THE FREE IDEMPOTENT GENERATED LOCALLY INVERSE SEMIGROUP

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Abstract. In this paper we construct a model for the free idempotent generated locally inverse semigroup on a set X. The elements of this model are special vertex-labeled bipartite trees with a pair of distinguished vertices. To describe this model, we need first to introduce a variation of a model for the free pseudosemilattice on a set X presented in [4]. A construction of a graph associated with a regular semigroup was presented in [5] in order to give a first example of a free regular idempotent generated semigroup on a biordered set E with non-free maximal subgroups. If G is the graph associated with the free pseudosemilattice on X, we shall see that the models we present for the free pseudosemilattice on X and for the free idempotent generated locally inverse semigroup on X are closely related with the graph G.

1. Introduction

Let x be an element of a semigroup S. An *inverse* of x is another element $x' \in S$ such that $xx'x = x$ and $x'xx' = x'$. A semigroup S is *regular* if every $x \in S$ has at least one inverse, and it is *inverse* if every $x \in S$ has exactly one inverse. A locally inverse semigroup is a regular semigroup where every local submonoid is an inverse monoid, that is, for every idempotent e of S, the subsemigroup eSe of S is an inverse monoid. Thus,

$$\mathbf{G} \subseteq \mathbf{I} \subseteq \mathbf{LI} \subseteq \mathbf{RS}$$

where $\mathbf{G}$, $\mathbf{I}$, $\mathbf{LI}$ and $\mathbf{RS}$ designate respectively the classes of all groups, all inverse semigroups, all locally inverse semigroups and all regular semigroups.

As usual, we shall designate by $E(S)$ the set of idempotents of S and by $\mathscr{R}$, $\mathscr{L}$, $\mathscr{D}$, $\mathscr{H}$ and $\mathscr{J}$ the five Green's relations on S. We refer the reader to [13] for usual concepts in Semigroup Theory left undefined in this paper. Let $\leq$ denote the usual natural partial order on a regular semigroup S. Consider the following two quasi-order relations on $E(S)$:

$$e \leq_{\mathscr{R}} f \Leftrightarrow e = fe \quad \text{and} \quad e \leq_{\mathscr{L}} f \Leftrightarrow e = ef$$

(these relations are often denoted by ω^r and ω^l [16]). Then the restrictions of $\mathscr{R}$, $\mathscr{L}$ and $\leq$ to the set of idempotents of a regular semigroup S are

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respectively the relations $\leq_{\mathscr{R}} \cap \geq_{\mathscr{R}}$, $\leq_{\mathscr{L}} \cap \geq_{\mathscr{L}}$ and $\leq_{\mathscr{R}} \cap \leq_{\mathscr{L}}$, for $\geq_{\mathscr{R}}$ and $\geq_{\mathscr{L}}$ the expected reverse relations corresponding to $\leq_{\mathscr{R}}$ and $\leq_{\mathscr{L}}$. However, the join of the relations $\mathscr{R}_{|E(S)}$ and $\mathscr{L}_{|E(S)}$ on $E(S)$ does not coincide with the restriction of $\mathscr{D}$ to the set $E(S)$ usually. Therefore, we shall use the symbols $\mathscr{R}$, $\mathscr{L}$ and $\leq$ also to denote the restriction of these relations to $E(S)$, but for the join $\mathscr{R}_{|E(S)} \vee \mathscr{L}_{|E(S)}$ on $E(S)$ we shall use the symbol $\mathscr{C}$ instead (a $\mathscr{D}_{|E(S)}$-class is always the union of $\mathscr{C}$-classes). Nevertheless, there is a particular instance where we know that $\mathscr{C}{=}\mathscr{D}_{|E(S)}$: if S is idempotent generated.

An *E-path* in S is a finite sequence $(e_0, e_1, \ldots, e_n)$ of idempotents of S such that $e_{i-1}\,(\mathscr{R} \cup \mathscr{L})\, e_i$ for $i = 1, \ldots, n$. If we consider the graph

$$(E(S), \mathscr{R} \cup \mathscr{L})$$

(that is, the graph with vertices $E(S)$ and edges $(e, f) \in \mathscr{R} \cup \mathscr{L}$; (e, f) and (f, e) represent the same edge), then the E-paths are the usual paths in this graph. Two idempotents are said to be *connected* if there exists an E-path starting in one of them and ending in the other one. Thus $e \ \mathscr{C} \ f$ if and only if e and f are connected. We shall call the $\mathscr{C}$-classes the *connected components* of $E(S)$.

We shall denote by $(f]_{\mathscr{R}}$ the principal ideal of $E(S)$ generated by f for the quasi-order $\leq_{\mathscr{R}}$, that is,

$$(f]_{\mathscr{R}} = \{e \in E(S) \mid e \leq_{\mathscr{R}} f\}\,.$$

We define similarly the principal ideals $(f]_{\mathscr{L}}$ and $(f]_{\leq}$ generated by f respectively for the relations $\leq_{\mathscr{L}}$ and $\leq$. The locally inverse semigroups are precisely the regular semigroups such that, for each ordered pair (e, f) of idempotents of S,

$$(e]_{\mathscr{R}} \cap (f]_{\mathscr{L}} = (g]_{\leq}$$

for some $g \in E(S)$. Since $\leq$ is a partial order, the idempotent g is unique for each ordered pair (e, f), and so we can introduce another binary operation $\wedge$ on $E(S)$ by setting $e \wedge f = g$. The new binary algebra $(E(S), \wedge)$ obtained from each locally inverse semigroup S is called the *pseudosemilattice of idempotents* of S [17]. If S is a locally inverse semigroup, then the quasi-orders $\leq_{\mathscr{R}}$ and $\leq_{\mathscr{L}}$ can be recovered from the operation $\wedge$:

$$e \leq_{\mathscr{R}} f \Leftrightarrow f \wedge e = e \quad \text{ and } \quad e \leq_{\mathscr{L}} f \Leftrightarrow e \wedge f = e\,.$$

Nambooripad [17] showed that the pseudosemilattices of idempotents of locally inverse semigroups satisfy the identities

$(PS1)$ $x \wedge x \approx x$;
$(PS2)$ $(x \wedge y) \wedge (x \wedge z) \approx (x \wedge y) \wedge z$;
$(PS3)$ $((x \wedge y) \wedge (x \wedge z)) \wedge (x \wedge w) \approx (x \wedge y) \wedge ((x \wedge z) \wedge (x \wedge w))$;

together with the left-right duals (PS2') and (PS3') of (PS2) and (PS3), respectively; and that any binary algebra satisfying these five identities is (isomorphic to) the pseudosemilattice of idempotents of some locally inverse semigroup. Thus, a *pseudosemilattice* is a binary algebra satisfying the five identities above; and the class of all pseudosemilattices constitutes a variety of algebras. We shall denote this variety by **PS**.

An e-variety of regular semigroups [11, 14] is a class of these algebras closed for taking homomorphic images, direct products and regular subsemigroups. The classes **G**, **I**, **LI** and **RS** are examples of e-varieties of regular semigroups. Auinger [3] extended Nambooripad's result stated above and proved that the mapping

$$\xi : \mathcal{L}_e(\mathbf{LI}) \longrightarrow \mathcal{L}(\mathbf{PS}), \quad \mathbf{V} \longmapsto \{(E(S), \wedge) \mid S \in \mathbf{V}\}$$

is a well defined complete homomorphism from the lattice $\mathcal{L}_e(\mathbf{LI})$ of e-varieties of locally inverse semigroups onto the lattice $\mathcal{L}(\mathbf{PS})$ of varieties of pseudosemilattices. A key ingredient in the proof of this result is the notion of free idempotent generated semigroup in $\mathbf{V}$ on a set X, for $\mathbf{V} \in \mathcal{L}_e(\mathbf{LI})$ (see section 2).

In the following section we shall recall some results and concepts about locally inverse semigroups and pseudosemilattices. In particular, we shall define the notion of free idempotent generated semigroup in $\mathbf{V}$ on a set X and we shall see how this semigroup is related with the E-chains from the free pseudosemilattice in $\mathbf{V}\xi$ on the set X.

In section 3 we shall present a variation of the model for the free pseudosemilattice described in [4]. In our opinion, the variation on the model presented here is more natural (see section 5) and captures better the structure of the free pseudosemilattice. For example, we can easily describe the relations $\mathscr{R}$, $\mathscr{L}$, $\mathscr{C}$, $\leq_{\mathscr{R}}$, $\leq_{\mathscr{L}}$ and $\leq$ on this new model. We shall use this model of the free pseudosemilattice on X to describe, in section 4, a model for the free idempotent generated locally inverse semigroup on X.

The set of idempotents of a regular semigroup has a structure of a partial algebras, called a *regular biordered set*. These partial algebras were introduced and characterized by Nambooripad [16]. In [5], a natural (oriented) graph G was associated with each biordered set. These graphs were used to construct a first example of a *free regular idempotent generated semigroup* on a biordered set E (see next section) with maximal subgroups not free. We shall see in section 5 that the graph G associated with the free pseudosemilattice on X is closely related with the models presented in sections 3 and 4.

2. Idempotent generated locally inverse semigroups

A *basic pair* in a semigroup S is a set $\{e, f\}$ of two idempotents such that

$$\{ef, fe\} \cap \{e, f\} \neq \emptyset\,.$$

Thus the products ef and fe are always idempotents if $\{e, f\}$ is a basic pair. The *biordered set of idempotents* of S is the partial algebra $E(S)$ with multiplication restricted to the basic pairs. This concept was introduced by Nambooripad in [16] who gave also an axiomatic characterization for them. Since we do not need this axiomatic characterization here, we refer the reader to [16] for more details.

The *sandwich set* $S(e,f)$ for each ordered pair (e,f) of idempotents of S is the set of idempotents

$$S(e,f) = \{h \in E(S) \mid ehf = ef,\ fhe = h\}.$$

This notion is intrinsic to the concept of biordered set, that is, the sandwich sets $S(e,f)$ can be defined using only the axioms for biordered sets (see [16]). A biordered set is called *regular* if all sandwich sets are nonempty. Nambooripad [16] proved that each regular biordered set is the biordered set of some regular semigroup, and vice-versa. Nambooripad's result was then extended to general biordered sets by Easdown [7] who proved that each biordered set is the biordered set of some semigroup.

