Presentations for inverse subsemigroups with finite complement

Isabel Araújo,

Departamento de Matemática, Universidade de Évora,
Rua Romão Ramalho, 59, 7000-671 Évora, Portugal
e-mail: imma@uevora.pt

N. Ruškuc

Mathematical Institute, University of St. Andrews
St. Andrews KY16 9SS, Scotland, U.K.
e-mail: nik@mcs.st-and.ac.uk

and Pedro V. Silva

Centro de Matemática, Fac. de Ciências, Univ. do Porto,
R. Campo Alegre 687, 4169-007 Porto, Portugal
e-mail: pvsilva@fc.up.pt

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ABSTRACT

Let T be an inverse subsemigroup of an inverse semigroup S, and suppose that the complement of T in S is finite. We show that T is finitely presented if and only if S is finitely presented (both in the sense of inverse semigroup presentations). In the case where T is an ideal we obtain a particularly simple, effectively computable presentation for it.

1 Introduction

Let S be a semigroup, and let T be a subsemigroup of S. The *index* of T in S is defined to be the size $|S \setminus T|$ of its complement in S.

This definition seems strange at a first glance: it certainly does not generalise the familiar notion of index for groups. Nevertheless, it turns out to share various properties with its group-theoretic counterpart. For example, Jura [7] proved the following result:

Theorem 1.1 *Let S be a semigroup, and let T be a subsemigroup of finite index in S. Then T is finitely generated if and only if S is finitely generated.*

Building on this, Ruškuc [11] (see also [2, 3]) proved:

Theorem 1.2 *Let S be a semigroup, and let T be a subsemigroup of finite index in S. Then T is finitely presented if and only if S is finitely presented.*

These two results may be considered as analogues of the classical results of Reidemeister and Schreier from combinatorial group theory; see [8].

Of course the question arises of an overarching notion of index, which would include both of the above as special cases. An initial study in this direction is attempted in [12], but many interesting open questions remain.

In this paper we will be concerned with inverse semigroups and their subsemigroups of finite index. Recall that a semigroup S is said to be inverse if every $s \in S$ has a unique (semigroup-theoretic) inverse s^{-1} (satisfying $ss^{-1}s = s$ and $s^{-1}ss^{-1} = s^{-1}$); or, equivalently, if S is regular and its idempotents commute. Inverse semigroups, considered as algebraic structures with one binary operation $\cdot$ and one unary operation $^{-1}$, form a variety (with defining identities $(x^{-1})^{-1} = x$, $xx^{-1}x = x$, $xx^{-1}yy^{-1} = yy^{-1}xx^{-1}$). Therefore, free inverse semigroups exist, and one can use them to define arbitrary inverse semigroups by means of presentations (generators and defining relations). It turns out that inverse semigroup presentations (or i-presentations for short) are somewhat different in nature from both semigroup and group presentations. This is perhaps best illustrated by the fact that the free inverse semigroup on one generator is not finitely presented as an (ordinary) semigroup; see [13].

The purpose of this paper is to prove the analogue of Theorem 1.2 for inverse semigroups:

Main Theorem *Let S be an inverse semigroup, and let T be an inverse subsemigroup of finite index in S. Then T is finitely i-presented if and only if S is finitely i-presented.*

Of course, the analogue of Theorem 1.1 (finite generation) also holds. In fact, unlike the Main Theorem, it follows directly from Theorem 1.1, because an inverse semigroup S is finitely generated as an inverse semigroup if and only if it is finitely generated as an (ordinary) semigroup. Indeed, if A is a (finite) inverse semigroup generating set for S, then the set $A \cup A^{-1}$ generates S as a semigroup.

The paper is organised as follows. Sections 2 and 3 introduce basic definitions and notation: the former on inverse semigroups and their presentations; the latter on automata and Stephen's procedure for inverse semigroup presentations. In Section 4 we prove two useful lemmas involving Green's relations. Section 5 contains the proof of the Main Theorem. Finally, in Section 6 we give another proof of the Main Theorem in the case where T is an ideal; this prof establishes a natural, constructive presentation for T.

2 Preliminaries: inverse semigroups and presentations

For notation and basic results on semigroups see [6]. For an introduction to inverse semigroups see [10].

The natural partial order on an inverse semigroup S is defined by

$$a \leqslant b \iff \exists e \in E(S): \; a = eb,$$

where $E(S)$ denotes the set of idempotents of S. Green's equivalence relations also play an important role in the study of inverse semigroups. We recall their definition on an inverse semigroup S. Thus, for $a, b \in S$, we have

$$\begin{aligned}
a\mathcal{L}b &\iff Sa = Sb;\\
a\mathcal{R}b &\iff aS = bS;\\
a\mathcal{J}b &\iff SaS = SbS;\\
a\mathcal{D}b &\iff \exists c \in S: \; a\mathcal{R}c \text{ and } c\mathcal{L}b;\\
a\mathcal{H}b &\iff a\mathcal{L}b \text{ and } a\mathcal{R}b.
\end{aligned}$$

We denote by $\mathcal{L}_a, \mathcal{R}_a, \mathcal{J}_a, \mathcal{D}_a$ and $\mathcal{H}_a$ the $\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{D}$ and $\mathcal{H}$-class of $a \in S$, respectively.

Let X be a set. The set $\widetilde{X}$ is the set $X \cup X^{-1}$, where X^{-1} is a set in one-one correspondence with X, disjoint from X. The formal inverse u^{-1} is defined for every $u \in \widetilde{X}^+$ according to

$$\begin{aligned}
(x^{-1})^{-1} &= x \quad (x \in X),\\
(x_1 \dots x_n)^{-1} &= x_n^{-1} \dots x_1^{-1} \quad (x_i \in \widetilde{X}).
\end{aligned}$$

An *i-presentation* is a formal expression of the form $\mathrm{Inv}\langle X \mid R\rangle$, where X is a set and R is a binary relation on $\widetilde{X}^+$. The i-presentation is said to be *finite* if both X and R are finite. The inverse semigroup defined by the i-presentation $\mathrm{Inv}\langle X \mid R\rangle$ is the quotient

$$\widetilde{X}^+/(\rho \cup R)^\sharp,$$

where ρ is the relation on $\widetilde{X}^+$ given by

$$\{(uu^{-1}u, u) \mid u \in \widetilde{X}^+\} \cup \{(uu^{-1}vv^{-1}, vv^{-1}uu^{-1}) \mid u, v \in \widetilde{X}^+\}.$$

We usually write $\tau = (\rho \cup R)^\sharp$. An element $(r, s) \in R \cup \rho$ is often written as $r = s$. Furthermore, if $u\tau = v\tau$ we say that $u = v$ *holds* in $\widetilde{X}^+/\tau$ or that $u = v$ is a *consequence* of relations in $\rho \cup R$. This is equivalent to the existence of a sequence of words

$$u = x_0, x_1, \dots, x_n = v,$$

where x_i is obtained from x_{i-1} by application of one relation from $R \cup \rho \cup (R \cup \rho)^{-1}$, where we use the notation $R^{-1} = \{(u, v) \mid (v, u) \in R\}$. It is easy to see that any inverse semigroup may be defined by an i-presentation.

We say that S is *finitely i-presented* if it can be defined by a finite i-presentation. It is well known (see, for example, [10]) that ρ is not a finitely generated congruence on $\widetilde{X}^+$. Thus, the concepts of finite presentability and finite i-presentability do not coincide in general. That is, there exist finitely i-presented inverse semigroups which are not finitely presented as semigroups. On the other hand it is clear that the concepts of finite generation as an inverse semigroup and as a semigroup coincide.

3 Preliminaries: automata

Let X be a set. An $\widetilde{X}$-*automaton* is a triple $\mathcal{A} = (I, \Gamma, F)$ such that Γ is a graph with vertex set $V(\mathcal{A})$ and edge set $E(\mathcal{A}) \subseteq V(\mathcal{A}) \times \widetilde{X} \times V(\mathcal{A})$ and $I, F \subseteq V(\mathcal{A})$. The set I is called the set of *initial vertices* and the set F is called the set of *final vertices* of $\mathcal{A}$. For an introduction on automata see [4] or [5].

An $\widetilde{X}$-automaton is *dual* if its graph is connected and

$$(p, x, q) \in E(\mathcal{A}) \Longleftrightarrow (q, x^{-1}, p) \in E(\mathcal{A})$$

for all $p, q \in \mathcal{A}$ and $x \in \widetilde{X}$. Edges (p, x, q) and (q, x^{-1}, p) are said to be *dual*. When dealing with dual automata we usually represent only one edge for each pair of dual edges. An $\widetilde{X}$-automaton is *deterministic* if there is only one initial vertex and

$$(p, x, q), (p, x, q') \in E(\mathcal{A}) \Longrightarrow q = q'$$

for all $p, q, q' \in V(\mathcal{A})$ and $x \in \widetilde{X}$. An $\widetilde{X}$-automaton is said to be *inverse* if it is dual and deterministic.

A path on $\mathcal{A}$ is a sequence

$$q_0 \xrightarrow{x_1} q_1 \xrightarrow{x_2} \cdots \xrightarrow{x_n} q_n$$

where $n \geqslant 0$ and $(q_{j-1}, x_j, q_j) \in E(\mathcal{A})$ for all $j \in \{1, \ldots, n\}$. The word $x_1 \cdots x_n$ is the *label* of the path. A word $u \in \widetilde{X}^*$ is said to be *accepted* by $\mathcal{A}$ if there is a path on $\mathcal{A}$ whose label is u, whose initial vertex is in I and whose final vertex is in F. The set of all words accepted by $\mathcal{A}$ is $L(\mathcal{A}) \subseteq \widetilde{X}^*$.

Let $\mathcal{A}$ and $\mathcal{A}'$ be $\widetilde{X}$-automata. An $\widetilde{X}$-*automaton morphism* $\phi : \mathcal{A} \to \mathcal{A}'$ is a map $\phi : V(\mathcal{A}) \to V(\mathcal{A}')$ such that

$$(p, a, q) \in E(\mathcal{A}) \Longrightarrow (p\phi, a, q\phi) \in E(\mathcal{A}')$$

and the initial and final vertices are preserved. If ϕ is injective we say that $\mathcal{A}$ *embeds* in $\mathcal{A}'$. If ϕ is bijective and ϕ^{-1} is also a morphism we say that $\mathcal{A}$ and $\mathcal{A}'$ are *isomorphic* and we write $\mathcal{A} \cong \mathcal{A}'$.

From now on we will consider $\widetilde{X}$-automata with only one initial vertex. We present a process to transform a finite dual $\widetilde{X}$-automaton into a finite inverse $\widetilde{X}$-automaton. Thus we consider a finite dual $\widetilde{X}$-automaton $\mathcal{A}$ and we build a new $\widetilde{X}$-automaton by identifying certain vertices of $\mathcal{A}$. Indeed, if in $\mathcal{A}$ we have two edges $(p, x, q), (p, x, q')$ we obtain a finite dual $\widetilde{X}$-automaton $\mathcal{A}'$ by identifying respectively:

- the vertices q and q';
- all pairs of edges of the form $(q, y, r), (q', y, r)$, with $y \in \widetilde{X}, r \in V(\mathcal{A})$;
- all pairs of edges of the form $(r, y, q), (r, y, q')$, with $y \in \widetilde{X}, r \in V(\mathcal{A})$.

We say that $\mathcal{A}'$ is *obtained from* $\mathcal{A}$ *by an elementary reduction*. It should be clear that an $\widetilde{X}$-automaton is deterministic if and only if it does not admit any reductions. If a deterministic $\widetilde{X}$-automaton $\mathcal{A}'$ is obtained from an $\widetilde{X}$-automaton $\mathcal{A}$ by a finite number of elementary reductions we call the process a *complete reduction*. If an $\widetilde{X}$-automaton $\mathcal{A}'$ is obtained from an $\widetilde{X}$-automaton $\mathcal{A}$ by a sequence of elementary reductions we say that $\mathcal{A}'$ is a *quotient* of $\mathcal{A}$.

