DONOHO-STARK AND PALEY-WIENER THEOREMS FOR THE G-TRANSFORM

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Abstract

This paper deals with a class of G-transforms

$$g(x) = G(f)(x) := \int_0^\infty G^{m,n}_{2n,2m}\left(xy \,\middle|\, \begin{matrix} (a_n), -(\overline{a}_n) \\ (b_m), -(\overline{b}_m) \end{matrix}\right) f(y)\, dy$$

with the Meijer G-function as the kernel. We prove the Donoho-Stark uncertainty principle for the G-transform. We also give a characterization of the image of functions that have compact support (Paley-Wiener theorem) under this class of transforms in case $n = 0$.

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1 G-transform

As it is known [9], [18] the Meijer G-function is defined by the following Mellin-Barnes integral

$$G^{m,n}_{p,q}\left(x \,\middle|\, \begin{matrix} (a_p) \\ (b_q) \end{matrix}\right) = \frac{1}{2\pi i}\int_c \frac{\prod_{k=1}^{m}\Gamma(b_k+s)\prod_{j=1}^{n}\Gamma(1-a_j-s)}{\prod_{k=m+1}^{q}\Gamma(1-b_k-s)\prod_{j=n+1}^{p}\Gamma(a_j+s)} x^{-s} ds, \tag{1.1}$$

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where c is a suitable contour in $\mathbb{C}$, the orders (m, n, p, q) are nonnegative integers, $0 \le m \le q$, $0 \le n \le p$, and the parameters $a_j \in \mathbb{C}$, $\quad j = 1, 2, \ldots, p$; $\quad b_k \in \mathbb{C}$, $\quad k = 1, 2, \ldots, q$; are such that all the points $-b_k - l$, $\quad k = 1, 2, \ldots, m$; $\quad l = 0, 1, 2, \ldots$; are positioned to the left of the contour c, and all the points $1 - a_j + l'$, $\quad j = 1, 2, \ldots, n$; $\quad l' = 0, 1, 2, \ldots$; are situated to the right of the contour c.
Meijer's G- function satisfies the following differential equation

$$\left[(-1)^{p-m-n} x \prod_{j=1}^{p} (\delta - a_j + 1) - \prod_{j=1}^{q} (\delta - b_j)\right] y = 0, \qquad \delta = x \frac{d}{dx}.$$

Consider the following G-transform [11], [17], [18]

$$g(x) = (Gf)(x) := \int_0^\infty G_{2n,2m}^{m,n}\left(xy \,\middle|\, \begin{array}{c} (a_n), -(\overline{a}_n) \\ (b_m), -(\overline{b}_m) \end{array} \right) f(y)\, dy, \tag{1.2}$$

where the integral in (1.2) is convergent in the mean square sense, $\Re(b_j) > -\frac{1}{2}$, $\quad j = 1, \cdots, m$, and $\Re(a_j) < \frac{1}{2}$, $\quad j = 1, \cdots, n$. The last conditions on $\Re(a_j)$ and $\Re(b_j)$ guarantee that the contour c in the definition (1.1) of the G-function can be chosen as the vertical line $\sigma = \left(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty\right)$. On this vertical line we have

$$\left| \frac{\prod_{k=1}^{m} \Gamma(b_k + s) \prod_{j=1}^{n} \Gamma(1 - a_j - s)}{\prod_{k=1}^{m} \Gamma\left(1 + \overline{b}_k - s\right) \prod_{j=1}^{n} \Gamma(-\overline{a}_j + s)} \right| = 1, \qquad s \in \sigma,$$

hence, G-transform is an unitary isomorphism on $L_2(\mathbb{R}_+)$, and the Plancherel theorem and Parseval identity hold for the G-transform (for details see [18], [13])

$$\|Gf\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}. \tag{1.3}$$

Denote the Mellin transform as follows [9], [13]

$$f^*(s) = \int_0^\infty x^{s-1} f(x)\, dx, \tag{1.4}$$

then

$$G^*(s) = \frac{\prod_{k=1}^{m} \Gamma(b_k + s) \prod_{j=1}^{n} \Gamma(1 - a_j - s)}{\prod_{k=1}^{m} \Gamma\left(1 + \overline{b}_k - s\right) \prod_{j=1}^{n} \Gamma(-\overline{a}_j + s)}$$

is the Mellin transform of $G_{2n,2m}^{m,n}\left(x \,\middle|\, \begin{array}{c} (a_n), -(\overline{a}_n) \\ (b_m), -(\overline{b}_m) \end{array} \right)$. Applying the Mellin transform (1.4) to the G-transform (1.2) yields