There are many different ways to characterize locally inverse semigroups as regular semigroups with special properties (see [21, Corollary 2.3]). One of them is a characterization in terms of its biordered set of idempotents: a regular semigroup S is locally inverse if and only if the sandwich sets $S(e,f)$ in the biordered set $E(S)$ are singletons. If we define a binary operation $\wedge$ on $E(S)$ by setting $e \wedge f$ to be the unique element of $S(e,f)$, we recover the notion of pseudosemilattice of idempotents of S introduced earlier [16]. Further, if $\{e,f\}$ is a basic pair, then $ef = e \wedge f$. Thus, pseudosemilattices are associated with special biordered sets where the partial multiplication can be extended naturally to a full binary operation.

Fix a regular biordered set E. Let $\mathrm{RIG}(E)$ be the semigroup given by the following presentation:

$$\begin{aligned}\mathrm{RIG}(E) = \langle E \,:\, & e^2 = e \text{ for all } e \in E,\ ef = e \cdot f \text{ if } \{e,f\} \text{ is a basic pair} \\ & \text{in } E,\ ef = ehf \text{ for all } e,f \in E \text{ and } h \in S(e,f)\,\rangle\end{aligned}$$

(the notation $e \cdot f$ represents here the product in the biorderd set E). The semigroup $\mathrm{RIG}(E)$ is called the *free regular idempotent generated semigroup* on E due to the property stated in the next theorem proved by Nambooripad [16] and Pastijn [20]. We should mention that there exists also the *free idempotent generated semigroup* [7] on a general biordered set, but since we will not need it here, we refer the reader to [7] for further details.

Theorem 2.1. [16, 20] *If E is a regular biordered set, then* $\mathrm{RIG}(E)$ *is a regular semigroup with biordered set of idempotents E. If S is an idempotent generated regular semigroup with biordered set isomorphic to E (in the category of biordered sets), then the natural map $E \to S$ extends uniquely to a homomorphism* $\mathrm{RIG}(E) \to S$.

In particular, if E is the biordered set of a locally inverse semigroup, then $\mathrm{RIG}(E)$ is a locally inverse semigroup with pseudosemilattice of idempotents $(E,\wedge)$. Thus, we can talk about the free regular idempotent generated semigroup on a pseudosemilattice $(E,\wedge)$. We shall call this semigroup the *free idempotent generated locally inverse semigroup* on the pseudosemilattice E.

An idempotent e_i of an E-path $(e_1, e_2, \ldots, e_n)$ is called *inessential* if either

(*i*) $i = 1$ and $e_1 \mathscr{R} e_2$ or $i = n$ and $e_{n-1} \mathscr{L} e_n$; or

(*ii*) $e_{i-1} \mathscr{R} e_i \mathscr{R} e_{i+1}$ or $e_{i-1} \mathscr{L} e_i \mathscr{L} e_{i+1}$ for $1 < i < n$.

Consider the equivalence relation on the set of E-paths induced by adding or removing inessential vertices. An E-chain, as introduced in [16], is considered to be the equivalence class of an E-path for this equivalence relation. Each E-chain contains a unique E-path with no inessential vertex (see [16]). For the purpose of this paper, we shall consider an *E-chain* as an E-path without inessential vertices. Thus, there is a natural bijection between Nambooripad's concept of E-chain and the concept of E-chain we consider for this paper. An E-chain is then an E-path $(e_1, e_2, \cdots, e_n)$ with distinct consecutive idempotents such that $e_1 \mathscr{L} e_2$, $e_{n-1} \mathscr{R} e_n$, and $e_{i-1} \mathscr{R} e_i$ if and only if $e_i \mathscr{L} e_{i+1}$.

Let $\mathcal{G}(E)$ be the set of E-chains in a regular biordered set E. We can look at $\mathcal{G}(E)$ as a groupoid with objects the set E and morphisms the E-chains: an E-chain $(e_1, \ldots, e_n)$ represents a morphism from e_1 to e_n; the product $(e_1, \ldots, e_n)(f_1, \ldots, f_m)$ of two E-chains is defined (and equal to the unique E-chain in the equivalence class of the E-path $(e_1, \ldots, e_n, f_1, \ldots, f_m)$) if and only if $e_n = f_1$. An equivalence relation $\sim$ on the set of E-chains was introduced in [16] and used to show that the maximal subgroup of $\mathrm{RIG}(E)$ containing an idempotent $e \in E$ is isomorphic to the local group of $\mathcal{G}(E)/\sim$ at the object e. This result becomes particularly interesting if E is a pseudosemilattice: the equivalence relation $\sim$ becomes trivial, and the maximal subgroups of $\mathrm{RIG}(E)$ are isomorphic to the local subgroups of $\mathcal{G}(E)$. Thus, if the E-chains (or the connected components) of a pseudosemilattice E are 'loop-free', then $\mathrm{RIG}(E)$ has trivial subgroups.

Lemma 2.2. *Let E be a pseudosemilattice with all E-chains loop-free. Then* $\mathrm{RIG}(E)$ *is combinatorial.*

It was conjectured that the maximal subgroups of $\mathrm{RIG}(E)$ for a regular biordered set E were always free, and in fact this conjecture was proved to be true for some classes of biordered sets [18, 19]. One such example is the class of all pseudosemilattices [18]. However, a first example were this conjecture fails appears in [5] (see [6, 9, 10] for further developments on this topic). The construction presented in [5] is based on adding a 2-complex structure on top of another type of (oriented) graph G that can be associated with a biordered set E. The vertices of the graph G are the $\mathscr{L}$-classes and the $\mathscr{R}$-classes of E; and there is an oriented edge from an $\mathscr{L}$-class L to an $\mathscr{R}$-class R if and only if $L \cap R$ has an idempotent of E. Since we do not need for this paper the 2-complex structure added on top of G, we refer the reader to [5] for more details. We should mention also that the graphs G were introduced independently by Graham [8] and Houghton [12] for finite 0-simple semigroups in order to study them.

In section 5 we shall consider the graph G associated with the free pseudosemilattice on X, say E, but with the edges being non-oriented. We shall introduce a labeling for the vertices of G and define a category $\mathscr{G}(E)$ whose objects are the connected components of G (including labels) and whose morphisms are special label preserving graph homomorphisms. We shall give more details in section 5. In that section we shall see that the models for the free pseudosemilattices and for the free idempotent generated locally inverse semigroups constructed in the next two sections are closely related with the category $\mathscr{G}(E)$.

Let $\mathbf{V}$ be an e-variety of regular semigroups. The *free idempotent generated semigroup* in $\mathbf{V}$ on X is a semigroup $\mathrm{FI}\,\mathbf{V}(X) \in \mathbf{V}$, together with a mapping $\iota : X \longrightarrow E(\mathrm{FI}\,\mathbf{V}(X))$, such that any other mapping $\theta : X \longrightarrow E(S)$ with $S \in \mathbf{V}$ is uniquely extended to a homomorphism $\overline{\theta} : \mathrm{FI}\,\mathbf{V}(X) \longrightarrow S$. Auinger [2, 3] showed the existence (and uniqueness) of free idempotent generated semigroups on X in every e-variety $\mathbf{V}$ of locally inverse semigroups. Further, in this case, the pseudosemilattice of idempotents of $\mathrm{FI}\,\mathbf{V}(X)$ is the free object $\mathrm{F}(\mathbf{V}\xi)(X)$ on X in the variety $\mathbf{V}\xi$ of pseudosemilattices. We shall denote by $\mathrm{FILI}(X)$ the free idempotent generated locally inverse semigroup on X, that is, the free idempotent generated semigroup in $\mathbf{LI}$ on X; and by $\mathrm{FPS}(X)$ the free pseudosemilattice on X, that is, the free object on X in the variety $\mathbf{PS}$ of all pseudosemilattices.

For e-varieties of locally inverse semigroups, the two concepts of free idempotent generated semigroup in $\mathbf{V}$ and of free idempotent generated locally inverse semigroup on a pseudosemilattice E are related by the following [3]: if $\mathbf{E}$ is a variety of pseudosemilattices and $\mathbf{V}$ is the largest e-variety such that $\mathbf{E} = \mathbf{V}\xi$, then

$$\mathrm{FI}\,\mathbf{V}(X) \cong \mathrm{RIG}(\mathrm{F}\,\mathbf{E}(X))\,.$$

A particular instance is the case of the variety $\mathbf{LI}$:

$$\mathrm{FILI}(X) \cong \mathrm{RIG}(\mathrm{FPS}(X))\,.$$

3. The free pseudosemilattice on a set X

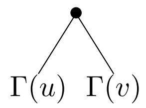
Let $(F_2(X), \wedge)$ be the absolutely free binary algebra on X. Thus, the elements of $F_2(X)$ are well-formed words on the alphabet $X \cup \{(,), \wedge\}$. In [4] we associated a graph $\Delta(u)$ to each word $u \in F_2(X) \setminus X$. The process to obtain $\Delta(u)$ shall be recalled next, alongside with the illustration of an example to make the construction more clear. We shall consider the word

$$w = ((x \wedge (t \wedge z)) \wedge x) \wedge ((y \wedge z) \wedge ((y \wedge t) \wedge y))$$

for our example. This word shall accompany us throughout this paper in several illustrative examples.

We need to introduce first another graph $\Gamma(u)$ for each word $u \in F_2(X)$. The graph $\Gamma(u)$ is the *downward* (connected) tree obtained by setting $\Gamma(x) =$

$\underset{x}{\bullet}$ for each $x \in X$ and using inductively the rule

$\Gamma(u \wedge v) :=$ 

for $u, v \in F_2(X)$. For the word w of our example, the graph $\Gamma(w)$ is depicted in Figure 1. The graphs $\Gamma(u)$ have always a top vertex, the *root*. If $u \notin X$ and a is a non-root vertex of $\Gamma(u)$, then a has a unique predecessor b, that is, has a unique vertex adjacent to a but placed above a in the graph $\Gamma(u)$. We shall call the vertex a a *left* vertex of $\Gamma(u)$ if a is placed to the left of b in $\Gamma(u)$; if a is placed to the right of b then a is called a *right* vertex of $\Gamma(u)$. The root is considered neither a left vertex nor a right vertex.