For the remainder of the section we let S be an inverse semigroup and $\mathrm{Inv}\langle X \mid R\rangle$ be a presentation for S.

The *linear automaton* $\mathrm{Lin}(u)$ of $u = x_1x_2\cdots x_n \in \widetilde{X}^+$ is given by

$$\to q_0 \xrightarrow{x_1} q_1 \xrightarrow{x_2} \cdots \xrightarrow{x_n} q_n \to .$$

The inverse $\widetilde{X}$-automaton obtained by complete reduction of $\mathrm{Lin}(u)$ is called the *Munn tree* of u and it is denoted by $\mathrm{MT}(u)$. These automata were introduced by Munn in [9] to solve the word problem for the free inverse monoid. Indeed, $u\rho = v\rho$ if and only if $\mathrm{MT}(u) \cong \mathrm{MT}(v)$.

For $u \in \widetilde{X}^+$ the *Schützenberger automaton* of u relative to $\mathrm{Inv}\langle X \mid R\rangle$ is the $\widetilde{X}$-automaton

$$\mathcal{A}_R(u) = ((uu^{-1})\tau, \Gamma(u), u\tau),$$

where $\Gamma(u)$ is the graph whose set of vertices is $\mathcal{R}_{u\tau}$ and whose set of edges is

$$\{(v\tau, x, w\tau) \in \mathcal{R}_{u\tau} \times \widetilde{X} \times \mathcal{R}_{u\tau} \mid w\tau = (vx)\tau\}.$$

Given $u, v \in \widetilde{X}^+$, we have

$$v \in L(\mathcal{A}_R(u)) \Leftrightarrow v\tau \geq u\tau \tag{1}$$

(see [14]). In particular,

$$v\tau = u\tau \Leftrightarrow (v \in L(\mathcal{A}_R(u)) \wedge u \in L(\mathcal{A}_R(v))). \tag{2}$$

Let $\mathcal{A}$ be a finite dual $\widetilde{X}$-automaton and suppose that there are $(r, s) \in R \cup R^{-1}$ and $p, q \in V(\mathcal{A})$ such that

(i) there is a path $p \xrightarrow{r} q$ in $\mathcal{A}$; and

(ii) there is no path $p \xrightarrow{s} q$ in $\mathcal{A}$.

We define a finite dual $\widetilde{X}$-automaton $\mathcal{A}'$ by adding to $\mathcal{A}$ new vertices and edges to obtain a path

$$\to p = t_0 \xrightarrow{x_1} t_1 \xrightarrow{x_2} \cdots \xrightarrow{x_n} t_n = q \to,$$

where $s = x_1 x_2 \cdots x_n$. We say that $\mathcal{A}'$ is obtained from $\mathcal{A}$ by a *simple R-expansion*. Consider now simultaneously all $(r, s) \in R \cup R^{-1}$, and $p, q \in V(\mathcal{A})$ such that (i) and (ii) are satisfied. If we add to $\mathcal{A}$, for each such case, a path according to the single expansion rules, we obtain a finite dual $\widetilde{X}$-automaton $\mathcal{B}$ which is said to be obtained from $\mathcal{A}$ by a *complete R-expansion*.

The Schützenberger automaton of $u \in \widetilde{X}^+$ can be obtained as the limit of a sequence of finite inverse $\widetilde{X}$-automata $(\mathcal{A}_k(u))_{k \geqslant 1}$ which we will now define. We will simultaneously define a sequence $(\mathcal{B}_k(u))_{k \geqslant 1}$ of finite dual $\widetilde{X}$-automata. Both sequences are defined as follows:

1. $\mathcal{A}_1(u) = \mathrm{MT}(u)$;

2. $\mathcal{B}_k$ is obtained from $\mathcal{A}_k$ by a complete R-expansion;

3. $\mathcal{A}_{k+1}$ is obtained from $\mathcal{B}_k$ by a complete reduction.

The sequence $(\mathcal{A}_k(u))_{k \geqslant 1}$ is usually referred to as the *Stephen's sequence* of u relative to $\mathrm{Inv}\langle X \mid R \rangle$ (see [14]).

4 Green's classes of inverse semigroups

We present in this section some lemmas on Green's classes of inverse semigroups that will prove useful in forthcoming sections.

Lemma 4.1 *Let D be a finite $\mathcal{D}$-class of an inverse semigroup S. Then D is a $\mathcal{J}$-class.*

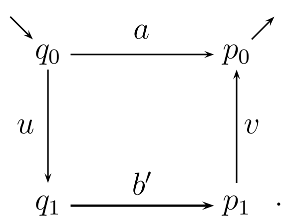
Proof Let $a \in D$ and consider $b \mathcal{J} a$. Then there exist $x, y \in S$ such that $b = xay$. Let $b' = x^{-1} b y^{-1}$. It follows easily that

$$b' b'^{-1} = x^{-1} b b^{-1} x \text{ and } b^{-1} x x^{-1} b = b^{-1} b.$$

Hence $b' \mathcal{R} x^{-1} b \mathcal{L} b$ and $b \mathcal{D} b'$. Moreover

$$b' = x^{-1} b y^{-1} = x^{-1} x a y y^{-1} \leqslant a.$$

Since $b' \mathcal{D} b \mathcal{J} a$ we have $b' \mathcal{J} a$ and so $a = u b' v$ for some $u, v \in S$. Let $\mathrm{Inv}\langle S \mid R \rangle$ be the i-presentation of S induced by its multiplication table. By (1), $\mathcal{A}_R(a)$ admits paths



Clearly, $\mathcal{A}_R(b')$ maps (homomorphically) into $\mathcal{A}_R(a)$ at vertex q_1. Since $b' \leqslant a$, we have $a \in L(\mathcal{A}_R(b'))$ and so we also have $q_1 \xrightarrow{a} p_1$. Thus, in $\mathcal{A}_R(a)$, we have

$$\begin{array}{ccc} q_1 & \xrightarrow{\;b',\,a\;} & p_1 \\ {\scriptstyle u}\big\downarrow & & \big\uparrow{\scriptstyle v} \\ q_2 & \xrightarrow{\;\;b'\;\;} & p_2 \end{array}$$

and we can continue indefinitely. Since $\mathcal{A}_R(a)$ is finite, we eventually obtain a repetition on the q_i sequence. Thus

$$q_0 \xrightarrow{u} \dots \xrightarrow{u} q_i \xrightarrow{u} \dots \xrightarrow{u} q_n = q_i$$

and $\mathcal{A}_R(a)$ deterministic implies that $i = 0$, otherwise $q_{n-1} = q_{i-1}$. Thus we have a path

$$q_0 = q_n \xrightarrow{b'} p_n$$

and a path $q_0 = q_n \xrightarrow{a} p_n$. Since $\mathcal{A}_R(a)$ is deterministic, we have $p_n = p_0$, yielding $q_0 \xrightarrow{b'} p_0$. Thus $b' \geqslant a$ by (1) and so $b' = a$. Therefore $b\mathcal{D}b' = a$ and the $\mathcal{J}$-class of a is exactly D. □

The above result no longer holds if either of the conditions that D be finite or that S be inverse is omitted, as the following two examples show. In the first one we present a (non-inverse) semigroup with a finite $\mathcal{D}$-class contained in an infinite $\mathcal{J}$-class.

Example 4.2 *Let $X = \{a, b, c, d, e, f\}$ and let S be the semigroup defined by the semigroup presentation $\langle X \mid R\rangle$, with*

$$R = \{a^2 = a, b = cad, a = ebf, adc = dca, bfe = feb\}.$$

Let $\sigma = R^\sharp$. Then $\mathcal{D}_{b\sigma} = \{b\sigma\}$ but $\mathcal{J}_{b\sigma}$ is infinite.

For every $n \in \mathbb{N}$, we have

$$a\sigma = a^n\sigma = (ebf)^n\sigma = (e(bfe)^{n-1}bf)\sigma = (eb^n(fe)^{n-1}f)\sigma$$

and $b^n\sigma = (b^{n-1}cad)\sigma$, hence $b^n\sigma \;\mathcal{J}\; a\sigma$ for every $n \in \mathbb{N}$.

For all $u \in X^+$ and $x \in X$, we denote by $|u|_x$ the number of occurrences of x in u. Next we remark that

$$u\,\sigma\,v \Rightarrow |u|_b + |u|_c - |u|_e = |v|_b + |v|_c - |v|_e. \tag{3}$$

Indeed, we may assume without loss of generality that $(u, v) \in R$ and the claim follows from direct verification.

We can conclude from (3) that $b^n\sigma \neq b^m\sigma$ whenever $n \neq m$, thus $b\sigma, b^2\sigma, b^3\sigma, \dots$ represent infinitely many distinct elements of $\mathcal{J}_{b\sigma}$.

To prove that $\mathcal{D}_{b\sigma} = \{b\sigma\}$, we define a homomorphism $\varphi : X^* \to \mathbb{Z}$ by $w\varphi = |w|_c + |w|_e - |w|_d - |w|_f$. We show that, for every prefix v of $u \in b\sigma$,

$$v\varphi \geq 0 \text{ and } (v\varphi = 0 \iff v \in \{1, u\}). \tag{4}$$

The claim holds trivially for $u = b$. Assume now that the claim holds for $prq \in b\sigma$ and $(r, s) \in R \cup R^{-1}$. We must show that the claim holds for psq as well. This can be done systematically in two steps: proving that $r\varphi = s\varphi$ (we omit the direct verification) and $(ps')\varphi > 0$ for every nonempty proper prefix s' of s, all other cases following from the hypothesis on prq. It is therefore enough to consider all the different cases for $|s| > 1$:

Case I: $(r, s) = (a, a^2)$.

Since $a\sigma = a^2\sigma$ and $b\sigma \neq b^2\sigma$, we must have $pq \neq 1$. Since $p = 1$ would imply $a\varphi = 0$ for the proper prefix a of prq, contradicting our assumption on prq, we may assume that $p \neq 1$ and so $p\varphi > 0$. Since $(pa)\varphi = p\varphi$, the claim now follows from the hypothesis on prq.

Case II: $(r, s) = (b, cad)$.

We only need to observe that $(pc)\varphi, (pca)\varphi > p\varphi \geq 0$.

Case III: $(r, s) = (adc, dca)$.

Since pad is a proper prefix of prq, we have $(pad)\varphi > 0$ and so $p\varphi \geq 2$. Thus $(pd)\varphi, (pdc)\varphi > 0$ and the claim follows.

Case IV: $(r, s) = (dca, adc)$.

Since pd is a proper prefix of prq, we have $(pd)\varphi > 0$ and so $p\varphi \geq 2$. Thus $(pa)\varphi, (pad)\varphi > 0$ and the claim follows.

The cases $(r, s) = (a, ebf)$, $(r, s) = (bfe, feb)$ and $(r, s) = (feb, bfe)$ are analogous to cases II, III and IV, respectively.

Therefore (4) holds. Suppose now that $b\sigma \; \mathcal{R} \; w\sigma$. Then $b\sigma = (wy)\sigma$ and $w\sigma = (bz)\sigma$ for some $y, z \in X^*$. It follows that $b\sigma = (bzy)\sigma$ and so b is a nonempty prefix of $bzy \in b\sigma$ satisfying $b\varphi = 0$. By (4), we must have $bzy = b$, hence $z = y = 1$ and $w = b$. Thus $\mathcal{R}_{b\sigma} = \{b\sigma\}$.

Suppose now that $b\sigma \, \mathcal{L} \, w\sigma$. Then $b\sigma = (yw)\sigma$ and $w\sigma = (zb)\sigma$ for some $y, z \in X^*$. It follows that $b\sigma = (yzb)\sigma$ and so yz is a proper prefix of $yzb \in b\sigma$ satisfying $(yz)\varphi = (yzb)\varphi = b\varphi = 0$. By (4), we must have $z = y = 1$ and so $w = b$. Thus $\mathcal{L}_{b\sigma} = \{b\sigma\}$ and so $\mathcal{D}_{b\sigma} = \{b\sigma\}$ as claimed.