$$g^*(s) = G^*(s) f^*(1 - s). \tag{1.5}$$

Hence,

$$f^*(s) = \frac{1}{G^*(1 - s)} g^*(1 - s). \tag{1.6}$$

Since

$$\frac{1}{G^*(1-s)} = \overline{G}^*(s) = \frac{\prod_{k=1}^{m} \Gamma\left(\overline{b}_k + s\right) \prod_{j=1}^{n} \Gamma\left(1 - \overline{a}_j - s\right)}{\prod_{k=1}^{m} \Gamma\left(1 + b_k - s\right) \prod_{j=1}^{n} \Gamma\left(-a_j + s\right)},$$

the Mellin transform of $G^{m,n}_{2n,2m}\left(x \,\middle|\, \begin{matrix} (\overline{a}_n), -(a_n) \\ (\overline{b}_m), -(b_m) \end{matrix}\right)$, then $G^*(s)\overline{G}^*(1-s) = 1$, therefore, the inverse G-transform has the conjugate symmetrical form

$$f(x) = \overline{G}(g)(x) := \int_0^\infty G^{m,n}_{2n,2m}\left(xy \,\middle|\, \begin{matrix} (\overline{a}_n), -(a_n) \\ (\overline{b}_m), -(b_m) \end{matrix}\right) g(y)\, dy. \tag{1.7}$$

In particular, in case (a_n), (b_m) are real vectors, then the inverse G-transform has symmetrical form. As $J_\nu(2\sqrt{x}) = G^{1,0}_{0,2}\left(x \,\middle|\, \begin{matrix} \cdot\,,\,\cdot \\ \nu/2, -\nu/2 \end{matrix}\right)$, the Hankel transform [13] is a special case of the symmetric G-transform. For other special cases of the G-transform see [18].

2 Donoho-Stark Uncertainty Principle

Following [5] we say that f is ϵ-concentrated on a measurable set $\mathbb{E}$ if

$$\|f - \chi_{\mathbb{E}} f\|_2 < \epsilon,$$

where $\chi_{\mathbb{E}}$ is the characteristic function of the set $\mathbb{E}$. Donoho and Stark [5] show that if f of unit L_2 norm is $\epsilon_{\mathbb{T}}$-concentrated on a measurable set $\mathbb{T}$ and its Fourier transform $\hat{f}$ is $\epsilon_{\mathbb{W}}$-concentrated on a measurable set $\mathbb{W}$, then

$$|\mathbb{W}|.|\mathbb{T}| \geq (1 - \epsilon_{\mathbb{T}} - \epsilon_{\mathbb{W}})^2. \tag{2.1}$$

Here $|\mathbb{T}|$ is the Lebesque measure of the set $\mathbb{T}$. This inequality has been slightly improved in [7] to

$$|\mathbb{W}|.|\mathbb{T}| \geq (1 - (\epsilon_{\mathbb{T}}^2 + \epsilon_{\mathbb{W}}^2)^{1/2})^2. \tag{2.2}$$

In this section we will extend the Donoho-Stark uncertainty principle to the G-transform (1.2) in the space $L_2(\mathbb{R}_+)$. Concerning other uncertainty principles for integral transforms, see also in [6, 12, 2, 16].

Let $P_{\mathbb{E}}$ denote the time-limiting operator

$$(P_{\mathbb{E}} f)(x) = \begin{cases} f(x), & x \in \mathbb{E} \\ 0, & x \notin \mathbb{E} \end{cases}.$$

This operator cuts off the part of f outside $\mathbb{E}$. So f is ϵ-concentrated on a set $\mathbb{E}$ if, and only if

$$\|f - P_{\mathbb{E}} f\|_{L_2(\mathbb{R}_+)} \leq \epsilon.$$

Clearly $\|P_{\mathbb{E}}\| = 1$. The second operator is the frequency-limiting operator

$$(Q_{\mathbb{E}} f)(x) := \int_{\mathbb{E}} G^{m,n}_{2n,2m}\left(xy \,\middle|\, \begin{matrix} (\overline{a}_n), -(a_n) \\ (\overline{b}_m), -(b_m) \end{matrix}\right) (Gf)(y)\, dy.$$

