition 2 holds.

3. Suppose that (7.3) holds, that $|\ell(x^k) - l(x^{\lceil k/2 \rceil})| \leqslant n$. By the induction hypothesis with $h = \lceil k/2 \rceil$, one of the following holds

$$\ell(x^{\lceil k/2 \rceil}) \leqslant n\lceil \log_2 \lceil k/2 \rceil \rceil \tag{7.6}$$

$$|\ell(x^{\lceil k/2 \rceil}) - \ell(x)| \leqslant n\lceil \log_2 \lceil k/2 \rceil \rceil. \tag{7.7}$$

The reasoning parallels the previous case, except that one calls on (7.6) and (7.7) rather than (7.4) and (7.5). 7.2

THEOREM 7.3. *Any unary FA-presentable semigroup satisfies a Burnside identity.*

Proof of 7.3. Let S be a unary FA-presentable semigroup. By Theorem 3.1, let (a^*, ϕ) be an injective unary automatic presentation for S.

Let $s \in S$. Then, by Lemma 7.2, for any $k \in \mathbb{N}$, one of the following holds:

$$\ell(s^k) \leqslant n\lceil \log_2 k \rceil$$

$$|\ell(s^k) - \ell(s)| \leqslant n\lceil \log_2 k \rceil,$$

where n is the constant of Lemma 7.1.

Choose h such that $h > 3n\lceil \log_2 h \rceil$. Then for each $k < h$, there are only $3n\lceil \log_2 h \rceil$ possible values for $\ell(s^k)$, since $\ell(s^k)$ is either within $n\lceil \log_2 h \rceil$ of $\ell(s)$ or at most $n\lceil \log_2 h \rceil$. Since h exceeds $3n\lceil \log_2 h \rceil$, by the pigeon-hole principle there exist k_s and k_s', with $k_s < k_s' < h$, such that $\ell(s^{k_s}) = \ell(s^{k_s'})$. Let $m_s = k_s' - k_s$; then $0 < m_s < h$ and $\ell(s^{k_s}) = \ell(s^{k_s + m_s})$. So $s^{k_s} = s^{k_s + m_s}$, and it follows that the index and period of s are less than h, which is dependent only on (L, ϕ). Let $k = \max\{k_s : s \in S\}$ and $m = \operatorname{lcm}\{m_s : s \in S\}$. Since there are only finitely many possibilities for k_s and m_s, both k and m exist. Then $s^k = s^{k+m}$ for any element of s, and so S satisfies the Burnside identity $x^k = x^{k+m}$. 7.3

Margolis [Personal communication] posed the following question:

QUESTION 7.4. Do all FA-presentable semigroups satisfy some non-trivial semigroup identity?

All known classes of FA-presentable semigroup satisfy some semigroup identity; see the various examples in [CORT09]. Additionally, those semigroup constructions under which the class of FA-presentable semigroups is known to be closed [CORT10] are also constructions under which the class of semigroups satisfying non-trivial identities is closed. Theorem 7.3 is further, albeit limited, evidence in favour of a positive answer to this question.

8 GREEN'S RELATIONS & SCHÜTZENBERGER GROUPS

This section is devoted to describing the Green's relations $\mathcal{H}$, $\mathcal{R}$, $\mathcal{L}$, $\mathcal{D}$, and $\mathcal{J}$ for unary FA-presentable semigroups. The reader is assumed to be familiar with the definitions and basic theory of Green's relations; for background information, see [How95, Ch. 2].

The following result is immediate:

COROLLARY 8.1. *In a unary FA-presentable semigroup, Green's relations $\mathcal{D}$ and $\mathcal{J}$ coincide.*

Proof of 8.1. A unary FA-presentable semigroup is periodic by Theorem 7.3, and $\mathcal{D} = \mathcal{J}$ in periodic semgroups by [How95, Proposition 2.1.4]. 8.1

Since all the Green's relations are equivalence relations, the following result is an immediate consequence of Theorem 2.8:

PROPOSITION 8.2. *In a unary FA-presentable semigroup, there are only finitely many infinite $\mathcal{J}$-, $\mathcal{D}$-, $\mathcal{R}$-, $\mathcal{L}$-, and $\mathcal{H}$-classes, and the finite ones are of bounded size.*

The next result says, essentially, that the eggbox diagram for a $\mathcal{D}$-class (see [How95, § 2.2]) cannot have both infinitely many rows and infinitely many columns:

PROPOSITION 8.3. *In a unary FA-presentable semigroup, a $\mathcal{D}$-class cannot contain both infinitely many $\mathcal{R}$-classes and infinitely many $\mathcal{L}$-classes.*

Proof of 8.3. Suppose a unary FA-presentable semigroup S has some $\mathcal{D}$-class D that contains infinitely many $\mathcal{R}$-classes and infinitely many $\mathcal{L}$-classes. Then since there are infinitely many $\mathcal{L}$-classes in D and every $\mathcal{H}$-class contains at least one element, every $\mathcal{R}$-class of D is infinite. So there are infinitely many infinite $\mathcal{R}$-classes in S. Since $\mathcal{R}$ is an equivalence relation, this contradicts Theorem 2.8. So no such $\mathcal{D}$-class can exist. 8.3

In order to strengthen Proposition 8.2 to show that the $\mathcal{H}$-classes of a unary FA-presentable semigroup are always finite, and in fact of bounded size, ideas from the theory of Schützenberger groups are required. The necessary definitions are recalled here; see [CP61, § 2.4] for further background.

DEFINITION 8.4. Let S be a semigroup. Let H be an $\mathcal{H}$-class of S and let h_0 be an arbitrary element of H. The semigroup S acts by right multiplication on the set of $\mathcal{H}$-classes in the $\mathcal{R}$-class containing H with a sink adjoined. The right stabilizer of H is denoted $\mathrm{Stab}(H)$:

$$\mathrm{Stab}(H) = \{s \in S : Hs = H\} = \{s \in S : h_0 s \,\mathcal{H}\, h_0\}. \tag{8.1}$$

Define a relation $\sigma(H)$ on $\text{Stab}(H)$ by

$$(s, t) \in \sigma(H) \iff (\forall h \in H)(hs = ht).$$

This relation is a congruence, and its definition is equivalent to

$$(s, t) \in \sigma(H) \iff (h_0 s = h_0 t). \tag{8.2}$$

The factor semigroup $\Gamma(H) = \text{Stab}(H)/\sigma(H)$ is actually a group, called the *Schützenberger group* of H. The group $\Gamma(H)$ acts regularly on H; thus $|H| = |\Gamma(H)|$, and if H is a group then $H \simeq \Gamma(H)$.

