SYMMETRIES OF PROJECTED SYMMETRIC PATTERNS
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Abstract. In this paper we describe how the symmetry of a function is transformed by a projection or by a restriction — two operations that reduce the dimension of its domain.

We work with functions defined in $(n+1)$-dimensional domains and that are periodic along a lattice $\mathcal{L}$ containing $n+1$ noncolinear directions, *i.e.*, functions invariant under the action of a crystallographic group Γ, a subgroup of the Euclidean group $\mathbf{E}(n+1)$.

After projection into $\mathbf{R}^n$ (or restriction) we obtain functions that are invariant under the action of a subgroup Σ of $\mathbf{E}(n)$, related to the original crystallographic group Γ. Our main result describes a bijection relating Σ to a subgroup of Γ, the symmetries before and after the reduction in dimension.

We use both algebraic and analytic tools: the Fourier expansion of the Γ-invariant functions and the induced action of Γ in the space of Fourier coefficients are both essential in dealing with the two perspectives. Intermediate results relate the symmetry groups to the dual lattice $\mathcal{L}^*$.

1. Introduction

Patterns observed in reaction-diffusion experiments on thin layers are often explained by two-dimensional models. However there are patterns observed experimentally that are not expected in two-dimensions. Gomes [3] proposes that some of these arise as the projection into the plane of a three-dimensional repetitive solution.

We may ask in general how a projection transforms repetitive patterns — the projection may be seen either as a physical phenomenon or as a mathematical tool. The first perspective would be used whenever we observe solutions that are the integration, along some variables, of a solution in a higher dimensional space. As a mathematical tool, projection is a way of lowering the dimension in order to obtain desirable properties. This happens in the theoretical construction of quasicrystals, where the quasiperiodic three-dimensional structure may be obtained projecting a periodic structure in $\mathbf{R}^5$, see Senechal [9, section 2.6] for a description of this method.

1.1. **aim.** We study real functions with domain $\mathbf{R}^{n+1}$, periodic along $n+1$ noncolinear directions. The elements of $\mathbf{E}(n+1)$ that leave these functions invariant form a group Γ with a subgroup of translations corresponding to periods. These are called crystallographic groups since they are analogous, in the $(n+1)$-dimensional space, to the symmetry groups of crystals in $\mathbf{R}^3$.

We explore two ways of obtaining a new function on a n-dimensional subspace: either we project into the subspace (figure 1) or we take the restriction (figure 2).

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In both cases some structure will remain due to the initial symmetry. Our aim is to describe the symmetries of the functions defined in a lower dimension.

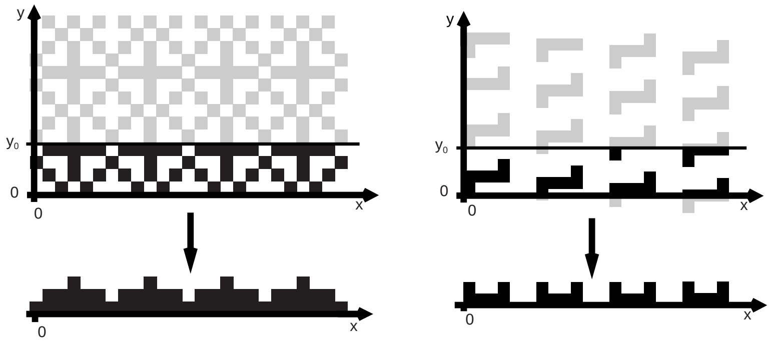

Figure 1. The projection of periodic patterns in $\mathbf{R}^2$ restricted to a strip of width y_0 defines functions with domain $\mathbf{R}$.

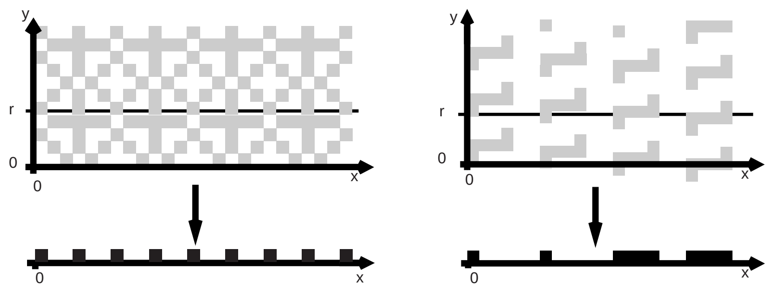

Figure 2. The restriction of periodic patterns in $\mathbf{R}^2$ to the line $y = r$ are functions with domain $\mathbf{R}$.

The symmetries of Γ-invariant functions are transformed by the projection. Some elements of Γ will yeld symmetries of the projected functions. Others may give rise to some other structure of the projected functions that is not a symmetry, *i.e.*, a structure that cannot be described as the invariance of the projected functions under the action of some element in $\mathbf{E}(n)$. An illustrative example of the second case is the quasiperiodic structure obtained by the canonical projection of a periodic one, see Senechal [9, section 2.6].

Our main result, Theorem 3.1, describes the elements of Γ that contribute to the symmetry after projection and the way these relevant elements of Γ are transformed by the projection. Formally, it states necessary and sufficient conditions upon Γ ensuring that all the functions in a space X_Γ are invariant, after projection, under the action of some element in $\mathbf{E}(n)$.

An analogous result for the restriction is presented in Theorem 7.1.

1.2. **how to.** Although we state Theorem 3.1 for a space of functions, it establishes a homomorphism between two symmetry groups. Thus it induces a projection in the space of crystallographic groups.

This relation between algebra and analysis is always present along the article. Our tools are either algebraic, like crystallographic groups and modules, or analytical, as Fourier expansion. The two perspectives are held together mostly by two reasons. By the induced action of the symmetry group Γ in the space of the Fourier coefficients of Γ-invariant functions, the symmetry is visible in the space of Fourier coefficients as equations that can be traced also after projection. The formulation of the results for sufficiently large spaces of Γ-invariant functions highlights their common characteristic, the symmetry.

Along the proof of Theorem 3.1 several interesting results arise. Proposition 4.1 states necessary and sufficient conditions on a crystallographic group Γ and its dual lattice, for the projected Γ-invariant functions to be symmetric. Proposition 4.1 is the generalization of the two-dimensional result presented in Labouriau and Pinho [4]. After this Proposition the structure of the dual lattice and its relation to the lattice associated to Γ are studied.

Theorem 7.1 presents, for the restriction, an analogue of Theorem 3.1. Its proof uses the same arguments developed for the projection and is simpler, since it has fewer conditions.

This paper extends and generalizes previous results obtained by the same authors [5] for the particular case $n = 1$, where Γ is a plane crystallographic group (wallpaper group). Some geometrical arguments cannot be easily extended to the present case, so we use a different treatment.

Periodicity and the remaining symmetries (parity) are studied separately in the first paper following an approach that is more intuitive. Here the aims are concision, simplicity and generality. All symmetries are treated together allowing us to deal with the higher complexity of the problem due to the dimension. The separate treatment would also be cumbersome since we would have to deal with more cases here.

2. Notation and Preliminaries

We work with real functions $f : \mathbf{R}^{n+1} \longrightarrow \mathbf{R}$, where $n \in \mathbf{N}$, and we use the notation $(x, y) \in \mathbf{R}^{n+1}$, with $x \in \mathbf{R}^n$ and $y \in \mathbf{R}$. The reader is referred to Armstrong [1, chapters 24, 25 and 26], for results on Euclidean and plane crystallographic groups, and to Senechal [9, chapter 2] and Miller [7, chapter 2] for results on lattices and crystallographic groups. A detailed description is also made in Pinho [8, chapter 2].

2.1. **the Euclidean group.** Consider the $(n+1)$-dimensional *Euclidean group* as the semi-direct product $\mathbf{E}(n+1) \cong \mathbf{R}^{n+1} \ltimes \mathbf{O}(n+1)$ with elements $\gamma = (v, \delta)$, where $v \in \mathbf{R}^{n+1}$ and $\delta \in \mathbf{O}(n+1)$. The group operation is $(v_1, \delta_1) \cdot (v_2, \delta_2) = (v_1 + \delta_1 v_2, \delta_1\delta_2)$, for (v_1, δ_1), $(v_2, \delta_2) \in \mathbf{E}(n+1)$ and the *action* of $(v, \delta) \in \mathbf{E}(n+1)$ on $(x, y) \in \mathbf{R}^{n+1}$ is given by $(v, \delta) \cdot (x, y) = v + \delta(x, y)$.

2.2. **crystallographic groups.** Let $\Gamma \leq \mathbf{E}(n+1)$ be a *crystallographic group* with *lattice* $\mathcal{L}$. Thus, by definition, the orbit, on $\mathbf{R}^{n+1}$, of the origin under the action of the subgroup of translations $\{v : (v, Id_{n+1}) \in \Gamma\}$, is a $\mathbf{Z}$-module generated by $n+1$ noncolinear vectors $l_1, \ldots, l_{n+1} \in \mathbf{R}^{n+1}$:

$$\mathcal{L} = \{l_1, \ldots, l_{n+1}\}_{\mathbf{Z}} = \left\{ \sum_{i=1}^{n+1} m_i l_i : m_i \in \mathbf{Z} \right\}.$$

We also use the symbol $\mathcal{L}$ for the subgroup of translations of Γ, since it is isomorphic to the group $(\mathcal{L}, +)$.

The projection $(v, \delta) \longmapsto \delta$, of Γ into $\mathbf{O}(n+1)$, has kernel $\mathcal{L}$ and image $\mathbf{J} = \{\delta : (v, \delta) \in \Gamma \text{ for some } v \in \mathbf{R}^{n+1}\}$, which is isomorphic to the finite quotient $\Gamma/\mathcal{L}$. Group $\mathbf{J}$ is called the *point group* of $\mathcal{L}$ and is a subgroup of the *holohedry* of $\mathcal{L}$: the largest subgroup of $\mathbf{O}(n+1)$ that leaves $\mathcal{L}$ invariant. Thus, $\mathbf{J}\mathcal{L} = \{\delta l : \delta \in \mathbf{J}, l \in \mathcal{L}\} = \mathcal{L}$.

The set of all the elements in Γ with *orthogonal component* $\delta \in \mathbf{J}$ is the coset $\mathcal{L} \cdot (v, \delta) = \{(l + v, \delta) : l \in \mathcal{L}\}$ for any $v \in \mathbf{R}^{n+1}$ such that $(v, \delta) \in \Gamma$. We use the symbol (v_δ, δ) for any element of that coset, *i.e.*, v_δ is the *non-orthogonal component* of $(v, \delta) \in \Gamma$ defined up to elements of $\mathcal{L}$.

Group Γ is thus characterized by the $n + 1$ generators of $\mathcal{L}$ plus a finite number of elements (v_δ, δ), with $\delta \in \mathbf{J}$.

2.3. Γ **acting on functions.** The action of Γ in $\mathbf{R}^{n+1}$ induces the scalar action: $(\gamma \cdot f)(x, y) = f(\gamma^{-1} \cdot (x, y))$ for $\gamma \in \Gamma$ and $(x, y) \in \mathbf{R}^{n+1}$, see Melbourne [6, section 2.1]. A function f is Γ-*invariant* if $(\gamma \cdot f)(x, y) = f(x, y)$, for all $\gamma \in \Gamma$ and all $(x, y) \in \mathbf{R}^{n+1}$.

We will work in X_Γ, a space of Γ-invariant functions f, for the scalar action. In particular, f is $\mathcal{L}$-invariant or a $\mathcal{L}$-*periodic* function, the generalization, for any dimension, of functions on the plane whose level curves form a periodic tiling.

2.4. **dual lattices.** We study Γ-invariant functions that have formal Fourier expansion in terms of the *waves* $\omega_k(x, y) = \mathrm{e}^{2\pi i < k,(x,y)>}$, where $k \in \mathbf{R}^{n+1}$ and $< \cdot, \cdot >$ is the usual inner product in $\mathbf{R}^{n+1}$. The set of all the elements $k \in \mathbf{R}^{n+1}$ such that ω_k is a $\mathcal{L}$-periodic function is the *dual lattice* of $\mathcal{L}$:

$$\mathcal{L}^* = \{k \in \mathbf{R}^{n+1} :< k, l_i >\in \mathbf{Z},\ i = 1, \ldots, n + 1\}.$$

It may be written as $\mathcal{L}^* = \{l_1^*, \ldots, l_{n+1}^*\}_{\mathbf{Z}}$, where $l_i^* \in \mathbf{R}^{n+1}$ and $< l_i^*, l_j >= \delta_{ij}$ for all $i, j \in \{1, \ldots, n + 1\}$.

The lattices $\mathcal{L}$ and $\mathcal{L}^*$ have the same holohedry and the *matrices of their bases*, respectively

$$M = \begin{pmatrix} l_1 \\ \vdots \\ l_{n+1} \end{pmatrix} \quad \text{and} \quad M^* = \begin{pmatrix} l_1^* \\ \vdots \\ l_{n+1}^* \end{pmatrix},$$

are related by $M^* = \left(M^{-1}\right)^T$.

2.5. **formal Fourier expansion.** The *formal Fourier expansion* of a function $f \in X_\Gamma$ is

$$f(x, y) = \sum_{k \in \mathcal{L}^*} \omega_k(x, y) C(k)$$

where $C : \mathcal{L}^* \longrightarrow \mathbf{C}$ are the *Fourier coefficients*. We assume that in X_Γ this expansion is unique. For a real function f, the coefficients have the restriction $\overline{C(k)} = C(-k)$.

2.6. Γ **acting on the Fourier coefficients.** From the action of Γ on X_Γ we get:

$$\begin{aligned} (v_\delta, \delta) \cdot f(x, y) &= \textstyle\sum_{k \in \mathcal{L}^*} \omega_k(\delta^{-1}(x, y))\omega_k(-\delta^{-1}v_\delta)C(k) \\ &= \textstyle\sum_{k \in \mathcal{L}^*} \omega_{\delta k}(x, y)\omega_{\delta k}(-v_\delta)C(k), \text{ by orthogonality of } \delta, \\ &= \textstyle\sum_{k \in \mathcal{L}^*} \omega_k(x, y)\omega_k(-v_\delta)C(\delta^{-1}k), \text{ because } \delta\mathcal{L}^* = \mathcal{L}^*. \end{aligned}$$

By the unicity of the Fourier expansion, this induces an action of Γ on the space of Fourier coefficients $(v_\delta, \delta) \cdot C(k) = \omega_k(-v_\delta)C(\delta^{-1}k)$. Analogously, the (v_δ, δ)-invariance of f implies $C(k) = \omega_k(-v_\delta)C(\delta^{-1}k)$ for all its Fourier coefficients.