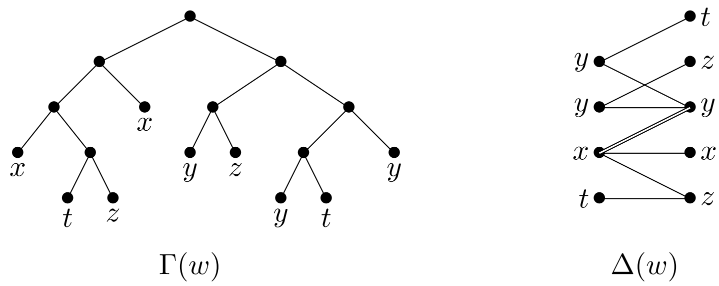


FIGURE 1. The graphs $\Gamma(w)$ and $\Delta(w)$.

Let $u \in F_2(X) \backslash X$. If a is a vertex of $\Gamma(u)$, denote by $\Gamma(u, a)$ the subgraph of $\Gamma(u)$ with top vertex a. Thus $\Gamma(u) = \Gamma(u, a)$ if a is the root of $\Gamma(u)$. We construct the graph $\Delta(u)$ as follows:

(i) set the vertices of $\Delta(u)$ as the leaves (degree 1 vertices) of $\Gamma(u)$ (the vertices keep the labels in $\Delta(u)$);
(ii) for each non-leaf vertex a of $\Gamma(u)$ add an edge (b, c) to $\Delta(u)$ where b and c are respectively the leftmost leaf and the rightmost leaf of $\Gamma(u, a)$; and
(iii) distinguish, in $\Delta(u)$, the leftmost leaf and the rightmost leaf of $\Gamma(u)$ from the other vertices (so, these distinguished vertices are always adjacent).

The vertices of $\Delta(u)$ that are left leaves of $\Gamma(u)$ shall be called *left vertices* of $\Delta(u)$, while the others shall be called *right vertices* of $\Delta(u)$. For the word w of our example, the graph $\Delta(w)$ is depicted also in Figure 1: the left column contains the left vertices of $\Delta(u)$ while the right column contains the right vertices of $\Delta(u)$; the two distinguished vertices are identified by drawing with a double line the edge connecting them.

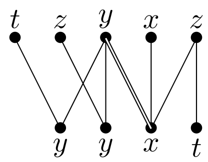
A graph γ is called a (vertex) *X-labeled bipartite tree* if it is a tree with the vertices labeled with letters from X and partitioned into two sets, say

L and R, such that the set of edges is contained in $L \times R$. Thus $\Delta(u)$ is an X-labeled bipartite tree if we consider L as the set of left vertices of $\Delta(u)$ and R as the set of right vertices of $\Delta(u)$. In fact, if

$$\mathfrak{B}'(X) = \{\Delta(u) \mid u \in F_2(X) \setminus X\},$$

then $\mathfrak{B}'(X)$ can be described abstractly as the set of all X-labeled bipartite trees with a pair of adjacent vertices distinguished (see [4]).

We can draw any graph $\gamma \in \mathfrak{B}'(X)$ following the conventions introduced above for our example $\Delta(w)$: left vertices of γ appear in a left column while the right vertices appear in a right column; the two distinguished vertices are marked by drawing with a double line the edge connecting them. However, it is often convenient to represent the vertices of γ into two rows instead of two columns. When representing the vertices of γ into two rows, we convention that the bottom row and the top row correspond respectively to the left vertices and the right vertices of γ. In our example, the graph $\Delta(w)$ would be depicted as follows:

$\Delta(w) :=$ 

We shall denote by L_γ and R_γ respectively the set of left vertices and right vertices of a graph $\gamma \in \mathfrak{B}'(X)$. The left distinguished vertex of γ shall be called the *left root* of γ and shall be denoted by $\mathfrak{l}_\gamma$; similarly, the right distinguished vertex is called the *right root* of γ and it is denoted by $\mathfrak{r}_\gamma$. Given a vertex a of a (vertex) labeled graph, we shall denote by $\mathbf{c}_a$ the label of a. We shall use also the notation $\mathrm{l}(\gamma)$ and $\mathrm{r}(\gamma)$ respectively for $\mathbf{c}_{\mathfrak{l}_\gamma}$ and $\mathbf{c}_{\mathfrak{r}_\gamma}$. Let ${}^L\gamma$ be the graph γ but now only with the left root as a distinguished vertex. We define γ^R dually and consider $\check{\gamma}$ to be the graph γ without distinguished vertices. Note that we continue to look at the vertices of ${}^L\gamma$, γ^R and $\check{\gamma}$ as left and right vertices as in γ. We introduce a binary operation $\sqcap$ on $\mathfrak{B}'(X)$ by setting

$$\alpha \sqcap \beta = {}^L\alpha \,\dot{\cup}\, \{(\mathfrak{l}_\alpha, \mathfrak{r}_\beta)\} \,\dot{\cup}\, \beta^R$$

for $\alpha, \beta \in \mathfrak{B}'(X)$, that is, we construct $\alpha \sqcap \beta$ by taking the disjoint union of α and β, adding the edge $(\mathfrak{l}_\alpha, \mathfrak{r}_\beta)$, and setting $\mathfrak{l}_\alpha$ and $\mathfrak{r}_\beta$ respectively as the left root and the right root of $\alpha \sqcap \beta$. We illustrate an example of $\Delta(w) \sqcap \alpha$ for a graph $\alpha \in \mathfrak{B}'(X)$ in Figure 2.

A vertex a from a graph $\gamma \in \mathfrak{B}'(X)$ is *paired* if there exists another vertex in γ adjacent to a and with the same label. A *thorn* of γ is an ordered pair (a, e) where a is a degree 1 paired vertex and e is the only edge having a at one of its endpoints. The thorn (a, e) is called *essential* if a is one of the distinguished vertices of γ; otherwise it is called an *inessential* thorn. We introduce now the following two reduction rules:

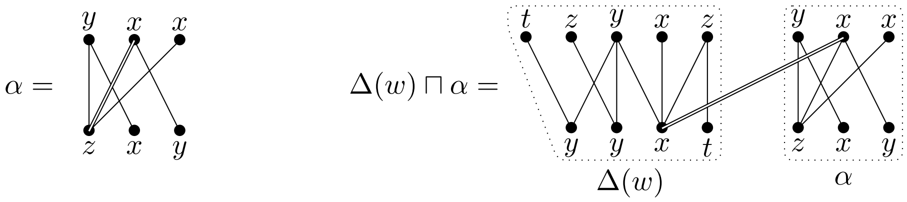
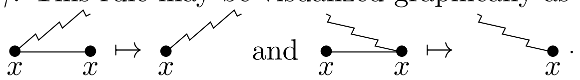
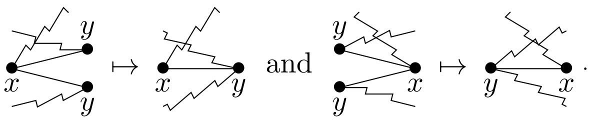


FIGURE 2. An example of $\Delta(w) \sqcap \alpha$.

(i) If (a, e) is an inessential thorn, then delete the vertex a and the edge e from γ. This rule may be visualized graphically as



(ii) Suppose that two edges e and f have a vertex in common and that the two other (distinct) vertices a and b have the same label; then identify the two edges e and f and the vertices a and b (and retain their label). If one of the merged vertices happens to be a distinguished one then so is the resulting vertex. Graphically, this rule may be visualized as



Rule (i) is referred to as the *deletion of a thorn* while rule (ii) is called an *edge-folding*. We illustrate in Figure 3 an example of a reduction sequence for our graph $\Delta(w)$ until we get a graph where no other reduction can be applied.

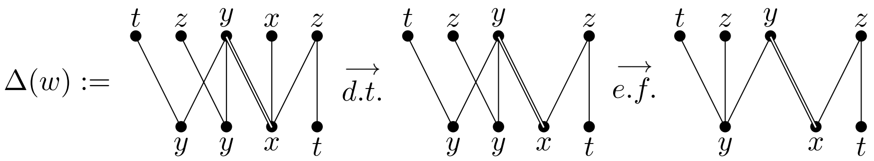


FIGURE 3. A reduction sequence for $\Delta(w)$.

A graph $\gamma \in \mathfrak{B}'(X)$ is called *reduced* if none of the two rules above can be applied to it. Let $\mathfrak{A}(X)$ be the set of all reduced graphs from $\mathfrak{B}'(X)$. Clearly, the order of reductions that we can apply to a given $\gamma \in \mathfrak{B}'(X)$ is not unique. Nevertheless, independently of the order we choose to apply the reductions, we always get the same reduced graph $\alpha \in \mathfrak{A}(X)$ from γ. We shall denote α by $\overline{\gamma}$. In fact, since the deletion of a thorn from an edge-folding reduced graph originates another edge-folding reduced graph, we can divide this reduction process into a two step process

$$\gamma \to \widetilde{\gamma} \to \overline{\gamma}\,,$$

where $\widetilde{\gamma}$ is the edge-folding reduced graph obtained from γ by applying all possible edge-foldings, and then $\overline{\gamma}$ is obtained from $\widetilde{\gamma}$ by applying only deletions of thorns. Note further that the edge-foldings constitute a noetherian locally confluent system of reduction rules on $\mathfrak{B}'(X)$ (see [1, page 110]), whence we always get the same edge-folding reduced graph $\widetilde{\gamma}$ from γ independently of the order of edge-foldings we choose to apply to γ.