PROPOSITION 8.5. *Any unary FA-presentable semigroup has a bound on the size of its $\mathcal{H}$-classes.*

Proof of 8.5. Let (L, ϕ) be a unary automatic presentation for S. Choose $w \in L$. Let $h_0 = w\phi$; the aim is to show that H_{h_0} is finite.

The set $\text{Stab}(H_{h_0})$ is first-order definable by (8.1); thus the set of words $K = \{w \in L : w\phi \in \text{Stab}(H_{h_0})\}$ is regular. Thus $(K, \phi|_K)$ is a unary automatic presentation for the subsemigroup $\text{Stab}(H_{h_0})$.

The congruence $\sigma(H_{h_0})$ is first-order definable by (8.2). Thus the Schützenberger group $\Gamma(H_{h_0}) = \text{Stab}(H_{h_0})/\sigma(H_{h_0})$ admits a unary automatic presentation $(K, \phi|_K \sigma^\#)$, where $\sigma^\#$ is the natural map from $\text{Stab}(H_{h_0})$ to $\text{Stab}(H_{h_0})/\sigma(H_{h_0})$. Thus, by Proposition 4.1, the group $\Gamma(H_{h_0})$ is finite.

Since $w \in L$ (and thus $h_0 \in S$) was arbitrary, every Schützenberger group of an $\mathcal{H}$-class of S is finite. Thus every $\mathcal{H}$-class of S is finite. Since $\mathcal{H}$ is an equivalence relation on S, there is a bound on the size of the $\mathcal{H}$-classes of S by Theorem 2.8. $\boxed{8.5}$

PROPOSITION 8.6. *The principal factor arising from any $\mathcal{J}$-class of a unary FA-presentable semigroup is either completely 0-simple or a null semigroup.*

Proof of 8.6. Let T be some principal factor of a unary FA-presentable semigroup S. By [How95, Theorem 3.1.6(2)], T is either 0-simple or null. If it is null, there is nothing more to prove. So suppose T is 0-simple. Since S is periodic by Theorem 7.3, so is T. In particular, T is group-bound. Thus, by [How95, Theorem 3.2.11], T is completely 0-simple. $\boxed{8.6}$

The following example shows that there do exist unary FA-presentable semigroups with an arbitrary finite number of infinite $\mathcal{D}$-classes and an infinite number of finite ones.

EXAMPLE 8.7. Let S be a countable right zero semigroup, which is unary FA-presentable by Example 4.2.

Let T be the countable chain $\{t_0, t_1, \ldots\}$ with ordering $t_i \leqslant t_j$ if and only if $i \leqslant j$. Let $\psi : a^* \to T$ be defined by $a^n\psi = t_n$. Then

$$\Lambda(=, \psi) = \{(a^p, a^p) : p \in \mathbb{N} \cup \{0\}\}$$

and

$$\begin{aligned}
\Lambda(\circ, \psi) &= \{(a^p, a^q, a^r) : a^p\psi \circ a^q\psi = a^r\psi \; p, q, r \in \mathbb{N} \cup \{0\}\} \\
&= \{(a^p, a^q, a^r) : t_p \circ t_q = t_r, \; p, q, r \in \mathbb{N} \cup \{0\}\} \\
&= \{(a^p, a^q, a^r) : t_q = t_r, \; t_q \leqslant t_p, \; p, q, r \in \mathbb{N} \cup \{0\}\} \\
&\quad \cup \{(a^p, a^q, a^r) : t_p = t_r, \; t_p \leqslant t_q, \; p, q, r \in \mathbb{N} \cup \{0\}\} \\
&= \{(a^p, a^q, a^q) : p, q \in \mathbb{N} \cup \{0\}, \; q \leqslant p\} \\
&\quad \cup \{(a^p, a^q, a^p) : p, q \in \mathbb{N} \cup \{0\}, \; p \leqslant q\}
\end{aligned}$$

and so $\Lambda(=,\psi)$ and $\Lambda(\circ,\psi)$ are regular. Thus (a^*,ψ) is a unary automatic presentation for T.

Let $k \in \mathbb{N} \cup \{0\}$. Let $U_0 = T$. For each $i = 1,\ldots,k$, let S_i be a copy of S and let U_i be the ordinal sum of S_i and U_{i-1} with respect to the ordering $S > U_i$. (See § 9.1 for the definition of ordinal sums.) Then by iterated applicaton of Proposition 9.1, U_k is unary FA-presentable.

Now, in U_k, products in each subsemigroup U_i are as before, and if $x \in U_i$ and $y \in U_j$ with $i < j$, then $xy = yx = x$. So in U_k, the $\mathcal{R}$-class, and thus the $\mathcal{D}$-class of any element of S_i is the whole of S_i, and the $\mathcal{D}$-class of any element $t \in T$ is the singleton set $\{t\}$. So U_k contains countably many finite (singleton) $\mathcal{D}$-classes inside T, and k countable $\mathcal{D}$-classes, namely the S_i.

Although the results in this section describe the possible $\mathcal{J}$-, $\mathcal{D}$-, $\mathcal{R}$-, $\mathcal{L}$-, and $\mathcal{H}$-classes and principal factors of a unary FA-presentable semigroup, what is lacking is a description of how these interact. In particular, no characterization is yet known of unary FA-presentable semilattices (where all Green's relations are simply the equality relation). This seems to be the major obstacle on the way to a complete characterization of unary FA-presentable semigroups.

9 CONSTRUCTIONS

This section examines the interaction of the class of unary FA-presentable semigroups and four semigroup constructions: extensions and subsemigroups, Rees matrix semigroups, direct products, and free products.

9.1 *Extensions and subsemigroups*

The *ordinal sum* of two semigroup S and T with respect to the ordering $S > T$, is the disjoint union of S and T with the multiplication of two elements of S or two elements of T as before and the product of $s \in S$ and $t \in T$ defined to be t: that is, $st = ts = t$ for all $s \in S$ and $t \in T$. So this ordinal sum is a particular ideal extension of T by S. (The notion of an ordinal sum is due to Clifford [Cli54], who defined it for an arbitrary collection of semigroups indexed by a totally ordered semigroup, and with each semigroup admitting a particular type of total order.)

PROPOSITION 9.1. *The ordinal sum of two unary FA-presentable semigroups is itself unary FA-presentable.*

Proof of 9.1. Let S and T be semigroups admitting unary automatic presentations (K,ϕ) (where $K \subseteq a^*$) and (L,ψ) (where $L \subseteq b^*$) respectively. (Note that Theorem 3.1 cannot be applied here because one or both of S and T may be finite.) Let U be the ordinal sum of S and T with respect to the ordering $S > T$.