The induced action of Γ in the space of Fourier coefficients connects the algebraic and analytical perspectives. It translates into analytic language those properties of a function arising from symmetry. Moreover it allows the separate study of the periodicity, whose information is carried by the waves, and the remaining relevant elements (v_δ, δ), which impose restrictions on the Fourier coefficients.

2.7. **the functions I_k.** The simplest Γ-invariant functions are the real and imaginary components of I_k, for $k \in \mathcal{L}^*$, given by:

$$I_k(x,y) = \sum_{\delta \in \mathbf{J}} \omega_{\delta k}(x,y)\omega_{\delta k}(-v_\delta)$$

and we will assume that they lie in X_Γ. Each function I_k, for $k \in \mathcal{L}^*$, is the sum of all the elements in the orbit of ω_k under the action of Γ.

2.8. **the projection operator.** For $y_0 > 0$, consider the restriction of f to the region between the *hyperplanes* $y = 0$ and $y = y_0$. The *projection operator* Π_{y_0} integrates this restriction of f along the width y_0, yielding a new function with domain $\mathbf{R}^n$:

$$\Pi_{y_0}(f)(x) = \int_0^{y_0} f(x,y)dy.$$

The region between $y = 0$ and $y = y_0$ is called the *projected band* or the *projection band*, and y_0 is called the *width of projection* or the *width of the projected band.*

2.9. **functions after projection.** If $f \in X_\Gamma$ then the *projected function* satisfies $\Pi_{y_0}(f)(x) = \int_0^{y_0} \sum_{k\in\mathcal{L}^*} \omega_k(x,y)C(k)dy$ and, when the integral and the summation commute,

$$\begin{aligned}\Pi_{y_0}(f)(x) &= \textstyle\sum_{k\in\mathcal{L}^*} \int_0^{y_0} \omega_k(x,y)C(k)dy \\ &= \textstyle\sum_{k\in\mathcal{L}^*} \omega_{k_1}(x)C(k_1,k_2)\int_0^{y_0} \omega_{k_2}(y)dy,\end{aligned}$$

where $k = (k_1, k_2)$, with $k_1 \in \mathbf{R}^n$ and $k_2 \in \mathbf{R}$. Grouping the terms with common n first components in $\mathcal{L}^*$, we obtain

$$\begin{aligned}\Pi_{y_0}(f)(x) &= \textstyle\sum_{k_1\in\mathcal{L}_1^*} \omega_{k_1}(x)\sum_{k_2:(k_1,k_2)\in\mathcal{L}^*} C(k_1,k_2)\int_0^{y_0} \omega_{k_2}(y)dy \\ &= \textstyle\sum_{k_1\in\mathcal{L}_1^*} \omega_{k_1}(x)D(k_1),\end{aligned}$$

where $\mathcal{L}_1^* = \{k_1 : (k_1,k_2) \in \mathcal{L}^*\}$ and $D(k_1) = \sum_{k_2:(k_1,k_2)\in\mathcal{L}^*} C(k_1,k_2)\int_0^{y_0} \omega_{k_2}(y)dy$. The coefficients $D(k_1)$ could be written as $D_{y_0}(k_1)$ since they depend on y_0. We avoid that notation in order to simplify the formalism.

2.10. **symmetry of projected functions.** The functions $\Pi_{y_0}(f)$ may be invariant under the action of some elements of the group $\mathbf{E}(n) \cong \mathbf{R}^n \ltimes \mathbf{O}(n)$. Using a notation similar to the $(n+1)$-dimensional case, $(v_\alpha, \alpha) \in \mathbf{E}(n)$ is a symmetry of $\Pi_{y_0}(f)$ if

$$(v_\alpha,\alpha)\cdot\Pi_{y_0}(f)(x) = \Pi_{y_0}(f)(x) \quad \forall x \in \mathbf{R}^n.$$

For $f \in X_\Gamma$ this is equivalent to

$$\sum_{k_1\in\mathcal{L}_1^*} \omega_{k_1}(x)D(k_1) = \sum_{k_1\in\mathcal{L}_1^*} \omega_{k_1}(\alpha^{-1}x)\omega_{k_1}(-\alpha^{-1}v_\alpha)D(k_1).$$

This equation imposes restrictions on the coefficients $D(k_1)$, see Lemma 5.1.

2.11. **restriction.** Let Φ_r be the operator that restricts the functions to the hyperplane $y = r$,

$$\Phi_r(f)(x) = f(x,r).$$

If $f \in X_\Gamma$ then, for $D(k_1) = \sum_{k_2:(k_1,k_2)\in\mathcal{L}^*} C(k_1,k_2)\omega_{k_2}(r)$, the restriction of f is

$$\Phi_r(f)(x) = \sum_{k\in\mathcal{L}^*} \omega_k(x,r)C(k) = \sum_{k_1\in\mathcal{L}_1^*} \omega_{k_1}(x)D(k_1).$$

The functions $\Phi_r(f)$ may be invariant under the action of some elements of the group $\mathbf{E}(n) \cong \mathbf{R}^n \ltimes \mathbf{O}(n)$, as described above for the projected functions.

2.12. **characterization of the space X_Γ.** We assume X_Γ is a vector space of functions such that:

(1) Γ is a $(n+1)$-dimensional crystallographic group with lattice $\mathcal{L}$, dual lattice $\mathcal{L}^*$ and point group $\mathbf{J}$,
(2) if $f \in X_\Gamma$ then:
  (i) $f : \mathbf{R}^{n+1} \longrightarrow \mathbf{R}$,
  (ii) f is a Γ-invariant function,
  (iii) f has a unique formal Fourier expansion in waves $\omega_k(x,y)$, $k \in \mathcal{L}^*$,
  (iv) the integral and the summation commute in the projection of f,
(3) $Re(I_k), Im(I_k) \in X_\Gamma$ for all $k \in \mathcal{L}^*$, and $I_k(x,y) = \sum_{\delta\in\mathbf{J}} \omega_{\delta k}(x,y)\omega_{\delta k}(-v_\delta)$.

3. Symmetry of $\Pi_{y_0}(X_\Gamma)$ related to Γ and $\mathcal{L}$

Now we state our main result, Theorem 3.1, relating the symmetry of the functions f in the space X_Γ to the symmetry of the projected functions $\Pi_{y_0}(f)$ in the space $\Pi_{y_0}(X_\Gamma)$.

For $\alpha \in \mathbf{O}(n)$, we define the elements of $\mathbf{O}(n+1)$:

$$\sigma = \begin{pmatrix} Id_n & 0 \\ 0 & -1 \end{pmatrix}, \quad \alpha_+ = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \alpha_- = \sigma\alpha_+ = \begin{pmatrix} \alpha & 0 \\ 0 & -1 \end{pmatrix}.$$

For simplicity of notation we write (v_+, α_+) for (v_{α_+}, α_+) and (v_-, α_-) for (v_{α_-}, α_-).

Theorem 3.1. *All functions in $\Pi_{y_0}(X_\Gamma)$ are invariant under the action of $(v_\alpha, \alpha) \in \mathbf{R}^n \ltimes \mathbf{O}(n)$ if and only if one of the following conditions holds:*

(I) $((v_\alpha, 0), \alpha_+) \in \Gamma$,
(II) $((v_\alpha, y_0), \alpha_-) \in \Gamma$,
(III) $(0, y_0) \in \mathcal{L}$ *and either* $((v_\alpha, y_1), \alpha_+) \in \Gamma$ *or* $((v_\alpha, y_1), \alpha_-) \in \Gamma$, *for some* $y_1 \in \mathbf{R}$.

3.1. **interpretation of Theorem 3.1.** Let $(x, y) \in \mathbf{R}^{n+1}$ with $x \in \mathbf{R}^n$ and $y \in \mathbf{R}$. Theorem 3.1 concerns those elements of Γ whose orthogonal component acts separately on the direction of projection ($x = 0$), and on the domain ($y = 0$) of the projected functions: both α_+ and α_- leave these subspaces invariant. Thus, Theorem 3.1 states that the symmetry of $\Pi_{y_0}(X_\Gamma)$ depends strongly on the way elements of Γ are related to the direction of projection.

After projection, and under some extra conditions, both α_+ and α_- ensure a symmetry with orthogonal component $\alpha \in \mathbf{O}(n)$. The translations associated to either α_+ or α_- are related by the conditions of Theorem 3.1 to the projection width. For each one of the three cases we give a brief intuitive description of how the symmetries that remain after projection correspond to the n first components of the original symmetries.

We present in figures 3, 4 and 5 some examples illustrating the conditions in Theorem 3.1, in the form of patterns. These may be interpreted as the level sets of functions $f : \mathbf{R}^2 \longrightarrow \mathbf{R}$, taking only the values 0 and 1, with $f(x, y) = 0$ on the white regions. After projection we obtain a function whose value for each $x \in \mathbf{R}$ is the width of the black region above it.

(I) Elements of Γ whose orthogonal component fixes the one-dimensional subspace $x = 0$, *i.e.*, those elements with orthogonal part α_+ and translation $(v_\alpha, 0)$, act effectively on the subspace $y = 0$ (see figure 3). This symmetry remains after the projection, independently of the projection width.

(II) Elements with orthogonal part α_- will contribute to the symmetry of $\Pi_{y_0}(X_\Gamma)$ if the associated translation is (v_α, y_0), *i.e.*, if its last component equals the width of the projection (figure 4). This happens because the restriction, f_{y_0}, of f to the

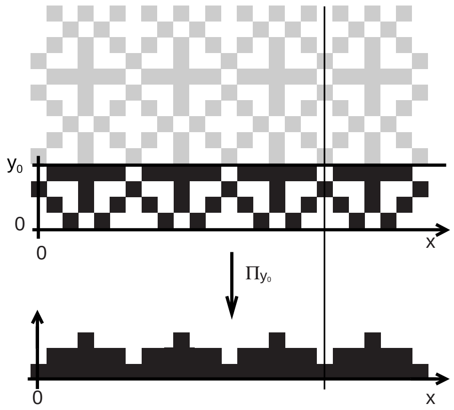

FIGURE 3. The invariance of the pattern under the reflection in the vertical line ensures an analogous symmetry after projection, for all y_0, by condition (I) of Theorem 3.1.

projection band is invariant under the action of these elements. Moreover their action does not mix the direction of projection with the other directions. Thus, when the coordinate y is collapsed by Π_{y_0}, what remains is a function with symmetry (v_α, α), the part of the original symmetry concerning the n first coordinates of f_{y_0}. If the width of the projection is changed, then these elements of Γ no longer induce symmetries of the projected functions.

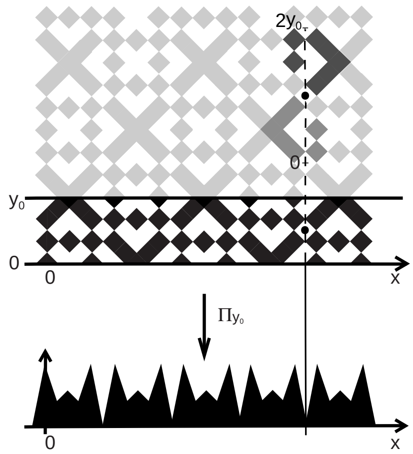

FIGURE 4. The rotations of π around the black dots are symmetries of the pattern. The rotation of π around the lower black dot leaves f_{y_0} (black region) invariant and, by condition (II) of Theorem 3.1, acts as a reflection for the projected pattern.

(III) If $\mathcal{L}$ has an element in the direction of the projection whose norm coincides with the projection width, *i.e.*, if $(0, y_0) \in \mathcal{L}$, then any element of Γ whose orthogonal component is either α_+ or α_- will induce some symmetry on all projected functions (figure 5). This happens because the pattern of f in $\mathbf{R}^{n+1}$ can be built by the repetition of f_{y_0}. Therefore, all the symmetries of f are somehow inscribed in f_{y_0}. Symmetries that act separately in the n first coordinates, remain when the last coordinate is removed.

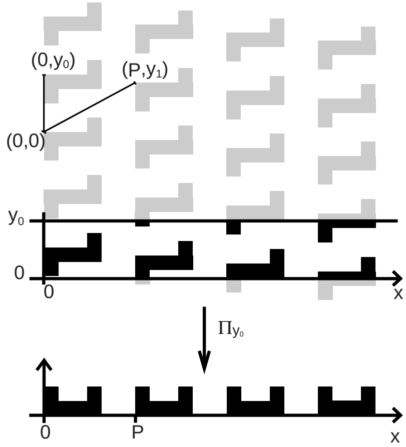

FIGURE 5. If $(0, y_0) \in \mathcal{L}$ then repeated vertical translations of the projected band of width y_0 (black region) cover the whole pattern. After the projection, (P, y_1) acts as a translation by P (condition (III) of Theorem 3.1).

For $n = 1$ we recover the results of Labouriau and Pinho [5] both on periodicity ($\alpha = 1$) and on parity of the projected functions ($\alpha = -1$).

3.2. **reformulation of Theorem 3.1 for symmetry groups.** Consider the subgroup Γ' of elements of Γ with orthogonal part $\alpha_\pm$ for some $\alpha \in \mathbf{O}(n)$, i.e., of all elements in Γ of the form

$$\left((v_\alpha, y_\alpha), \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\right) \qquad \text{with} \qquad \beta = \pm 1.$$

Theorem 3.1 states that the elements in Γ that effectively contribute to the symmetry of $\Pi_{y_0}(X_\Gamma)$ are those in a subgroup Γ'' of Γ'. If $(0, y_0) \notin \mathcal{L}$ then Γ'' contains those elements of Γ' where either $y_\alpha = 0$ with $\beta = 1$ or where $y_\alpha = y_0$ with $\beta = -1$. If $(0, y_0) \in \mathcal{L}$ then $\Gamma'' = \Gamma'$.