If β is obtained from α by applying an edge-folding, then the natural induced mapping from α onto β is a graph homomorphism preserving the labels and the distinguished vertices. This observation can be obviously generalized to any sequence of edge-foldings applied to α, that is, if β is obtained from α by applying a sequence of edge-foldings, then the natural mapping from α onto β is a graph homomorphism preserving the labels and the distinguished vertices. A particular instance of this observation is the case of $\beta = \widetilde{\alpha}$. Thus $\widetilde{\alpha}$ is always a homomorphic image of α (under the natural mapping which also preserves labels and distinguished vertices).

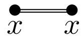
Let σ be the least equivalence relation on $\mathfrak{B}'(X)$ containing the relations induced by the two reduction rules introduced above. Thus $\alpha\sigma\beta$ if and only if $\overline{\alpha} = \overline{\beta}$; and $\mathfrak{A}(X)$ has a unique representative for each σ-class, namely for $\gamma \in \mathfrak{B}'(X)$, $\overline{\gamma}$ is the unique representative in $\mathfrak{A}(X)$ of the σ-class of γ. Further, σ is a congruence relation of the binary algebra $(\mathfrak{B}'(X), \sqcap)$ because

$$\overline{\alpha \sqcap \beta} = \overline{\overline{\alpha} \sqcap \overline{\beta}}$$

for all $\alpha, \beta \in \mathfrak{B}'(X)$ since the order of reductions applied to a given $\gamma \in \mathfrak{B}'(X)$ to obtain $\overline{\gamma}$ is irrelevant. Now, introduce a binary operation $\wedge$ on $\mathfrak{A}(X)$ by setting

$$\alpha \wedge \beta = \overline{\alpha \sqcap \beta}$$

and observe that the mapping $\mathfrak{B}'(X)/\sigma \to \mathfrak{A}(X)$ which associates to each σ-class the unique representative from $\mathfrak{A}(X)$ is an isomorphism. Hence $(\mathfrak{B}'(X)/\sigma, \sqcap)$ and $(\mathfrak{A}(X), \wedge)$ are isomorphic algebras.

Theorem 3.1. [4, Theorem 3.11] *The algebra* $(\mathfrak{A}(X), \wedge)$ *is a model for the free pseudosemilattice on* X *if we identify each* $x \in X$ *with* 

.

Next, we shall introduce a variation of the model $(\mathfrak{A}(X), \wedge)$ by choosing a different representative for each σ-class of $\mathfrak{B}'(X)$. We shall call a graph $\gamma \in \mathfrak{B}'(X)$ *full* if every vertex of γ is paired. Let

$$\mathfrak{A}_f(X) = \{\gamma \in \mathfrak{B}'(X) \mid \gamma \text{ is full and reduced for edge-folding}\}\,.$$

If $\beta \in \mathfrak{A}_f(X)$ then $\alpha = \overline{\beta}$ is obtained from β only by applying deletions of thorns; whence β is obtained from α by adding thorns only. Note that we must add a thorn to each non-paired vertex of α to obtain a full graph; and adding all these thorns yields a full edge-folding reduced graph. On the other hand, if we add a thorn to a paired vertex of α, the new graph is no longer edge-folding reduced. Thus β has to be the graph obtained from α by adding a thorn to each non-paired vertex of α (and only those thorns); and we shall denote β by α_f. In particular, we can conclude that each σ-class has a unique representative in $\mathfrak{A}_f(X)$. More precisely, if $\gamma \in \mathfrak{B}'(X)$ then $\overline{\gamma}_f$ is the unique representative of the σ-class of γ in $\mathfrak{A}_f(X)$. We illustrate in Figure 4 the unique representative of the σ-class of $\Delta(w)$ in $\mathfrak{A}_f(X)$ for w the word that has been considered in our examples (note that $\overline{\Delta(w)}$ is the rightmost graph of Figure 3).

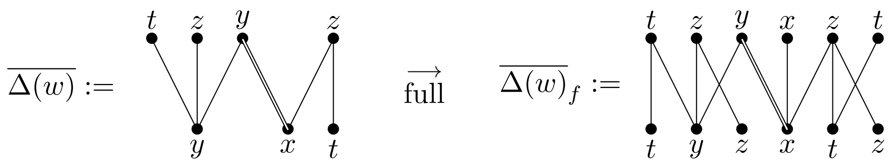


FIGURE 4. The full edge-folding reduced graph $\overline{\Delta(w)}_f$.

The property of being full is preserved under the operation $\sqcap$ and the edge-folding procedure, that is:

(1) if $\alpha, \beta \in \mathfrak{B}'(X)$ are full then so is $\alpha \sqcap \beta$; and
(2) if $\alpha \in \mathfrak{B}'(X)$ is full and β is obtained from α by applying edge-foldings, then β is full.

Thus, if $\alpha, \beta \in \mathfrak{A}_f(X)$ then $\widetilde{\alpha \sqcap \beta} \in \mathfrak{A}_f(X)$ and $\widetilde{\alpha \sqcap \beta}$ is the representative of the σ-class of $\alpha \sqcap \beta$ in $\mathfrak{A}_f(X)$. We can introduce now a binary operation $\wedge$ on $\mathfrak{A}_f(X)$ by setting

$$\alpha \wedge \beta = \widetilde{\alpha \sqcap \beta}\,.$$

The new binary algebra $(\mathfrak{A}_f(X), \wedge)$ obtained is obviously isomorphic to $\mathfrak{B}'(X)/\sigma$, and we have proven the following theorem.

Theorem 3.2. *The algebra $(\mathfrak{A}_f(X), \wedge)$ is a model for the free pseudosemilattice on X if we identify each $x \in X$ with* $\underset{x\quad x}{\bullet\!\!=\!\!=\!\!\bullet}$.

The relations $\leq_{\mathscr{R}}$, $\leq_{\mathscr{L}}$, $\leq$, $\mathscr{R}$, $\mathscr{L}$ and $\mathscr{C}$ on $\mathfrak{A}(X)$ were characterized in subsection 3.3 of [4]. Note that this characterization needed the introduction of three more graphs for each $\alpha \in \mathfrak{A}(X)$, namely ${}^{l}\alpha$, α^r and $\widehat{\alpha}$ (see [4]). Now, in our variation $\mathfrak{A}_f(X)$ of the model $\mathfrak{A}(x)$, the description of these relations becomes easier and more transparent. The following result is an immediate consequence of Theorem 3.16, Corollary 3.17 and Proposition 3.19 of [4].

Proposition 3.3. *Let $\alpha, \beta \in \mathfrak{A}_f(X)$.*

(1) *$\beta \leq_{\mathscr{R}} \alpha$ if and only if α is a 'left-rooted' subtree of β (that is, there is a (label preserving) embedding of α into β such that the image of $\mathfrak{l}_\alpha$ is $\mathfrak{l}_\beta$).*
(2) *$\beta \leq_{\mathscr{L}} \alpha$ if and only if α is a 'right-rooted' subtree of β.*

(3) $\beta \leq \alpha$ *if and only if* α *is a 'bi-rooted' subtree of* β.
(4) $\beta \mathscr{R} \alpha$ *if and only if* $\breve{\alpha} = \breve{\beta}$ *and* $\mathfrak{l}_\alpha = \mathfrak{l}_\beta$ *(recall that* $\breve{\alpha}$ *is just the graph* α *with no distinguished vertices).*
(5) $\beta \mathscr{L} \alpha$ *if and only if* $\breve{\alpha} = \breve{\beta}$ *and* $\mathfrak{r}_\alpha = \mathfrak{r}_\beta$.
(6) $\beta \mathscr{C} \alpha$ *if and only if* $\breve{\alpha} = \breve{\beta}$.

The relation $\leq_{\mathscr{R}}$ could be described alternatively as follows: $\beta \leq_{\mathscr{R}} \alpha$ if and only if ${}^L\alpha$ is a subtree of ${}^L\beta$ in the category of all left-rooted labeled bipartite trees (similar characterizations could be given for $\leq_{\mathscr{L}}$ and $\leq$). Also, the relation $\mathscr{R}$ could be described as follows: $\alpha \mathscr{R} \beta$ if and only if ${}^L\alpha = {}^L\beta$ (a similar characterization could be given for $\mathscr{L}$). The previous result also makes it obvious and easier to describe the size of each $\mathscr{R}$-class, each $\mathscr{L}$-class and each $\mathscr{C}$-class:

Corollary 3.4. *Let* $\alpha \in \mathfrak{A}_f(X)$.
(1) *The size of the* $\mathscr{R}$*-class of* α *is the degree of the vertex* $\mathfrak{l}_\alpha$ *in* α.
(2) *The size of the* $\mathscr{L}$*-class of* α *is the degree of the vertex* $\mathfrak{r}_\alpha$ *in* α.
(3) *The size of the* $\mathscr{C}$*-class of* α *is the number of edges of* α.
(4) *The number of distinct* $\mathscr{R}$*-classes in the connected component of* α *is the number of left vertices of* α.
(5) *The number of distinct* $\mathscr{L}$*-classes in the connected component of* α *is the number of right vertices of* α.

There are two further results about the structure of $\mathfrak{A}_f(X)$ that we state next, which are obvious translations to $\mathfrak{A}_f(X)$ of results from [4] (Proposition 3.13 and Corollary 3.14 of [4]). For $x, y \in X$, let

(*i*) $\mathfrak{R}_x(X) = \{\alpha \in \mathfrak{A}_f(X) \mid \mathrm{l}(\alpha) = x\}$;
(*ii*) $\mathfrak{L}_x(X) = \{\alpha \in \mathfrak{A}_f(X) \mid \mathrm{r}(\alpha) = x\}$;
(*iii*) $\mathfrak{S}_{x,y}(X) = \mathfrak{R}_x(X) \cap \mathfrak{L}_y(X) = \{\alpha \in \mathfrak{A}_f(X) \mid (\mathrm{l}(\alpha), \mathrm{r}(\alpha)) = (x, y)\}$.