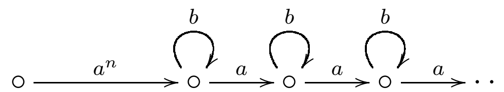
The next example exhibits an inverse monoid with a $\mathcal{J}$-class containing infinitely many $\mathcal{D}$-classes. We remark that in a monoid i-presentation $\mathrm{Invm}\langle X \mid R\rangle$ the binary relation R is taken on $\widetilde{X}^*$ and the inverse monoid is defined by the quotient $\widetilde{X}^*/(\rho \cup R)^\sharp$.

Example 4.3 *Let* $X = \{a, b\}$ *and let* M *be the inverse monoid defined by the i-presentation* $\mathrm{Invm}\langle X \mid R\rangle$, *with*

$$R = \{aa^{-1} = 1, b^2 = b, ba = bab\}.$$

Let $\sigma = R^\sharp$. Then $\mathcal{J}_{b\sigma}$ is the union of infinitely many $\mathcal{D}$-classes.

It is a simple exercise to construct the Schützenberger graph for the word $a^n b$, obtaining the following:



Clearly, these graphs are non-isomorphic for different values of n, and so $\mathcal{D}_{b\sigma}$, $\mathcal{D}_{(ab)\sigma}$, $\mathcal{D}_{(a^2b)\sigma}$, ... are all distinct $\mathcal{D}$-classes ($\mathcal{D}$-equivalent elements have the same Schützenberger graphs [14]). However, since they clearly embed in each other, their representatives must all belong to the same $\mathcal{J}$-class: indeed, if $\mathcal{A}_R(u)$ embeds in $\mathcal{A}_R(v)$, then $wuw' \in L(\mathcal{A}_R(v))$ for some $w, w' \in \widetilde{X}^*$ and so $(wuw')\tau \geq v\tau$ by (1), yielding $v\tau \in M(u\tau)M$. Therefore $\mathcal{J}_{b\sigma}$ is the union of infinitely many $\mathcal{D}$-classes.

Lemma 4.4 *Let T be a finite index inverse subsemigroup of an inverse semigroup S and let D be a $\mathcal{D}$-class of S such that $D \cap (S \backslash T) \neq \emptyset$. Then*

(i) *D is finite;*

(ii) *$S \backslash T$ intersects every $\mathcal{R}$-class and every $\mathcal{L}$-class of D; and*

(iii) *for each $x \in D \cap T$ there exist $s \in S \backslash T$ and $d \in D$ such that $xs \in S \backslash T$, $sd \in T$ and $xsd = x$.*

Proof (i) We start by proving that any $\mathcal{H}$-class of D is finite. It is of course enough to prove that, for some $s \in D \cap (S \backslash T)$, the $\mathcal{H}$-class of s is finite. Suppose first that there exists $s \in D \cap (S \backslash T)$ such that $\mathcal{H}_s$ is a group $\mathcal{H}$-class. We have that $\mathcal{H}_s \cap (S \backslash T)$ is certainly finite and for all $r \in T \cap \mathcal{H}_s$ we have

$$sr \in \mathcal{H}_s \cap (S \backslash T)$$

for otherwise $s = srr^{-1} \in T$. So $\mathcal{H}_s \cap (S \backslash T)$ is a union of cosets of $T \cap \mathcal{H}_s$ and therefore $T \cap \mathcal{H}_s$ must be finite. Therefore we conclude that a group $\mathcal{H}$-class which contains elements of $S \backslash T$ is finite.

Suppose now that no group $\mathcal{H}$-class of D contains elements of $S \backslash T$. Let $s \in D \cap (S \backslash T)$ and let $\mathcal{H}_s$ be its $\mathcal{H}$-class. If $T \cap \mathcal{H}_s \neq \emptyset$, and $r \in T \cap \mathcal{H}_s$, then from $sr^{-1} \mathcal{H} ss^{-1}$ we obtain that $sr^{-1} \in T$ and thus

$$s = ss^{-1}s = sr^{-1}r \in T$$

which is a contradiction. Hence $\mathcal{H}_s \subseteq S \backslash T$ and therefore it is finite.

We now prove that D has only finitely many $\mathcal{R}$-classes and $\mathcal{L}$-classes. Suppose there are infinitely many $\mathcal{R}$-classes in D. Let $s \in D \cap (S \backslash T)$. Then there exists $r \in D$ such that $r \mathcal{R} s$, $\mathcal{L}_r \subseteq T$ and $\mathcal{R}_{r^{-1}r} \subseteq T$. Hence $s = ru$, for some $u \in S$, and thus

$$s = ru = rr^{-1}ru = rr^{-1}s \in T$$

since $r^{-1}s\mathcal{R}r^{-1}r$. We therefore conclude that D has a finite number of $\mathcal{R}$ and $\mathcal{L}$-classes.
(ii) Suppose that there exists $a \in D$ such that $\mathcal{R}_a \subseteq T$. Let $b \in D$ and take $c \in \mathcal{R}_a \cap \mathcal{L}_b$. Then

$$b = bb^{-1}b = bc^{-1}c.$$

Since $c \in \mathcal{R}_a$ we have that $c \in T$. On the other hand

$$(cb^{-1})(cb^{-1})^{-1} = cb^{-1}bc^{-1} = cc^{-1}cc^{-1} = cc^{-1} = aa^{-1}$$

so that $cb^{-1} \in \mathcal{R}_a \subseteq T$ and so $bc^{-1} = (cb^{-1})^{-1} \in T$. Thus $b = (bc^{-1})c \in T$ and $D \subseteq T$ which is a contradiction. If, on the other hand, $\mathcal{L}_a \subseteq T$ then $\mathcal{R}_{a^{-1}} = (\mathcal{L}_a)^{-1} \subseteq T$.
(iii) Let $x \in D \cap T$. By (ii), $S\backslash T$ intersects $\mathcal{R}_x$ and hence there exist $s, y \in S\backslash T$, $u \in S$ such that $xs = y \in S\backslash T$ and $yu = x$. Since $x = xsu$ we may assume that $s = x^{-1}xs$, hence $su\mathcal{J}x$ and so $su \in \mathcal{J}_x = D$ by Lemma 4.1. Since $x = x(su)^n$ for every n, and D is finite by (i), we have $x = x(su)^n$ for some n such that $(su)^n$ is idempotent. Thus we may assume that su is an idempotent, replacing s by $(su)^{n-1}s$ if necessary. Since $x = xsu$ we have that $su \geqslant_{\mathcal{L}} x$. Since $su\mathcal{D}x$ and D is finite, we obtain $su\mathcal{L}x$ (see, for example, [1, Ex. 5.1.3]) and so $x^{-1}x = (su)^{-1}(su) = su$ (since su is idempotent). Thus $su \in T$.

Write $d = ux^{-1}x$. Clearly, $x = xsu$ yields $x = xsd$ and $x, su \in T$ together imply $sd \in T$. Since $d = ux^{-1}x$ and $x = xsd$ imply $d \in \mathcal{L}_x \subseteq D$, the lemma holds. □

5 Inverse subsemigroups of finite index

In this section we prove our main theorem:

Theorem 5.1 *Let T be a finite index inverse subsemigroup of an inverse semigroup S. Then T is finitely i-presented if and only if S is finitely i-presented.*

Proof Assume first that $\mathrm{Inv}\langle A \mid R\rangle$ is a finite i-presentation for S, and let $\pi : \widetilde{A}^+ \to S$ be the associated canonical morphism. Without loss of generality we may assume that $S\backslash T \subseteq D$, where D is a finite $\mathcal{D}$-class (hence also a $\mathcal{J}$-class by Lemma 4.1) of S: indeed, every $\mathcal{D}$-class of S intersecting $S \setminus T$ is a finite $\mathcal{J}$-class by Lemmas 4.4 and 4.1. If D is $\geq_{\mathcal{J}}$-maximal among those classes, then $T \cup (S \setminus D)$ is a proper inverse subsemigroup of S. Iteration of this argument produces a finite chain

$$T = T_0 < T_1 < \ldots < T_k = S$$

where $T_i \setminus T_{i-1}$ is contained in a finite $\mathcal{D}$-class of T_i.

We may assume that there is a generator in A for each $d \in D$ and we denote that unique generator by $\overline{d}$. If $u \in \widetilde{A}^+$ is such that u represents an element of D then we also write $\overline{u}$ to represent $\overline{u\pi}$. We also write $\overline{\varepsilon} = \varepsilon$ for the empty word ε. The generators in A corresponding to elements of $S\backslash T \subseteq D$ form the set Y. Using [11, Theorem 1.1], we take as generating set for T the set $X\pi$, where

$$X = \{yay' \mid y, y' \in Y^1;\ a \in \widetilde{A};\ ya, yay' \in T\pi^{-1}\},$$

where $Y^1 = Y \cup \{\varepsilon\}$. Now we introduce an alphabet

$$B = \{[y, a, y'] \mid yay' \in X\}$$

in one-one correspondence with the set X, and a homomorphism $\psi : \widetilde{B}^+ \to \widetilde{A}^+$ defined by

$$[y, a, y']^m \psi = (yay')^m,$$

for $[y, a, y'] \in B$ and $m \in \{1, -1\}$. Thus ψ "interprets" each word in $\widetilde{B}^+$ as a word from $T\pi^{-1} \subseteq \widetilde{A}^+$ representing the same element of T. On the other hand, it is clear that any element of T may be represented by a word in $\widetilde{B}^+$. Thus, we define a mapping $\phi : T\pi^{-1} \to \widetilde{B}^+$ as follows: given $w \in T\pi^{-1}$ write $w = \alpha a \beta$, where αa is the shortest prefix of w in $T\pi^{-1}$ ($\alpha, \beta \in \widetilde{A}^*$, $a \in \widetilde{A}$). Then

$$w\phi = \begin{cases} [\overline{\alpha}, a, \varepsilon](\beta\phi) \text{ if } \beta \in T\pi^{-1} \\ [\overline{\alpha}, a, \overline{\beta}] \qquad \text{otherwise} \end{cases}.$$

Note that $w\phi\psi\pi = w\pi$ for every $w \in T\pi^{-1}$. It follows from [2, Theorem 2.1] that a presentation for T is given by

$$\mathrm{Inv}\langle B \mid R_1 \cup R_2 \rangle$$

where

$$\begin{aligned} R_1 = &\{[y, a, y']\psi\phi = [y, a, y'] \mid y, y' \in Y^1,\ a \in \widetilde{A},\ ya, yay' \in T\pi^{-1}\} \\ &\cup\{(w_1\, w_2)\phi = (w_1\phi)\,(w_2\phi) \mid w_1, w_2 \in T\pi^{-1}\} \\ &\cup\{(w_3\, u\, w_4)\phi = (w_3\, v\, w_4)\phi \mid (u, v) \in R,\ w_3, w_4 \in \widetilde{A}^*,\ w_3\, u\, w_4 \in T\pi^{-1}\} \end{aligned} \tag{5}$$

and

$$R_2 = \{(w_3\, u\, w_4)\phi = (w_3\, v\, w_4)\phi \mid (u, v) \in \rho,\ w_3, w_4 \in \widetilde{A}^*,\ w_3\, u\, w_4 \in T\pi^{-1}\}$$

Moreover, straightforward checking allows us to conclude, from the proof of [11, Theorem 6.1], that there exists a finite set of relations R_1' on $\widetilde{B}^+$ such that

$${R_1'}^\sharp = R_1^\sharp$$

and

$$(\alpha\, \beta\, \gamma)\phi\, {R_1'}^\sharp\, (\alpha\, \overline{\beta}\, \gamma)\phi \tag{6}$$

whenever $\alpha\, \beta\, \gamma \in T\pi^{-1}$ and $\beta \notin T\pi^{-1}$. In particular if $y \in Y$ is such that $\overline{y^{-1}} = y'$, with $y' \in Y$, then

$$(\alpha\, y^{-1}\, \gamma)\phi\, {R_1'}^\sharp\, (\alpha\, y'\, \gamma)\phi, \tag{7}$$

for $\alpha\, y^{-1}\, \gamma \in T\pi^{-1}$.