$Q_{\mathbb{E}}f$ is a partial reconstruction of f using only Gf on $\mathbb{E}$. If $h = Q_{\mathbb{E}}f$, then $\overline{G}(h)$ vanishes outside $\mathbb{E}$, and $g = Gf$ is ϵ-concentrated on $\mathbb{E}$ if, and only if,

$$\|f - Q_{\mathbb{E}}f\|_{L_2(\mathbb{R}_+)} \leq \epsilon.$$

Using the Parseval formula $\|f\|_{L_2(\mathbb{R}_+)} = \|Gf\|_{L_2(\mathbb{R}_+)}$ one can show that $\|Q_{\mathbb{E}}\| = 1$. We have

$$\begin{aligned} (P_{\mathbb{X}}Q_{\mathbb{Y}}f)(x) &= P_{\mathbb{X}} \int_{\mathbb{Y}} G^{m,n}_{2n,2m}\left(xy \,\middle|\, \begin{matrix} (\overline{a}_n), -(a_n) \\ (\overline{b}_m), -(b_m) \end{matrix}\right) \int_0^\infty G^{m,n}_{2n,2m}\left(yt \,\middle|\, \begin{matrix} (a_n), -(\overline{a}_n) \\ (b_m), -(\overline{b}_m) \end{matrix}\right) f(t)\, dt dy \\ &= P_{\mathbb{X}} \int_0^\infty f(t) \int_{\mathbb{Y}} G^{m,n}_{2n,2m}\left(xy \,\middle|\, \begin{matrix} (\overline{a}_n), -(a_n) \\ (\overline{b}_m), -(b_m) \end{matrix}\right) G^{m,n}_{2n,2m}\left(yt \,\middle|\, \begin{matrix} (a_n), -(\overline{a}_n) \\ (b_m), -(\overline{b}_m) \end{matrix}\right) dy\, dt \\ &= \int_0^\infty q(x,t) f(t)\, dt, \end{aligned}$$

where

$$q(x,t) := \begin{cases} \int_{\mathbb{Y}} G^{m,n}_{2n,2m}\left(xy \,\middle|\, \begin{matrix} (\overline{a}_n), -(a_n) \\ (\overline{b}_m), -(b_m) \end{matrix}\right) G^{m,n}_{2n,2m}\left(yt \,\middle|\, \begin{matrix} (a_n), -(\overline{a}_n) \\ (b_m), -(\overline{b}_m) \end{matrix}\right) dy &, \quad x \in \mathbb{X} \\ 0 &, \quad x \notin \mathbb{X} \end{cases}.$$

The Hilbert-Schmidt norm of $P_{\mathbb{X}}Q_{\mathbb{Y}}$ is $\left(\int_0^\infty \int_0^\infty |q(x,t)|^2 dt dx\right)^{1/2}$. The norm $\|P_{\mathbb{X}}Q_{\mathbb{Y}}\|$ does not exceed the Hilbert-Schmidt norm of $P_{\mathbb{X}}Q_{\mathbb{Y}}$, therefore,

$$\begin{aligned} \|P_{\mathbb{X}}Q_{\mathbb{Y}}\|^2 &\leq \int_0^\infty \int_0^\infty |q(x,t)|^2\, dt dx \\ &= \int_{\mathbb{X}} \int_0^\infty |q(x,t)|^2\, dx dt. \end{aligned}$$

Notice that, for a fixed x,

$$q(x,t) = G\left(G^{m,n}_{2n,2m}\left(xy \,\middle|\, \begin{matrix} (\overline{a}_n), -(a_n) \\ (\overline{b}_m), -(b_m) \end{matrix}\right) \chi_{\mathbb{Y}} \right),$$

is the G-transform (1.2) of $G^{m,n}_{2n,2m}\left(xy \,\middle|\, \begin{matrix} (\overline{a}_n), -(a_n) \\ (\overline{b}_m), -(b_m) \end{matrix}\right) \chi_{\mathbb{Y}}$ with respect to y, where $\chi_{\mathbb{Y}}$ is the characteristic function of the set $\mathbb{Y}$. Thus the Parseval identity (1.3) for the G-transform yields

$$\int_0^\infty |q(x,t)|^2 dt = \int_{\mathbb{Y}} \left| G^{m,n}_{2n,2m}\left(xy \,\middle|\, \begin{matrix} (\overline{a}_n), -(a_n) \\ (\overline{b}_m), -(b_m) \end{matrix}\right) \right|^2 dy.$$

Consequently,

$$\|P_{\mathbb{X}}Q_{\mathbb{Y}}\|^2 \leq \int_{\mathbb{X}} \int_{\mathbb{Y}} \left| G^{m,n}_{2n,2m}\left(xy \,\middle|\, \begin{matrix} (\overline{a}_n), -(a_n) \\ (\overline{b}_m), -(b_m) \end{matrix}\right) \right|^2 dy dx.$$