The map

$$\begin{array}{ccc} \Gamma' & \longrightarrow & \mathbf{E}(n) \cong \mathbf{R}^n \ltimes \mathbf{O}(n) \\ \left((v_\alpha, y_\alpha), \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\right) & \longmapsto & (v_\alpha, \alpha) \end{array}$$

defines a group homomorphism whose kernel is given by elements such that $v_\alpha = 0$ and $\alpha = Id_n$. Let Σ be the group of symmetries of $\Pi_{y_0}(X_\Gamma)$, i.e., the largest subgroup of $\mathbf{E}(n) \cong \mathbf{R}^n \ltimes \mathbf{O}(n)$ that fixes all the elements in $\Pi_{y_0}(X_\Gamma)$. Theorem 3.1 states that Σ is the image of Γ'' by this homomorphism.

3.3. **structure for the proof of Theorem 3.1.** Each one of the conditions (I), (II) and (III) of Theorem 3.1 is sufficient by basic properties of the integrals. Thus, we omit the proof of sufficiency for Theorem 3.1 and refer to Pinho [8] for details.

Proving that the conditions of Theorem 3.1 are necessary will be our main assignment. First, in Proposition 4.1 below we establish an equivalence between the (v_α, α)-invariance of all the functions in $\Pi_{y_0}(X_\Gamma)$ and a set of properties of the group Γ and of the dual lattice $\mathcal{L}^*$. Then, in section 6, we show that these properties impose restrictions on Γ and $\mathcal{L}$. More specifically, they will imply the presence of some particular elements in Γ, as in Theorem 3.1.

4. Symmetry of $\Pi_{y_0}(X_\Gamma)$ Related to Γ and $\mathcal{L}^*$

4.1. **structure of $\mathcal{L}^*$ due to σ.** The simultaneous presence of the reflection (v_σ,σ) and of (v_+,α_+) in a group Γ imposes strong restrictions on $\mathcal{L}^*$. One of these restrictions is fundamental for the formulation and the proof of Proposition 4.1 and is presented in next Lemma.

Lemma 4.1. *If both $(v_\sigma,\sigma)\in\Gamma$ and $(v_+,\alpha_+)\in\Gamma$ then $2(\sigma v_+ - v_+)\in\mathcal{L}$.*

Proof. Since $(v_\sigma,\sigma)\cdot(v_+,\alpha_+) = (v_\sigma+\sigma v_+,\alpha_-)$ and $(v_+,\alpha_+)\cdot(v_\sigma,\sigma) = (v_+ + \alpha_+ v_\sigma,\alpha_-)$, then $v = v_\sigma + \sigma v_+ - v_+ - \alpha_+ v_\sigma \in \mathcal{L}$. As $\sigma\mathcal{L} = \mathcal{L}$ then $v - \sigma v = 2(\sigma v_+ - v_+) + (Id_{n+1} - \alpha_+ - \sigma + \alpha_-)v_\sigma$ also belongs to $\mathcal{L}$. Using $-\alpha_+ - \sigma + \alpha_- = -Id_{n+1}$ we get $v - \sigma v = 2(\sigma v_+ - v_+)$ or, equivalently, $2 < k, \sigma v_+ - v_+ >\in \mathbf{Z}$ for all $k\in\mathcal{L}^*$. □

4.2. **how symmetry restricts the dual lattice $\mathcal{L}^*$.**

Proposition 4.1. *All functions in $\Pi_{y_0}(X_\Gamma)$ are invariant under the action of $(v_\alpha,\alpha)\in\mathbf{R}^n\ltimes\mathbf{O}(n)$ if and only if one of the following conditions holds:*

- (A) *$(v_+,\alpha_+)\in\Gamma$ and*
  for each $k\in\mathcal{L}^*$ either $< k,(0,y_0) >\in\mathbf{Z}-\{0\}$ or $< k, v_+ - (v_\alpha,0) >\in\mathbf{Z}$,
- (B) *$(v_-,\alpha_-)\in\Gamma$ and*
  for each $k\in\mathcal{L}^*$ either $< k,(0,y_0) >\in\mathbf{Z}-\{0\}$ or $< k, v_- - (v_\alpha,y_0) >\in\mathbf{Z}$,
- (C) *both $(v_\sigma,\sigma)\in\Gamma$ and $(v_+,\alpha_+)\in\Gamma$. Moreover, if $< k,\sigma v_+ - v_+ >\in\mathbf{Z}$ then one of the conditions (Ci), (Cii) or (Ciii) below holds and, if $< k,\sigma v_+ - v_+ > + \frac{1}{2}\in\mathbf{Z}$, one of the conditions (Ci) or (Civ) holds:*
  - (i) *$< k,(0,y_0) >\in\mathbf{Z}-\{0\}$,*
  - (ii) *$< k, v_+ - (v_\alpha,0) >\in\mathbf{Z}$,*
  - (iii) *$< k, v_\sigma - (0,y_0) > + \frac{1}{2}\in\mathbf{Z}$,*
  - (iv) *$< k, v_- - (v_\alpha,y_0) >\in\mathbf{Z}$ and*
    either $< k, v_\sigma - (0,y_0) > + \frac{1}{4}\in\mathbf{Z}$ or $< k, v_\sigma - (0,y_0) > - \frac{1}{4}\in\mathbf{Z}$.

4.3. **subsets of the dual lattice.** A more concise formulation of this result is possible using the subsets of $\mathcal{L}^*$ that we proceed to define. Let $\mathcal{M}^*$, $\mathcal{M}^*_+$ and $\mathcal{M}^*_-$ be the modules

$$\mathcal{M}^* = \{k\in\mathcal{L}^* : < k,\sigma v_+ - v_+ >\in\mathbf{Z}\}$$
$$\mathcal{M}^*_+ = \{k\in\mathcal{L}^* : < k, v_+ - (v_\alpha,0) >\in\mathbf{Z}\}$$
$$\mathcal{M}^*_- = \{k\in\mathcal{L}^* : < k, v_- - (v_\alpha,y_0) >\in\mathbf{Z}\}$$

and let

$$\mathcal{N}^* = \left\{k\in\mathcal{L}^* : < k,\sigma v_+ - v_+ > + \frac{1}{2}\in\mathbf{Z}\right\}$$
$$\mathcal{N}^*_{y_0} = \{k\in\mathcal{L}^* : < k,(0,y_0) >\in\mathbf{Z}-\{0\}\}$$
$$\mathcal{N}^*_\sigma = \left\{k\in\mathcal{L}^* : < k, v_\sigma - (0,y_0) > + \frac{1}{2}\in\mathbf{Z}\right\}$$
$$\mathcal{N}^*_{\tilde\sigma} = \left\{k\in\mathcal{L}^* : < k, v_\sigma - (0,y_0) > \pm\frac{1}{4}\in\mathbf{Z}\right\}.$$

The last four sets are not modules. The smallest modules generated by each of them are, respectively, $\overline{\mathcal{N}^*} = \mathcal{N}^*\cup\mathcal{M}^*$, which equals $\mathcal{L}^*$ under the conditions of Lemma 4.1,

$$\overline{\mathcal{N}^*_{y_0}} = \mathcal{N}^*_{y_0}\cup\mathcal{M}^*_{y_0},\quad \overline{\mathcal{N}^*_\sigma} = \mathcal{N}^*_\sigma\cup\mathcal{M}^*_\sigma \quad\text{and}\quad \overline{\mathcal{N}^*_{\tilde\sigma}} = \mathcal{N}^*_{\tilde\sigma}\cup\overline{\mathcal{N}^*_\sigma},$$

where all the unions are disjoint and $\mathcal{M}^*_{y_0}$ and $\mathcal{M}^*_\sigma$ are the modules

$$\mathcal{M}^*_{y_0} = \{k\in\mathcal{L}^* : < k,(0,y_0) >= 0\} \text{ and } \mathcal{M}^*_\sigma = \{k\in\mathcal{L}^* : < k, v_\sigma - (0,y_0) >\in\mathbf{Z}\}.$$

We summarize below some properties of $\mathcal{N}_\sigma^*$ and $\mathcal{N}_{\tilde\sigma}^*$ that will be used in the sequel.

Properties of $\mathcal{N}_\sigma^*$ and $\mathcal{N}_{\tilde\sigma}^*$. *Let $m_1, m_2 \in \mathbf{Z}$.*

(1) *If $g_1, g_2 \in \mathcal{N}_\sigma^*$ then* $m_1g_1 + m_2g_2 \in \begin{cases} \mathcal{M}_\sigma^* & \text{if} \quad m_1+m_2 \text{ even} \\ \mathcal{N}_\sigma^* & \text{if} \quad m_1+m_2 \text{ odd} \end{cases}$.

(2) *If $g_1, g_2 \in \mathcal{N}_{\tilde\sigma}^*$ then* $m_1g_1 + m_2g_2 \in \begin{cases} \overline{\mathcal{N}_\sigma^*} & \text{if} \quad m_1+m_2 \text{ even} \\ \mathcal{N}_{\tilde\sigma}^* & \text{if} \quad m_1+m_2 \text{ odd} \end{cases}$.

4.4. **reformulation of Proposition 4.1.** With this notation Proposition 4.1 may be written the following equivalent way:

Proposition 4.2. *All functions in $\Pi_{y_0}(X_\Gamma)$ are invariant under the action of $(v_\alpha, \alpha) \in \mathbf{R}^n \ltimes \mathbf{O}(n)$ if and only if one of the following conditions holds:*

(A) $(v_+, \alpha_+) \in \Gamma$ *and* $\mathcal{L}^* = \mathcal{N}_{y_0}^* \cup \mathcal{M}_+^*$,

(B) $(v_-, \alpha_-) \in \Gamma$ *and* $\mathcal{L}^* = \mathcal{N}_{y_0}^* \cup \mathcal{M}_-^*$,

(C) *both (v_σ, σ) and (v_+, α_+) belong to Γ and, moreover,* $\mathcal{M}^* \subset \left(\mathcal{N}_{y_0}^* \cup \mathcal{M}_+^* \cup \mathcal{N}_\sigma^*\right)$ *and* $\mathcal{N}^* \subset \left(\mathcal{N}_{y_0}^* \cup \left(\mathcal{M}_-^* \cap \mathcal{N}_{\tilde\sigma}^*\right)\right)$.

5. Proof of Proposition 4.1

5.1. **structure for the proof of Proposition 4.1.** There are three main steps in the proof of Proposition 4.1. First, in Lemma 5.1, we write the (v_α, α)-invariance of the projection of $f \in X_\Gamma$ as conditions relating the operator Π_{y_0} to the projection of the dual lattice $\mathcal{L}^*$ and to the coefficients of the formal Fourier expansion of f in waves. Second, we prove that the conditions (A), (B) and (C) are sufficient, writing explicitly the restrictions they impose on $\mathcal{L}^*$ and on the Fourier coefficients. Finally we conclude that the conditions of Proposition 4.1 are also necessary by the (v_α, α)-invariance of the projection of the functions I_k, whose real and imaginary components are the simplest Γ-invariant functions. This last part is divided into lemmas.

The main tools used in this proof are properties of waves and of Fourier coefficients, due to the symmetries in Γ and to the symmetry $(v_\alpha, \alpha) \in \mathbf{R}^n \ltimes \mathbf{O}(n)$, together with properties of the modules and subsets of $\mathcal{L}^*$ defined above.

5.2. **symmetry of $\Pi_{y_0}(X_\Gamma)$ related to $\mathcal{L}_1^*$.** For $\alpha\mathcal{L}_1^* = \{\alpha k_1 : k_1 \in \mathcal{L}_1^*\}$, we have:

Lemma 5.1. *Let $f \in X_\Gamma$ and $(v_\alpha, \alpha) \in \mathbf{R}^n \ltimes \mathbf{O}(n)$. The projection $\Pi_{y_0}(f)(x)$ is (v_α, α)-invariant if and only if for each $k_1 \in \mathcal{L}_1^*$ the following conditions hold:*

(1) *if $k_1 \in \mathcal{L}_1^* \cap \alpha\mathcal{L}_1^*$ then $D(k_1) = \omega_{k_1}(-v_\alpha)D(\alpha^{-1}k_1)$,*

(2) *if $k_1 \notin \mathcal{L}_1^* \cap \alpha\mathcal{L}_1^*$ then $D(k_1) = 0$.*

Proof. Notice first that the equality

$$\Pi_{y_0}(f)(x) = (v_\alpha, \alpha) \cdot \Pi_{y_0}(f)(x) = \Pi_{y_0}(f)(\alpha^{-1}x - \alpha^{-1}v_\alpha)$$

is equivalent to

$$\sum_{k_1 \in \mathcal{L}_1^*} \omega_{k_1}(x)D(k_1) = \sum_{k_1 \in \mathcal{L}_1^*} \omega_{k_1}(\alpha^{-1}x)\omega_{k_1}(-\alpha^{-1}v_\alpha)D(k_1), \tag{5.1}$$

where, by orthogonality, the right hand side equals $\sum_{k_1 \in \mathcal{L}_1^*} \omega_{\alpha k_1}(x)\omega_{\alpha k_1}(-v_\alpha)D(k_1)$ and, for $\tilde k_1 = \alpha k_1$, is given by $\sum_{\tilde k_1 \in \alpha\mathcal{L}_1^*} \omega_{\tilde k_1}(x)\omega_{\tilde k_1}(-v_\alpha)D(\alpha^{-1}\tilde k_1)$. Thus, by the unicity of the Fourier expansion, expression (5.1) is valid for all $x \in \mathbf{R}^n$ if and only if, for any $k_1 \in \mathcal{L}_1^*$, the conditions hold. □

5.3. **proof of sufficiency in Proposition 4.1.** For each case we write $D(k_1) - \omega_{k_1}(-v_\alpha)D(\alpha^{-1}k_1)$, in the form

$$\sum_{k_2:(k_1,k_2)\in\mathcal{L}^*} C(k_1,k_2)G(k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy,$$

and show that it is zero, by an explicit computation of $G(k_1,k_2)$.