It remains to show that there is a finite subset R_2' of R_2 such that the relations of R_2 are consequences of R_1 (and hence of R_1' as well), R_2' and ρ_B, where ρ_B is the set of inverse semigroup relations of $\mathrm{Inv}\langle B \mid R_1 \cup R_2 \rangle$.

We start by proving the following

Claim The relations

$$\beta\phi = \overline{\beta}\phi,$$

where $\beta \in T\pi^{-1} \cap D\pi^{-1}$ is such that at least one letter from $\widetilde{Y}$ occurs in β, are a consequence of finitely many relations from $R_1 \cup R_2 \cup \rho_B$.

Proof[of Claim] Let $\beta \in T\pi^{-1} \cap D\pi^{-1}$ be such that at least one letter from $\widetilde{Y}$ occurs in β. By Lemma 4.4 (iii), there exist $a \in Y$ and $c \in \widetilde{A}$ such that $\beta a \notin T\pi^{-1}$, $ac \in T\pi^{-1}$ and $\beta\pi = (\beta ac)\pi$. Now, using (5) and (6), we have that $(\beta ac)\phi = \beta\phi\,(ac)\phi$ and $(\beta ac)\phi = (\overline{\beta ac})\phi$ are consequences of R_1'. Thus

$$(\overline{\beta ac})\phi = \beta\phi\,(ac)\phi$$

is also a consequence of R_1'. Therefore it is enough to prove that the relations

$$(\overline{\beta ac})\phi = \overline{\beta}\phi \text{ and } \beta\phi\,(ac)\phi = \beta\phi$$

are consequences of finitely many relations from $R_1 \cup R_2 \cup \rho_B$. Since there are just finitely many choices for a and c, we may assume them fixed.

The first part is easy: since $(\overline{\beta ac})\pi = \overline{\beta}\pi \in T$, we have

$$(\overline{\beta ac})\phi\psi\pi = (\overline{\beta ac})\pi = \overline{\beta}\pi = \overline{\beta}\phi\psi\pi$$

and so $((\overline{\beta ac})\phi, \overline{\beta}\phi) \in (R_1 \cup R_2 \cup \rho_B)^\sharp$. Thus each relation $(\overline{\beta ac})\phi = \overline{\beta}\phi$ may be derived from finitely many relations in $R_1 \cup R_2 \cup \rho_B$. Since there are finitely many relations of the form $(\overline{\beta ac})\phi = \overline{\beta}\phi$ to consider, we are done.

For the second type write

$$\beta = \delta_1 a_1 \delta_2 a_2 \cdots \delta_k a_k \delta_{k+1},$$

where $a_i \in \widetilde{A}$, $\delta_i \in \widetilde{A}^*$ are such that:

- $\delta_i a_i$ is the shortest prefix of $\delta_i a_i \delta_{i+1} \cdots \delta_k a_k \delta_{k+1}$ in $T\pi^{-1}$;
- no nonempty prefix of δ_{k+1} belongs to $T\pi^{-1}$.

Then

$$\beta\phi = [\overline{\delta_1}, a_1, \varepsilon] \cdots [\overline{\delta_{k-1}}, a_{k-1}, \varepsilon]\, [\overline{\delta_k}, a_k, \overline{\delta_{k+1}}]$$

and

$$\beta\phi\,(ac)\phi = [\overline{\delta_1}, a_1, \varepsilon] \cdots [\overline{\delta_{k-1}}, a_{k-1}, \varepsilon]\, [\overline{\delta_k}, a_k, \overline{\delta_{k+1}}]\, [a, c, \varepsilon].$$

If $\overline{\delta_k} a_k \overline{\delta_{k+1}} \in D\pi^{-1}$ then, since $(\overline{\delta_k} a_k \overline{\delta_{k+1}})\pi \geqslant_{\mathcal{L}} \beta\pi$ and D is finite, we have as before $(\overline{\delta_k} a_k \overline{\delta_{k+1}})\pi \mathcal{L} \beta\pi$. Since $(ac)\pi$ is a right identity for $\beta\pi$, it follows that $(\overline{\delta_k} a_k \overline{\delta_{k+1}} ac)\pi = (\overline{\delta_k} a_k \overline{\delta_{k+1}})\pi$. Thus, in this case, $\beta\phi = \beta\phi\,(ac)\phi$ is a consequence of relations of the form

$$[y, b, y'] = [y, b, y'][a, c, \varepsilon]$$

where $[y,b,y'] \in B$. Notice that these relations hold in S and that there are finitely many of them. Hence $\beta\phi = (\beta ac)\phi$ is a consequence of finitely many relations from $R_1 \cup R_2 \cup \rho_B$.

If on the other hand $(\overline{\delta_k} a_k \overline{\delta_{k+1}})\pi \notin D$ then $\beta\pi \in D$ implies $(\overline{\delta_k} a_k \overline{\delta_{k+1}})\pi >_{\mathcal{J}} D$ and so $\delta_k, \delta_{k+1} = \varepsilon$ and $a_k \notin \widetilde{Y}$. Hence $\beta\phi$ ends by

$$[\overline{\delta_i}, a_i, \varepsilon][\varepsilon, a_{i+1}, \varepsilon] \cdots [\varepsilon, a_k, \varepsilon]$$

where i is the smallest subscript such that $\delta_i \neq \varepsilon$. Note that such subscript exists because of our assumption that at least one letter of $\widetilde{Y}$ occurs in β. Now, $(\overline{\delta_i} a_i a_{i+1})\pi \in D$ for otherwise $(\overline{\delta_i} a_i a_{i+1})\pi$ is strictly $\mathcal{J}$-above D and therefore it could not have letters from $\widetilde{Y}$. Thus, by Lemma 4.4 (ii), there exists $z \in Y$ such that

$$(\overline{\delta_i} a_i a_{i+1})\pi \mathcal{L} z\pi,$$

that is, there is $u \in \widetilde{A}^+$ such that $(\overline{\delta_i} a_i a_{i+1})\pi = (uz)\pi$. We then have

$$(uz)\phi = w[\overline{\delta}, d, \overline{\delta'}]$$

where $w \in \widetilde{B}^*$ and $\overline{\delta} d \overline{\delta'} \in \widetilde{A}^* Y \widetilde{A}^*$. Hence, we may consider the set of relations of the form

$$[y, a, y_1][\varepsilon, b, \varepsilon] = w[y_2, d, y_3],$$

where $[y,a,y_1], [\varepsilon, b, \varepsilon] \in B$ and $w, [y_2, d, y_3]$ are fixed as in the previous paragraph, with $yay_1, y_2dy_3 \in \widetilde{A}^* Y \widetilde{A}^*$. Note that these relations hold in S. Moreover, this is a finite set. Using these relations we are able to rewrite $\beta\phi$ obtaining a word of the form $z[y,d,y']$ (with $z \in \widetilde{B}^+$ and $ydy' \in \widetilde{A}^* Y \widetilde{A}^*$). Now, $(ydy')\pi \in D$ and therefore, using the first case, we may conclude that relations $\beta\phi = \beta\phi\,(ac)\phi$ are a consequence of finitely many relations from $R_1 \cup R_2 \cup \rho_B$. □

Let R_2' be the subset of R_2 which is the union of the relations of R_2 used in the claim with the set

$$\{(ww^{-1}w)\phi = w\phi \mid w \in T\pi^{-1}, |w| \leqslant 3\} \cup$$
$$\cup \{(yy'zz')\phi = (zz'yy')\phi \mid y, z \in Y, y' = \overline{y^{-1}}, z' = \overline{z^{-1}}, yy'zz' \in T\pi^{-1}\}.$$

Let $\Lambda = R_1' \cup R_2' \cup \rho_B$ and notice that $R_1' \cup R_2'$ is finite.

We show that the relations in R_2 are consequences of relations in Λ. We start with the relations $w\phi = (w\,w^{-1}\,w)\phi$, for $w \in T\pi^{-1}$. We use induction on $|w|$. For $|w| \leqslant 3$ the relations of the form $w\phi = (w\,w^{-1}\,w)\phi$ are all in R_2'. Now let $|w| \geqslant 4$. We may assume that the only factors of w in $(S \backslash T)\pi^{-1}$ are letters from $\widetilde{Y}$, for otherwise we may write $w = \alpha\,\beta\,\gamma$, with $\alpha, \gamma \in \widetilde{A}^*$, $\beta \in (S \backslash T)\pi^{-1}$, $|\beta| \geqslant 2$, and

$$\begin{aligned}
(w\,w^{-1}\,w)\phi &= (\alpha\beta\gamma\gamma^{-1}\beta^{-1}\alpha^{-1}\alpha\beta\gamma)\phi \\
&\Lambda^\sharp\ (\alpha\overline{\beta}\gamma\gamma^{-1}\overline{\beta^{-1}}\alpha^{-1}\alpha\overline{\beta}\gamma)\phi && \text{(by (6))} \\
&\Lambda^\sharp\ (\alpha\overline{\beta}\gamma\gamma^{-1}\overline{\beta}^{-1}\alpha^{-1}\alpha\overline{\beta}\gamma)\phi && \text{(by (7))} \\
&\Lambda^\sharp\ (\alpha\overline{\beta}\gamma)\phi && \text{(by induction hypothesis)} \\
&\Lambda^\sharp\ (\alpha\beta\gamma)\phi && \text{(by (6))} \\
&= w\phi.
\end{aligned}$$

Since $|w| \geqslant 4$, we can split $w = uv$ with $|u|, |v| \geqslant 2$. Thus $u, v \in T\pi^{-1}$. Moreover

$$\begin{array}{rll} (vv^{-1})\phi\,(vv^{-1})\phi & \Lambda^\sharp\,(vv^{-1}v)\phi\,(v^{-1}\phi) & \text{(using (5))} \\ & \Lambda^\sharp\,(v\phi)\,(v^{-1}\phi) & \text{(by induction hypothesis)} \\ & \Lambda^\sharp\,(vv^{-1})\phi & \text{(using (5))} \end{array}$$

and thus $(vv^{-1})\phi$ is an idempotent modulo $\Lambda^\sharp$. Similarly, so is $(u^{-1}u)\phi$. Thus

$$\begin{array}{rll} (ww^{-1}w)\phi = & (uvv^{-1}u^{-1}uv)\phi & \text{(since } w = uv\text{)} \\ & \Lambda^\sharp\,u\phi\,(vv^{-1})\phi\,(u^{-1}u)\phi\,v\phi & \text{(using (5))} \\ & \Lambda^\sharp\,u\phi\,(u^{-1}u)\phi\,(vv^{-1})\phi\,v\phi & \text{(commuting idempotents)} \\ & \Lambda^\sharp\,(uu^{-1}u)\phi\,(vv^{-1}v)\phi & \text{(using (5))} \\ & \Lambda^\sharp\,u\phi\,v\phi & \text{(by induction hypothesis)} \\ & \Lambda^\sharp\,(uv)\phi = w\phi. & \end{array}$$

We now consider the relations $(uu^{-1}vv^{-1})\phi = (vv^{-1}uu^{-1})\phi$ with $uu^{-1}vv^{-1} \in T\pi^{-1}$. Notice that for each $w \in T\pi^{-1}$, $(ww^{-1})\phi$ is an idempotent modulo $\Lambda^\sharp$, since $(ww^{-1}w)\phi\,\Lambda^\sharp\,w\phi$. We consider the following cases:

(I) $u, v \in T\pi^{-1}$;

(II) $u, v \notin T\pi^{-1}$,

(III) $u \in T\pi^{-1}$ and $v \notin T\pi^{-1}$.