We get now an upper bound for $G^{m,n}_{2n,2m}\left(x \,\middle|\, \begin{matrix} (a_n), -(\overline{a}_n) \\ (b_m), -(\overline{b}_m) \end{matrix}\right)$, under some extra assumptions

$$m - n \geq 2, \quad \Re(b_j) > 0,\ j = 1, \cdots, m; \quad \Re(a_j) < \frac{1}{2},\ j = 1, \cdots, n.$$

We have

$$G^{m,n}_{2n,2m}\left(x \,\middle|\, \begin{matrix}(a_n),-(\overline{a}_n)\\(b_m),-(\overline{b}_m)\end{matrix}\right)=\frac{1}{2\pi i}\int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{\prod\limits_{k=1}^{m}\Gamma\left(b_k+s\right)\prod\limits_{k=1}^{n}\Gamma\left(1-a_k-s\right)}{\prod\limits_{k=1}^{n}\Gamma\left(-\overline{a}_k+s\right)\prod\limits_{k=1}^{m}\Gamma\left(1+\overline{b}_k-s\right)}x^{-s}ds.$$

Under the assumptions

$$\Re(b_j)>0,\ j=1,\cdots,m;\quad \Re(a_j)<\frac{1}{2},\ j=1,\cdots,n, \tag{2.3}$$

the function

$$\frac{\prod\limits_{k=1}^{m}\Gamma\left(b_k+s\right)\prod\limits_{k=1}^{n}\Gamma\left(1-a_k-s\right)}{\prod\limits_{k=1}^{n}\Gamma\left(-\overline{a}_k+s\right)\prod\limits_{k=1}^{m}\Gamma\left(1+\overline{b}_k-s\right)}x^{-s}$$

has no pole in the strip $0\le\Re(s)\le\frac{1}{2}$. Hence, the contour $\sigma=\left(\frac{1}{2}-i\infty,\frac{1}{2}+i\infty\right)$ in the definition of G-function can be replaced by the vertical line $(-i\infty,i\infty)$. Consequently,

$$G^{m,n}_{2n,2m}\left(x \,\middle|\, \begin{matrix}(a_n),-(\overline{a}_n)\\(b_m),-(\overline{b}_m)\end{matrix}\right)=\frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty}\frac{\prod\limits_{k=1}^{m}\Gamma\left(b_k+s\right)\prod\limits_{k=1}^{n}\Gamma\left(1-a_k-s\right)}{\prod\limits_{k=1}^{n}\Gamma\left(-\overline{a}_k+s\right)\prod\limits_{k=1}^{m}\Gamma\left(1+\overline{b}_k-s\right)}x^{-s}ds. \tag{2.4}$$

Let's estimate the integral. By replacing $s=it,\ t\in\mathbb{R}$, we have

$$\left|\frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty}\frac{\prod\limits_{k=1}^{m}\Gamma\left(b_k+s\right)\prod\limits_{k=1}^{n}\Gamma\left(1-a_k-s\right)}{\prod\limits_{k=1}^{n}\Gamma\left(-\overline{a}_k+s\right)\prod\limits_{k=1}^{m}\Gamma\left(1+\overline{b}_k-s\right)}x^{-s}ds\right|$$

$$\le\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\prod\limits_{k=1}^{m}\left|\Gamma\left(b_k+it\right)\right|\prod\limits_{k=1}^{n}\left|\Gamma\left(1-a_k-it\right)\right|}{\prod\limits_{k=1}^{n}\left|\Gamma\left(-\overline{a}_k+it\right)\right|\prod\limits_{k=1}^{m}\left|\Gamma\left(1+\overline{b}_k-it\right)\right|}\left|x^{-it}\right|dt$$

$$=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\prod\limits_{k=1}^{n}\left|(-a_k-it)\right|\prod\limits_{k=1}^{m}\left|\Gamma\left(b_k+it\right)\right|\prod\limits_{k=1}^{n}\left|\Gamma\left(-a_k-it\right)\right|}{\prod\limits_{k=1}^{m}\left|\left(\overline{b}_k-it\right)\right|\prod\limits_{k=1}^{n}\left|\Gamma\left(-\overline{a}_k+it\right)\right|\prod\limits_{k=1}^{m}\left|\Gamma\left(\overline{b}_k-it\right)\right|}dt$$

$$=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\prod\limits_{k=1}^{n}\left|(a_k+it)\right|}{\prod\limits_{k=1}^{m}\left|\left(\overline{b}_k-it\right)\right|}dt=:C((a_n),(b_m)).$$