Define the following homomorphisms:

$$\eta : a^* \to c^*, \quad a \mapsto c^2,$$
$$\vartheta : b^* \to c^*, \quad b \mapsto c^2.$$

Since regularity is preserved under homomorphism, $K' = K\eta$ and $L' = L\vartheta$ are regular. Notice that $K', L' \subseteq \{c^2\}^*$, so K' and cL' are disjoint. Let $M = K' \cup cL'$. Now define a map

$$\chi : M \to U, \quad \begin{cases} c^{2k} \mapsto a^k\phi \\ c^{2k+1} \mapsto b^k\psi. \end{cases}$$

By the definition of M, this map is well-defined.

Let $\mathfrak{A}$ recognize $\operatorname{conv}(\Lambda(\circ, \phi))$ and $\mathfrak{B}$ recognize $\operatorname{conv}(\Lambda(\circ, \psi))$. In $\mathfrak{A}$, each edge is labelled by a triple whose components are either a or \$. On every edge, replace each component a with c^2 and each component \$ with $\$^2$. Call the resulting automaton $\mathfrak{A}'$. Similarly, on every edge of $\mathfrak{B}$, replace each component b with c^2 and each component \$ with $\$^2$ to obtain an automaton $\mathfrak{B}'$. It is easy to see that

$$\operatorname{conv}\big(\Lambda(\circ, \chi|_{S\chi^{-1}})\big) = L(\mathfrak{A}') \text{ and } \operatorname{conv}\big(\Lambda(\circ, \chi|_{T\chi^{-1}})\big) = (c, c, c)L(\mathfrak{B}').$$

So $\Lambda(\circ, \chi|_{S\chi^{-1}})$ and $\Lambda(\circ, \chi|_{T\chi^{-1}})$ are both regular. Now,

$$\begin{aligned}
&\Lambda(\circ, \chi) \\
&= \{(u, v, w) : u, v, w \in c^* : (u\chi) \circ (v\chi) = w\chi\} \\
&= \Lambda(\circ, \chi|_{S\chi^{-1}}) \cup \Lambda(\circ, \chi|_{T\chi^{-1}}) \\
&\qquad\qquad \cup \{(c^{2k}, c^{2m+1}, c^{2m+1}), (c^{2m+1}, c^{2k}, c^{2m+1}) : k, m \in \mathbb{N} \cup \{0\}\},
\end{aligned}$$

so $\Lambda(\circ, \chi)$ is regular. Thus (c^*, χ) is a unary automatic presentation for U. 9.1

Recall that a subsemigroup T of a semigroup S has finite Rees index if the set $S - T$ is finite.

PROPOSITION 9.2. *The class of unary FA-presentable semigroups is closed under passing to subsemigroups of finite Rees index.*

Proof of 9.2. Let S be a unary FA-presentable semigroup and let T be a subsemigroup of S of finite Rees index. Let (a^*, ϕ) be an injective unary automatic presentation for S. Let $K = (S - T)\phi^{-1}$. Since $S - T$ is finite and ϕ is injective, K is a finite subset of a^* and therefore regular. So $L = a^* - K$ is regular, and $L\phi|_L = T$. Finally,

$$\begin{aligned}
\Lambda(=, \phi|_L) &= \Lambda(=, \phi) \cap (L \times L), \\
\Lambda(\circ, \phi|_L) &= \Lambda(\circ, \phi) \cap (L \times L \times L),
\end{aligned}$$

and so $(L, \phi|_L)$ is a unary automatic presentation for T. 9.2

COROLLARY 9.3. *Let S be a semigroup. Then S is unary FA-presentable if and only if S^0 is unary FA-presentable.*

Proof of 9.3. For any semigroup S, the semigroup S^0 is the ordinal sum of S and the trivial semigroup $\{0\}$ with respect to the ordering $S > \{0\}$. Thus, by Proposition 9.1, S^0 is unary FA-presentable if S is. In the other direction, S is a finite Rees index subsemigroup of S^0 and so S is unary FA-presentable if S^0 is by Proposition 9.2. 9.3

[Proposition 5.1 could also be deduced from Proposition 9.2 and Proposition 9.1 (since S^1 is the ordinal sum of the semigroup S and trivial semigroup $\{1\}$ with respect to the ordering $\{1\} > S$) in a manner similar to Corollary 9.3.]

The converse of Proposition 9.2 does not hold: the following example gives an example of a semigroup S with a subsemigroup T of finite Rees index (indeed, $|S - T| = 1$) with T admitting a unary automatic presentation and S not admitting *any* automatic presentation, unary or otherwise.

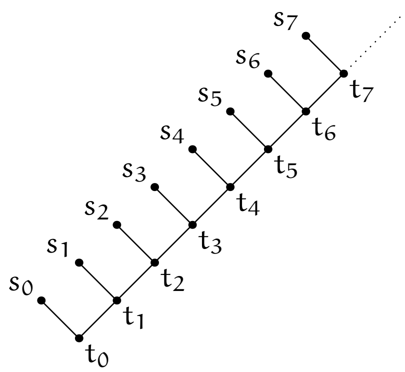


FIGURE 1. Hasse diagram for $(S, \leqslant)$.

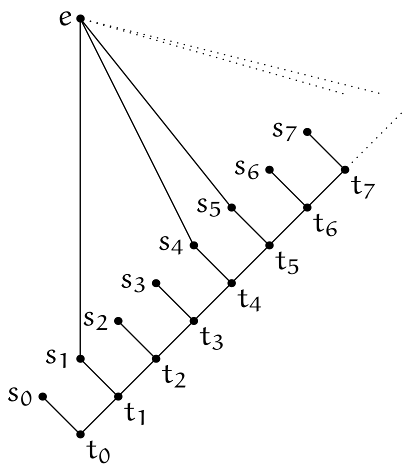


FIGURE 2. Hasse diagram for $(U, \leqslant)$, assuming for the sake of illustration that $1, 4, 5$ lie in Y.

EXAMPLE 9.4. Define a semilattice S as follows. The set of elements is $\{s_i, t_i : i \in \mathbb{N} \cup \{0\}\}$, and the order $\leqslant$ is defined on S as follows: for all $i, j \in \mathbb{N}$,

$$\begin{aligned} t_i \leqslant t_j &\iff i \leqslant j \\ t_i \leqslant s_j &\iff i \leqslant j \\ s_i \leqslant s_j &\iff i = j \\ s_i \nleqslant t_j. \end{aligned}$$

The Hasse diagram for $(S, \leqslant)$ is as illustrated in Figure 1.