Proof. Suppose either condition (A) or condition (B) happens. Since either α_+ or α_-, $\alpha_\pm = \begin{pmatrix}\alpha & 0\\ 0 & \pm 1\end{pmatrix}$, belongs to $\mathbf{J}$ then $\alpha_\pm\mathcal{L}^* = \{\alpha_\pm k : k\in\mathcal{L}^*\} = \mathcal{L}^*$, which implies $\alpha\mathcal{L}_1^* = \mathcal{L}_1^*$. Therefore, for any $f\in X_\Gamma$, the projection $\Pi_{y_0}(f)(x) = \sum_{k_1\in\mathcal{L}_1^*}\omega_{k_1}(x)D(k_1)$ is (v_α,α)-invariant if and only if condition (1) of Lemma 5.1 is valid for all $k_1\in\mathcal{L}_1^*$.

The $(v_\pm,\alpha_\pm)$-invariance of f implies $C(k) = \omega_k(-v_\pm)C(\alpha_\pm^{-1}k)$ for all its Fourier coefficients. Writing $\mathcal{L}_\pm^* = \{k_2 : (\alpha^{-1}k_1, \pm k_2)\in\mathcal{L}^*\}$ then $D(\alpha^{-1}k_1)$ is

$$\sum_{k_2\in\mathcal{L}_\pm^*} C(\alpha^{-1}k_1,\pm k_2)\int_0^{y_0}\omega_{\pm k_2}(y)dy = \sum_{k_2\in\mathcal{L}_\pm^*}\omega_k(v_\pm)C(k_1,k_2)\int_0^{y_0}\omega_{\pm k_2}(y)dy.$$

As $\{k_2 : (k_1,k_2)\in\mathcal{L}^*\} = \mathcal{L}_\pm^*$ and using, in the minus sign case, the property

$$\int_0^{y_0}\omega_{-k_2}(y)dy = \omega_{k_2}(-y_0)\int_0^{y_0}\omega_{k_2}(y)dy, \tag{5.2}$$

the above expressions equal either

$$\sum_{k_2:(k_1,k_2)\in\mathcal{L}^*}\omega_k(v_+)C(k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy$$

or

$$\sum_{k_2:(k_1,k_2)\in\mathcal{L}^*}\omega_k(v_-)\omega_{k_2}(-y_0)C(k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy.$$

Thus $D(k_1) - \omega_{k_1}(-v_\alpha)D(\alpha^{-1}k_1)$, for all $k_1\in\mathcal{L}_1^*$, is

$$\sum_{k_2:(k_1,k_2)\in\mathcal{L}^*}\left(1-\omega_k(v_\pm - (v_\alpha,\beta_\pm))\right)C(k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy,$$

with $\beta_+ = 0$ and $\beta_- = y_0$, which is zero because either $\int_0^{y_0}\omega_{k_2}(y)dy = 0$, if $< k,(0,y_0) >\in\mathbf{Z}-\{0\}$, or $1-\omega_k(v_\pm-(v_\alpha,\beta_\pm)) = 0$ for $< k, v_\pm - (v_\alpha,\beta_\pm) >\in\mathbf{Z}$.

When (C) happens then $\sigma = \begin{pmatrix} I_n & 0\\ 0 & -1\end{pmatrix}\in\mathbf{J}$ and so $(k_1,-k_2)\in\mathcal{L}^*$ if $(k_1,k_2)\in\mathcal{L}^*$. Thus $D(k_1)$ is

$$\begin{aligned}
&\frac{1}{2}\sum_{k_2:(k_1,k_2)\in\mathcal{L}^*}\left(C(k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy + C(k_1,-k_2)\int_0^{y_0}\omega_{-k_2}(y)dy\right)\\
&\quad = \frac{1}{2}\sum_{k_2:(k_1,k_2)\in\mathcal{L}^*}\left(1+\omega_k(v_\sigma)\omega_{k_2}(-y_0)\right)C(k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy,
\end{aligned}$$

by property (5.2), and $D(\alpha^{-1}k_1)$ equals

$$\begin{aligned}
&\frac{1}{2}\sum_{k_2:(\alpha^{-1}k_1,k_2)\in\mathcal{L}^*}\left(C(\alpha^{-1}k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy + C(\alpha^{-1}k_1,-k_2)\int_0^{y_0}\omega_{-k_2}(y)dy\right)\\
&\quad = \frac{1}{2}\sum_{k_2:(\alpha^{-1}k_1,k_2)\in\mathcal{L}^*}\left(\omega_k(v_+)+\omega_k(v_-)\omega_{k_2}(-y_0)\right)C(k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy
\end{aligned}$$

by the invariance of f under the action of (v_+,α_+) and (v_-,α_-), as $\alpha_- = \sigma\alpha_+\in\mathbf{J}$.

Condition (1) of Lemma 5.1 is valid for all $k_1 \in \mathcal{L}_1^*$ because the terms of the summation in the expression below vanish:

$$D(k_1) - \omega_{k_1}(-v_\alpha)D(\alpha^{-1}k_1) = \sum_{k_2:(k_1,k_2)\in\mathcal{L}^*} G(k_1,k_2)C(k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy,$$

where

$$G(k_1,k_2) = 1 + \omega_k(v_\sigma)\omega_{k_2}(-y_0) - \omega_{k_1}(-v_\alpha)\left(\omega_k(v_+) + \omega_k(v_-)\omega_{k_2}(-y_0)\right).$$

If (Ci) happens then $\int_0^{y_0}\omega_{k_2}(y)dy = 0$, otherwise, we show below that $G(k_1,k_2) = 0$ if k verifies any other condition of case (C).

First notice that the hypotheses of Lemma 4.1 are valid and (see the proof of Lemma 4.1)

$$\omega_k(v_-) = \omega_k(v_\sigma)\omega_k(\sigma v_+). \tag{5.3}$$

If $< k, \sigma v_+ - v_+ > \in \mathbf{Z}$ then $\omega_k(\sigma v_+ - v_+) = 1$ and $G(k_1,k_2)$ equals, using (5.3),

$$\begin{aligned} 1 + \omega_k(v_\sigma)\omega_{k_2}(-y_0) - \omega_{k_1}(-v_\alpha)\omega_k(v_+)\left(1 + \omega_k(\sigma v_+ - v_+)\omega_k(v_\sigma)\omega_{k_2}(-y_0)\right) \\ = \left(1 - \omega_k(v_+ - (v_\alpha,0))\right)\left(1 + \omega_k(v_\sigma - (0,y_0))\right) = 0 \end{aligned}$$

because either $1-\omega_k(v_+-(v_\alpha,0)) = 0$, by condition (Cii), or $1+\omega_k(v_\sigma-(0,y_0)) = 0$, by (Ciii).

If $< k, \sigma v_+ - v_+ > +\frac{1}{2} \in \mathbf{Z}$ then $\omega_k(\sigma v_+)\omega_k(-v_+) = -1$ and

$$\begin{array}{rcl} \omega_{k_1}(-v_\alpha)\omega_k(v_+) & = & -\omega_{k_1}(-v_\alpha)\omega_k(\sigma v_+) \\ & = & -\omega_{k_1}(-v_\alpha)\omega_k(v_-)\omega_k(-v_\sigma), \text{ by expression (5.3)} \\ & = & -\omega_k(v_- - (v_\alpha,y_0))\omega_k(-v_\sigma + (0,y_0)). \end{array}$$

Thus $G(k_1,k_2)$ is $1+\omega_k(v_\sigma-(0,y_0))+\omega_k(v_- - (v_\alpha,y_0))\left(\omega_k(-v_\sigma + (0,y_0)) - 1\right) = 0$ because, by condition (Civ), $\omega_k(v_\sigma - (0,y_0)) = \pm i$ and $\omega_k(v_- - (v_\alpha,y_0)) = 1$. Notice that we use the property $\omega_k(-v) = \overline{\omega_k(v)}$ in order to obtain this result. □

5.4. **proof of necessity in Proposition 4.1.** We want to show that if the hypothesis of Proposition 4.1 holds for the projection of the simplest invariant functions in X_Γ, the real and imaginary parts of

$$I_k(x,y) = \sum_{\delta\in\mathbf{J}}\omega_{\delta k}(-v_\delta)\omega_{\delta k}(x,y),$$

then one of the three conditions (A), (B) or (C) must hold.

The functions I_k are given by a summation over $\mathbf{J}$, which corresponds to an orbit on $\mathcal{L}^*$. This orbit is projected into $\mathcal{L}_1^*$ as a new orbit that may be used as an index for the summation of $\Pi_{y_0}(I_k)$ wirting it in a form suitable for the use of Lemma 5.1.

Proof. For $\delta \in \mathbf{O}(n+1)$ and $k \in \mathcal{L}^*$, let $\delta k = (\tilde{k}_1, \tilde{k}_2)$, where $\tilde{k}_1 \in \mathbf{R}^n$ and $\tilde{k}_2 \in \mathbf{R}$. With the notation $\delta k|_1 = \tilde{k}_1$ and $\delta k|_2 = \tilde{k}_2$, the projections of the I_k, with $k \in \mathcal{L}^*$, have the form $\Pi_{y_0}(I_k)(x) = \sum_{\delta\in\mathbf{J}}\omega_{\delta k|_1}(x)D'(\delta,k)$, where $D'(\delta,k) = \omega_{\delta k}(-v_\delta)\int_0^{y_0}\omega_{\delta k|_2}(y)dy$. This corresponds to a summation over the projection of the orbit $\mathbf{J}k$ given by: $\mathbf{J}k|_1 = \{\delta k|_1 : \delta \in \mathbf{J}\} \subset \mathcal{L}_1^*$. Grouping the terms with the same first n components, we obtain

$$\Pi_{y_0}(I_k)(x) = \sum_{\tilde{k}_1\in\mathbf{J}k|_1}\omega_{\tilde{k}_1}(x)\sum_{\tilde{k}_2:(\tilde{k}_1,\tilde{k}_2)\in\mathbf{J}k} D'(\delta,\tilde{k}).$$

In particular, for $k = (k_1,k_2)$, the Fourier coefficient of $\Pi_{y_0}(I_k)$ associated to ω_{k_1} is $\sum_{\delta\in \mathrm{J}^{Id}(k)} D'(\delta,k)$, where $\mathrm{J}^{Id}(k)$ is the subset of $\mathbf{J}$ which preserves k_1, $\mathrm{J}^{Id}(k) = \{\delta \in \mathbf{J} : \delta k|_1 = k_1\}$. Analogously, we define $\mathrm{J}^\alpha(k) = \left\{\delta \in \mathbf{J} : \delta k|_1 = \alpha^{-1}k_1\right\}$.

Since $\Pi_{y_0}(I_k)$ is (v_α,α)-invariant, by hypothesis, then Lemma 5.1 holds and therefore, for all $k = (k_1,k_2) \in \mathcal{L}^*$, we have:

(a) if $k_1 \in \mathcal{L}_1^* \cap \alpha\mathcal{L}_1^*$ then $\sum_{\delta\in \mathrm{J}^{Id}(k)} D'(\delta,k) = \omega_{k_1}(-v_\alpha)\sum_{\delta\in\mathrm{J}^{\alpha}(k)} D'(\delta,k)$,
(b) if $k_1 \notin \mathcal{L}_1^* \cap \alpha\mathcal{L}_1^*$ then $\sum_{\delta\in \mathrm{J}^{Id}(k)} D'(\delta,k) = 0$.

The rest of the proof is divided in three Lemmas. Although these conditions involve the sets $\mathrm{J}^{Id}(k)$ and $\mathrm{J}^{\alpha}(k)$ for all $k \in \mathcal{L}^*$, we show in Lemmas 5.2, 5.3 and 5.4 below that for this proof we will only need the elements of $\mathbf{J}$ that lie in the following subsets:

$$\mathbf{J}^{Id} = \{Id_{n+1},\sigma\}\cap\mathbf{J} \text{ and } \mathbf{J}^{\alpha} = \left\{\alpha_+^{-1},\alpha_-^{-1}\right\}\cap\mathbf{J}.$$

In Lemma 5.2 we describe all the possibilities for $\mathbf{J}^{Id}$ and $\mathbf{J}^{\alpha}$ and obtain in each case some consequences for $\mathcal{L}^*$ in terms of the subsets defined before the statement of Proposition 4.2. In Lemma 5.3 we study the set of all $k \in \mathcal{L}^*$ such that either $\mathrm{J}^{Id}(k) \neq \mathbf{J}^{Id}$ or $\mathrm{J}^{\alpha}(k) \neq \mathbf{J}^{\alpha}$. Finally, conditions (A), (B) and (C) are obtained in Lemma 5.4. □

5.5. **the sets $\mathbf{J}^{Id}$ and $\mathbf{J}^{\alpha}$.**

Properties of $\mathrm{J}^{Id}(k)$ and $\mathrm{J}^{\alpha}(k)$. *Let $k \in \mathcal{L}^*$.*

(1) $\mathrm{J}^{Id}(k) = \{\delta\in\mathbf{J}:\ \delta k = k \vee \delta k = \sigma k\}$ *and*
$\mathrm{J}^{\alpha}(k) = \left\{\delta\in\mathbf{J}:\ \delta k = \alpha_+^{-1}k \vee \delta k = \alpha_-^{-1}k\right\}$.
(2) $\mathbf{J}^{Id} \subset \mathrm{J}^{Id}(k)$, $\mathbf{J}^{\alpha} \subset \mathrm{J}^{\alpha}(k)$ *and* $\mathrm{J}^{Id}(0,0) = \mathrm{J}^{\alpha}(0,0) = \mathbf{J}$.

Proof. Property (1), for $\mathrm{J}^{Id}(k)$, follows by orthogonality of $\mathbf{J}$, since any element of the orbit $\mathbf{J}(k_1,k_2)$ whose n first components equal k_1 is of the form $(k_1,\pm k_2)$. For $\mathrm{J}^{\alpha}(k)$, the elements on $\mathbf{J}(k_1,k_2)$ with n first components $\alpha^{-1}k_1$ are of the form $(\alpha^{-1}k_1,\pm k_2)$, by orthogonality of $\mathbf{J}$ and of α.