Proposition 3.5. *The sets* $\mathfrak{R}_x(X)$ *and* $\mathfrak{L}_x(X)$ *for* $x \in X$ *are respectively the maximal right and left normal subbands of* $\mathfrak{A}_f(X)$*, while the sets* $\mathfrak{S}_{x,y}(X)$ *for* $x, y \in X$ *are the maximal subsemilattices of* $\mathfrak{A}_f(X)$.

Corollary 3.6. *Two elements* $\alpha, \beta \in \mathfrak{A}_f(X)$ *commute if and only if*

$$(\mathrm{l}(\alpha), \mathrm{r}(\alpha)) = (\mathrm{l}(\beta), \mathrm{r}(\beta)).$$

4. The free idempotent generated locally inverse semigroup

In this section we shall construct a model for the free idempotent generated locally inverse semigroup $\mathrm{FILI}(X)$ on a set X based on the model $\mathfrak{A}_f(X)$ for the free pseudosemilattice on X. Recall that

$$\mathrm{FILI}(X) \cong \mathrm{RIG}(\mathrm{FPS}(X)) \cong \mathrm{RIG}(\mathfrak{A}_f(X)) .$$

We begin with the following two lemmas:

Lemma 4.1. *The free idempotent generated locally inverse semigroups are combinatorial.*

Proof. Let $(\alpha_1, \alpha_2, \dots, \alpha_n)$ be a non-trivial E-chain in $\mathfrak{A}_f(X)$ $(n \geq 2)$ and let $\gamma = \check{\alpha_1}$. Then

$$\gamma = \check{\alpha_1} = \check{\alpha_2} = \cdots = \check{\alpha_n}$$

by Proposition 3.3. Further, there are vertices $a_1, a_2, \dots, a_n, a_{n+1}$ in γ with $a_{i-1} \neq a_{i+1}$ such that $\mathfrak{l}_{\alpha_i} = a_i$ and $\mathfrak{r}_{\alpha_i} = a_{i+1}$ if i odd, and $\mathfrak{l}_{\alpha_i} = a_{i+1}$ and $\mathfrak{r}_{\alpha_i} = a_i$ if i even. Thus $a_1, a_2, \dots, a_n$ is a geodesic path in γ from the left root a_1 of α_1 to the right root a_{n+1} of α_n; whence $\alpha_1 \neq \alpha_n$ since γ is a tree. We can now conclude that the E-chains of $\mathfrak{A}_f(X)$ are loop-free, and so $\mathrm{RIG}(\mathfrak{A}_f(X))$ is combinatorial by Lemma 2.2. We have proved this lemma. □

Lemma 4.2. *If S is an idempotent generated locally inverse semigroup with pseudosemilattice of idempotents isomorphic to* $\mathrm{FPS}(X)$*, then S is isomorphic to* $\mathrm{FILI}(X)$.

Proof. Let S be an idempotent generated locally inverse semigroup with pseudosemilattice of idempotents isomorphic to $\mathrm{FPS}(X)$, and let

$$\varphi : \mathrm{FILI}(X) \to S$$

be the only homomorphism extending the identity mapping on X. Then $\varphi_{|\,\mathrm{FPS}(X)} : \mathrm{FPS}(X) \to E(S)$ is a (pseudosemilattice) isomorphism; and φ is an idempotent-separating (that is, if $e\varphi = f\varphi$ for idempotents e and f, then $e = f$) onto homomorphism since both $\mathrm{FILI}(X)$ and S are idempotent generated. Thus the kernel congruence $\ker\varphi$ on $\mathrm{FILI}(X)$ induced by φ is an idempotent-separating congruence (that is, no congruence class contains more than one idempotent). But every idempotent-separating congruence on a regular semigroup is contained in $\mathscr{H}$ [15]; whence $\ker\varphi$ is the identity relation and φ is an isomorphism. □

The model for $\mathrm{FILI}(X)$ we shall present in this section will be obtained by considering a natural generalization of the graphs from $\mathfrak{B}'(X)$: the distinguished vertices may not be adjacent, although one of them has to be a left vertex and the other one has to be a right vertex. By Lemma 4.2, we just need to prove that the binary algebra we shall describe is in fact an idempotent generated locally inverse semigroup with $\mathfrak{A}_f(X)$ as its pseudosemilattice of idempotents.

So, let $\mathfrak{T}'(X)$ be the set of all (vertex) X-labeled bipartite trees with a left vertex and a right vertex distinguished. We shall continue to use for the graphs in $\mathfrak{T}'(X)$ the terminology and notation introduced earlier for the graphs in $\mathfrak{B}'(X)$. For example, if $\gamma \in \mathfrak{T}'(X)$ then $\mathfrak{l}_\gamma$ and $\mathfrak{r}_\gamma$ denote respectively the *left root* and the *right root* of γ. We need to make also a convention on the graphical representation of a graph $\gamma \in \mathfrak{T}'(X)$ since the two vertices $\mathfrak{l}_\gamma$ and $\mathfrak{r}_\gamma$ are no longer adjacent necessarily. We shall represent the distinguished vertices of $\gamma \in \mathfrak{T}'(X)$ as "encircled bullets" like this ◉.

We introduce a binary operation $\odot$ on the set $\mathfrak{T}'(X)$ as follows: for $\alpha, \beta \in \mathfrak{T}'(X)$ set

$$\alpha \odot \beta = {}^{L}\alpha \,\dot\cup\, \{(\mathfrak{l}_\beta, \mathfrak{r}_\alpha)\} \,\dot\cup\, \beta^{R}\,.$$

Thus $\alpha \odot \beta$ is just the graph obtained by taking the disjoint union of α and β, adding the edge $(\mathfrak{l}_\beta, \mathfrak{r}_\alpha)$, and setting $\mathfrak{l}_\alpha$ and $\mathfrak{r}_\beta$ respectively as the left root and the right root of $\alpha \odot \beta$. If we compare $\alpha \odot \beta$ with $\alpha \sqcap \beta$ for $\alpha, \beta \in \mathfrak{B}'(X)$, we see that the only difference between them is on the new edge added: we add $(\mathfrak{l}_\alpha, \mathfrak{r}_\beta)$ to $\alpha \sqcap \beta$ while we add $(\mathfrak{l}_\beta, \mathfrak{r}_\alpha)$ to $\alpha \odot \beta$. To make the operation $\odot$ clearer, we illustrate the graph $\Delta(w) \odot \alpha$ in Figure 5 for w the word considered in our examples and α the graph of Figure 2 (compare the graph $\Delta(w) \odot \alpha$ with the graph of $\Delta(w) \sqcap \alpha$ depicted in Figure 2).

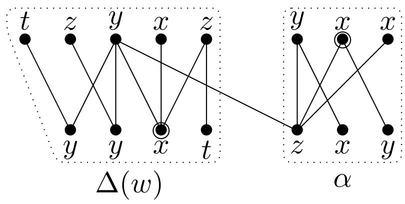


FIGURE 5. The graph $\Delta(w) \odot \alpha$.

The binary operation $\odot$ is clearly associative (although $\sqcap$ is not), and so $(\mathfrak{T}'(X), \odot)$ is a semigroup. This semigroup has no idempotents since the number of vertices in $\alpha \odot \alpha$ duplicates in comparison with α. Further, the graph $\alpha \odot \beta$ is clearly full if both α and β are full. Thus, for

$$\mathfrak{T}(X) = \{\gamma \in \mathfrak{T}'(X) \mid \gamma \text{ is full}\},$$

$(\mathfrak{T}(X), \odot)$ is a subsemigroup of $\mathfrak{T}'(X)$.

The edge-folding reducing rule introduced for graphs $\gamma \in \mathfrak{B}'(X)$ does not interfere with the distinguished vertices (if one of the two vertices folded together is a distinguished one, then the resulting vertex becomes a distinguished vertex). Thus, we can consider this same reduction rule for the graphs in $\mathfrak{T}(X)$. Further, for any given $\gamma \in \mathfrak{T}(X)$, we continue to have a unique edge-folding reduced graph $\widetilde{\gamma} \in \mathfrak{T}(X)$ that can be obtained from γ by applying edge-foldings only. In other words, independently of the order we choose to apply the edge-foldings to γ, at the end, when we cannot apply any further edge-folding, we always get the same graph $\widetilde{\gamma} \in \mathfrak{T}(X)$. We can continue to see $\widetilde{\gamma}$ as a homomorphic image of γ under a natural homomorphism that preserves labels and distinguished vertices.

Let τ be the least equivalence relation on $\mathfrak{T}(X)$ containing the binary relation induced by the edge-folding reducing rule. Then

$$\alpha \tau \beta \quad \text{if and only if} \quad \widetilde{\alpha} = \widetilde{\beta}$$

for any $\alpha, \beta \in \mathfrak{T}(X)$. Since the order of edge-folding reductions applied to a given graph is irrelevant, we conclude that

$$\widetilde{\alpha \odot \beta} = \widetilde{\widetilde{\alpha} \odot \widetilde{\beta}}$$

for any $\alpha, \beta \in \mathfrak{T}(X)$. Thus τ is in fact a congruence on $\mathfrak{T}(X)$; and we have the quotient semigroup $(T(X)/\tau, \odot)$. Let

$$\mathfrak{S}(X) = \{\gamma \in \mathfrak{T}(X) \mid \gamma \text{ is reduced for edge-folding}\}.$$

Then each τ-class has a unique representative in $\mathfrak{S}(X)$, namely if $\gamma \in \mathfrak{T}(X)$ then $\widetilde{\gamma}$ is the unique representative of the τ-class of γ in $\mathfrak{S}(X)$. Further, if $\alpha, \beta \in \mathfrak{S}(X)$, then $\widetilde{\alpha \odot \beta}$ is the unique representative in $\mathfrak{S}(X)$ of the τ-class of $\alpha \odot \beta$. Hence, if we introduce a binary operation $\cdot$ on $\mathfrak{S}(X)$ as follows:

$$\alpha \cdot \beta = \widetilde{\alpha \odot \beta},$$

then $(\mathfrak{S}(X), \cdot)$ becomes a semigroup isomorphic to $(\mathfrak{T}(X)/\tau, \odot)$.