Let $Y \subseteq \mathbb{N} \cup \{0\}$ be non-recursively enumerable. Let $U = S \cup \{e\}$ and extend the relation $\leqslant$ to S as follows: for $i \in \mathbb{N}$, by defining

$$\begin{aligned} t_i &\leqslant e \\ s_i \leqslant e &\iff i \in Y. \end{aligned}$$

The Hasse diagram for $(S, \leqslant)$ is as illustrated in Figure 2.

Define a mapping

$$\phi : a^* \to S, \quad \begin{cases} a^{2i} \mapsto s_i \\ a^{2i+1} \mapsto t_i. \end{cases}$$

First, notice that ϕ is injective, so $\Lambda(=, \phi) = \{a^n, a^n : n \in \mathbb{N} \cup \{0\}\}$. Furthermore,

$$\begin{aligned}
&\Lambda(\leqslant, \phi) \\
&= \{(a^m, a^n) : m, n \in \mathbb{N} \cup \{0\}, a^m\phi \leqslant a^n\phi\} \\
&= \{(a^{2i}, a^{2j}) : i, j \in \mathbb{N} \cup \{0\}, a^{2i}\phi \leqslant a^{2j}\phi\} \\
&\quad \cup \{(a^{2i}, a^{2j+1}) : i, j \in \mathbb{N} \cup \{0\}, a^{2i}\phi \leqslant a^{2j+1}\phi\} \\
&\qquad \cup \{(a^{2i+1}, a^{2j+1}) : i, j \in \mathbb{N} \cup \{0\}, a^{2i+1}\phi \leqslant a^{2j+1}\phi\} \\
&\qquad\quad \cup \{(a^{2i+1}, a^{2j}) : i, j \in \mathbb{N} \cup \{0\}, a^{2i+1}\phi \leqslant a^{2j}\phi\} \\
&= \{(a^{2i}, a^{2j}) : i, j \in \mathbb{N} \cup \{0\}, s_i \leqslant s_j\} \\
&\quad \cup \{(a^{2i}, a^{2j+1}) : i, j \in \mathbb{N} \cup \{0\}, s_i \leqslant t_j\} \\
&\qquad \cup \{(a^{2i+1}, a^{2j+1}) : i, j \in \mathbb{N} \cup \{0\}, t_i \leqslant t_j\} \\
&\qquad\quad \cup \{(a^{2i+1}, a^{2j}) : i, j \in \mathbb{N} \cup \{0\}, t_i \leqslant s_j\} \\
&= \{(a^{2i}, a^{2j}) : i, j \in \mathbb{N} \cup \{0\}, i = j\} \\
&\quad \cup \emptyset \qquad\qquad (\text{since } s_i \not\leqslant t_j) \\
&\qquad \cup \{(a^{2i+1}, a^{2j+1}) : i, j \in \mathbb{N} \cup \{0\}, i \leqslant j\} \\
&\qquad\quad \cup \{(a^{2i+1}, a^{2j}) : i, j \in \mathbb{N} \cup \{0\}, i \leqslant j\},
\end{aligned}$$

which is regular. Thus (a^*, ϕ) is a unary automatic presentation for $(S, \leqslant)$.

Suppose for *reductio ad absurdum* that $(U, \leqslant)$ admits an automatic presentation (K, ϕ). The aim is obtain a contradiction by showing that the set Y is effectively enumerable. Without loss of generality, assume by Proposition 2.5 that ϕ is injective. Let

$$\sigma(x, y) = (x < y) \wedge (\forall z \in U)(x < z \implies y \leqslant z).$$

For any $x \in U$, let $\Sigma(x)$ be the set of elements $y \in U$ such that $\sigma(x, y)$. Then $\Sigma(x)$ consists of the set of minimal elements lying above x in the semilattice. That is,

$$\begin{aligned}
\Sigma(s_i) &= \begin{cases} \{e\} & \text{if } i \in Y \\ \emptyset & \text{if } i \notin Y \end{cases} \\
\Sigma(t_i) &= \{t_{i+1}, s_i\} \\
\Sigma(e) &= \emptyset.
\end{aligned}$$

Since σ is a first-order formula, given a word representing some element x, a set of at most two words representing the elements of the set $\Sigma(x)$ can be found effectively.

First, let $u_0 \in K$ and $v \in K$ be the unique words with $u_0\phi = t_0$ and $v\phi = e$. The procedure enumerating Y stores a word u_i and the subscript i between iterations.

Each iteration of the procedure is as follows: For a word u_i representing t_i, find the set of words representing $\Sigma(t_i)$. This set consists of two words w_1, w_2, one representing t_{i+1} and one representing s_i. Find words representing the elements of the sets $\Sigma(w_1\phi)$ and $\Sigma(w_2\phi)$; whichever word w_j has $\Sigma(w_j\phi)$ consisting of exactly two words must represent t_{i+1}. Set $u_{i+1} = w_j$. The other word represents s_i and so the set of words representing $\Sigma(s_i)$ can be effectively calculated. This set is non-empty if and only if $i \in Y$: in this case, output the subscript i. This completes the iteration and the procedure continues from the start of this paragraph.

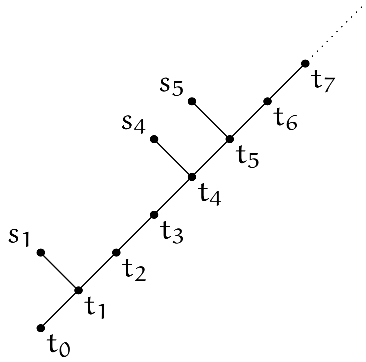


FIGURE 3. Hasse diagram for $(T, \leqslant)$, assuming for the sake of illustration that 1, 4, 5 lie in Y.

This procedure enumerates the elements of Y. This is a contradiction since Y is not recursively enumerable, and so $(U, \leqslant)$ cannot admit an automatic presentation.

EXAMPLE 9.5. Let $(S, \leqslant)$ be the semilattice from Example 9.4. Let $Y \subseteq \mathbb{N} \cup \{0\}$ be non-recursively enumerable and let $T = \{t_i : i \in \mathbb{N} \cup \{0\}\} \cup \{s_i : i \in Y\}$. Then T is a subsemilattice of S, and the Hasse diagram of $(T, \leqslant)$ is as illustrated in Figure 3.