Consequently,

$$\left|G^{m,n}_{2n,2m}\left(x \,\middle|\, \begin{matrix}(a_n),-(\overline{a}_n)\\(b_m),-(\overline{b}_m)\end{matrix}\right)\right|\le C((a_n),(b_m)).$$

Hence, if $\Re(b_j) > 0$ for $j = 1, \cdots, m$, then $\left| G^{m,n}_{2n,2m}\left(x \,\middle|\, \begin{array}{c}(a_n), -(\overline{a}_n)\\ (b_m), -(\overline{b}_m)\end{array}\right)\right|$ is uniformly bounded by constant $C((a_n),(b_m))$. The constant $C((a_n),(b_m))$ can also be estimated. For example, if $n = 0$,

$$C(.,(b_m)) \leq \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{|(b+it)^m|}dt = (2b)^{1-m}\frac{(m-2)!}{\Gamma^2\left(\frac{m}{2}\right)},$$

where $b = \min\{\Re(b_1), \cdots, \Re(b_m)\}$.
Thus we get

$$\|P_{\mathbb{X}}Q_{\mathbb{Y}}\|^2 \leq \int_{\mathbb{X}}\int_{\mathbb{Y}}\left| G^{m,n}_{2n,2m}\left(xy \,\middle|\, \begin{array}{c}(\overline{a}_n), -(a_n)\\ (\overline{b}_m), -(b_m)\end{array}\right)\right|^2 dydx \leq |\mathbb{X}|.|\mathbb{Y}|\, C^2((a_n),(b_m)).$$

Following [8, 5] the Donoho-Stark uncertainty principle for the G-transform now can be stated as follows

Theorem 1 *(Donoho-Stark-de Jeu) Let*

$$m - n \geq 2, \quad \Re(b_j) > 0,\ j = 1, \cdots, m; \quad \Re(a_j) < \frac{1}{2},\ j = 1, \cdots, n.$$

Then if f of unit norm is $\epsilon_{\mathbb{X}}$-concentrated on a measurable set $\mathbb{X}$, and Gf is $\epsilon_{\mathbb{Y}}$-concentrated on a measurable set $\mathbb{Y}$, then

$$|\mathbb{X}|.|\mathbb{Y}|\, C^2((a_n),(b_m)) \geq \left(1 - (\epsilon^2_{\mathbb{X}} + \epsilon^2_{\mathbb{Y}})^{1/2}\right)^2.$$

Proof. The proof is mimicked from [7]. Let $|\mathbb{X}|.|\mathbb{Y}| < C^{-2}((a_n),(b_m))$, otherwise it is trivial. Then $\|P_{\mathbb{X}}Q_{\mathbb{Y}}\| < 1$, and therefore, $I - P_{\mathbb{X}}Q_{\mathbb{Y}}$ is invertible with

$$\begin{aligned}\|(I - P_{\mathbb{X}}Q_{\mathbb{Y}})^{-1}\| &\leq \sum_{k=0}^{\infty}\|P_{\mathbb{X}}Q_{\mathbb{Y}}\|^k \\ &\leq \sum_{k=0}^{\infty}\left[|\mathbb{X}|.|\mathbb{Y}|\, C^2((a_n),(b_m))\right]^{k/2} \\ &= \frac{1}{1 - |\mathbb{X}|^{1/2}.|\mathbb{Y}|^{1/2}\, C((a_n),(b_m))}.\end{aligned}$$

We have

$$I = P_{\mathbb{X}} + P_{\mathbb{X}^c} = P_{\mathbb{X}}Q_{\mathbb{Y}} + P_{\mathbb{X}}Q_{\mathbb{Y}^c} + P_{\mathbb{X}^c},$$

where $\mathbb{X}^c = \mathbb{R}_+\backslash\mathbb{X}$, the complement of $\mathbb{X}$. The orthogonality of $P_{\mathbb{X}}$ and $P_{\mathbb{X}^c}$ gives

$$\|P_{\mathbb{X}}Q_{\mathbb{Y}^c}f\|^2_{L_2(\mathbb{R}_+)} + \|P_{\mathbb{X}^c}f\|^2_{L_2(\mathbb{R}_+)} = \|P_{\mathbb{X}}Q_{\mathbb{Y}^c}f + P_{\mathbb{X}^c}f\|^2_{L_2(\mathbb{R}_+)}.$$