Property (2) follows directly from the previous one and from the definitions of $\mathbf{J}^{Id}$ and $\mathbf{J}^{\alpha}$. □

5.6. **the set $\mathcal{O}^*$.** Next lemma describes, under the hypothesis of Proposition 4.1, the structure of the set

$$\mathcal{O}^* = \left\{k\in\mathcal{L}^*:\ \mathrm{J}^{Id}(k) = \mathbf{J}^{Id}\ \wedge\ \mathrm{J}^{\alpha}(k) = \mathbf{J}^{\alpha}\right\}$$

according to each of the possible cases for $\mathbf{J}^{Id}$ and $\mathbf{J}^{\alpha}$. In its proof we use the definition of $\mathcal{O}^*$ in order to simplify conditions (a) and (b). Applying the properties of the waves we will be able to restate these conditions in terms of the submodules and subsets of $\mathcal{L}^*$ previously defined.

Lemma 5.2. *Suppose that*

(a) *if* $k_1 \in \mathcal{L}_1^* \cap \alpha\mathcal{L}_1^*$ *then* $\sum_{\delta\in \mathrm{J}^{Id}(k)} D'(\delta,k) = \omega_{k_1}(-v_\alpha)\sum_{\delta\in\mathrm{J}^{\alpha}(k)} D'(\delta,k)$ *and*
(b) *if* $k_1 \notin \mathcal{L}_1^* \cap \alpha\mathcal{L}_1^*$ *then* $\sum_{\delta\in \mathrm{J}^{Id}(k)} D'(\delta,k) = 0$,

for all $k = (k_1,k_2) \in \mathcal{L}^*$. Then one of the following cases holds:

(1) $\mathbf{J}^{Id} = \{Id_{n+1}\}$, $\mathbf{J}^{\alpha} = \emptyset$ *and* $\mathcal{O}^* \subset \mathcal{N}_{y_0}^*$,
(2) $\mathbf{J}^{Id} = \{Id_{n+1},\sigma\}$, $\mathbf{J}^{\alpha} = \emptyset$ *and* $\mathcal{O}^* \subset \left(\mathcal{N}_{y_0}^* \cup \mathcal{N}_{\sigma}^*\right)$,
(3) $\mathbf{J}^{Id} = \{Id_{n+1}\}$, $\mathbf{J}^{\alpha} = \{\alpha_+^{-1}\}$ *and* $\mathcal{O}^* \subset \left(\mathcal{N}_{y_0}^* \cup \mathcal{M}_{+}^*\right)$,
(4) $\mathbf{J}^{Id} = \{Id_{n+1}\}$, $\mathbf{J}^{\alpha} = \{\alpha_-^{-1}\}$ *and* $\mathcal{O}^* \subset \left(\mathcal{N}_{y_0}^* \cup \mathcal{M}_{-}^*\right)$,
(5) $\mathbf{J}^{Id} = \{Id_{n+1},\sigma\}$, $\mathbf{J}^{\alpha} = \{\alpha_+^{-1},\alpha_-^{-1}\}$,
$(\mathcal{O}^*\cap\mathcal{M}^*) \subset \left(\mathcal{N}_{y_0}^* \cup \mathcal{M}_{+}^* \cup \mathcal{N}_{\sigma}^*\right)$ *and* $(\mathcal{O}^*\cap\mathcal{N}^*) \subset \left(\mathcal{N}_{y_0}^* \cup \left(\mathcal{M}_{-}^* \cap \mathcal{N}_{\tilde{\sigma}}^*\right)\right)$.

Proof. Cases (1) to (5) enumerate all the possibilities for $\mathbf{J}^{Id}$ and $\mathbf{J}^{\alpha}$. This happens because $\mathbf{J}^{Id}$ is a group; if $\alpha_+^{-1},\alpha_-^{-1} \in \mathbf{J}$ then $\alpha_+\alpha_-^{-1} = \sigma \in \mathbf{J}$ and if $\sigma \in \mathbf{J}$ then either $\mathbf{J}^{\alpha} = \emptyset$ or $\mathbf{J}^{\alpha}$ has two elements.

In this proof we will use the property

$$\omega_k(-\sigma v_\sigma) = \omega_k(v_\sigma) \quad \text{if } k\in\mathcal{L}^* \text{ and } (v_\sigma,\sigma)\in\Gamma, \tag{5.4}$$

and this hods because if $(v_\sigma,\sigma)\in\Gamma$ then $(v_\sigma,\sigma)\cdot(v_\sigma,\sigma)=(v_\sigma+\sigma v_\sigma,I)\in\Gamma$ implying $v_\sigma+\sigma v_\sigma\in\mathcal{L}$.

If $\mathbf{J}^\alpha=\emptyset$ then, for all $k=(k_1,k_2)\in\mathcal{O}^*$, the conditions in the hypothesis of the lemma become $\sum_{\delta\in\mathbf{J}^{Id}}D'(\delta,k)=0$. Thus, either $\int_0^{y_0}\omega_{k_2}(y)dy=0$ or $\int_0^{y_0}\omega_{k_2}(y)dy+\omega_{\sigma k}(-v_\sigma)\int_0^{y_0}\omega_{-k_2}(y)dy=0$, according to the absence or presence of σ in $\mathbf{J}$. Using (5.2), the second case is equivalent to

$$(1+\omega_{\sigma k}(-v_\sigma)\omega_{k_2}(-y_0))\int_0^{y_0}\omega_{k_2}(y)dy=0\Leftrightarrow$$

$$\Leftrightarrow(1+\omega_k(v_\sigma-(0,y_0)))\int_0^{y_0}\omega_{k_2}(y)dy=0,$$

by orthogonality and property (5.4). Cases (1) and (2) follow because $\int_0^{y_0}\omega_{k_2}(y)dy=0$ implies $k\in\mathcal{N}^*_{y_0}$ and $1+\omega_k(v_\sigma-(0,y_0))=0$ implies $k\in\mathcal{N}^*_\sigma$.

In the remaining cases either α_+ or α_- belong to $\mathbf{J}$. Thus, $\alpha\mathcal{L}_1^*=\mathcal{L}_1^*$ and the first condition in the hypothesis of the lemma must be verified for all $k_1\in\mathcal{L}_1^*$. For $k\in\mathcal{O}^*$ this condition becomes

$$\sum_{\delta\in\mathbf{J}^{Id}}D'(\delta,k)=\omega_{k_1}(-v_\alpha)\sum_{\delta\in\mathbf{J}^\alpha}D'(\delta,k). \tag{5.5}$$

Recall that $(v_\delta,\delta)^{-1}=(-\delta^{-1}v_\delta,\delta^{-1})$ when writing $D'(\delta,k)$ explicitly, below. In case (3), the condition above is $\left(1-\omega_{k_1}(-v_\alpha)\omega_{\alpha_+^{-1}k}(\alpha_+^{-1}v_+)\right)\int_0^{y_0}\omega_{k_2}(y)dy=0$ and, by orthogonality, is equivalent to $(1-\omega_{k_1}(-v_\alpha)\omega_k(v_+))\int_0^{y_0}\omega_{k_2}(y)dy=0$. The result follows because $1-\omega_{k_1}(-v_\alpha)\omega_k(v_+)=0$ implies $k\in\mathcal{M}_+^*$.

For case (4), condition (5.5) is equivalent to

$$\int_0^{y_0}\omega_{k_2}(y)dy-\omega_{k_1}(-v_\alpha)\omega_{\alpha_-^{-1}k}(\alpha_-^{-1}v_-)\int_0^{y_0}\omega_{-k_2}(y)dy=0\Leftrightarrow$$

$$\Leftrightarrow(1-\omega_{k_1}(-v_\alpha)\omega_k(v_-)\omega_{k_2}(-y_0))\int_0^{y_0}\omega_{k_2}(y)dy=0.$$

Thus, either $k\in\mathcal{N}^*_{y_0}$ or $1-\omega_{k_1}(-v_\alpha)\omega_k(v_-)\omega_{k_2}(-y_0)=0$, which implies $k\in\mathcal{M}_-^*$.

Condition (5.5) is, for case (5), equivalent to

$$\int_0^{y_0}\omega_{k_2}(y)dy+\omega_{\sigma k}(-v_\sigma)\int_0^{y_0}\omega_{-k_2}(y)dy-$$

$$-\omega_{k_1}(-v_\alpha)\left(\omega_{\alpha_+^{-1}k}(\alpha_+^{-1}v_+)\int_0^{y_0}\omega_{k_2}(y)dy+\omega_{\alpha_-^{-1}k}(\alpha_-^{-1}v_-)\int_0^{y_0}\omega_{-k_2}(y)dy\right)=0,$$

which, by orthogonality and properties (5.2) and (5.4), has the form $G(k_1,k_2)\int_0^{y_0}\omega_{k_2}(y)dy=0$, where $G(k_1,k_2)=1+\omega_k(v_\sigma)\omega_{k_2}(-y_0)-\omega_{k_1}(-v_\alpha)\left(\omega_k(v_+)+\omega_k(v_-)\omega_{k_2}(-y_0)\right)$, as in the proof of Proposition 4.1. Therefore, either $k\in\mathcal{N}^*_{y_0}$ or $G(k_1,k_2)=0$.

In case (5) we are under the conditions of Lemma 4.1 and so $\mathcal{O}^*\subset(\mathcal{M}^*\cup\mathcal{N}^*)$. If $k=(k_1,k_2)\in\mathcal{M}^*$ then $G(k_1,k_2)=0$ is equivalent, as shown in the proof of Proposition 4.1, to $(1-\omega_k(v_+-(v_\alpha,0)))(1+\omega_k(v_\sigma-(0,y_0)))=0$ and the result follows. For $k=(k_1,k_2)\in\mathcal{N}^*$, the term $G(k_1,k_2)$ equals, by the proof of Proposition 4.1, $1+\omega_k(v_\sigma-(0,y_0))+\omega_k(v_--(v_\alpha,y_0))\left(\overline{\omega_k(v_\sigma-(0,y_0))}-1\right)$. Equation $G(k_1,k_2)=0$ is equivalent, for $\omega_k(v_\sigma-(0,y_0))=z_1$ and $\omega_k(v_--(v_\alpha,y_0))=z_2$, to $(1+z_1)/(1-\overline{z_1})=z_2$ because $z_1=1$ is not a solution of $G(k_1,k_2)=0$. Therefore, $|(1+z_1)/(1-\overline{z_1})|=1$ which implies $\mathrm{Re}(z_1)=0\Leftrightarrow\omega_k(v_\sigma-(0,y_0))=\pm i$ and $z_2=\omega_k(v_--(v_\alpha,y_0))=1$, leading to $k\in\left(\mathcal{M}_-^*\cap\mathcal{N}_\sigma^*\right)$. □

5.7. **the set $\mathcal{P}^*$.** Let $\mathcal{P}^*$ be the complement of $\mathcal{O}^*$ in $\mathcal{L}^*$:

$$\mathcal{P}^* = \left\{k \in \mathcal{L}^* :\ \mathrm{J}^{Id}(k) \neq \mathbf{J}^{Id}\ \vee\ \mathrm{J}^{\alpha}(k) \neq \mathbf{J}^{\alpha}\right\}.$$

In Lemma 5.4 we reformulate the cases of Lemma 5.2 in terms of $\mathcal{L}^*$ instead of $\mathcal{O}^*$. We show that the set $\mathcal{P}^*$ is very small. In the first two cases of Lemma 5.2 it is too small so these cases cannot occur. In the remaining cases we show that $\mathcal{P}^*$ may be ignored and, therefore, that $\mathcal{L}^*$ can replace $\mathcal{O}^*$ in the expressions given. Thus, the estimate of the size of $\mathcal{P}^*$ in the next lemma is an essential step.

Lemma 5.3. *$\mathcal{P}^*$ is contained in the union of a finite number of vector subspaces of $\mathbf{R}^{n+1}$ with codimension at least one.*

Proof. $\mathcal{P}^*$ is the union of the submodules

$$\bigcup_{\delta \in \mathbf{J}-\mathbf{J}^{Id}} \mathcal{M}^*_{\delta,Id}\ \cup \bigcup_{\delta \in \mathbf{J}-\mathbf{J}^{\alpha}} \mathcal{M}^*_{\delta,\alpha}$$

where $\mathcal{M}^*_{\delta,Id} = \left\{k \in \mathcal{L}^* : \delta \in \mathrm{J}^{Id}(k)\right\}$ and $\mathcal{M}^*_{\delta,\alpha} = \{k \in \mathcal{L}^* : \delta \in \mathrm{J}^{\alpha}(k)\}$. This union is finite because $\mathbf{J}$ is a finite group. Moreover, for all $\xi \in \mathbf{O}(n+1)$, $\mathrm{Fix}(\xi) = \left\{(x,y) \in \mathbf{R}^{n+1} : \xi(x,y) = (x,y)\right\}$ is a vector subspace of $\mathbf{R}^{n+1}$ and $\mathrm{Fix}(\xi) = \mathbf{R}^{n+1} \Leftrightarrow \xi = Id_{n+1}$.

Let $\delta \in \mathbf{J} - \mathbf{J}^{Id}$. If $k \in \mathcal{M}^*_{\delta,Id}$ then either $\delta k = k$ or $\delta k = \sigma k \Leftrightarrow \sigma\delta k = k$, which implies $\mathcal{M}^*_{\delta,Id} \subset (\mathrm{Fix}(\delta) \cup \mathrm{Fix}(\sigma\delta))$. Moreover, neither $\delta = Id_{n+1}$ nor $\sigma\delta = Id_{n+1}$, by the hypothesis $\delta \in \mathbf{J} - \mathbf{J}^{Id}$. Thus, the codimensions of subspaces the $\mathrm{Fix}(\delta)$ and $\mathrm{Fix}(\sigma\delta)$ are at least one.