Suppose for *reductio ad absurdum* that $(T, \leqslant)$ admits an automatic presentation (K, ϕ). The aim is obtain a contradiction by showing that the set Y is effectively enumerable. Without loss of generality, assume that ϕ is injective. Let

$$\sigma(x, y) = (x < y) \wedge (\forall z \in U)(x < z \implies y \leqslant z).$$

For any $x \in U$, let $\Sigma(x)$ be the set of elements $y \in U$ such that $\sigma(x, y)$. Then $\Sigma(x)$ consists of the set of minimal elements lying above x in the semilattice. That is

$$\Sigma(s_i) = \emptyset$$
$$\Sigma(t_i) = \begin{cases} \{t_{i+1}, s_i\} & \text{if } i \in Y. \\ \{t_{i+1}\} & \text{if } i \notin Y \end{cases}$$

Since σ is a first-order formula, given a word representing some element x, a set of at most two words representing the elements of the set $\Sigma(x)$ can be found effectively.

First, let $u_0 \in K$ be the unique word with $u_0\phi = t_0$. The procedure enumerating Y stores a word u_i and the subscript i between iterations.

Each iteration of the procedure is as follows: For a word u_i representing t_i, find the set of words representing $\Sigma(t_i)$. If this set consists of a single word w, set $u_{i+1} = w$ and continue from the start of this paragraph. If the set consists of two words w_1, w_2, then one of these words represents t_{i+1} and one represents s_i. Find words representing the elements of the sets $\Sigma(w_1\phi)$ and $\Sigma(w_2\phi)$; whichever word w_j has $\Sigma(w_j\phi)$ non-empty must represent t_{i+1}. Set $u_{i+1} = w_j$. Output the index i, since in this case $i \in Y$. This completes the iteration and the procedure continues from the start of this paragraph.

This procedure enumerates the elements of Y. This is a contradiction since Y is not recursively enumerable, and so $(T, \leqslant)$ cannot admit an automatic presentation.

The next two results show, respectively, that the class of unary FA-presentable semigroups is closed under forming finite-by-finite Rees matrix semigroups, and that it includes all finite-by-countable Rees matrix semigroups over finite semigroups. Recall that a Rees matrix semigroup $\mathcal{M}[T; I, J; P]$, where T is a semigroup, I and J are abstract (possibly infinite) index sets, and P is a $J \times I$ matrix with entries from T, is a semigroup with underlying set $I \times T \times J$ and multiplication given by

$$(i, t, j)(k, u, \ell) = (i, tp_{j,k}u, \ell).$$

(See [CP61, § 3.1] or [How95, § 3.2] for further information on Rees matrix semigroups.)

PROPOSITION 9.6. *Any finite-by-finite Rees matrix semigroup over a unary FA-presentable semigroup is unary FA-presentable. More precisely, if* $S = \mathcal{M}[T; I, J; P]$*, where* I *and* J *are finite,* T *is unary FA-presentable, and* P *is a* $J \times I$ *matrix over* T*, then* S *is unary FA-presentable.*

Proof of 9.6. If T is finite, so is S and so S is unary FA-presentable. So assume T is infinite. Then by Theorem 3.1, T admits a unary automatic presentation (a^*, ϕ). Suppose that $I = \{0, \ldots, n_i - 1\}$ and $J = \{0, \ldots, n_j - 1\}$.

Let $k = n_i n_j$. Define a map

$$\psi : b^* \to S, \quad b^\alpha \psi = \big((\alpha \bmod k) \bmod n_j, a^{\lfloor \alpha/k \rfloor}\phi, \lfloor (\alpha \bmod k)/n_j \rfloor\big),$$

where $\alpha \bmod k$ is interpreted as the unique $h \in \mathbb{N}$ with $0 \leqslant h < k$ and $h \equiv \alpha \pmod{k}$. Since $n_j \mid k$,

$$b^\alpha \psi = \big(\alpha \bmod n_j, a^{\lfloor \alpha/k \rfloor}\phi, \lfloor (\alpha \bmod k)/n_j \rfloor\big).$$

The idea of the map ψ is that $b^{mk}, b^{mk+1}, \ldots, b^{mk+(k-1)}$ represent all elements of S of the form $(i, a^m\phi, j)$, with the exponent taken modulo k determining i and j.

For all $i \in I, j \in J$, let $p_{j,i} \in G$ be the (j, i)-th element of P. The relation

$$R'_{j,i} = \big\{(a^{\beta_1}, a^{\beta_2}, a^{\beta_3}) : \beta_i \in \mathbb{N} \cup \{0\}, (a^{\beta_1}\phi)p_{j,i}(a^{\beta_2}\phi) = a^{\beta_3}\phi\big\}$$

is first-order definable in terms of ϕ and so is regular. From an automaton recognizing $\operatorname{conv} R'_{j,i}$ it is easy to construct one recognizing $\operatorname{conv} R_{j,i}$, where

$$\begin{aligned} R_{j,i} &= \big\{(b^{k\beta_1+\beta'_1}, b^{k\beta_2+\beta'_2}, b^{k\beta_3+\beta'_3}) : \beta_i \in \mathbb{N} \cup \{0\}, \beta'_i < k, (a^{\beta_1}\phi)p_{j,i}(a^{\beta_2}\phi) = a^{\beta_3}\phi\big\} \\ &= \big\{(b^{\alpha_1}, b^{\alpha_2}, b^{\alpha_3}) : \alpha_i \in \mathbb{N} \cup \{0\}, (a^{\lfloor \alpha_1/k \rfloor}\phi)p_{j,i}(a^{\lfloor \alpha_2/k \rfloor}\phi) = a^{\lfloor \alpha_3/k \rfloor}\phi\big\}. \end{aligned}$$