The main result of this section states that $(\mathfrak{S}(X), \cdot)$ is a model for the free idempotent generated locally inverse semigroup FILI(X) on X. By Lemma 4.2 it suffices to show the following two claims:

- (i) $\mathfrak{S}(X)$ is an idempotent generated locally inverse semigroup with set of idempotents $\mathfrak{A}_f(X)$.
- (ii) The pseudosemilattice operation on $\mathfrak{A}_f(X)$ induced by the semigroup multiplication $\cdot$ on $\mathfrak{S}(X)$ coincides with the operation $\wedge$ on $\mathfrak{A}_f(X)$ introduced in the previous section.

We prove claim (i) in Proposition 4.6 and claim (ii) in Proposition 4.7. Our first result in this direction is a technical but crucial lemma about the product of elements in $\mathfrak{S}(X)$.

Lemma 4.3. *Let $\alpha, \beta \in \mathfrak{S}(X)$ be such that $\check{\beta}$ is a subgraph of $\check{\alpha}$ with $\mathfrak{l}_\beta$ adjacent to $\mathfrak{r}_\alpha$ in α. Then $\alpha \cdot \beta = \alpha_1$ with $\check{\alpha}_1 = \check{\alpha}$, $\mathfrak{l}_{\alpha_1} = \mathfrak{l}_\alpha$ and $\mathfrak{r}_{\alpha_1} = \mathfrak{r}_\beta$.*

Proof. To prove this lemma it is convenient to consider the vertices from β disjoint from the vertices of α. So, let $\pi : \beta \to \alpha$ be the (graph) embedding of $\check{\beta}$ into $\check{\alpha}$ such that $\mathfrak{l}_\beta \pi$ is adjacent to $\mathfrak{r}_\alpha$. If b is a vertex of β, then we shall denote the vertex $b\pi$ of α by b'. Thus $\mathfrak{l}'_\beta$ is adjacent to $\mathfrak{r}_\alpha$ in α, and each vertex b of β has the same label as the vertex b' of α.

Take $\gamma = \alpha \odot \beta$ and let b be a vertex of β. Let $\mathfrak{l}_\beta = b_1, b_2, \cdots, b_n = b$ be the geodesic path in β connecting $\mathfrak{l}_\beta$ with b. Then

$$\mathfrak{r}_\alpha, b_1, b_2, \cdots, b_n = b \quad \text{and} \quad \mathfrak{r}_\alpha, b'_1, b'_2, \cdots, b'_n = b'$$

are two isomorphic paths (including labels) in γ, each connecting $\mathfrak{r}_\alpha$ with respectively b and b'. We can now apply a first edge-folding to γ and merge b_1 with b'_1; then we can apply another edge-folding to the resulting graph to merge b_2 with b'_2. The process just described can be continued until b is merged to b' by a last edge-folding. Summing up, we can merge in γ each vertex b from β with its counterpart b' from α using a sequence of edge-foldings.

Let $\alpha_1 = \alpha \cdot \beta = \widetilde{\gamma}$. Since $\check{\alpha}$ is reduced for edge-folding, $\check{\alpha}_1$ must contain a copy of $\check{\alpha}$ as a subgraph. However, by the previous paragraph, $\check{\alpha}_1$ is a subgraph of $\check{\widetilde{\alpha}} = \check{\alpha}$. Therefore $\check{\alpha} = \check{\alpha}_1$. Further $\mathfrak{l}_{\alpha_1} = \mathfrak{l}_\alpha$ and $\mathfrak{r}_{\alpha_1} = \mathfrak{r}'_\beta$ since $\mathfrak{r}_\beta$ is merged with $\mathfrak{r}'_\beta$ in α_1. We have proved this lemma. $\square$

The next result identifies the idempotents of $\mathfrak{S}(X)$ as the set $\mathfrak{A}_f(X)$ and shows that $\mathfrak{S}(X)$ is idempotent generated.

Lemma 4.4. *The semigroup $\mathfrak{S}(X)$ is generated (as a semigroup) by the set $\mathfrak{A}_f(X)$ which is precisely the set of idempotents of $\mathfrak{S}(X)$.*

Proof. First note that $\mathfrak{A}_f(X) \subseteq \mathfrak{S}(X)$ and that, by the previous lemma, all graphs from $\mathfrak{A}_f(X)$ are idempotent elements of $\mathfrak{S}(X)$. Consider now $\alpha \in \mathfrak{S}(X)$ such that $\alpha \notin \mathfrak{A}_f(X)$ and let us prove that α is not an idempotent of $\mathfrak{S}(X)$. Let p be the geodesic path in α connecting $\mathfrak{l}_\alpha$ with $\mathfrak{r}_\alpha$, say

$$p := \quad \mathfrak{l}_\alpha = a_1, a_2, \cdots, a_{2n-1}, a_{2n} = \mathfrak{r}_\alpha \,.$$

Then $n \geq 2$ since $\alpha \notin \mathfrak{A}_f(X)$. Let β be another disjoint copy of α and let

$$q := \quad \mathfrak{l}_\beta = b_1, b_2, \cdots, b_{2n-1}, b_{2n} = \mathfrak{r}_\beta$$

be the copy of the geodesic path p in β. Let $\gamma = \alpha \cdot \beta$ and let a'_{2n}, b'_1 and b'_{2n} be the vertices of γ corresponding respectively to the vertices a_{2n}, b_1 and b_{2n}. Then a'_{2n} is adjacent to b'_1, there is a geodesic path in γ isomorphic to q connecting b'_1 to b'_{2n}, and b'_{2n} is the right root of γ. Thus $a'_{2n} \neq b'_{2n}$ since γ is a tree, and so $\gamma \neq \alpha$ (if $\gamma = \alpha$ then $a'_{2n} = b'_{2n}$). Thereby, α is not an idempotent.

In the previous paragraph we proved that $\mathfrak{A}_f(X)$ is the set of idempotents of $\mathfrak{S}(X)$. Let us prove now that $\mathfrak{A}_f(X)$ generates $\mathfrak{S}(X)$ as a semigroup. Let $\alpha \in \mathfrak{S}(X)$ and let

$$\mathfrak{l}_\alpha = a_1, a_2, \cdots, a_{2n} = \mathfrak{r}_\alpha$$

be the geodesic path in α connecting $\mathfrak{l}_\alpha$ with $\mathfrak{r}_\alpha$. If $n = 1$ then $\alpha \in \mathfrak{A}_f(X)$. So, assume that $n > 1$. For each $i = 1, \ldots, n$ let α_i be the graph α but with the vertices a_{2i-1} and a_{2i} as the left root and the right root, respectively. Thus all α_i belong to $\mathfrak{A}_f(X)$. By Lemma 4.3, the graph $\alpha'_2 = \alpha_1 \cdot \alpha_2$ is just the graph α but with distinguished vertices a_1 and a_4; and $\alpha'_3 = \alpha'_2 \cdot \alpha_3$ is just the graph α but with distinguished vertices a_1 and a_6. Continuing this process one can conclude that $\alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_n$ is just the graph α but with distinguished vertices a_1 and a_{2n}, that is, it is precisely α. We have shown that

$$\alpha = \alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_n \,,$$

and therefore $\mathfrak{S}(X)$ is generated by the set of idempotents $\mathfrak{A}_f(X)$. □

We can prove now that $\mathfrak{S}(X)$ is an idempotent generated regular semigroup.

Lemma 4.5. *The semigroup $\mathfrak{S}(X)$ is an idempotent generated regular semigroup with set of idempotents $\mathfrak{A}_f(X)$. Further, for $\alpha, \beta \in \mathfrak{S}(X)$,*

(1) *$\alpha \,\mathscr{R}\, \beta$ if and only if $\breve{\alpha} = \breve{\beta}$ and $\mathfrak{l}_\alpha = \mathfrak{l}_\beta$.*
(2) *$\alpha \,\mathscr{L}\, \beta$ if and only if $\breve{\alpha} = \breve{\beta}$ and $\mathfrak{r}_\alpha = \mathfrak{r}_\beta$.*
(3) *$\alpha \,\mathscr{D}\, \beta$ if and only if $\breve{\alpha} = \breve{\beta}$.*

Proof. We claim we just need to prove (1). Indeed, (2) is the dual statement of (1) and (3) is an obvious conclusion from (1) and (2). Further, $\mathfrak{S}(X)$ is an idempotent generated semigroup with set of idempotents $\mathfrak{A}_f(X)$ by Lemma 4.4. Finally, by (1), every $\beta \in \mathfrak{S}(X)$ is $\mathscr{R}$-related to some idempotent of $\mathfrak{S}(X)$; thus $\mathfrak{S}(X)$ is regular.

So, let us prove (1). The 'if' part follows from Lemma 4.3. To prove the 'only if' part we begin with the following observation: if $\beta = \alpha \cdot \gamma$, then the graph ${}^L\alpha$ is a left-rooted subgraph of ${}^L\beta$ since α is edge-folding reduced. It is clear now that if $\alpha \mathrel{\mathscr{R}} \beta$ then ${}^L\alpha = {}^L\beta$, which is equivalently to say that $\check{\alpha} = \check{\beta}$ and $\mathfrak{l}_\alpha = \mathfrak{l}_\beta$. □

We are able now to prove our first claim (i) stated earlier.

Proposition 4.6. *The semigroup $\mathfrak{S}(X)$ is an idempotent generated locally inverse semigroup with set of idempotents $\mathfrak{A}_f(X)$.*

Proof. We just need to prove that the local submonoids $\alpha \cdot \mathfrak{S}(X) \cdot \alpha$ for $\alpha \in \mathfrak{A}_f(X)$ are inverse semigroups due to Lemma 4.5. Let $\alpha \in \mathfrak{A}_f(X)$ with $\mathrm{l}(\alpha) = x$ and $\mathrm{r}(\alpha) = y$, and set

$$\beta := \quad x \bullet\!\!-\!\!\!-\!\!\bullet\, y \ .$$

Clearly $\beta \cdot \alpha = \alpha$ and $\alpha \cdot \beta = \alpha$, and so $\alpha \cdot \mathfrak{S}(X) \cdot \alpha$ is a regular subsemigroup of $\beta \cdot \mathfrak{S}(X) \cdot \beta$. The set of idempotents of $\beta \cdot \mathfrak{S}(X) \cdot \beta$ is the subset $\mathfrak{S}_{x,y}(X)$ of $\mathfrak{A}_f(X)$ introduced at the end of section 3. We just need to show that $\mathfrak{S}_{x,y}(X)$ is a semilattice under the semigroup operation $\cdot$ because then $\beta \cdot \mathfrak{S}(X) \cdot \beta$ is an inverse semigroup, and the same holds true for $\alpha \cdot \mathfrak{S}(X) \cdot \alpha$; whence $\mathfrak{S}(X)$ is a locally inverse semigroup.