Analogously, if $\delta \in \mathbf{J} - \mathbf{J}^{\alpha}$ and $k \in \mathcal{M}^*_{\delta,\alpha}$ then either $\delta k = \alpha_+^{-1} k \Leftrightarrow \alpha_+\delta k = k$ or $\delta k = \alpha_-^{-1} k \Leftrightarrow \alpha_-\delta k = k$. Therefore, $\mathcal{M}^*_{\delta,\alpha} \subset (\mathrm{Fix}(\alpha_+\delta) \cup \mathrm{Fix}(\alpha_-\delta))$, where both $\mathrm{Fix}(\alpha_+\delta)$ and $\mathrm{Fix}(\alpha_-\delta)$ have codimensions at least one due to the hypothesis $\delta \in \mathbf{J} - \mathbf{J}^{\alpha}$. □

Lemma 5.4. *Suppose that*

(a) *if $k_1 \in \mathcal{L}_1^* \cap \alpha\mathcal{L}_1^*$ then $\sum_{\delta \in \mathrm{J}^{Id}(k)} D'(\delta,k) = \omega_{k_1}(-v_\alpha)\sum_{\delta \in \mathrm{J}^{\alpha}(k)} D'(\delta,k)$ and*
(b) *if $k_1 \notin \mathcal{L}_1^* \cap \alpha\mathcal{L}_1^*$ then $\sum_{\delta \in \mathrm{J}^{Id}(k)} D'(\delta,k) = 0$,*

for all $k = (k_1,k_2) \in \mathcal{L}^*$. Then one of the following cases holds:

(A) $\mathbf{J}^{\alpha} = \{\alpha_+^{-1}\}$ *and* $\mathcal{L}^* = \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+$,
(B) $\mathbf{J}^{\alpha} = \{\alpha_-^{-1}\}$ *and* $\mathcal{L}^* = \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_-$,
(C) $\mathbf{J}^{\alpha} = \{\alpha_+^{-1}, \alpha_-^{-1}\}$,
$\mathcal{M}^* \subset \left(\mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+ \cup \mathcal{N}^*_{\sigma}\right)$ *and* $\mathcal{N}^* \subset \left(\mathcal{N}^*_{y_0} \cup \left(\mathcal{M}^*_- \cap \mathcal{N}^*_{\tilde{\sigma}}\right)\right)$.

Notice that the conditions in Lemma 5.4 are the same of Proposition 4.1 as $\delta^{-1} \in \mathbf{J}^{\alpha}$ is equivalent to $(v_\delta, \delta) \in \Gamma$ for some $v_\delta \in \mathbf{R}^{n+1}$, by definition.

Proof. At first, we prove that

$$\left(\mathcal{M}^*_{y_0} \cap \mathcal{P}^*\right) - \{(0,0)\} = \emptyset. \tag{5.6}$$

If $k \in \mathcal{M}^*_{y_0}$ then $k = (k_1, 0)$ for some $k_1 \in \mathbf{R}^n$. If, moreover, $k \in \mathcal{P}^*$ then either $\delta(k_1,0) = (k_1,0)$, for some $\delta \in \mathbf{J} - \mathbf{J}^{Id}$, or $\delta(k_1,0) = (\alpha^{-1}k_1, 0)$, for some $\delta \in \mathbf{J} - \mathbf{J}^{\alpha}$, by the definition of $\mathcal{P}^*$ and the properties of $\mathrm{J}^{Id}(k)$ and $\mathrm{J}^{\alpha}(k)$. By orthogonality of δ the first case implies, for $k_1 \neq 0$, either $\delta = I$ or $\delta = \sigma$, which is equivalent to $\delta \in \mathbf{J}^{Id}$. Similarly, for $k_1 \neq 0$, the second case implies $\delta \in \mathbf{J}^{\alpha}$, by orthogonality of δ and α.

For any element $k \neq (0,0)$ of the dual lattice $\mathcal{L}^*$, let $g \neq (0,0)$ be the smallest element of $\mathcal{L}^*$ in the direction of k. Thus, there are elements $g_1, \ldots, g_n \in \mathcal{L}^*$ such that $\mathcal{L}^* = \{g, g_1, \ldots, g_n\}_{\mathbf{Z}}$.

Let $\mathcal{M}_k^*$ be the submodule $\mathcal{M}_k^* = \{g_1, g_2, \ldots, g_n\}_{\mathbf{Z}} \subset \mathcal{L}^*$ and, given $h \in \mathcal{M}_k^*$, let $\mathcal{Q}_{k,h}^*$ be the set $\mathcal{Q}_{k,h}^* = \{k + mh : m \in \mathbf{Z}\}$.

We claim that there is some $h \in \mathcal{M}_k^*$ such that $\mathcal{Q}_{k,h}^* \cap \mathcal{P}^*$ is a finite set. Lemma 5.3 asserts that $\mathcal{P}^* \subset \bigcup_{i=1}^m H_i$, where each H_i is a codimension one subspace of $\mathbf{R}^{n+1}$. Let $p \in \mathbf{N}$ and consider the subset of $k + \mathcal{M}_k^*$ with p^n elements:

$$W_p = \{k + m_1 g_1 + \cdots + m_n g_n : m_i \in \mathbf{Z}, 1 \leq m_i \leq p\}.$$

Each H_i has at most p^{n-1} elements in W_p and so $W_p \cap \bigcup_{i=1}^m H_i$ has, at most, mp^{n-1} elements. For $p > m$ we have $p^n > mp^{n-1}$ and there is some $h \in \mathcal{M}^*$ such that $k + h \notin \bigcup_{i=1}^m H_i$. For this h, let r be a line containing $\mathcal{Q}_{k,h}^*$. Since for each i, $r \cap H_i$ is either r or a finite set, and r contains at least the element $k + h \notin H_i$, it follows that $\bigcup_{i=1}^m (r \cap H_i)$ is a finite set. The claim is proved because $\mathcal{Q}_{k,h}^* \cap \mathcal{P}^*$ is a subset of $\bigcup_{i=1}^m (r \cap H_i)$.

Let k be any element of $\mathcal{L}^* - \{(0,0)\}$ and choose some $h \in \mathcal{M}_k^*$ such that $\mathcal{Q}_{k,h}^* \cap \mathcal{P}^*$ is a finite set. For simplicity of notation we write $\mathcal{Q}^*$ instead of $\mathcal{Q}_{k,h}^*$.

The intersection $\mathcal{Q}^* \cap \overline{\mathcal{N}_{y_0}^*}$ is either the empty set or a set with only a point or an infinite set of equally spaced points. This happens because $\overline{\mathcal{N}_{y_0}^*}$ is a module and the existence of any two distinct elements of $\mathcal{Q}^* \cap \overline{\mathcal{N}_{y_0}^*}$, $k + m_1 h$ and $k + m_2 h$, implies $(m_2 - m_1)h \in \overline{\mathcal{N}_{y_0}^*}$ and

$$\{k + m_1 h + m(m_2 - m_1)h : m \in \mathbf{Z}\} \subset \left(\mathcal{Q}^* \cap \overline{\mathcal{N}_{y_0}^*}\right).$$

A characteristic period, τ_{y_0}, is given by the smallest difference between two elements of $\mathcal{Q}^* \cap \overline{\mathcal{N}_{y_0}^*}$.

For the set $\mathcal{Q}^* \cap \mathcal{N}_\sigma^*$ there are also the three possible results. Although $\mathcal{N}_\sigma^*$ is not a module, the smallest difference between two elements of $\mathcal{Q}^* \cap \mathcal{N}_\sigma^*$ defines a period $\tau_\sigma \in \mathcal{M}_\sigma^*$, by properties of $\mathcal{N}_\sigma^*$, in subsection 4.3. Thus, whenever $\mathcal{Q}^* \cap \mathcal{N}_\sigma^*$ has more than one element, if $k + m_1 h \in \mathcal{N}_\sigma^*$ then

$$\{k + m_1 h + m\tau_\sigma : m \in \mathbf{Z}\} = \mathcal{Q}^* \cap \mathcal{N}_\sigma^*.$$

An analogous construction may be done for the sets $\mathcal{Q}^* \cap \mathcal{M}_+^*$ and $\mathcal{Q}^* \cap \mathcal{M}_-^*$. Thus, if these sets have more than one element we may define, respectively, characteristic periods τ_+ and τ_-.

If the set $\mathcal{Q}^* \cap \left(\mathcal{M}_-^* \cap \mathcal{N}_{\tilde{\sigma}}^*\right)$ has two distinct elements, $k + m_1 h$ and $k + m_2 h$, then $(m_2 - m_1)h \in \left(\mathcal{M}_-^* \cap \overline{\mathcal{N}_\sigma^*}\right)$ and

$$\{k + m_1 h + m(m_2 - m_1)h : m \in \mathbf{Z}\} \subset \left(\mathcal{Q}^* \cap \left(\mathcal{M}_-^* \cap \mathcal{N}_{\tilde{\sigma}}^*\right)\right),$$

by the module structure of $\mathcal{M}_-^*$ and by the properties of $\mathcal{N}_{\tilde{\sigma}}^*$, in subsection 4.3. As above, this set has also a period, $\tau_{\tilde{\sigma}}$.

Under the hypothesis of the Lemma, one of the cases (1) to (5) of Lemma 5.2 must happen.

If case (1) happens then $\mathcal{L}^* = \mathcal{N}_{y_0}^* \cup \mathcal{P}^*$, which implies $\mathcal{M}_{y_0}^* \subset \mathcal{P}^*$ and, by (5.6), $\mathcal{M}_{y_0}^* = \{(0,0)\}$. Moreover, $\mathcal{Q}^* \cap \mathcal{N}_{y_0}^*$ must be an infinite set because $\mathcal{Q}^* \cap \mathcal{P}^*$ is, by construction, finite. Thus, there exists the period τ_{y_0} implying that $\mathcal{Q}^* - \overline{\mathcal{N}_{y_0}^*}$ is either the empty set or an infinite set. Since $\left(\mathcal{Q}^* - \overline{\mathcal{N}_{y_0}^*}\right) \subset (\mathcal{Q}^* \cap \mathcal{P}^*)$ is finite, it follows that $\mathcal{L}^* = \overline{\mathcal{N}_{y_0}^*}$. However, by property (2) of the bases, in the next section, in this case $\mathcal{M}_{y_0}^* \neq \{(0,0)\}$ and so case (1) cannot occur.

In case (2), $\mathcal{L}^* = \mathcal{N}_{y_0}^* \cup \mathcal{N}_\sigma^* \cup \mathcal{P}^*$ which, by (5.6), implies $\mathcal{M}_{y_0}^* \subset (\mathcal{N}_\sigma^* \cup \{(0,0)\})$. Moreover, $\mathcal{M}_{y_0}^* \neq \{(0,0)\}$ due to the existence of σ in $\mathbf{J}$, (see properties (2) and (3) of the bases. Suppose $\tilde{k} \in \mathcal{M}_{y_0}^*$ and $\tilde{k} \neq (0,0)$. Thus, $\tilde{k} \in \mathcal{N}_\sigma^*$ and $2\tilde{k} \in \mathcal{M}_{y_0}^*$. However, by properties of $\mathcal{N}_\sigma^*$ in subsection 4.3, $2\tilde{k} \notin \mathcal{N}_\sigma^*$ and so case (2) is also impossible.

For case (3) we follow the arguments of case (1). As $\mathcal{L}^* = \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+ \cup \mathcal{P}^*$ then $\mathcal{Q}^* \cap \left(\mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+\right)$ is an infinite set and at least one of the periods τ_{y_0} or τ_+ must exist. The least common multiple of the existing periods is a period of $\mathcal{Q}^* \cap \left(\mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+\right)$ which implies that $\mathcal{Q}^* - \left(\mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+\right)$ is the empty set. Therefore $k \in \left(\mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+\right)$ and condition (A) follows by definition of k and because $(0,0) \in \mathcal{M}^*_+$.

In a similar way, with $\mathcal{M}^*_-$ and τ_- instead of $\mathcal{M}^*_+$ and τ_+, we prove that case (4) of Lemma 5.2 leads to condition (B).

In case (5) $(\mathcal{Q}^* \cap \mathcal{M}^*) - \left(\mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+ \cup \mathcal{N}^*_\sigma\right)$ must be the empty set by the necessary existence of, at least, one of the periods τ_{y_0}, τ_+ or τ_σ and, analogously, $(\mathcal{Q}^* \cap \mathcal{N}^*) - \left(\mathcal{N}^*_{y_0} \cup \left(\mathcal{M}^*_- \cap \mathcal{N}^*_{\tilde{\sigma}}\right)\right)$ is empty due to the least common multiple of the periods τ_{y_0} and $\tau_{\tilde{\sigma}}$. Besides, either $k \in (\mathcal{Q}^* \cap \mathcal{M}^*)$ or $k \in (\mathcal{Q}^* \cap \mathcal{N}^*)$ and, as $(0,0) \notin \mathcal{N}^*$, condition (C) follows. □

This completes the proof of Propositions 4.1 and 4.2.

6. Proof of Theorem 3.1

Theorem 3.1 will be proved when we show how conditions (A), (B) and (C) of Proposition 4.1 lead to the conclusion that one of the cases (I), (II) and (III) of the theorem must hold.

Proposition 4.1 states that the elements of Γ ensuring some symmetry after projection have orthogonal components α_+ or α_-. Information on the non-orthogonal components ($v_+, v_- \in \mathbf{R}^{n+1}$) appears as constraints on the structure of the dual lattice $\mathcal{L}^*$.

Our aim is then to translate the restrictions on the group Γ and its dual lattice $\mathcal{L}^*$ into restrictions on Γ and its lattice $\mathcal{L}$. One major tool will be to use the conditions of Proposition 4.2 to obtain restrictions on a basis of $\mathcal{L}^*$. This in turn is used to find a suitable basis for $\mathcal{L}$. The properties of the modules involved will also be extensively used.

We begin by stating some properties of the bases for the lattices. Then we show that the conditions in Proposition 4.2 imply those of Theorem 3.1. Each condition of Proposition 4.2 is treated in a separate lemma.

6.1. the lattices $\mathcal{L}$ and $\mathcal{L}^*$.