Then

$$\begin{aligned} &\Lambda(\circ, \psi) \\ &= \big\{(b^\alpha, b^\beta, b^\gamma) : \alpha, \beta, \gamma \in \mathbb{N} \cup \{0\}, (b^\alpha\psi)(b^\beta\psi) = (b^\gamma\psi)\big\} \\ &= \Big\{(b^\alpha, b^\beta, b^\gamma) : \alpha, \beta, \gamma \in \mathbb{N} \cup \{0\}, \\ &\qquad \big(\alpha \bmod n_j, a^{\lfloor \alpha/k \rfloor}\phi, \lfloor (\alpha \bmod k)/n_j \rfloor\big) \\ &\qquad\quad \circ \big(\beta \bmod n_j, a^{\lfloor \beta/k \rfloor}\phi, \lfloor (\beta \bmod k)/n_j \rfloor\big) \\ &\qquad\qquad = \big(\gamma \bmod n_j, a^{\lfloor \gamma/k \rfloor}\phi, \lfloor (\gamma \bmod k)/n_j \rfloor\big)\Big\} \end{aligned}$$

$$
\begin{aligned}
&= \Big\{ (b^\alpha, b^\beta, b^\gamma) : \alpha, \beta, \gamma \in \mathbb{N} \cup \{0\}, \\
&\qquad (\alpha \bmod n_j, (a^{\lfloor \alpha/k \rfloor}\phi) p_{\lfloor (\alpha \bmod k)/n_j \rfloor, \beta \bmod n_j}(a^{\lfloor \beta/k \rfloor}\phi), \lfloor (\beta \bmod k)/n_j \rfloor) \\
&\qquad\qquad = (\gamma \bmod n_j, a^{\lfloor \gamma/k \rfloor}\phi, \lfloor (\gamma \bmod k)/n_j \rfloor) \Big\} \\
&= \Big\{ (b^\alpha, b^\beta, b^\gamma) : \alpha, \beta, \gamma \in \mathbb{N} \cup \{0\}, \\
&\qquad \alpha \bmod n_j = \gamma \bmod n_j \wedge \lfloor (\beta \bmod k)/n_j \rfloor = \lfloor (\gamma \bmod k)/n_j \rfloor \\
&\qquad\qquad \wedge (a^{\lfloor \alpha/k \rfloor}\phi) p_{\lfloor (\alpha \bmod k)/n_j \rfloor, \beta \bmod n_j}(a^{\lfloor \beta/k \rfloor}\phi) = a^{\lfloor \gamma/k \rfloor}\phi \Big\} \\
&= \Big\{ (b^\alpha, b^\beta, b^\gamma) : \alpha, \beta, \gamma \in \mathbb{N} \cup \{0\}, \\
&\qquad \alpha \bmod n_j = \gamma \bmod n_j \wedge \lfloor (\beta \bmod k)/n_j \rfloor = \lfloor (\gamma \bmod k)/n_j \rfloor \\
&\qquad\qquad \wedge (a^{\lfloor \alpha/k \rfloor}, a^{\lfloor \beta/k \rfloor}, a^{\lfloor \gamma/k \rfloor}) \in R_{\lfloor (\alpha \bmod k)/n_j \rfloor, \beta \bmod n_j} \Big\}.
\end{aligned}
$$

Since the relations $R_{j,i}$ are all regular, and since a finite automaton can track integers modulo n_j and modulo k, it follows that $\Lambda(\circ, \psi)$ is regular, and hence (b^*, ψ) is an automatic presentation for S. 9.6

The following example, which is a modified version of a discussion in [CORT10, § 8], shows that the converse of Proposition 9.6 does not hold:

EXAMPLE 9.7. Let F be the free semigroup with basis $\{x\}$. Form the Rees matrix semigroup $S = \mathcal{M}[F^0; I, J; P]$, where $I = J = \{1\}$ and let P is the $J \times I$ matrix whose single entry is 0. So the underlying set of S is $\{1\} \times (\{0\} \cup \{x^\alpha : \alpha \in \mathbb{N}\}) \times \{1\}$, and every product in T is $(1, 0, 1)$ because the single entry of P is 0.

Define a map

$$
\phi : a^* \to S, \quad a^\alpha \mapsto \begin{cases} (1, 0, 1) & \text{if } \alpha = 0 \\ (1, x^\alpha, 1) & \text{if } \alpha \neq 0. \end{cases}
$$

Then ϕ is injective, so $\Lambda(=, \phi) = \{(a^\alpha, a^\alpha) : \alpha \in \mathbb{N} \cup \{0\}\}$, which is regular. Furthermore,

$$
\begin{aligned}
\Lambda(\circ, \phi) &= \{(a^\alpha, a^\beta, a^\gamma) : \alpha, \beta, \gamma \in \mathbb{N} \cup \{0\}, (a^\alpha\phi)(a^\beta\phi) = a^\gamma\phi\} \\
&= \{(a^\alpha, a^\beta, a^\gamma) : \alpha, \beta, \gamma \in \mathbb{N} \cup \{0\}, a^\gamma\phi = (1, 0, 1)\} \\
&= \{(a^\alpha, a^\beta, \varepsilon) : \alpha, \beta \in \mathbb{N} \cup \{0\}\},
\end{aligned}
$$

so that $\Lambda(\circ, \phi)$ is regular. Hence (a^*, ϕ) is a unary automatic presentation for S.

However, the base semigroup F^0 is finitely generated and infinite, and therefore cannot be unary FA-presentable by Corollary 6.2.

PROPOSITION 9.8. *Any finite-by-countable Rees matrix semigroup over a finite semigroup is unary FA-presentable. More precisely, if* $S = \mathcal{M}[T, I, J, P]$*, where one of* I *and* J *is finite and the other countable,* T *is finite, and* P *is a* $J \times I$ *matrix over* T*, then* S *is unary FA-presentable.*

Proof of 9.8. Let $S = \mathcal{M}[T, I, J, P]$. Assume that I is finite and J is countable, with $I = \{0, \ldots, n_i - 1\}$ and $J = \mathbb{N} \cup \{0\}$. There are only finitely many distinct rows of the $J \times I$ matrix P. So some rows will appear only finitely many times, some will appear infinitely many times. Permute the rows as follows. The p rows that appear only finitely many times are placed first, in rows 0 up to $p - 1$.

The q rows that appear infinitely many times are arranged periodically from p onwards, so that for any $j \geqslant p$, row j is identical to row $((j-p) \bmod q)+p$. Permuting the rows thus yields a semigroup isomorphic to the original Rees matrix semigroup, so assume without loss of generality that P has already been arranged in this way.

Let the elements of the finite semigroup T be $t_0, \ldots, t_{r-1}$. Let $k = n_i r$. Define a map

$$\phi : a^* \to S, \quad a^\alpha \mapsto \big(\lfloor(\alpha \bmod k)/r\rfloor, t_{\alpha \bmod r}, \lfloor \alpha/k\rfloor\big)$$

It is easy to see that ϕ is injective and so $\Lambda(=,\phi) = \{(a^\alpha, a^\alpha) : \alpha \in \mathbb{N} \cup \{0\}\}$, which is regular.