Together with $\|P_{\mathbb{X}}\| = 1$ it yields

$$\begin{aligned}\|f\|^2_{L_2(\mathbb{R}_+)} &\leq \|(I - P_{\mathbb{X}}Q_{\mathbb{Y}})^{-1}\|^2\|(I - P_{\mathbb{X}}Q_{\mathbb{Y}})f\|^2_{L_2(\mathbb{R}_+)} \\ &\leq \left(\frac{1}{1 - |\mathbb{X}|^{1/2}.|\mathbb{Y}|^{1/2}\, C((a_n),(b_m))}\right)^2\left[\|P_{\mathbb{X}}Q_{\mathbb{Y}^c}f\|^2_{L_2(\mathbb{R}_+)} + \|P_{\mathbb{X}^c}f\|^2_{L_2(\mathbb{R}_+)}\right] \\ &\leq \left(\frac{1}{1 - |\mathbb{X}|^{1/2}.|\mathbb{Y}|^{1/2}\, C((a_n),(b_m))}\right)^2\left[\|Q_{\mathbb{Y}^c}f\|^2_{L_2(\mathbb{R}_+)} + \|P_{\mathbb{X}^c}f\|^2_{L_2(\mathbb{R}_+)}\right].\end{aligned}$$

If f of unit norm is $\epsilon_{\mathbb{X}}$-concentrated on $\mathbb{X}$, then $\|P_{\mathbb{X}^c} f\|_{L_2(\mathbb{R}_+)} \leq \epsilon_{\mathbb{X}}$. If Gf of unit norm is $\epsilon_{\mathbb{Y}}$-concentrated on $\mathbb{Y}$, then $\|Q_{\mathbb{Y}^c} f\|_{L_2(\mathbb{R}_+)} \leq \epsilon_{\mathbb{Y}}$. Then if f of unit norm is $\epsilon_{\mathbb{X}}$-concentrated on $\mathbb{X}$, and Gf is $\epsilon_{\mathbb{Y}}$-concentrated on $\mathbb{Y}$, we have

$$1 \quad \leq \quad \left(\frac{1}{1 - |\mathbb{X}|^{1/2}.|\mathbb{Y}|^{1/2}\, C((a_n), (b_m))} \right)^2 \left(\epsilon_{\mathbb{Y}}^2 + \epsilon_{\mathbb{Y}}^2\right),$$

or

$$|\mathbb{X}|.|\mathbb{Y}| \geq C^{-2}((a_n), (b_m)) \left(1 - (\epsilon_{\mathbb{X}}^2 + \epsilon_{\mathbb{Y}}^2)^{1/2}\right)^2.$$

■

Let f be an $L_2(\mathbb{R}_+)$ function, that is bandlimited to the set $\mathbb{Y}$ (i.e. Gf vanishes off $\mathbb{Y}$). Suppose that f is corrupted by both noise $n \in L_2(\mathbb{R}_+)$ and unregistered values on $\mathbb{X}$. Thus the observable function r satisfies

$$r(x) = \begin{cases} f(x) + n(x), & x \notin \mathbb{X} \\ 0, & x \in \mathbb{X} \end{cases}.$$

Here we have assumed without loss of generality that $n = 0$ on $\mathbb{X}$. Equivalently,

$$r = (I - P_{\mathbb{X}})f + n.$$

We say that f can be stably reconstructed from r, if there exists a linear operator K and a constant C such that

$$\|f - Kr\|_{L_2(\mathbb{R}_+)} \leq C\|n\|_{L_2(\mathbb{R}_+)}. \tag{2.5}$$

Following [5] we obtain

Corollary 1 *If*

$$|\mathbb{X}|.|\mathbb{Y}| < C^{-2}((a_n), (b_m)),$$

then f can be stably reconstructed from r.

Proof. Although the proof is mimicked from [5] we present it here as some ingredients in the proof are needed for discussion later. Let $K = (I - P_{\mathbb{X}} Q_{\mathbb{Y}})^{-1}$. From $|\mathbb{X}|.|\mathbb{Y}| < C^{-2}((a_n), (b_m))$ it follows that $\|P_{\mathbb{X}} Q_{\mathbb{Y}}\| < 1$, hence the existence of K. Since $(I - P_{\mathbb{X}})f = (I - P_{\mathbb{X}} Q_{\mathbb{Y}})f$ for every f, bandlimited to $\mathbb{Y}$,

$$\begin{aligned} f - Kr &= f - K((I - P_{\mathbb{X}})f + n) \\ &= f - K(I - P_{\mathbb{X}} Q_{\mathbb{Y}})f - Kn \\ &= f - (I - P_{\mathbb{X}} Q_{\mathbb{Y}})^{-1}(I - P_{\mathbb{X}} Q_{\mathbb{Y}})f - Kn \\ &= 0 - Kn. \end{aligned}$$