Properties of the bases for $\mathcal{L}$ and $\mathcal{L}^*$ and notation.
Let $\{l_1, \ldots, l_{n+1}\}$ be a basis for $\mathcal{L}$ and $\left\{l^*_1, \ldots, l^*_{n+1}\right\}$ be its dual basis, i.e., a basis for $\mathcal{L}^*$ such that $< l^*_i, l_j >= \delta_{ij}$ for $i, j \in \{1, \ldots, n+1\}$.

For $M = \begin{pmatrix} l_1 \\ \vdots \\ l_{n+1} \end{pmatrix}$ then $M^* = \left(M^{-1}\right)^T = \begin{pmatrix} l^*_1 \\ \vdots \\ l^*_{n+1} \end{pmatrix}$ and the following properties hold:

1. *If $(v_\delta, \delta) \in \Gamma$ then, given the real numbers $r_1, \ldots, r_{n+1}$, we may write $v_\delta = \sum_{i=1}^{n+1} s_i l_i$ with $(s_i - r_i) \in [0, 1[$ for all $i \in \{1, \ldots, n+1\}$.*
2. *If $(0, a) \in \mathcal{L}$ for some $a \neq 0$ then we may choose the basis $\{l_1, \ldots, l_{n+1}\}$ for $\mathcal{L}$ such that*

(i) $M = \begin{pmatrix} A & B \\ 0 & b \end{pmatrix}$, *where* $b = \frac{a}{m}$ *for some* $m \in \mathbf{Z}$,

$$A = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \text{ with } a_i \in \mathbf{R}^n \text{ for } i \in \{1, \ldots, n\} \text{ and}$$

$$B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}, \text{ with } b_i \in \mathbf{R} \text{ for } i \in \{1, \ldots, n\}.$$

(ii) $M^* = \begin{pmatrix} A^* & 0 \\ -\frac{1}{b}B^T A^* & \frac{1}{b} \end{pmatrix}$, *where* $A^* = \left(A^{-1}\right)^T = \begin{pmatrix} a_1^* \\ \vdots \\ a_n^* \end{pmatrix}$, *with* $< a_i^*, a_j >= \delta_{ij}$ *for* $i, j \in \{1, \ldots, n\}$.

(iii) *The set* $\{a_1, \ldots, a_n\}$ *is a basis for a lattice in* $\mathbf{R}^n$ *and* $\{a_1^*, \ldots, a_n^*\}$ *is a basis for its dual.*

(iv) $l_i^* = (a_i^*, 0)$ *for* $i \in \{1, \ldots, n\}$ *and* $\mathcal{M}_{y_0}^* = \{l_1^*, \ldots, l_n^*\}_{\mathbf{Z}}$.

(3) *If* σ *belongs to the holohedry of* $\mathcal{L}$ *then there is some nonzero* $a \in \mathbf{R}$ *such that* $(0, a) \in \mathcal{L}$. *Moreover, each entry* b_i *of the matrix* B *defined above, may be taken to be either zero or* $b/2$.

Proof. (1) The set $\{l_1, \ldots, l_{n+1}\}$ is a basis for $\mathbf{R}^{n+1}$ and so $v_\delta = \sum_{i=1}^{n+1} s_i l_i$ with $s_i \in \mathbf{R}$ for all $i \in \{1, \ldots, n+1\}$. As v_δ is defined up to elements of $\mathcal{L}$ then we may restrict each s_i to an interval $[r_i, r_i + 1[$, where $r_i \in \mathbf{R}$.

(2) Given $(0, a) \in \mathcal{L}$, $a \neq 0$, let $(0, b)$, $b \neq 0$, be the smallest element of $\mathcal{L}$ in the direction of $(0, a)$. Thus $(0, b)$ is a generator and $(0, a) = m(0, b)$ for some $m \in \mathbf{Z}$. Moreover, there are elements $l_1, \ldots, l_n$ in $\mathcal{L}$ such that $\mathcal{L} = \{l_1, \ldots, l_n, (0, b)\}_{\mathbf{Z}}$. For $l_i = (a_i, b_i)$, with $i \in \{1, \ldots, n\}$, and $(0, b) = l_{n+1}$ we obtain the matrix M and it is easy to show that M^* has the form given in (2ii). Property (2iv) follows by definition of $\mathcal{M}_{y_0}^*$.

(3) There is some (c, d) in $\mathcal{L}$ with $d \neq 0$. If $\sigma\mathcal{L} = \mathcal{L}$ then $(c, d) - \sigma(c, d) = (0, 2d) \in \mathcal{L}$ and property (2) is valid. For $l_i = (a_i, b_i)$, with $i \in \{1, \ldots, n\}$, the elements $l_i - \sigma l_i = (0, 2b_i)$ belong to $\mathcal{L}$ and so $(0, 2b_i) = m(0, b)$ for some $m \in \mathbf{Z}$. Therefore $l_i = \left(a_i, \frac{mb}{2}\right)$, and either $l_i = (a_i, 0)$ or $l_i = \left(a_i, \frac{b}{2}\right)$ up to multiples of $(0, b) = l_{n+1}$. □

6.2. $\mathcal{L}^*$ constraining Γ.

Lemma 6.1. *If* $(v_+, \alpha_+) \in \Gamma$ *and* $\mathcal{L}^* = \mathcal{N}_{y_0}^* \cup \mathcal{M}_+^*$ *then one of the following conditions of Theorem 3.1 holds:*

(I) $((v_\alpha, 0), \alpha_+) \in \Gamma$,

(III) $(0, y_0) \in \mathcal{L}$ *and* $((v_\alpha, y_1), \alpha_+) \in \Gamma$ *for some* $y_1 \in \mathbf{R}$.

Proof. If $\mathcal{L}^* = \mathcal{N}_{y_0}^* \cup \mathcal{M}_+^*$ then either $\mathcal{L}^* = \overline{\mathcal{N}_{y_0}^*}$ or $\mathcal{L}^* = \mathcal{M}_+^*$. In the second case $< k, v_+ - (v_\alpha, 0) >\in \mathbf{Z}$ for all $k \in \mathcal{L}^*$, *i.e.* , $v_+ - (v_\alpha, 0) \in \mathcal{L}$, and so

$$(-v_+ + (v_\alpha, 0), I) \cdot (v_+, \alpha_+) = ((v_\alpha, 0), \alpha_+) \in \Gamma.$$

If $\mathcal{L}^* = \overline{\mathcal{N}_{y_0}^*}$ then $(0, y_0) \in \mathcal{L}$ and we may use the basis $\left\{l_1^*, \ldots, l_{n+1}^*\right\}$ for $\mathcal{L}^*$ having the properties (2) above. As $\mathcal{M}_{y_0}^* \subset \mathcal{M}_+^*$, it follows that $< l_i^*, v_+ - (v_\alpha, 0) >\in \mathbf{Z}$ for all $i \in \{1, \ldots n\}$. Now we show that $v_+ - (v_\alpha, y_1) \in \mathcal{L}$ for some $y_1 \in \mathbf{R}$. For any element $k \in \mathcal{L}^*$ and any $y_1 \in \mathbf{R}$,

$$\begin{aligned} < k, v_+ - (v_\alpha, y_1) > &= < k, v_+ - (v_\alpha, 0) > - < k, (0, y_1) > \\ &= m_1 + m_2 < l_{n+1}^*, v_+ - (v_\alpha, 0) > -m_2 \tfrac{m}{y_0} y_1, \end{aligned}$$

with $m_1, m_2 \in \mathbf{Z}$. Taking, for instance, $y_1 =< l^*_{n+1}, v_+ - (v_\alpha, 0) > \frac{y_0}{m}$ we obtain $< k, v_+ - (v_\alpha, y_1) > \in \mathbf{Z}$. Thus, $(-v_+ + (v_\alpha, y_1), I) \cdot (v_+, \alpha_+) = ((v_\alpha, y_1), \alpha_+) \in \Gamma$. $\square$

Lemma 6.2. *If $(v_-, \alpha_-) \in \Gamma$ and $\mathcal{L}^* = \mathcal{N}^*_{y_0} \cup \mathcal{M}^*_-$ then one of the following conditions of Theorem 3.1 holds:*
(II) $((v_\alpha, y_0), \alpha_-) \in \Gamma$,
(III) $(0, y_0) \in \mathcal{L}$ *and* $((v_\alpha, y_1), \alpha_-) \in \Gamma$ *for some* $y_1 \in \mathbf{R}$.

Proof. The proof of this lemma is analogous to the previous one with $v_- - (v_\alpha, y_0)$ instead of $v_+ - (v_\alpha, 0)$ and $y_1 =< l^*_{n+1}, v_- - (v_\alpha, y_0) > \frac{y_0}{m} + y_0$. $\square$

Lemma 6.3. *If both (v_σ, σ) and (v_+, α_+) belong to Γ, and if both*

$$\mathcal{M}^* \subset \left(\mathcal{N}^*_{y_0} \cup \mathcal{M}^*_+ \cup \mathcal{N}^*_\sigma\right) \text{ and } \mathcal{N}^* \subset \left(\mathcal{N}^*_{y_0} \cup \left(\mathcal{M}^*_- \cap \mathcal{N}^*_{\bar\sigma}\right)\right),$$

then one of the following conditions of Theorem 3.1 holds:
(I) $((v_\alpha, 0), \alpha_+) \in \Gamma$,
(II) $((v_\alpha, y_0), \alpha_-) \in \Gamma$,
(III) $(0, y_0) \in \mathcal{L}$ *and either* $((v_\alpha, y_1), \alpha_+) \in \Gamma$ *or* $((v_\alpha, y_1), \alpha_-) \in \Gamma$, *for some* $y_1 \in \mathbf{R}$.

The proof of Lemma 6.3 has three main steps. First we describe some properties of the linear components of (v_σ, σ) and (v_+, α_+), elements of Γ, and restrict the bases of $\mathcal{L}$ and $\mathcal{L}^*$ under the hypothesis of the Lemma. In particular we show that either $v_1 = v_\alpha$ or we may choose $a_1 = 2\,(v_1 - v_\alpha)$. Afterwards, we prove the result for each of these two cases.

Proof. Let $v_+ = (v_1, v_2)$ with $v_1 \in \mathbf{R}^n$ and $v_2 \in \mathbf{R}$ and notice that, since $\sigma \in \mathbf{J}$, the bases $\{l_1, \ldots, l_{n+1}\}$ and $\left\{l^*_1, \ldots, l^*_{n+1}\right\}$ have the properties (1) to (3) described above, where, in particular, $l_1 = (a_1, b_1)$ and $(0, b) \in \mathcal{L}$. We claim that the following properties are also verified:

(1) $v_\sigma + \sigma v_\sigma \in \mathcal{L}$.
(2) $\sigma v_+ - v_+ = -(0, 2v_2)$. Therefore
  (i) $(0, 4v_2) \in \mathcal{L}$ and
  (ii) if $(0, 2v_2) \in \mathcal{L}$ then $\mathcal{N}^* = \emptyset$.
(3) $\mathcal{M}^*_{y_0} \subset \left(\mathcal{M}^*_+ \cup \mathcal{N}^*_\sigma\right)$.
(4) Either $v_1 = v_\alpha$ or we may choose $a_1 = 2\,(v_1 - v_\alpha)$.
(5) In both cases of property (4), $l^*_i \in \mathcal{M}^*_+$ for all $i \in \{2, \ldots, n\}$.

We now prove the claims above.
(1) This property has already been used and proved, see (5.4).
(2) Lemma 4.1 is valid and these properties are a consequence of the definition of $\mathcal{M}^*$ and $\mathcal{N}^*$.
(3) The sets $\mathcal{M}^*_{y_0}$ and $\mathcal{N}^*$ are disjoint, by (2) and by the property (2iv) of the bases. Thus the hypothesis of Lemma 6.3 implies this result.
(4) The last property implies that for all $i \in \{1, \ldots, n\}$,

$$\text{either } < (a^*_i, 0), v_+ - (v_\alpha, 0) > \in \mathbf{Z} \text{ or } < (a^*_i, 0), v_\sigma - (0, y_0) > + \frac{1}{2} \in \mathbf{Z}.$$

If $l^*_i \in \mathcal{N}^*_\sigma$ then $2l^*_i \notin \mathcal{N}^*_\sigma$ and so, for all $i \in \{1, \ldots, n\}$,

$$2 < (a^*_i, 0), v_+ - (v_\alpha, 0) > = < a^*_i, 2(v_1 - v_\alpha) > \in \mathbf{Z}.$$

Therefore, $2(v_1 - v_\alpha) = \sum_{i=1}^n m_i a_i$ with $m_i \in \mathbf{Z}$ for all $i \in \{1, \ldots, n\}$ and, if $v_1 \neq v_\alpha$, we may choose $a_1 = \frac{2(v_1 - v_\alpha)}{m}$ for some $m \in \mathbf{Z}$, by the property (2iii) of the bases. If $v_\alpha = \sum_{i=1}^n r_i a_i$, with $r_i \in \mathbf{R}$, then, by property (1) of the bases, v_+ may be writen as $\sum_{i=1}^{n+1} s_i l_i$ such that $2\,(r_i - s_i) \in [0, 2[$ for all $i \in \{1, \ldots, n\}$. Thus, $m = 1$ and the result follows.

(5) $v_1 - v_\alpha$ is either zero or $a_1/2$. Therefore

$$< l_i^*, v_+ - (v_\alpha, 0) >=< a_i^*, v_1 - v_\alpha >= 0,$$

for $i \in \{2, \ldots, n\}$.

Property (4) above, divides Lemma 6.3 in two major cases that we consider separately.

Suppose $v_1 = v_\alpha$.

Thus $v_+ - (v_\alpha, 0) = (0, v_2)$ and all the elements $l_1^*, \ldots, l_n^*$ belong to the module $\mathcal{M}_+^*$.

If $l_{n+1}^* \in \mathcal{M}_+^*$ then $\mathcal{L}^* = \mathcal{M}_+^*$ and, as in the proof of Lemma 6.1, $((v_\alpha, 0), \alpha_+) \in \Gamma$, *i.e.*, condition (I) holds.