For all $i \in I, j \in \mathbb{N}$, let $p_{j,i} \in T$ be the (j,i)-th element of P. The relation

$$R_{j,i} = \big\{(a^\alpha, a^\beta, a^\gamma) : \alpha, \beta, \gamma \in \mathbb{N} \cup \{0\}, t_{\alpha \bmod r} p_{j,i} t_{\beta \bmod r} = t_{\gamma \bmod r}\big\}$$

is regular since a finite automaton can track the α, β, and γ modulo r. Notice further that for any $i \in I, j \in \mathbb{N}$ with $j \geqslant p$, the relations $R_{j,i}$ and $R_{((j-p) \bmod q)+p,i}$ are equal. For convenience later in the proof, define

$$\pi : \mathbb{N} \to \mathbb{N}, \quad j \mapsto \begin{cases} j & \text{if } j < p \\ ((j-p) \bmod q)+p & \text{if } j \geqslant p, \end{cases}$$

so that $R_{j,i}$ and $R_{j\pi,i}$ are equal for all $i \in I$ and $j \in \mathbb{N}$.

The relation

$$\begin{aligned} F_k &= \big\{(a^\beta, a^\gamma) : \beta, \gamma \in \mathbb{N} \cup \{0\}, \lfloor\beta/k\rfloor = \lfloor\gamma/k\rfloor\big\} \\ &= \big\{(a^{k\eta}, a^{k\eta}) : \eta \in \mathbb{N} \cup \{0\}\big\}\big\{(a^\mu, a^\nu) : \mu, \nu \in \{0, \ldots, k-1\}\big\} \end{aligned}$$

is also regular.

Furthermore,

$$\begin{aligned}
&\Lambda(\circ, \phi) \\
&= \{(a^\alpha, a^\beta, a^\gamma) : \alpha, \beta, \gamma \in \mathbb{N} \cup \{0\}, (a^\alpha\phi)(a^\beta\phi) = (a^\gamma\phi)\} \\
&= \big\{(a^\alpha, a^\beta, a^\gamma) : \alpha, \beta, \gamma \in \mathbb{N} \cup \{0\}, \\
&\qquad \big(\lfloor(\alpha \bmod k)/r\rfloor, t_{\alpha \bmod r}, \lfloor\alpha/k\rfloor\big)\big(\lfloor(\beta \bmod k)/r\rfloor, t_{\beta \bmod r}, \lfloor\beta/k\rfloor\big) \\
&\qquad\qquad = \big(\lfloor(\gamma \bmod k)/r\rfloor, t_{\gamma \bmod r}, \lfloor\gamma/k\rfloor\big)\big\} \\
&= \big\{(a^\alpha, a^\beta, a^\gamma) : \alpha, \beta, \gamma \in \mathbb{N} \cup \{0\}, \\
&\qquad \big(\lfloor(\alpha \bmod k)/r\rfloor, t_{\alpha \bmod r} p_{\lfloor\alpha/k\rfloor, \lfloor(\beta \bmod k)/r\rfloor} t_{\beta \bmod r}, \lfloor\beta/k\rfloor\big) \\
&\qquad\qquad = \big(\lfloor(\gamma \bmod k)/r\rfloor, t_{\gamma \bmod r}, \lfloor\gamma/k\rfloor\big)\big\} \\
&= \big\{(a^\alpha, a^\beta, a^\gamma) : \alpha, \beta, \gamma \in \mathbb{N} \cup \{0\}, \\
&\qquad \lfloor(\alpha \bmod k)/r\rfloor = \lfloor(\gamma \bmod k)/r\rfloor \wedge \lfloor\beta/k\rfloor = \lfloor\gamma/k\rfloor \\
&\qquad\qquad \wedge t_{\alpha \bmod r} p_{\lfloor\alpha/k\rfloor, \lfloor(\beta \bmod k)/r\rfloor} t_{\beta \bmod r} = t_{\gamma \bmod r}\big\} \\
&= \big\{(a^\alpha, a^\beta, a^\gamma) : \alpha, \beta, \gamma \in \mathbb{N} \cup \{0\}, \\
&\qquad \lfloor(\alpha \bmod k)/r\rfloor = \lfloor(\gamma \bmod k)/r\rfloor \wedge \lfloor\beta/k\rfloor = \lfloor\gamma/k\rfloor \\
&\qquad\qquad \wedge (a^\alpha, a^\beta, a^\gamma) \in R_{\lfloor\alpha/k\rfloor, \lfloor(\beta \bmod k)/r\rfloor}\big\} \\
&= \big\{(a^\alpha, a^\beta, a^\gamma) : \alpha, \beta, \gamma \in \mathbb{N} \cup \{0\}, \\
&\qquad \lfloor(\alpha \bmod k)/r\rfloor = \lfloor(\gamma \bmod k)/r\rfloor \wedge (a^\beta, a^\gamma) \in F_k \\
&\qquad\qquad \wedge (a^\alpha, a^\beta, a^\gamma) \in R_{\lfloor\alpha/k\rfloor\pi, \lfloor(\beta \bmod k)/r\rfloor}\big\}.
\end{aligned}$$

The relations $R_{j,i}$ and F_k are regular and an automaton can track integers modulo k and modulo p (the second being required by the definition of π). Thus the relation $\Lambda(\circ, \phi)$ is regular. Thus (a^*, ϕ) is a unary automatic presentation for S. 9.8

Proposition 9.8 does not extend to countable-by-countable Rees matrix semigroups as a consequence of Proposition 8.3, since if G is a group, $\mathcal{M}[G; I, J; P]$ consists of a single $\mathcal{D}$-class, and the $\mathcal{R}$- and $\mathcal{L}$-classes are respectively subsets of the form $\{i\} \times G \times J$ and $I \times G \times \{j\}$, for $i \in I$ and $j \in J$ (see [How95, §§ 3.1–2]).

Since every completely simple semigroup is isomorphic to a Rees matrix semigroup over a group by the Rees–Suschkewitsch theorem [How95, Theorem 3.3.1], Proposition 9.8 and the results of § 8 yield a complete classification of unary FA-presentable completely simple semigroups:

THEOREM 9.9. *A completely simple semigroup is unary FA-presentable if and only if it is either a finite semigroup or a finite-by-countable Rees matrix semigroup over a finite group.*

Proof of 9.9. First of all, let S be a unary FA-presentable completely simple semigroup; the aim is to show that S is of one of the two species given. Then $S = \mathcal{M}[G; I, J; P]$, where G is a group and P is a $J \times I$ matrix over G. By Proposition 8.5, the group G, being isomorphic to any $\mathcal{H}$-class of S, is finite. By Proposition 8.3, at least one of I and J is finite. Since S, like all FA-presentable structures, is either finite or countable, if one of I or J is infinite, it must be countable and so S is a finite-by-countable Rees matrix semigroup over the finite group G. If both I and J are finite, then S is finite.