So

$$\begin{aligned} \|f - Kr\|_{L_2(\mathbb{R}_+)} &= \|Kn\|_{L_2(\mathbb{R}_+)} \\ &\leq \|(I - P_{\mathbb{X}} Q_{\mathbb{Y}})^{-1}\| \, \|n\|_{L_2(\mathbb{R}_+)} \\ &\leq (1 - \|P_{\mathbb{X}} Q_{\mathbb{Y}}\|)^{-1}\| \, \|n\|_{L_2(\mathbb{R}_+)} \\ &\leq \frac{1}{1 - |\mathbb{X}|^{1/2}.|\mathbb{Y}|^{1/2}\, C((a_n), (b_m))} \|n\|_{L_2(\mathbb{R}_+)}. \end{aligned}$$

The constant C in (2.5) is therefore not larger than $\frac{1}{1-|\mathbb{X}|^{1/2}.|\mathbb{Y}|^{1/2}\,C((a_n),(b_m))}$. ■

The identity $K=\sum_{k=0}^{\infty}(P_{\mathbb{X}}Q_{\mathbb{Y}})^k$ suggests the following algorithm for computing Kr:

$$\begin{aligned} f^{(0)} &= r \\ f^{(k+1)} &= r+P_{\mathbb{X}}Q_{\mathbb{Y}}f^{(k)}. \end{aligned}$$

Then $f^{(k)}=\sum_{j=0}^{k}(P_{\mathbb{X}}Q_{\mathbb{Y}})^j r$ converges to Kr as $k\to\infty$. The iterate $f^{(k)}$ is the result of bandlimiting to $\mathbb{Y}$ and then timelimiting to $\mathbb{X}$ the previous $f^{(k-1)}$, then adding the result back to the original data r. On $\mathbb{X}^c$ where the data are given, $f^{(k)}=r$ at each iteration k, while on the unobserved set $\mathbb{X}$ the missing values are filled in by a gradual adjustment, iteration after iteration.

The main ingredient of stability recovery of f by Donoho-Stark operator K is that the norm of $\|P_{\mathbb{X}}Q_{\mathbb{Y}}\|$ is strictly less than 1, that has been proved only when $|\mathbb{X}|.|\mathbb{Y}|$ is small enough. Now we will show that $\|P_{\mathbb{X}}Q_{\mathbb{Y}}\|$ is **always** less than 1, if X and Y are bounded sets. From the proof one can see that it is still true, if one of the set is bounded, and the complement of the other set has nonzero measure. Moreover, we do not require that the kernel $k(x):=G^{m,n}_{2n,2m}\left(x\,\middle|\,\begin{matrix}(\overline{a}_n),-(a_n)\\(\overline{b}_m),-(b_m)\end{matrix}\right)$ is bounded.

Now we will consider the general case when the kernel $k(x)=G^{m,n}_{2n,2m}\left(x\,\middle|\,\begin{matrix}(\overline{a}_n),-(a_n)\\(\overline{b}_m),-(b_m)\end{matrix}\right)$ maybe unbounded. We require that (a_n), (b_m) are real vectors, so the G-transform (1.2) has symmetric inverse transform (1.7).
We have the following

Lemma 1 *Let $\mathbb{X}$ be a bounded set, and $|\mathbb{Y}^c|>0$. Then*

$$\|Q_{\mathbb{Y}}P_{\mathbb{X}}\|<1.$$

Proof. First we notice that the kernel $k(x)$ is analytic [9], therefore, if f has a compact support $\mathbb{X}$, then its G- transform Gf is an analytic function.
We have $Q_{\mathbb{Y}}=GP_{\mathbb{Y}}G$, therefore

$$\begin{aligned} (Q_{\mathbb{Y}}P_{\mathbb{X}}f)(t)=(GP_{\mathbb{Y}}GP_{\mathbb{X}}f)(t) &= \int_{\mathbb{Y}}k(ty)\int_{\mathbb{X}}k(yx)\,f(x)\,dx\,dy \\ &= \int_{\mathbb{X}}f(x)\int_{\mathbb{Y}}k(ty)\,k(yx)\,dy\,dx=\int_{\mathbb{X}}h(t,x)\,f(x)\,dx, \end{aligned}$$