<u>Case (I)</u>: $u, v \in T\pi^{-1}$. In this case we have

$$\begin{array}{rll} (uu^{-1}vv^{-1})\phi & \Lambda^\sharp\,(uu^{-1})\phi\,(vv^{-1})\phi & \text{(using (5))} \\ & \Lambda^\sharp\,(vv^{-1})\phi\,(uu^{-1})\phi & \text{(commuting idempotents)} \\ & \Lambda^\sharp\,(vv^{-1}uu^{-1})\phi & \text{(using (5)).} \end{array}$$

<u>Case II</u>: $u, v \notin T\pi^{-1}$. In this case we have, applying (6),

$$(uu^{-1}vv^{-1})\phi\,\Lambda^\sharp\,\left(\overline{u}\,\overline{u^{-1}}\,\overline{v}\,\overline{v^{-1}}\right)\phi \text{ and } (vv^{-1}uu^{-1})\phi\,\Lambda^\sharp\,\left(\overline{v}\,\overline{v^{-1}}\,\overline{u}\,\overline{u^{-1}}\right)\phi.$$

Since in R_2' we have relations of the form $(yy'zz')\phi = (zz'yy')\phi$, for $y, z \in Y$, we conclude that also in this case $(uu^{-1}vv^{-1})\phi\,\Lambda^\sharp\,(vv^{-1}uu^{-1})\phi$.
<u>Case III</u>: $u \in T$ and $v \notin T$. We may assume (by using relations (6) to replace v by $\overline{v}$ and v^{-1} by $\overline{v^{-1}}$) that $v = y \in Y$. If $uu^{-1}yy^{-1} \in D\pi^{-1}$ then of course $yy^{-1}uu^{-1} \in D\pi^{-1}$ and $\overline{uu^{-1}yy^{-1}} = \overline{yy^{-1}uu^{-1}}$. Hence

$$\begin{array}{rll} (uu^{-1}yy^{-1})\phi & \Lambda^\sharp\,\left(\overline{uu^{-1}yy^{-1}}\right)\phi & \text{(by the Claim)} \\ & \Lambda^\sharp\,\left(\overline{yy^{-1}uu^{-1}}\right)\phi & \text{(since } \overline{uu^{-1}yy^{-1}} = \overline{yy^{-1}uu^{-1}}\text{)} \\ & \Lambda^\sharp\,(yy^{-1}uu^{-1})\phi & \text{(by the Claim).} \end{array}$$

So let us suppose that $uu^{-1}yy^{-1} <_{\mathcal{J}} y$. From $(uu^{-1}u)\phi\,\Lambda^\sharp\,u\phi$ we have that $(uu^{-1})\phi$ is an idempotent modulo $\Lambda^\sharp$. Furthermore $(uu^{-1}yy^{-1})\phi$ is also an idempotent modulo $\Lambda^\sharp$. Indeed, since for every $w \in T\pi^{-1}$ we have $(ww^{-1}w)\phi\,\Lambda^\sharp\,w\phi$ and $u^{-1}yy^{-1} \in T\pi^{-1}$ (because $u^{-1}yy^{-1} <_{\mathcal{J}} y$) we obtain

$$\begin{array}{llr}
(uu^{-1}yy^{-1})\phi & \Lambda^\sharp\,(uu^{-1}yy^{-1}\,(uu^{-1}yy^{-1})^{-1}\,uu^{-1}yy^{-1})\phi & \\
& \Lambda^\sharp\,(uu^{-1}yy^{-1}\,yy^{-1}uu^{-1}\,uu^{-1}yy^{-1})\phi & \\
& \Lambda^\sharp\,(uu^{-1}yy^{-1}uu^{-1}uu^{-1}yy^{-1})\phi & \text{(see (+) below)} \\
& \Lambda^\sharp\,(uu^{-1}yy^{-1})\phi\,(uu^{-1}u)\phi\,(u^{-1}yy^{-1})\phi & \text{(using (5))} \\
& \Lambda^\sharp\,(uu^{-1}yy^{-1})\phi\,u\phi\,(u^{-1}yy^{-1})\phi & \text{(since } (uu^{-1}u)\phi = u\phi) \\
& \Lambda^\sharp\,(uu^{-1}yy^{-1})\phi\,(uu^{-1}yy^{-1})\phi & \text{(using (5))}
\end{array}$$

((+) since $\overline{yy^{-1}yy^{-1}} = \overline{yy^{-1}}$ if $yy^{-1} \notin T$ or using (6) if $yy^{-1} \in T$). Therefore

$$\begin{array}{llr}
(uu^{-1}yy^{-1})\phi & \Lambda^\sharp\,u\phi\,(u^{-1}yy^{-1})\phi & \text{(using (5))} \\
& \Lambda^\sharp\,(uu^{-1}u)\phi\,(u^{-1}yy^{-1})\phi & \text{(since } (uu^{-1}u)\phi = u\phi) \\
& \Lambda^\sharp\,(uu^{-1})\phi\,(uu^{-1}yy^{-1})\phi & \text{(using (5))} \\
& \Lambda^\sharp\,(uu^{-1}yy^{-1})\phi\,(uu^{-1})\phi & \text{(commuting idempotents)} \\
& \Lambda^\sharp\,(uu^{-1}yy^{-1}uu^{-1})\phi & \text{(using (5))} \\
& \Lambda^\sharp\,(uu^{-1})\phi\,(yy^{-1}uu^{-1})\phi & \text{(using (5))}
\end{array}$$

and so $(yy^{-1}uu^{-1})\phi \geqslant (uu^{-1}yy^{-1})\phi$ in $\widetilde{B}^+/\Lambda^\sharp$ (with respect to the natural partial order on an inverse semigroup). Similarly we obtain $(yy^{-1}uu^{-1})\phi \leqslant (uu^{-1}yy^{-1})\phi$ and so $(yy^{-1}uu^{-1})\phi\Lambda^\sharp(uu^{-1}yy^{-1})\phi$ holds.

To conclude the proof that T is finitely i-presented, we only have to show that

$$(w_3uw_4)\phi\,\Lambda^\sharp\,(w_3uu^{-1}uw_4)\phi$$

whenever $w_3uw_4 \in T\pi^{-1}$ and

$$(w_3uu^{-1}vv^{-1}w_4)\phi\,\Lambda^\sharp\,(w_3vv^{-1}uu^{-1}w_4)\phi$$

whenever $w_3uu^{-1}vv^{-1}w_4 \in T\pi^{-1}$. In order to do this we will prove that if w_3, w_4, u, v are such that $u\pi = v\pi$, $w_3uw_4 \in T\pi^{-1}$ and either $u\phi\,\Lambda^\sharp\,v\phi$ or $u \notin T\pi^{-1}$ then $(w_3uw_4)\phi\,\Lambda^\sharp\,(w_3vw_4)\phi$.

First suppose that $w_3uw_4 \in D\pi^{-1}$. If there are occurrences of letters of $\widetilde{Y}$ in both w_3uw_4 and w_3vw_4 (in particular, if $u \notin T\pi^{-1}$) then

$$\begin{array}{llr}
(w_3uw_4)\phi & \Lambda^\sharp\;(\overline{w_3uw_4})\phi & \text{(by Claim)} \\
& =\;(\overline{w_3vw_4})\phi & \text{(since } u\pi = v\pi \text{ also } (w_3uw_4)\pi = (w_3vw_4)\pi) \\
& \Lambda^\sharp\;(w_3vw_4)\phi & \text{(by Claim).}
\end{array}$$

On the other hand, if no letter of $\widetilde{Y}$ appears in w_3uw_4 then $w_3, u, w_4 \in T\pi^{-1}$ and hence also $v \in T\pi^{-1}$. Using (5), we obtain

$$(w_3uw_4)\phi\,\Lambda^\sharp\,(w_3\phi)\,(u\phi)\,(w_4\phi)\,\Lambda^\sharp\,(w_3\phi)\,(v\phi)\,(w_4\phi)\,\Lambda^\sharp\,(w_3vw_4)\phi.$$

Assume now that $w_3uw_4 \notin D\pi^{-1}$ (hence also $w_3vw_4 \notin D\pi^{-1}$). We may assume that $(w_3uw_4)\pi$ is $\mathcal{J}$-below D, otherwise we are done by (5). If $u \notin T\pi^{-1}$ then (6) yields

$$(w_3uw_4)\phi\,\Lambda^\sharp\,(w_3\overline{u}w_4)\phi = (w_3\overline{v}w_4)\phi\,\Lambda^\sharp\,(w_3vw_4)\phi.$$

On the other hand, if $u \in T\pi^{-1}$, then we may write

$$(w_3uw_4)\phi\,\Lambda^\sharp\,(w_3uw_4w_4^{-1}u^{-1}w_3^{-1}w_3uw_4w_4^{-1}u^{-1}w_3^{-1}w_3uw_4)\phi.$$

Since all $w_3uw_4w_4^{-1}u^{-1}w_3^{-1}w_3, u, w_4w_4^{-1}u^{-1}w_3^{-1}w_3uw_4 \in T\pi^{-1}$, we have

$$\begin{array}{llr}(w_3uw_4)\phi & \Lambda^\sharp\ (w_3uw_4w_4^{-1}u^{-1}w_3^{-1}w_3)\phi\,u\phi\,(w_4w_4^{-1}u^{-1}w_3^{-1}w_3uw_4)\phi & \text{(using (5))}\\ & \Lambda^\sharp\ (w_3uw_4w_4^{-1}u^{-1}w_3^{-1}w_3)\phi\,v\phi\,(w_4w_4^{-1}u^{-1}w_3^{-1}w_3uw_4)\phi & \text{(since } u\phi\,\Lambda^\sharp\,v\phi)\\ & \Lambda^\sharp\ (w_3uw_4w_4^{-1}u^{-1}w_3^{-1}w_3vw_4w_4^{-1}u^{-1}w_3^{-1}w_3uw_4)\phi & \text{(using (5))}\\ & \Lambda^\sharp\ (w_3uw_4w_4^{-1}u^{-1}w_3^{-1})\phi\,(w_3vw_4)\phi\,(w_4^{-1}u^{-1}w_3^{-1}w_3uw_4)\phi & \text{(using (5)).}\end{array}$$

Since $(w_3uw_4w_4^{-1}u^{-1}w_3^{-1})\phi$ and $(w_4^{-1}u^{-1}w_3^{-1}w_3uw_4)\phi$ are idempotents we have that $(w_3vw_4)\phi \geqslant (w_3uw_4)\phi$ in $\widetilde{B}^+/\Lambda^\sharp$. By symmetry we obtain $(w_3vw_4)\phi \leqslant (w_3uw_4)\phi$. Therefore $(w_3uw_4)\phi\,\Lambda^\sharp\,(w_3vw_4)\phi$ and T is finitely i-presented.

Conversely, we assume now that T is finitely i-presented. Given a finite presentation

$$\mathrm{Inv}\langle X \mid R\rangle$$

for T, we add a generator y for every element in $S\backslash T$, and the following relations (where Y denotes the set of all y):

$$R' = \{(yx, u_{y,x}) \mid y \in Y, x \in \widetilde{X}\} \cup \{(xy, v_{x,y}) \mid y \in Y, x \in \widetilde{X}\} \cup$$
$$\cup \{(yy', z_{y,y'}) \mid y, y' \in Y\} \cup \{(y^{-1}, p_y) \mid y \in Y\},$$

where $u_{y,x}, v_{x,y} \in \widetilde{X}^+ \cup Y$, $z_{y,y'} \in \widetilde{X}^+ \cup Y$ and $p_y \in Y$ are chosen so that all the relations of R' hold in S.

We show that

$$\langle X \cup Y \mid R \cup R'\rangle$$

is a (finite) presentation for S. If we map the elements of X to the corresponding elements of T, and each y to the element it represents in $S\backslash T$, it is immediate that the relations $R \cup R'$ are satisfied by the corresponding images in S (indeed, $u_{y,x}, v_{x,y}$ and $z_{y,y'}$ were chosen to make sure this happens). Thus we have an onto homomorphism

$$\phi : (\widetilde{X \cup Y})^+/(R \cup R')^\sharp \to S.$$

It follows from the definition of R' that every word in $(\widetilde{X \cup Y})^+$ is $(R')^\sharp$-equivalent to either a word in $\widetilde{X}^+$ or some element of Y. Since the restrictions

$$\widetilde{X}^+/(R \cup R')^\sharp \ \to T$$
$$Y \to S\backslash T$$

are clearly one-one, we conclude that ϕ is one-one and thus $\langle X \cup Y \mid R \cup R'\rangle$ is a finite presentation for S. □

6 Ideals of finite index

In this section we consider Main Theorem for ideals of inverse semigroups. Though this is undoubtedly a particular case, its proof is of independent interest, since the relations are efficiently computed.