Any finite semigroup is unary FA-presentable, and a finite-by-countable Rees matrix semigroup over a finite group is unary FA-presentable by Proposition 9.8. 9.9

9.3 *Direct products*

If G is the trivial group and I and J are countable, the Rees matrix semigroup $\mathcal{M}[G; I, J; P]$ is the countable-by-countable rectangular band, which is isomorphic to the direct product of a countable left zero semigroup and a countable right zero semigroup. Since countable left zero and right zero semigroups are unary FA-presentable Example 4.2, it follows that the class of unary FA-presentable semigroups is not closed under forming direct products. This contrasts the classes of general FA-presentable semigroups and general FA-presentable structures, both of which are closed under finite direct products [Blu99, Corollary 5.2.6(i)]. However, the class of unary FA-presentable semigroups is closed under forming direct products with finite semigroups:

PROPOSITION 9.10. *A direct product of a unary FA-presentable semigroup and a finite semigroup is itself unary FA-presentable.*

Proof of 9.10. Suppose S is a unary FA-presentable semigroup and T is finite. If S is finite, so is $S \times T$ and there is nothing to prove. So suppose S admits an injective unary automatic presentation (a^*, ϕ). Suppose the elements of T are $t_0, \ldots, t_{r-1}$.

Define a map

$$\psi : b^* \to S \times T, \quad b^\alpha \mapsto \left(a^{\lfloor \alpha/r \rfloor}\phi, \alpha \bmod r\right).$$

Then ψ is injective, so $\Lambda(=,\psi) = \{(b^\alpha, b^\alpha) : \alpha \in \mathbb{N} \cup \{0\}\}$ is regular. Reasoning similar to the proof of Proposition 9.6 shows that $\Lambda(\circ,\psi)$ is regular. So (b^*, ψ) is a unary automatic presentation for $S \times T$. 9.10

However, a direct product of two unary FA-presentable semigroups *may* be unary FA-presentable. For example, the direct product of two countable right zero semigroups is again a countable right zero semigroup.

QUESTION 9.11. Given unary automatic presentations for two semigroups, is it decidable whether their direct product is unary FA-presentable?

9.4 *Free products*

The *semigroup* free product of two semigroups never satisfies a non-trivial semigroup identity, so by Theorem 7.3, no semigroup free product is unary FA-presentable. A *monoid* free product of two monoids only satisfies a non-trivial semigroup identity if and only if one of the monoids is trivial and the other monoid satisfies a non-trivial semigroup identity. In this case, the free product is isomorphic to the second monoid. Therefore, no non-trivial free products are unary FA-presentable, which is perhaps unsurprising given how restricted is the class of semigroup or monoid free products that admit general FA-presentations [CORT10, § 4].

10 REFERENCES

[Blu99] A. Blumensath. *Automatic Structures*. Diploma thesis, RWTH Aachen, 1999. URL: www.mathematik.tu-darmstadt.de/~blumensath/Publications/AutStr.pdf.

[Cli54] A. H. Clifford. 'Naturally totally ordered commutative semigroups'. *Amer. J. Math.*, 76 (1954), pp. 631–646. URL: www.jstor.org/stable/2372706.

[CORT08] A. J. Cain, G. Oliver, N. Ruškuc, & R. M. Thomas. 'Automatic presentations for cancellative semigroups'. In C. Martín-Vide, H. Fernau, & F. Otto, eds, *Language and Automata Theory and Applications: Second International Conference, Tarragona, Spain, March 13–19, 2008*, no. 5196 in *Lecture Notes in Computer Science*, pp. 149–159. Springer, 2008. DOI: 10.1007/978-3-540-88282-4_15.

[CORT09] A. J. Cain, G. Oliver, N. Ruškuc, & R. M. Thomas. 'Automatic presentations for semigroups'. *Inform. and Comput.*, 207, no. 11 (2009), pp. 1156–1168. DOI: 10.1016/j.ic.2009.02.005.

[CORT10] A. J. Cain, G. Oliver, N. Ruškuc, & R. M. Thomas. 'Automatic presentations and semigroup constructions'. *Theory Comput. Syst.*, 47, no. 2 (2010), pp. 568–592. DOI: 10.1007/s00224-009-9216-4.

[CP61] A. H. Clifford & G. B. Preston. *The Algebraic Theory of Semigroups* (Vol. I). No. 7 in *Mathematical Surveys*. American Mathematical Society, Providence, R.I., 1961.

[Del04] C. Delhommé. 'Automaticité des ordinaux et des graphes homogènes'. *C. R. Math. Acad. Sci. Paris*, 339, no. 1 (2004), pp. 5–10. DOI: 10.1016/j.crma.2004.03.035.

[How95] J. M. Howie. *Fundamentals of Semigroup Theory*, vol. 12 of *London Mathematical Society Monographs* (New Series). Clarendon Press, Oxford University Press, New York, 1995.

[HU79] J. E. Hopcroft & J. D. Ullman. *Introduction to Automata Theory, Languages, and Computation*. Addison–Wesley Publishing Co., Reading, Mass., 1979.

[KN95] B. Khoussainov & A. Nerode. 'Automatic presentations of structures'. In *Logic and computational complexity (Indianapolis, IN, 1994)*, vol. 960 of *Lecture Notes in Computer Science*, pp. 367–392. Springer, Berlin, 1995. DOI: 10.1007/3-540-60178-3_93.

[KNRS04] B. Khoussainov, A. Nies, S. Rubin, & F. Stephan. 'Automatic structures: richness and limitations'. In *Proceedings of the 19th IEEE Symposium on Logic in Computer Science*, pp. 110–119. IEEE Computer Society, 2004. DOI: 10.1109/LICS.2004.1319599.

[NT08] A. Nies & R. M. Thomas. 'FA-presentable groups and rings'. *J. Algebra*, 320, no. 2 (2008), pp. 569–585. DOI: 10.1016/j.jalgebra.2007.04.015.

[OT05] G. P. Oliver & R. M. Thomas. 'Automatic presentations for finitely generated groups'. In V. Diekert & B. Durand, eds, *22nd Annual Symposium on Theoretical Aspects of Computer Science (STACS'05), Stuttgart, Germany*, vol. 3404 of *Lecture Notes in Comput. Sci.*, pp. 693–704, Berlin, 2005. Springer. DOI: 10.1007/978-3-540-31856-9_57.