Let $\alpha_1, \alpha_2 \in \mathfrak{S}_{x,y}(X)$ and set $\gamma_1 = \alpha_1 \odot \alpha_2$ and $\gamma_2 = \alpha_1 \sqcap \alpha_2$. Thus γ_1 differs from γ_2 by a single edge: γ_1 has the edge $(\mathfrak{l}_{\alpha_2}, \mathfrak{r}_{\alpha_1})$ while γ_2 has instead the edge $(\mathfrak{l}_{\alpha_1}, \mathfrak{r}_{\alpha_2})$. But since $\mathrm{l}(\alpha_1) = \mathrm{l}(\alpha_2) = x$ and $\mathrm{r}(\alpha_1) = \mathrm{r}(\alpha_2) = y$, we can perform two edge-foldings to each graph γ_1 and γ_2, merging $\mathfrak{l}_{\alpha_1}$ with $\mathfrak{l}_{\alpha_2}$ and $\mathfrak{r}_{\alpha_1}$ with $\mathfrak{r}_{\alpha_2}$ in both graphs. Further, we get the same graph γ from both cases. Thereby

$$\alpha_1 \cdot \alpha_2 = \widetilde{\gamma_1} = \widetilde{\gamma} = \widetilde{\gamma_2} = \alpha_1 \wedge \alpha_2 \, .$$

We have shown that $\cdot$ and $\wedge$ coincide in $\mathfrak{S}_{x,y}(X)$, whence $\mathfrak{S}_{x,y}(X)$ is a semilattice under the semigroup operation $\cdot$. □

Let $\alpha, \beta \in \mathfrak{A}_f(X)$ and recall the observation made in Lemma 4.5 to prove the 'only if' part of (1). Thus if $\beta = \alpha \cdot \beta$, then ${}^L\alpha$ is a left-rooted subgraph of ${}^L\beta$, or equivalently, $\check{\alpha}$ is a subgraph of $\check{\beta}$ with $\mathfrak{l}_\alpha = \mathfrak{l}_\beta$. Conversely, if $\check{\alpha}$ is a subgraph of $\check{\beta}$ with $\mathfrak{l}_\alpha = \mathfrak{l}_\beta$, then $\beta = \alpha \cdot \beta$ by the dual of Lemma 4.3 and since both α and β are idempotents. These conclusions and their duals constitute the second part of the following proposition. The first part is just our second claim (ii) stated earlier.

Proposition 4.7. *The pseudosemilattice operation on $\mathfrak{A}_f(X)$ induced by the semigroup multiplication $\cdot$ on $\mathfrak{S}(X)$ coincides with the operation $\wedge$ on $\mathfrak{A}_f(X)$*

introduced in the previous section. Further, if α and β are idempotents of $\mathfrak{S}(X)$, then

(1) *$\beta \leq_{\mathscr{R}} \alpha$ (in the semigroup $\mathfrak{S}(X)$) if and only if ${}^{L}\alpha$ is a left-rooted subgraph of ${}^{L}\beta$.*
(2) *$\beta \leq_{\mathscr{L}} \alpha$ (in the semigroup $\mathfrak{S}(X)$) if and only if α^{R} is a right-rooted subgraph of β^{R}.*
(3) *$\beta \leq \alpha$ (in the semigroup $\mathfrak{S}(X)$) if and only if α is a bi-rooted subgraph of β.*

Proof. Note that (1) was observed before the statement of this proposition, (2) is the dual of (1), and (3) is the combination of both (1) and (2). Hence, let us prove only the first part of this result, and we shall use (1), (2) and (3) for that propose.

Let $\alpha, \beta \in \mathfrak{A}_f(X)$ and let $\gamma = \alpha \wedge \beta$. We are done once we prove that

$$(\alpha]_{\mathscr{R}} \cap (\beta]_{\mathscr{L}} = (\gamma]_{\leq}$$

in the semigroup $\mathfrak{S}(X)$ since then $\wedge$ coincides with the pseudosemilattice operation on $\mathfrak{A}_f(X)$ induced by the semigroup multiplication $\cdot$. Since ${}^{L}\alpha$ is a left-rooted subgraph of ${}^{L}\gamma$ and β^{R} is a right-rooted subgraph of γ^{R}, we have $\gamma \in (\alpha]_{\mathscr{R}} \cap (\beta]_{\mathscr{L}}$ by (1) and (2).

Let $\gamma_1 \in (\alpha]_{\mathscr{R}} \cap (\beta]_{\mathscr{L}}$. Thus ${}^{L}\alpha$ is a left-rooted subgraph of ${}^{L}\gamma_1$ and β^{R} is a right-rooted subgraph of γ_1^{R}. Let

$$\pi_\alpha : {}^{L}\alpha \to {}^{L}\gamma_1$$

be the natural injective homomorphism (in the category of all left-rooted labeled bipartite graphs) from ${}^{L}\alpha$ to ${}^{L}\gamma_1$. Analogously, let

$$\pi_\beta : \beta^{R} \to \gamma_1^{R}$$

be the natural injective homomorphism (in the category of all right-rooted labeled bipartite graphs) from β^{R} to γ_1^{R}. Since $\mathfrak{l}_{\gamma_1}$ is adjacent to $\mathfrak{r}_{\gamma_1}$ ($\gamma_1 \in \mathfrak{A}_f(X)$), the mapping

$$\pi = \pi_\alpha \cup \pi_\beta : \alpha \sqcap \beta \to \gamma_1$$

is a bi-rooted homomorphism. If we perform an edge-folding reduction to $\alpha \sqcap \beta$, then π induces a new bi-rooted homomorphism from the new graph obtained from $\alpha \sqcap \beta$ to γ_1 since the graph γ_1 is reduced for edge-folding. Using this observation sequentially to each new edge-folding applied to $\alpha \sqcap \beta$, we can conclude that there exists a bi-rooted homomorphism

$$\widetilde{\pi} : \gamma = \widetilde{\alpha \sqcap \beta} \to \gamma_1$$

from γ to γ_1. Since both γ and γ_1 are reduced for edge-folding, the homomorphism $\widetilde{\pi}$ is injective, and so $\gamma_1 \in (\gamma]_{\leq}$ by (3). We have shown that $(\alpha]_{\mathscr{R}} \cap (\beta]_{\mathscr{L}} = (\gamma]_{\leq}$ and concluded the proof of this proposition. □

The main result of this section is now a corollary of the two previous propositions.

Theorem 4.8. *The semigroup $(\mathfrak{S}(X), \cdot)$ is a model for the free idempotent generated locally inverse semigroup on X if we identify each $x \in X$ with* x ●—● x *.*

The relations $\mathscr{R}$, $\mathscr{L}$ and $\mathscr{D}$ on $\mathfrak{S}(X)$ have been characterized in Lemma 4.5 while the relations $\leq_{\mathscr{R}}$, $\leq_{\mathscr{L}}$ and $\leq$ on the set of idempotents of $\mathfrak{S}(X)$ have been characterized in Proposition 4.7. It has been pointed out earlier that the $\mathscr{H}$-relation on FILI(X) (and so on $\mathfrak{S}(X)$) is the identity relation, and it is not hard to see that $\mathscr{J} = \mathscr{D}$. We should mention also that the relations $\leq_{\mathscr{R}}$, $\leq_{\mathscr{L}}$ and $\leq$ on the set of idempotents of $\mathfrak{S}(X)$ have natural extensions to the whole semigroup $\mathfrak{S}(X)$: for example, $\leq$ extends to the usual natural partial order. It is easily proven that those extensions are characterized by the same conditions presented in Proposition 4.7. We end the analysis of these relations on $\mathfrak{S}(X)$ with an obvious corollary about the number of elements in each $\mathscr{R}$-class, $\mathscr{L}$-class and $\mathscr{D}$-class.

Corollary 4.9. *Let $\alpha \in \mathfrak{S}(X)$ and let n_l, n_r and n_e be respectively the number of left vertices, right vertices and edges of α. Let also $\mathscr{D}_\alpha$, $\mathscr{R}_\alpha$ and $\mathscr{L}_\alpha$ denote respectively the $\mathscr{D}$-class, the $\mathscr{R}$-class and the $\mathscr{L}$-class of α.*

(1) *$\mathscr{D}_\alpha$ has n_l $\mathscr{R}$-classes, n_r $\mathscr{L}$-classes, $n_l n_r$ elements and n_e idempotents.*
(2) *In $\mathscr{D}_\alpha$ each $\mathscr{R}$-class has n_r elements and each $\mathscr{L}$-class has n_l elements.*
(3) *The number of idempotents in $\mathscr{R}_\alpha$ is the degree of $\mathfrak{l}_\alpha$ in α, while the number of idempotents in $\mathscr{L}_\alpha$ is the degree of $\mathfrak{r}_\alpha$ in α.*

5. The category $\mathscr{G}(\mathfrak{A}_f(X))$

Consider the graph G with vertices the $\mathscr{R}$-classes and the $\mathscr{L}$-classes of $\mathfrak{A}_f(X)$ and with edges connecting $\mathscr{R}$-classes with $\mathscr{L}$-classes: there is an edge connecting an $\mathscr{R}$-class R with an $\mathscr{L}$-class L if and only if $R \cap L$ has an idempotent. Thus G is the graph associated with the pseudosemilattice $\mathfrak{A}_f(X)$ as introduced in [5] but with the edges viewed as non-oriented. Hence G is a disconnected bipartite graph: the vertices are partitioned into the $\mathscr{R}$-classes and the $\mathscr{L}$-classes. We shall call the $\mathscr{R}$-classes the *left vertices* of G and the $\mathscr{L}$-classes the *right vertices* of G.