If $l_{n+1}^* \in \mathcal{N}_{y_0}^*$ then, by property (2iv) of the bases, $\mathcal{L}^* = \overline{\mathcal{N}_{y_0}^*}$, which implies $(0, y_0) \in \mathcal{L}$. Condition (III) follows since $((v_\alpha, v_2), \alpha_+) \in \Gamma$.

Now suppose that

$$\begin{array}{rcll} l_{n+1}^* & \notin & \left(\mathcal{M}_+^* \cup \mathcal{N}_{y_0}^*\right) & \text{and, consequently,} \\ l_i^* + l_{n+1}^* & \notin & \left(\mathcal{M}_+^* \cup \mathcal{N}_{y_0}^*\right) & \text{for all } i \in \{1, \ldots, n\}. \end{array} \tag{6.1}$$

If $(0, 2v_2) \in \mathcal{L}$ then we may choose $2v_2 = b$ and the above conditions, as $\mathcal{N}^* = \emptyset$, lead to

$$l_{n+1}^* \in \mathcal{N}_\sigma^* \quad \text{and} \quad l_i^* \in \mathcal{M}_\sigma^* \text{ for all } i \in \{1, \ldots, n\}$$

and, if $v_\sigma = \sum_{i=1}^{n+1} s_i l_i$, with $s_i \in [0, 1[$ for $i \in \{1, \ldots, n\}$, then

$$s_{n+1} - \frac{y_0}{b} + \frac{1}{2} \in \mathbf{Z} \quad \text{and} \quad s_i = 0 \text{ for all } i \in \{1, \ldots, n\}.$$

Therefore, up to multiples of $(0, b)$, we have $v_\sigma = (0, y_0 + b/2) = (0, y_0 + v_2)$ and

$$((0, y_0 + v_2), \sigma) \cdot ((v_\alpha, v_2), \alpha_+) = ((v_\alpha, y_0), \alpha_-) \in \Gamma,$$

i.e., condition (II).

If $(0, 2v_2) \notin \mathcal{L}$ then conditions (6.1) imply

$$l_{n+1}^* \in \mathcal{M}_-^* \quad \text{and} \quad l_i^* \in \mathcal{M}_-^* \text{ for all } i \in \{1, \ldots, n\}.$$

Thus, $\mathcal{L}^* = \mathcal{M}_-^*$ and, as in Lemma 6.2, $((v_\alpha, y_0), \alpha_-) \in \Gamma$, completing the proof in the case $v_1 = v_\alpha$.

Now suppose that $v_1 \neq v_\alpha$ and let $a_1 = 2(v_1 - v_\alpha)$.

Since $l_1^* \notin \mathcal{M}_+^*$, property (3) above implies

$$l_1^* \in \mathcal{N}_\sigma^* \quad \text{and} \quad l_i^* \in \mathcal{M}_\sigma^* \text{ for all } i \in \{2, \ldots, n\},$$

which, for $v_\sigma = \sum_{i=1}^{n+1} s_i l_i$, with $s_i \in [0, 1[$ for $i \in \{1, \ldots, n\}$, can be writen as

$$s_1 = \frac{1}{2} \quad \text{and} \quad s_i = 0 \text{ for all } i \in \{2, \ldots, n\}.$$

Thus, $v_\sigma = l_1/2 + s_{n+1}(0, b)$ and , by property (1), $(a_1, 0) \in \mathcal{L}$, *i.e.*, $b_1 = 0$. As $v_\sigma = (a_1/2, 0) + s_{n+1}(0, b) = v_+ - (v_\alpha, 0) + (0, s_{n+1}b - v_2)$, property (1) allows us to conclude that $(-\sigma v_+ + (v_\alpha, s_{n+1}b - v_2), \sigma) \in \Gamma$.

If $l_{n+1}^* \in \mathcal{N}_{y_0}^*$ then, as in the proof of the previous Lemma, $(0, y_0) \in \mathcal{L}$. Moreover,

$$(-\sigma v_+ + (v_\alpha, s_{n+1}b - v_2), \sigma) \cdot (v_+, \alpha_+) = ((v_\alpha, s_{n+1}b - v_2), \alpha_-) \in \Gamma$$

and condition (III) follows.

Now suppose that $l_{n+1}^* \notin \mathcal{N}_{y_0}^*$ and, consequently, that $l_i^* + l_{n+1}^* \notin \mathcal{N}_{y_0}^*$ for all $i \in \{1, \ldots, n\}$. If $l_{n+1}^* \in \mathcal{M}_+^*$ then $< l_{n+1}^*, l_1/2 + (0, v_2) >= v_2/b \in \mathbf{Z}$ and $(0, v_2) \in \mathcal{L}$, since $(0, b) \in \mathcal{L}$. Moreover, as $l_1^* \notin \mathcal{M}_+^*$, we must impose $l_1^* + l_{n+1}^* \in \mathcal{N}_\sigma^*$, which implies $s_{n+1} + y_0/b \in \mathbf{Z}$. Therefore, choosing $s_{n+1} = y_0/b$,

$$((0, v_2), I) \cdot (-\sigma v_+ + (v_\alpha, y_0 - v_2), \sigma) \cdot (v_+, \alpha_+) = ((v_\alpha, y_0), \alpha_-) \in \Gamma.$$

For $(0, 2v_2) \in \mathcal{L}$, the only missing case is $l^*_{n+1} \in \mathcal{N}^*_\sigma$, where $s_{n+1} + y_0/b + 1/2 \in \mathbf{Z}$ and, up to multiples of $(0, b)$, $s_{n+1}b - v_2 = y_0$. Condition (II) follows because

$$(-\sigma v_+ + (v_\alpha, y_0), \sigma) \cdot (v_+, \alpha_+) = ((v_\alpha, y_0), \alpha_-) \in \Gamma.$$

If $(0, 2v_2) \notin \mathcal{L}$ then both l^*_{n+1} and $l^*_i + l^*_{n+1}$ belong to $\mathcal{M}^*_-$ for $i \in \{1, \ldots, n\}$. Thus, as in the previous Lemma, condition (II) follows. □

7. Restriction

In this section we present, for the restriction of functions in X_Γ, results that are analogous to those obtained for the projection.

7.1. **Theorem for the restriction.** Recall that, for $r \in \mathbf{R}$, the operator Φ_r maps $f(x, y)$ to its restriction to the affine subspace $\{(x, r) : x \in \mathbf{R}^n\}$ given by $\Phi_r(f)(x) = f(x, r)$. If $f \in X_\Gamma$ then, formally,

$$\Phi_r(f)(x) = \sum_{k_1 \in \mathcal{L}^*_1} \omega_{k_1}(x) \sum_{k_2 : (k_1, k_2) \in \mathcal{L}^*} C(k_1, k_2)\omega_{k_2}(r),$$

where $\mathcal{L}^*_1 = \{k_1 : (k_1, k_2) \in \mathcal{L}^*\}$.

Theorem 7.1. *All functions in $\Phi_r(X_\Gamma)$ are invariant under the action of $(v_\alpha, \alpha) \in \mathbf{R}^n \ltimes \mathbf{O}(n)$ if and only if one of the following conditions holds:*

(I) $((v_\alpha, 0), \alpha_+) \in \Gamma$,
(II) $((v_\alpha, 2r), \alpha_-) \in \Gamma$.

7.2. **notes about the proof.** For any function $f \in X_\Gamma$ its restriction, $\Phi_r(f)$, and its projection, $\Pi_{y_0}(f)$, have analogous formal Fourier series. The difference lies on the term $\omega_{k_2}(r)$ of the restriction that corresponds, in the projection, to the integral $\int_0^{y_0} \omega_{k_2}(y)dy$. Thus, the results concerning the restriction Φ_r are similar to the ones proved throughout the previous sections for the projection Π_{y_0}.

We do not present the proof of Theorem 7.1 because it is analogous to the one for the projection, with $\omega_{k_2}(r)$ instead of $\int_0^{y_0} \omega_{k_2}(y)dy$. The condition $\omega_{k_2}(r) = 0$ is never verified and so the sets $\mathcal{N}^*_{y_0}$ and $\mathcal{M}^*_{y_0}$ disappear and we don't have an analogous to the condition $(0, y_0) \in \mathcal{L}$. Moreover, the expression

$$\int_0^{y_0} \omega_{k_2}(y)dy - \omega_{k_2}(y_0) \int_0^{y_0} \omega_{-k_2}(y)dy = 0$$

has the analogous

$$\omega_{k_2}(r) - \omega_{k_2}(2r)\omega_{-k_2}(r) = 0.$$

It follows that $2r$ appears where originally we had the variable y_0.

We now present some intermediate results in order to obtain Theorem 7.1 and make remarks about the proof for the analogous of Lemma 5.4.

Proposition 7.1. *All functions in $\Phi_r(X_\Gamma)$ are invariant under the action of $(v_\alpha, \alpha) \in \mathbf{R}^n \ltimes \mathbf{O}(n)$ if and only if one of the following conditions holds:*

(A) $(v_+, \alpha_+) \in \Gamma$ *and* $\mathcal{L}^* = \mathcal{M}^*_+$,
(B) $(v_-, \alpha_-) \in \Gamma$ *and* $\mathcal{L}^* = \mathcal{M}^*_-$,
(C) *both* (v_σ, σ) *and* (v_+, α_+) *belong to* Γ, $\mathcal{M}^* \subset \left(\mathcal{M}^*_+ \cup \mathcal{N}^*_\sigma\right)$ *and* $\mathcal{N}^* \subset \left(\mathcal{M}^*_- \cap \mathcal{N}^*_{\tilde{\sigma}}\right)$.

The analogous to Lemma 5.2 is, for $D'(\delta, k) = \omega_{\delta k}(-v_\delta)\omega_{\delta k|_2}(r)$:

Lemma 7.1. *Suppose that*

(a) *if* $k_1 \in \mathcal{L}^*_1 \cap \alpha\mathcal{L}^*_1$ *then* $\sum_{\delta \in \mathrm{J}^{Id}(k)} D'(\delta, k) = \omega_{k_1}(-v_\alpha) \sum_{\delta \in \mathrm{J}^\alpha(k)} D'(\delta, k)$ *and*
(b) *if* $k_1 \notin \mathcal{L}^*_1 \cap \alpha\mathcal{L}^*_1$ *then* $\sum_{\delta \in \mathrm{J}^{Id}(k)} D'(\delta, k) = 0$,

for all $k = (k_1, k_2) \in \mathcal{L}^*$. Then one of the following cases holds:

(1) $\mathbf{J}^{Id} = \{Id_{n+1}, \sigma\}$, $\mathbf{J}^{\alpha} = \emptyset$ *and* $\mathcal{O}^* \subset \mathcal{N}_\sigma^*$,
(2) $\mathbf{J}^{Id} = \{Id_{n+1}\}$, $\mathbf{J}^{\alpha} = \{\alpha_+^{-1}\}$ *and* $\mathcal{O}^* \subset \mathcal{M}_+^*$,
(3) $\mathbf{J}^{Id} = \{Id_{n+1}\}$, $\mathbf{J}^{\alpha} = \{\alpha_-^{-1}\}$ *and* $\mathcal{O}^* \subset \mathcal{M}_-^*$,
(4) $\mathbf{J}^{Id} = \{Id_{n+1}, \sigma\}$, $\mathbf{J}^{\alpha} = \{\alpha_+^{-1}, \alpha_-^{-1}\}$, $(\mathcal{O}^* \cap \mathcal{M}^*) \subset \left(\mathcal{M}_+^* \cup \mathcal{N}_\sigma^*\right)$ *and* $(\mathcal{O}^* \cap \mathcal{N}^*) \subset \left(\mathcal{M}_-^* \cap \mathcal{N}_{\tilde{\sigma}}^*\right)$.

In a similar way, we may state a lemma analogous to Lemma 5.4. Under the conditions for the restriction, in the proof of Lemma 5.4, property (5.6) concerning the set $\mathcal{M}_{y_0}^*$ does not hold. By Lemma 7.1 above, the paragraph corresponding to case (1) disappear and in the paragraph corresponding to case (2), the dual lattice is $\mathcal{L}^* = \mathcal{N}_\sigma^* \cup \mathcal{P}^*$. In this paragraph the arguments concerning $\mathcal{M}_{y_0}^*$ and $\mathcal{N}_{y_0}^*$ must be replaced by: let $\tilde{k} \notin \mathcal{P}^*$ and, thus, $\tilde{k} \in \mathcal{N}_\sigma^*$. However both $2\tilde{k} \notin \mathcal{P}^*$ and $2\tilde{k} \notin \mathcal{N}_\sigma^*$, by definition of $\mathcal{P}^*$ and the properties of $\mathcal{N}_\sigma^*$, and so this case is not possible.

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References

[1] Armstrong, M. A., *Groups and Symmetry*, Springer-Verlag, 1988
[2] Golubitsky, M., Stewart, I. and Schaeffer, D. G., *Singularities and Groups in Bifurcation Theory* - vol II, Appl. Math. Sci. **69**, Springer-Verlag, 1988
[3] Gomes, M. G. M., *Black-eye patterns: A representation of three-dimensional symmetries in thin domains*, Phys. Rev. E **60** (1999) 3741-3747
[4] Labouriau, I. S. and Pinho, E. M., *Projected wallpaper patterns*, in Brasselet, J-P; Ruas, M. A.S. (Eds.), *Real and Complex Singularities —- São Carlos Workshop 2004* Birkhäuser, Series: Trends in Mathematics (2006) pages 209–217
[5] Labouriau, I. S. and Pinho, E. M., *Symmetries of projected wallpaper patterns*, Math. Proc. Camb. Philos. Soc. **141** (2006) 421–441
[6] Melbourne, I., *Steady-state bifurcation with Euclidean symmetry*, Trans. Amer. Math. Soc. **351** (1999) 1575-1603
[7] Miller, W., *Symmetry Groups and their Applications*, Academic Press, 1972
[8] Pinho, E. M., *Symmetries of Projected Symmetric Patterns*, Thesis, University of Porto, 2006
[9] Senechal, M., *Quasicrystals and Geometry*, Cambridge University Press, 1995

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