Theorem 6.1 *Let S be an inverse semigroup and let T be a finite index ideal of S. Then T is finitely i-presented if and only if S is finitely i-presented.*

We start by proving the converse part of the theorem. Thus suppose that S is finitely i-presented. Since, by [11, Theorem 1.1], T is a finitely generated subsemigroup of S and $S\backslash T$ is finite, S is defined by a finite presentation $\mathrm{Inv}\langle X \cup Y \mid R\rangle$ where X generates T, the elements of Y represent the elements of $S\backslash T$, and Y is in one-one correspondence with $S\backslash T$.

Given $x \in \widetilde{X}$ and $y, y' \in Y$ we fix words $u_{y,x}, v_{x,y}, w_{x,y,y'} \in \widetilde{X}^+$ and $z_{y,y'} \in \widetilde{X}^+ \cup Y$ such that

$$yx R^\sharp u_{y,x},\ \ xy R^\sharp v_{x,y}, \tag{8}$$

$$yy' R^\sharp z_{y,y'} \tag{9}$$

and

$$w_{x,y,y'} = \begin{cases} xz_{y,y'} \text{ if } z_{y,y'} \in \widetilde{X}^+ \\ v_{x,y''} \text{ if } z_{y,y'} = y'' \in Y \end{cases}. \tag{10}$$

Indeed, notice that there exist words $u_{y,x}, v_{x,y} \in \widetilde{X}^+$ satisfying (8) since T is an ideal of S and X generates T. It is also clear that there exists $z_{y,y'}$ satisfying (9). Finally notice that it is clear, from (10), that

$$w_{x,y,y'} R^\sharp x z_{y,y'}.$$

We start by proving the following

Lemma 6.2 *The semigroup S can be defined by a finite presentation of the form*

$$\mathrm{Inv}\langle X \cup Y \mid R_X \cup N \cup Q\rangle$$

where

(i) X generates T;

(ii) the elements of Y represent the elements of $S\backslash T$ and Y is in one-one correspondence with $S\backslash T$;

(iii) $R_X \subseteq \widetilde{X}^+ \times \widetilde{X}^+$;

(iv) $N = \{yx = u_{y,x},\ xy = v_{x,y},\ yy' = z_{y,y'} \mid y, y' \in Y, x \in \widetilde{X}^+\}$;

(v) $Q = \{(v_{x,y}u_{y',x'}, w_{x,y,y'}x') \mid y, y' \in Y, x, x' \in \widetilde{X}\}$.

Proof The presentation $\langle X \cup Y \mid R \rangle$ for S satisfies (i) and (ii). We shall now modify it in order to satisfy the remaining conditions as well.

From the definition of $u_{y,x}, v_{x,y}$ and $z_{y,y'}$ it is clear that $\mathrm{Inv}\langle X \cup Y \mid R \cup N \rangle$, where $N = \{yx = u_{y,x},\ xy = v_{x,y},\ yy' = z_{y,y'} \mid y, y' \in Y,\ x \in \widetilde{X}\}$, is still a finite presentation for S.

Notice that, for every $y \in Y$, we have

$$y^{-1}\, R^{\sharp} y',$$

for some $y' \in Y$. Moreover, the relation $y^{-1} = y'$ can be derived from relations in N since

$$y^{-1}\, R^{\sharp} y' \Longleftrightarrow \begin{cases} yy'y\, N^{\sharp}\, y \\ y'yy'\, N^{\sharp}\, y' \end{cases}$$

and

$$yy'y\, N^{\sharp}\, y \Longleftrightarrow \begin{cases} yy'\, N\, y'' \\ y''y\, N\, y \end{cases},$$

for some $y'' \in Y$.

We now replace each relation in R involving letters from Y by some relation on $\widetilde{X}^+$ as follows: given $(a, b) \in R$, we replace every occurrence of some letter y^{-1} $(y \in Y)$ by y' such that $y^{-1}\, R^{\sharp}\, y'$. Notice that in order to do this we only need relations from N. Hence (a, b) is equivalent, modulo N, to some relation on $(\widetilde{X} \cup Y)^+$. Also, by using relations from N, we can reduce any word in $(\widetilde{X} \cup Y)^+$ to either a word on $\widetilde{X}^+$ or some letter in Y. Thus (a, b) must reduce to some relation of the form

$$u = v,\ y = u \text{ or } y = y'\ (u, v \in \widetilde{X}^+, y, y' \in Y).$$

The second case is impossible since y does not represent an element of T. The third case is either trivial (if y and y' are the same generator) or impossible. Thus, any relation in R is equivalent, modulo N, to some relation on $\widetilde{X}^+$.

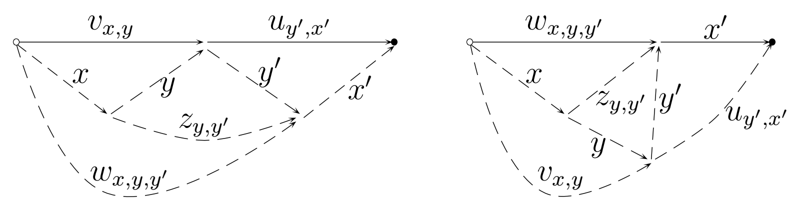
Therefore, S is defined by the presentation

$$\mathrm{Inv}\langle X \cup Y \mid R_X \cup N \rangle$$

which satisfies (i), (ii), (iii) and (iv).

Our next step is to extend $R_X \cup N$ by adding the finitely many relations $Q \subseteq \widetilde{X}^+ \times \widetilde{X}^+$. These relations will substitute the relations of the form $(yy', z_{yy'})$ in the computation of the Schützenberger automata for words in $\widetilde{X}^+$.

In order to prove that Q is contained in $(R_X \cup N)^{\sharp}$ we prove that the right hand side of each relation is recognized by the Schützenberger automaton (relative to $R_X \cup N$) of the left hand side and vice versa. It follows easily from the diagrams that, starting from any side of the relation, we need at most three expansions in the Stephen's sequence to recognize the other word:



Therefore $\langle X \cup Y \mid R_X \cup N \cup Q \rangle$ is a presentation for S which satisfies the stated conditions. $\square$

Let $\mathrm{Inv}\langle X \cup Y \mid R_X \cup N \cup Q \rangle$ be the presentation for S given in the above lemma. Let $R' = R_X \cup N \cup Q$ and let $R'' = R' \backslash \{(yy', z_{y,y'}) \mid y, y' \in Y\}$. We then have:

Lemma 6.3 *For R' and R'' as above,*

$$\mathcal{A}_{R'}(u) = \mathcal{A}_{R''}(u)$$

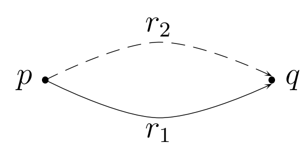
for all $u \in \widetilde{X}^+$.

Proof Since $\mathcal{A}_{R'}(u)$ and $\mathcal{A}_{R''}(u)$ are inverse and therefore minimal automata, to prove that $\mathcal{A}_{R'}(u) = \mathcal{A}_{R''}(u)$ is the same as showing that

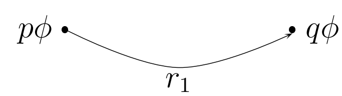
$$L(\mathcal{A}_{R'}(u)) = L(\mathcal{A}_{R''}(u)).$$

Since $R'' \subseteq R'$, we have $L(\mathcal{A}_{R''}(u)) \subseteq L(\mathcal{A}_{R'}(u))$ trivially. It remains to show that $L(\mathcal{A}_{R'}(u)) \subseteq L(\mathcal{A}_{R''}(u))$. Let $(\mathcal{A}_n(u))_{n\in\mathbb{N}}$ denote the Stephen's sequence of u relative to R'. We show, by induction on n, that $L(\mathcal{A}_n(u)) \subseteq L(\mathcal{A}''_R(u))$ for all n. Since $L(\mathcal{A}_{R'}(u)) = \bigcup_{n\geqslant 1} L(\mathcal{A}_n(u))$, this implies that $L(\mathcal{A}_{R'}(u)) \subseteq L(\mathcal{A}_{R''}(u))$. Since R' is finite, we may assume that each iteration of the sequence is obtained by a single expansion followed by complete reduction.

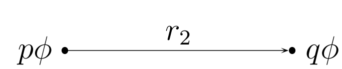
The case $n = 1$ is trivial since $\mathcal{A}_1(u)$ is the Munn tree of u, hence $L(\mathcal{A}_1(u)) \subseteq L(\mathcal{A}_{R''}(u))$. Assume now that $L(\mathcal{A}_n(u)) \subseteq L(\mathcal{A}_{R''}(u))$ and that $\mathcal{A}_{n+1}(u)$ is obtained from $\mathcal{A}_n(u)$ by applying the expansion



(where $(r_1, r_2) \in R' \cup (R')^{-1}$) to $\mathcal{A}_n(u)$ followed by complete reduction. Since $\mathcal{A}_n(u)$, $\mathcal{A}_{R''}(u)$ are inverse, $L(\mathcal{A}_n(u)) \subseteq L(\mathcal{A}_{R''}(u))$ is equivalent to saying that there is an automaton morphism $\phi : \mathcal{A}_n(u) \to \mathcal{A}_{R''}(u)$ (see [14]). Thus we have a path



in $\mathcal{A}_{R''}(u)$. It suffices to show that there is a path



(11)

in $\mathcal{A}_{R''}(u)$: in fact, this will imply at once that $L(\mathcal{B}_n(u)) \subseteq L(\mathcal{A}_{R''}(u))$, where $\mathcal{B}_n(u)$ denotes the dual automaton obtained after performing the expansion on $\mathcal{A}_n(u)$. Now, every $v \in L(\mathcal{A}_{n+1}(u))$ is such that $v \geqslant v'$, in the free inverse semigroup on X, for some $v' \in L(\mathcal{B}_n(u))$: indeed, due to the nature of reduction, every path in $\mathcal{A}_{n+1}(u)$ can be lifted to a path in $\mathcal{B}_n(u)$ by successively inserting words of the form aa^{-1}. Since $\mathcal{A}_{R''}(u)$ is inverse, idempotents of the free inverse semigroup must always label closed paths and so

$$(v' \in L(\mathcal{A}_{R''}(u)) \wedge v\rho^\sharp \geq v'\rho^\sharp) \Rightarrow v \in L(\mathcal{A}_{R''}(u)).$$

We will then conclude that $L(\mathcal{A}_{n+1}(u)) \subseteq L(\mathcal{A}_{R''}(u))$ as required.

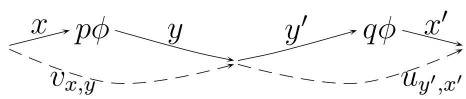
Let us now show the existence of the path (11). Since R'' contains all the relations of R' except those of the form $(yy', z_{y,y'})$, for $y, y' \in Y$, we only have to consider the following two cases:

(a) $r_1 = yy'$, $r_2 = z_{y,y'}$;

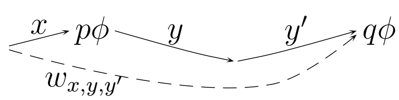
(b) $r_1 = z_{y,y'}$, $r_2 = yy'$.