Let R be an $\mathscr{R}$-class of $\mathfrak{A}_f(X)$. By Proposition 3.3, the left roots of the elements from R all have the same label, say x. Then, we shall label the vertex R of G with the letter x from X. We label all other left vertices of G similarly. This labeling can be extended to any right vertex L of G by using instead the label of the right roots of the graphs from L.

We shall define now the category $\mathscr{G}(\mathfrak{A}_f(X))$. The objects of $\mathscr{G}(\mathfrak{A}_f(X))$ are the connected components of G with the vertices labeled by letters of X as described above. Thus, each object γ from $\mathscr{G}(\mathfrak{A}_f(X))$ is a finite connected graph, represents a connected component C of the pseudosemilattice $\mathfrak{A}_f(X)$, and its vertices represent the $\mathscr{R}$-classes and the $\mathscr{L}$-classes of C. The vertices of γ are naturally partitioned into the $\mathscr{R}$-classes and the $\mathscr{L}$-classes of C; whence γ is seen as a bipartite graph. As for the graph G, we shall consider the $\mathscr{R}$-classes of C as the *left vertices* of γ and the $\mathscr{L}$-classes of C as the *right vertices* of γ. The morphisms from $\mathscr{G}(\mathfrak{A}_f(X))$ are the label preserving graph homomorphisms sending left vertices into left vertices and right vertices into right vertices.

Before we proceed we should make a remark. In this paper we opt to consider the graph G as a non-oriented graph and to consider the morphisms from $\mathscr{G}(\mathfrak{A}_f(X))$ as graph homomorphisms that send left vertices into left vertices and right vertices into right vertices. The same effect would be obtained if we considered G as an oriented graph and the morphisms from $\mathscr{G}(\mathfrak{A}_f(X))$ as oriented graph homomorphisms. The reason for our choice is merely because it highlights the connection between the graph G and the elements in our models presented in the two previous sections.

Consider the mapping

$$\nu : \mathfrak{S}(X) \longrightarrow \mathscr{G}(\mathfrak{A}_f(X))$$

that maps each $\alpha \in \mathfrak{S}(X)$ to the object from the category $\mathscr{G}(\mathfrak{A}_f(X))$ associated with the connected component of $\mathfrak{A}_f(X)$ containing an idempotent $\mathscr{R}$-related (or $\mathscr{L}$-related) with α. Thus any object from $\mathscr{G}(\mathfrak{A}_f(X))$ is the image of some element from $\mathfrak{S}(X)$ under the mapping ν.

Lemma 5.1. *For $\alpha \in \mathfrak{S}(X)$, $\alpha\nu = \breve{\alpha}$.*

Proof. It is obvious that we can assume that $\alpha \in \mathfrak{A}_f(X)$ without losing generality. Let C be the connected component of α. By Proposition 3.3 the $\mathscr{R}$-classes of C are in one-to-one correspondence with the left vertices of α and the $\mathscr{L}$-classes of C are in one-to-one correspondence with the right vertices of α. Thus, let

$$\pi : \alpha\nu \longrightarrow \breve{\alpha}$$

be the natural bijection from the vertices of $\alpha\nu$ onto the vertices of $\breve{\alpha}$. By the definition of the labeling in $\alpha\nu$, the mapping π preserves also the labels. Further, an $\mathscr{R}$-class R is adjacent to an $\mathscr{L}$-class L in $\alpha\nu$ if and only if $R \cap L$ contains an idempotent, and so if and only if the left vertex $R\pi$ is adjacent to the right vertex $L\pi$ in $\breve{\alpha}$. Hence π is a label preserving (graph) isomorphism from $\alpha\nu$ onto $\breve{\alpha}$. □

The next result is now an obvious consequence of the previous lemma and Lemma 4.5.

Corollary 5.2. *The objects of the category $\mathscr{G}(\mathfrak{A}_f(X))$ are the finite (vertex) X-labeled full bipartite trees reduced for edge-folding, and the relation* $\ker\nu$ *on $\mathfrak{S}(X)$ coincides with the Green's relation $\mathscr{D}$.*

Let $\pi : \alpha \to \beta$ be a morphism from $\mathscr{G}(\mathfrak{A}_f(X))$. Since β is a tree and π maps left vertices into left vertices and right vertices into right vertices, if π was not injective, then there would exist two vertices a and b in α adjacent to another vertex c such that $a\pi = b\pi$; whence a and b could be merged together by an edge-folding in α. Consequently, π must be injective. We have shown the following result.

Corollary 5.3. *The morphisms from* $\mathscr{G}(\mathfrak{A}_f(X))$ *are injective graph homomorphisms that preserve labels and left and right vertices.*

Consider now the partially ordered set $(\mathfrak{S}(X), \leq)$ where $\leq$ denotes the natural partial order on $\mathfrak{S}(X)$. As observed at the end of the previous section, for $\alpha, \beta \in \mathfrak{S}(X)$, $\beta \leq \alpha$ if and only if α is a bi-rooted subgraph of β. But since both α and β are bipartite trees reduced for edge-folding, the latter condition is equivalent to say that there exists an injective graph homomorphism $\pi : \alpha \to \beta$ preserving the labels and the left and right vertices, and such that $\mathfrak{l}_\alpha \pi = \mathfrak{l}_\beta$ and $\mathfrak{r}_\alpha \pi = \mathfrak{r}_\beta$. Let $\mathrm{cat}(\mathfrak{S}(X))$ be the category whose objects are the elements of $\mathfrak{S}(X)$ and whose morphisms are the injective graph homomorphisms that preserve labels and the left and right roots, and send left vertices into left vertices and right vertices into right vertices. Thus $\beta \leq \alpha$ in $(\mathfrak{S}(X), \leq)$ if and only if there is a morphism from α to β in the category $\mathrm{cat}(\mathfrak{S}(X))$. It is easy to see now that the mapping ν introduced above is extendable to a functor

$$\overline{\nu} : \mathrm{cat}(\mathfrak{S}(X)) \longrightarrow \mathscr{G}(\mathfrak{A}_f(X)),$$

namely by defining $\alpha\overline{\nu} = \check{\alpha}$ and letting $\pi\overline{\nu} : \check{\alpha} \to \check{\beta}$ be such that $\pi\overline{\nu} = \pi$ for any morphism $\pi : \alpha \to \beta$ from $\mathrm{cat}(\mathfrak{S}(X))$. In other words, the functor $\overline{\nu}$ just tell us to "forget" the distinguished vertices.

Proposition 5.4. $\overline{\nu} : \mathrm{cat}(\mathfrak{S}(X)) \to \mathscr{G}(\mathfrak{A}_f(X))$ *is a functor.*

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References

[1] J. Almeida, *Finite semigroups and universal algebra*, World Scientific, Singapore, 1994.

[2] K. Auinger, The bifree locally inverse semigroup on a set, *J. Algebra* **166** (1994), 630–650.

[3] K. Auinger, On the lattice of existence varieties of locally inverse semigroups, *Canad. Math. Bull.* **37** (1994), 13–20.

[4] K. Auinger and L. Oliveira, On the variety of strict pseudosemilattices, *Studia Sci. Math. Hungarica* **50** (2013), 207–241.

[5] M. Brittenham, S. Margolis and J. Meakin, Subgroups of free idempotent generated semigroups need not be free, *J. Algebra* **321** (2009), 3026–3042.

[6] I. Dolinka and N. Ruškuc, Every group is a maximal subgroup of the free idempotent generated semigroup over a band, *Internat. J. Algebra Comput.* **23** (2013), 573–581.

[7] D. Easdown, Biordered sets come from semigroups, *J. Algebra* **96** (1985), 581–591.

[8] R. L. Graham, On finite 0-simple semigroups and graph theory, *Math. Syst. Theory* **2** (1968), 325–339.

[9] R. Gray and N. Ruskuc, On maximal subgroups of free idempotent generated semigroups, *Israel J. Math.* **189** (2012), 147–176.

[10] R. Gray and N. Ruškuc, Maximal subgroups of free idempotent-generated semigroups over the full transformation monoid, *Proc. Lond. Math. Soc. (3)* **104** (2012), 997–1018.

[11] T. Hall, Identities for existence varieties of regular semigroups, *Bull. Austral. Math. Soc.* **40** (1989), 59–77.

[12] C. H. Houghton, Completely 0-simple semigroups and their associated graphs and groups, *Semigroup Forum* **14** (1977), 41–67.

[13] J. Howie, Fundamentals of Semigroup Theory, *Oxford University Press, New York*, 1995.

[14] J. Kaďourek and M. B. Szendrei, A new approach in the theory of orthodox semigroups, *Semigroup Forum* **40** (1990), 257–296.

[15] G. Lallement, Demi-groupes reguliers, *Ann. Mat. Pura Appl.* **77** (1967), 47–129.

[16] K. S. S. Nambooripad, Structure of regular semigroups. *Mem. Amer. Math. Soc.* **224** (1979).

[17] K. S. S. Nambooripad, Pseudo-semilattices and biordered sets I, *Simon Stevin* **55** (1981), 103–110.

[18] K. S. S. Nambooripad and F. Pastijn, Subgroups of free idempotent generated regular semigroups, *Semigroup Forum* **21** (1980), 1–7.

[19] F. Pastijn, Idempotent generated completely 0-simple semigroups, *Semigroup Forum* **15** (1977), 41–50.

[20] F. Pastijn, The biorder on the partial groupoid of idempotents of a semigroup, *J. Algebra* **65** (1980), 147–187.

[21] F. Pastijn and L. Oliveira, Maximal Dense Ideal Extensions of Locally Inverse Semigroups, *Semigroup Forum* **72** (2006), 441–458.

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