We claim that all vertices in $\mathcal{A}_{R''}(u)$ can be connected through $\widetilde{X}$-edges proving that the claim holds for all the iterations of the Stephen's sequence. This is certainly true for the first iteration (the Munn tree of u) since $u \in \widetilde{X}^+$. For the induction step, we note that in any relation from R'' there is at most one occurrence of a letter y not in $\widetilde{X}$. Using such a relation for an expansion in the Stephen's iterative procedure preserves our property, as the unique occurrence of y (if there is one) can be "bridged" by using the rest of the relation. Reductions clearly preserve our property, and hence the claim holds. In particular, every vertex is adjacent to some $\widetilde{X}$-edge.

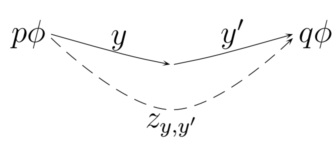
So, suppose we are in case (a). Using the preceding remark, we have in $\mathcal{A}_{R''}(u)$ a situation of the form



for some $x, x' \in \widetilde{X}$. Because of the relation $(v_{x,y}u_{y',x'}, w_{x,y,y'}x') \in R''$, we have a path

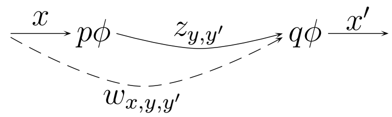


and since $(xz_{y,y'}, w_{x,y,y'}) \in R''$ (if $z_{y,y'} \in Y$) or $xz_{y,y'} = w_{x,y,y'}$ (if $z_{y,y'} \in \widetilde{X}^+$), we obtain

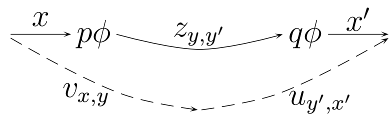


as required.

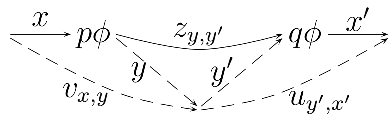
Suppose now that we are in case (b). Once again, we may write



with $x, x' \in \widetilde{X}$. So



finally obtaining



Therefore $L(\mathcal{A}_{n+1}(u)) \subseteq L(\mathcal{A}_{R''}(u))$ which concludes the proof. $\square$

Consider the presentation

$$\langle X \cup Y \mid R'' \rangle.$$

This finite presentation is no longer a presentation for S. For our purposes, however, it is enough to have the property proved in Lemma 6.3. We may write $R''_X = R'' \cap (\widetilde{X}^+ \times \widetilde{X}^+)$ and thus

$$\begin{aligned} R'' = R''_X &\cup \{(yx, u_{y,x}) \mid y \in Y,\ x \in \widetilde{X}\} \\ &\cup \{(xy, v_{x,y}) \mid y \in Y,\ x \in \widetilde{X}\}. \end{aligned}$$

We now give a finite presentation for T.

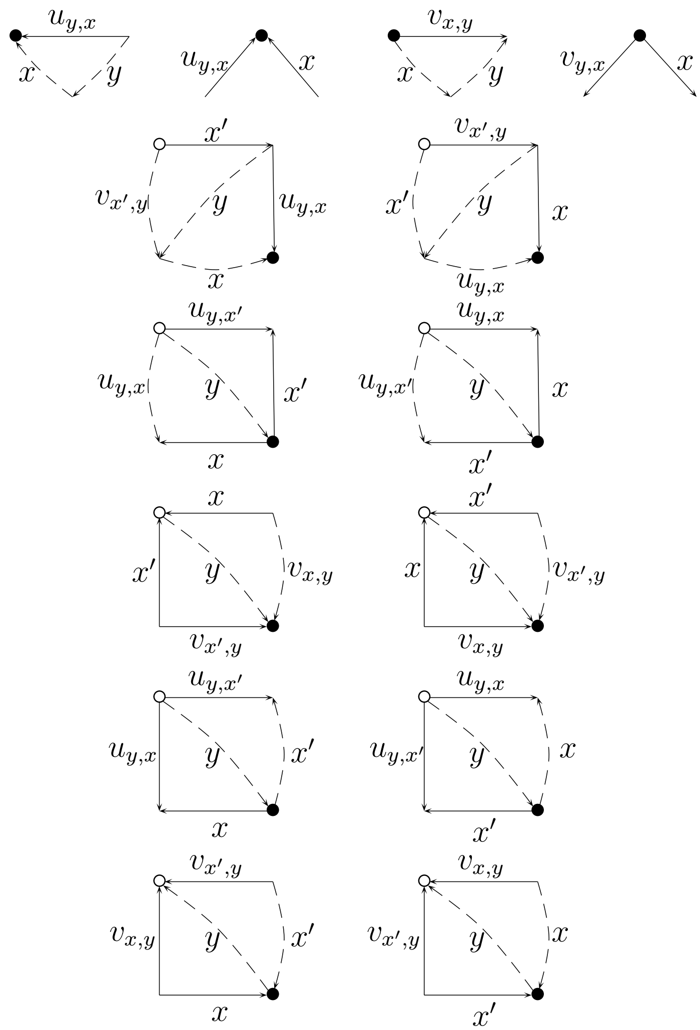
Lemma 6.4 *With the above notation, T is defined by the finite i-presentation*

$$\mathrm{Inv}\langle X \mid R''' \rangle$$

where

$$\begin{aligned} R''' = R''_X &\cup \{(u_{y,x}^{-1}u_{y,x}, x^{-1}xu_{y,x}^{-1}u_{y,x}) \mid y \in Y, x \in \widetilde{X}\} \\ &\cup \{(v_{x,y}v_{x,y}^{-1}, xx^{-1}v_{x,y}v_{x,y}^{-1}) \mid y \in Y, x \in \widetilde{X}\} \\ &\cup \{(x'u_{y,x}, v_{x',y}x) \mid y \in Y, x, x' \in \widetilde{X}\} \\ &\cup \{(u_{y,x'}x'^{-1}xx^{-1}, u_{y,x}x^{-1}x'x'^{-1}) \mid y \in Y, x, x' \in \widetilde{X}\} \\ &\cup \{(x^{-1}xx'^{-1}v_{x',y}, x'^{-1}x'x^{-1}v_{x,y}) \mid y \in Y, x, x' \in \widetilde{X}\} \\ &\cup \{(u_{y,x'}u_{y,x'}^{-1}u_{y,x}x^{-1}, u_{y,x}u_{y,x}^{-1}u_{y,x'}x'^{-1}) \mid y \in Y, x, x' \in \widetilde{X}\} \\ &\cup \{(v_{x',y}^{-1}v_{x',y}v_{x,y}^{-1}x, v_{x,y}^{-1}v_{x,y}v_{x',y}^{-1}x') \mid y \in Y, x, x' \in \widetilde{X}\}. \end{aligned}$$

Proof Clearly R''' is a finite relation on $\widetilde{X}^+$. We show that $R''' \subseteq (R')^\sharp$ by proving that the right hand side of each relation in R''' is recognized by the Schützenberger automaton (relative to R') of the left hand side and vice versa.



Thus $R''' \subseteq (R')^\sharp$.

Since X generates T in the presentation $\langle X \cup Y \mid R' \rangle$ of S, and $R''' \subseteq (R')^\sharp$, the canonical homomorphism

$$\begin{aligned} \theta : \widetilde{X}^+/(R''')^\sharp &\rightarrow \quad T \\ u(R''')^\sharp &\mapsto u(R')^\sharp \end{aligned}$$

is well defined and onto. It remains to show that it is injective. Let $u, v \in \widetilde{X}^+$ and suppose that $u(R')^\sharp v$. Then $\mathcal{A}_{R'}(u) \cong \mathcal{A}_{R'}(v)$. Since, by Lemma 6.3, $\mathcal{A}_{R'}(w) = \mathcal{A}_{R''}(w)$, whenever $w \in \widetilde{X}^+$, we conclude that $\mathcal{A}_{R''}(u) \cong \mathcal{A}_{R''}(v)$. Thus $L(\mathcal{A}_{R''}(u)) = L(\mathcal{A}_{R''}(v))$.

Let $(\mathcal{A}_n(w))_n$ denote the Stephen's sequence of $w \in \widetilde{X}^+$ relative to R''. We prove the following

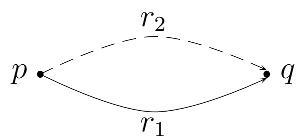
Claim For all n, there exists an $\widetilde{X}$-automaton morphism $\phi : \mathcal{A}_n(w) \rightarrow \mathcal{A}_{R'''}(w)$.

Proof[of Claim] Notice that we can view $\mathcal{A}_n(w)$ as an $\widetilde{X}$-automaton by ignoring the Y-edges. In fact, $\mathcal{A}_n(w)$, viewed as an $\widetilde{X}$-automaton, is inverse: it is obviously dual and connectedness follows from the fact that any expansion performed in an $\widetilde{X}$-connected inverse $(\widetilde{X} \cup Y)$-automaton using a relation in R'' produces an $\widetilde{X}$-connected automaton.

We prove the claim by induction on n. The case $n = 1$ being trivial ($\mathcal{A}_1(w)$ is the Munn tree of w), we assume that

$$\phi : \mathcal{A}_n(w) \to \mathcal{A}_{R'''}(w)$$

is an $\widetilde{X}$-automaton morphism. Since R'' is finite, we may assume that $\mathcal{A}_{n+1}(w)$ is obtained from $\mathcal{A}_n(w)$ by performing a single expansion (leading to intermediate automaton $\mathcal{B}_n(w)$) followed by complete reduction. Assume that our expansion is

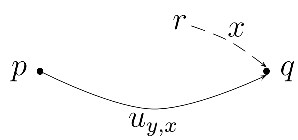


(where $(r_1, r_2) \in R'' \cup (R'')^{-1}$).

We extend ϕ to $\mathcal{B}_n(w)$ considering different cases:

Case $r_1, r_2 \in \widetilde{X}^+$. In this case $(r_1, r_2) \in R''' \cup (R''')^{-1}$. Since we have by hypothesis a path $p\phi \xrightarrow{r_1} q\phi$ in $\mathcal{A}_{R'''}(w)$, then we have also a path $p\phi \xrightarrow{r_2} q\phi$ and so we have an extension $\phi : \mathcal{B}_n(w) \to \mathcal{A}_{R'''}(w)$ as required.

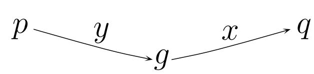
Case $r_1 = u_{y,x}$, $r_2 = yx$ ($y \in Y$, $x \in \widetilde{Y}$). We have a path $p\phi \xrightarrow{u_{y,x}} q\phi$ in $\mathcal{A}_{R'''}(w)$. Since $(u_{y,x}^{-1} u_{y,x}, x^{-1} x u_{y,x}^{-1} u_{y,x}) \in R'''$, we have an edge $r' \xrightarrow{x} q\phi$ in $\mathcal{A}_{R'''}(w)$. Now, $\mathcal{B}_n(w)$, as an $\widetilde{X}$-automaton, is obtained from $\mathcal{A}_n(w)$ by adding an edge



Defining $r\phi = r'$, we have the required extension $\phi : \mathcal{B}_n(w) \to \mathcal{A}_{R'''}(w)$.

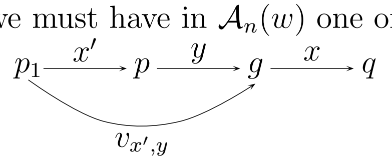
Case $r_1 = v_{x,y}$, $r_2 = xy$ ($y \in Y$, $x \in \widetilde{X}$). This case is similar to the previous case, using the relation $(v_{x,y} v_{x,y}^{-1}, x x^{-1} v_{x,y} v_{x,y}^{-1}) \in R'''$.

Case $r_1 = yx$, $r_2 = u_{y,x}$ ($y \in Y$, $x \in \widetilde{Y}$). We have a path

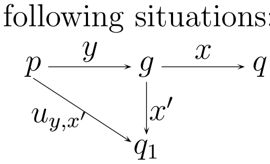


in $\mathcal{A}_n(w)$. Notice that the existence of the edge $p \xrightarrow{y} g$ in $\mathcal{A}_n(w)$ follows from the performance (at some previous moment in the sequence) of some expansion relative to some relation in R''. This relation is not the relation $(yx, u_{y,x})$, otherwise there would be a path $p \xrightarrow{u_{y,x}} q$ in $\mathcal{A}_n(w)$.

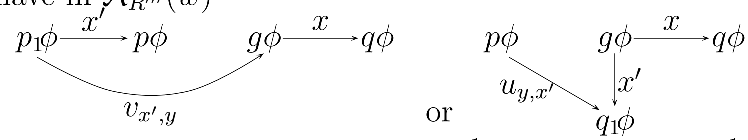
Therefore we must have in $\mathcal{A}_n(w)$ one of the following situations:



or

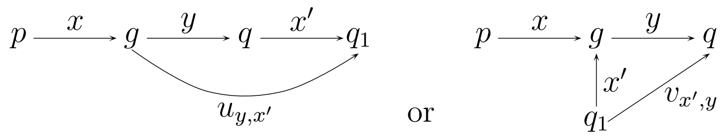


Therefore we have in $\mathcal{A}_{R'''}(w)$

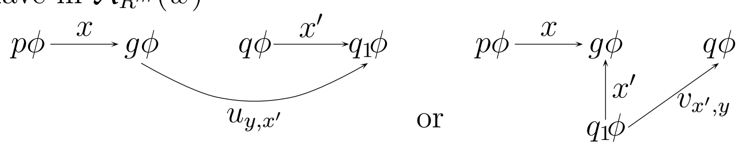


Applying the relations $(x'u_{y,x}, v_{x',y}x)$ or $(u_{y,x'}x'^{-1}xx^{-1}, u_{y,x}x^{-1}x'x'^{-1})$ in R''', respectively, we conclude that we have a path $p\phi \xrightarrow{u_{y,x}} q\phi$ in $\mathcal{A}'''_R(w)$. Thus we may map the path $p \xrightarrow{u_{y,x}} q$ in $\mathcal{B}_n(w)$ onto the path $p\phi \xrightarrow{u_{y,x}} q\phi$ in $\mathcal{A}_{R'''}(w)$ and extend ϕ to a morphism $\phi : \mathcal{B}_n(w) \to \mathcal{A}_{R'''}(w)$.

Case $r_1 = xy$, $r_2 = v_{x,y}$ ($y \in Y$, $x \in \widetilde{Y}$). Similarly to the previous case, we have in $\mathcal{A}_n(w)$ either



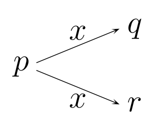
Therefore we have in $\mathcal{A}_{R'''}(w)$



Applying the relations $(xu_{y,x'}, v_{x,y}x')$ or $(x^{-1}xx'^{-1}v_{x',y}, x'^{-1}x'x^{-1}v_{x,y})$ in R''', respectively, we conclude that there is a path $p\phi \xrightarrow{v_{x,y}} q\phi$ in $\mathcal{A}_n(w)$. Thus we may map the path $p \xrightarrow{v_{x,y}} q$ in $\mathcal{B}_n(w)$ onto the path $p\phi \xrightarrow{v_{x,y}} q\phi$ in $\mathcal{A}_{R'''}(w)$ and extend ϕ to a morphism $\phi : \mathcal{B}_n(w) \to \mathcal{A}_{R'''}(w)$.

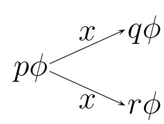
We will now extend $\phi : \mathcal{B}_n(w) \to \mathcal{A}_{R'''}(w)$ to $\mathcal{A}_{n+1}(w)$. Let $\mathcal{B}$ be a quotient of $\mathcal{B}_n(w)$. It suffices to show that if $\phi : \mathcal{B} \to \mathcal{A}_{R'''}(w)$ is an $\widetilde{X}$-automaton morphism then we can define an $\widetilde{X}$-automaton morphism $\overline{\phi} : \mathcal{B}' \to \mathcal{A}_{R'''}(w)$, where $\mathcal{B}'$ is obtained from $\mathcal{B}$ by an elementary reduction. Indeed, since $\mathcal{A}_{n+1}(w)$ is obtained from $\mathcal{B}_n(w)$ by performing finitely many elementary reductions, the claim will follow.

Suppose first that $\mathcal{B}'$ is obtained from $\mathcal{B}$ by performing the elementary reduction induced by



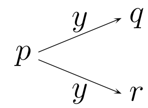
$(x \in \widetilde{X})$.

Since we have



in $\mathcal{A}_{R'''}(w)$, and $\mathcal{A}_{R'''}(w)$ is deterministic, we have $q\phi = r\phi$ and so ϕ induces naturally a quotient morphism $\overline{\phi} : \mathcal{B}' \to \mathcal{A}_{R'''}(w)$.

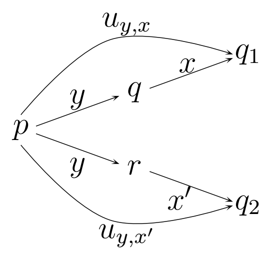
Suppose now that $\mathcal{B}'$ is obtained from $\mathcal{B}$ by performing the elementary reduction induced by



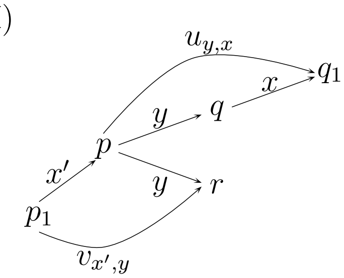
$(y \in Y)$.

Since $\mathcal{B}$ is a quotient of $\mathcal{B}_n(w)$, each edge labeled y appeared as the result of an expansion of some relation in R''. Thus we must have in $\mathcal{B}$ one of the following four situations:

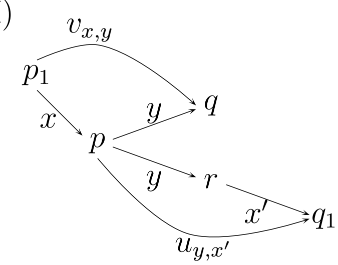
(I)



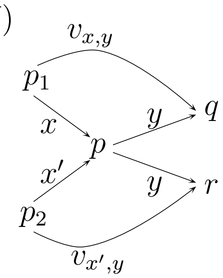
(II)



(III)

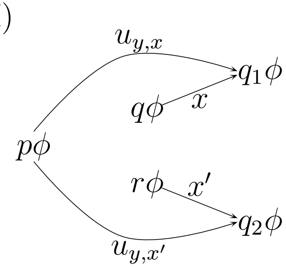


(IV)

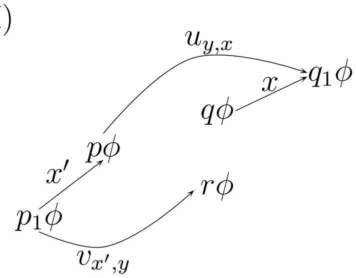


This implies that we have in $\mathcal{A}_{R'''}(w)$ one of the following situations:

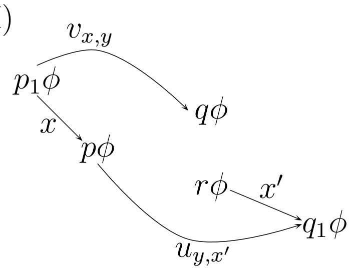
(I)



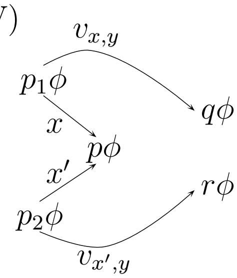
(II)



(III)



(IV)



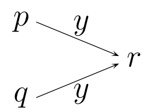
Considering the following relations of R''':

$$(u_{y,x'}u_{y,x'}^{-1}u_{y,x}x^{-1}, u_{y,x}u_{y,x}^{-1}u_{y,x'}x'^{-1}), (x'u_{y,x}, v_{x',y}x), (xu_{y,x'}, v_{x,y}x'),$$

$$(x^{-1}xx'^{-1}v_{x',y}, x'^{-1}x'x^{-1}v_{x,y}),$$

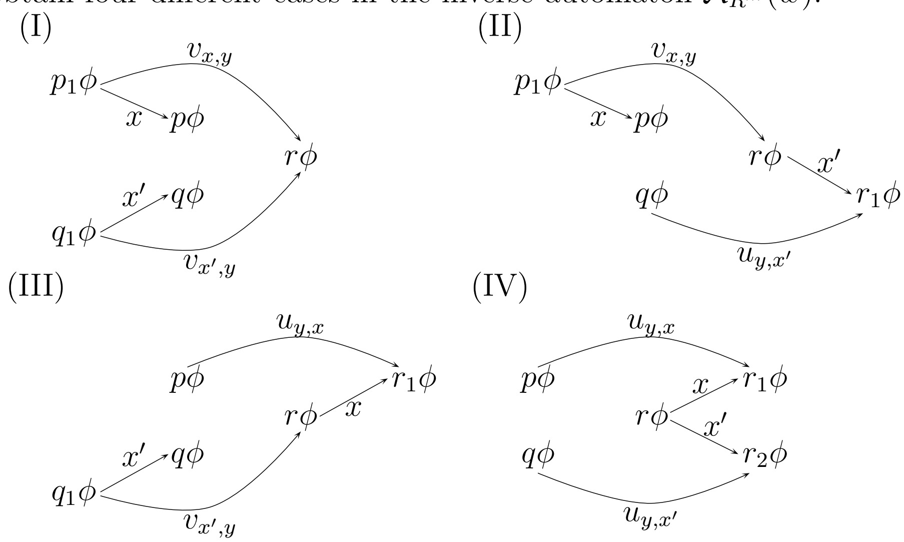
respectively, we end up in every case by concluding that $q\phi = r\phi$ and so $\phi : \mathcal{B} \to \mathcal{A}_{R'''}(w)$ factors to a quotient morphism $\overline{\phi} : \mathcal{B}' \to \mathcal{A}_{R'''}(w)$.

Finally, we suppose that $\mathcal{B}'$ is obtained from $\mathcal{B}$ by performing the elementary reduction induced by



$(y \in Y)$.

Once again, we may consider the different possibilities for the origin of these edges and we obtain four different cases in the inverse automaton $\mathcal{A}_{R'''}(w)$:



Considering the following relations of R'''

$$(v_{x',y}^{-1}v_{x',y}v_{x,y}^{-1}x, v_{x,y}^{-1}v_{x,y}v_{x',y}^{-1}x'), (xu_{y,x'}, v_{x,y}x'), (x'u_{y,x}, v_{x',y}x),$$

$$(u_{y,x'}x'^{-1}xx^{-1}, u_{y,x}x^{-1}x'x'^{-1}),$$

respectively, we conclude in every case that $p\phi = q\phi$. Hence $\phi : \mathcal{B} \to \mathcal{A}_{R'''}(w)$ factors to a quotient morphism $\overline{\phi} : \mathcal{B}' \to \mathcal{A}_{R'''}(w)$ also in this last case, which completes the proof of the claim. □

Since $\mathcal{A}_n(w)$ is inverse as an $\widetilde{X}$-automaton, we have

$$L(\mathcal{A}_n(w)) \cap \widetilde{X}^* \subseteq L(\mathcal{A}_{R'''}(w))$$

by the claim. Thus

$$L(\mathcal{A}_{R''}(w)) \cap \widetilde{X}^* \subseteq L(\mathcal{A}_{R'''}(w)).$$

Since $L(\mathcal{A}_{R''}(u)) = L(\mathcal{A}_{R''}(v))$, we have that $u \in L(\mathcal{A}_{R''}(v))$ and $v \in L(\mathcal{A}_{R''}(u))$. Since $u, v \in \widetilde{X}^+$, it follows that $u \in L(\mathcal{A}_{R'''}(v))$ and $v \in L(\mathcal{A}_{R'''}(u))$. Thus $u(R''')^\sharp v$ by (2) and so θ is injective. This shows that $\langle X \mid R''' \rangle$ is a finite presentation for T.

The direct part of the Theorem follows of course from Theorem 5.1. □

Acknowledgements

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