where

$$h(t,x)=\int_{\mathbb{Y}}k(ty)\,k(yx)\,dy$$

is a symmetric function $h(t,x)=h(x,t)$. Consider the symmetric integral operator $M:L_2(\mathbb{X})\to L_2(\mathbb{X})$

$$g(t)=(Mf)(t)=\int_{\mathbb{X}}h(t,x)\,f(x)\,dx,\quad t\in\mathbb{X},$$

the restriction of $Q_{\mathbb{Y}}P_{\mathbb{X}}$ on $\mathbb{X}$. Let $m(t)=(GP_{\mathbb{Y}}GP_{\mathbb{X}}f)(t)$, $0<t<\infty$. Then on $\mathbb{X}$ we have $g(t)=m(t)$. Hence,

$$\|g\|_{L_2(\mathbb{X})}\le\|m\|_{L_2(\mathbb{R}_+)}=\|GP_{\mathbb{Y}}GP_{\mathbb{X}}f\|_{L_2(\mathbb{R}_+)}=\|P_{\mathbb{Y}}GP_{\mathbb{X}}f\|_{L_2(\mathbb{R}_+)}$$

$$\leq \|GP_{\mathbb{X}}f\|_{L_2(\mathbb{R}_+)} = \|P_{\mathbb{X}}f\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{X})}.$$

Hence M is a bounded operator in $L_2(\mathbb{X})$ with the norm $\|M\| \leq 1$. We will show that $\|M\| < 1$. Since M is an integral operator with symmetric kernel, it has only real eigenvalues. Let λ be the largest by absolute value eigenvalue of M. Then $|\lambda| = \|M\| \leq 1$. We will prove now that $|\lambda| < 1$. Assume the contrary $|\lambda| = 1$. Then $\lambda = 1$ or $\lambda = -1$.
Consider the first case $\lambda = 1$. Let l be the eigenfunction associated with the eigenvalue $\lambda = 1$.

$$l(t) = (Ml)(t) = \int_{\mathbb{X}} h(t,x)\, l(x)\, dx.$$

Denote

$$n(t) = (GP_{\mathbb{Y}}GP_{\mathbb{X}}l)(t),\ 0 < t < \infty.$$

Then $n(t) = l(t)$ on $\mathbb{X}$. We consider two cases:
a) $n \neq P_{\mathbb{X}}n$. Then $\|P_{\mathbb{X}}n\|_{L_2(\mathbb{R}_+)} < \|n\|_{L_2(\mathbb{R}_+)}$. On the other hand,

$$\|n\|_{L_2(\mathbb{R}_+)} = \|GP_{\mathbb{Y}}GP_{\mathbb{X}}n\|_{L_2(\mathbb{R}_+)} = \|P_{\mathbb{Y}}GP_{\mathbb{X}}n\|_{L_2(\mathbb{R}_+)}$$

$$\leq \|GP_{\mathbb{X}}n\|_{L_2(\mathbb{R}_+)} = \|P_{\mathbb{X}}n\|_{L_2(\mathbb{R}_+)} < \|n\|_{L_2(\mathbb{R}_+)}.$$

Contradiction.
b) $n = P_{\mathbb{X}}n$. Then n has a compact support. But the G transform of n is $P_{\mathbb{Y}}GP_{\mathbb{X}}l$, that is zero on a set of positive measure $\mathbb{Y}^c$, that is impossible since Gn is analytic and nonzero.
Similarly, one can show that $\lambda \neq -1$. Hence $\|M\| < 1$.
Now let λ be any eigenvalue of M, and l be the associated eigenfunction

$$Ml = \lambda l.$$

The left hand side is an integral on $\mathbb{X}$, but it defines a function on the whole $(0,\infty)$. Denote that function on $(0,\infty)$ again by λl, then

$$Q_{\mathbb{Y}}P_{\mathbb{X}}l = \lambda l,$$

that means λ and l are the eigenvalue and the associated eigenfunction of $Q_{\mathbb{Y}}P_{\mathbb{X}}$. Similarly, if λ and l are an eigenvalue and the associated eigenfunction of $Q_{\mathbb{Y}}P_{\mathbb{X}}$, then λ and $P_{\mathbb{X}}l$ are the eigenvalue and eigenfunction of M. Hence, the largest by absolute value eigenvalue of $Q_{\mathbb{Y}}P_{\mathbb{X}}$ is less than 1, and actually, $\|Q_{\mathbb{Y}}P_{\mathbb{X}}\| = \|M\| < 1$. ■

Since $P_{\mathbb{X}}Q_{\mathbb{Y}}$ is the transpose of $Q_{\mathbb{Y}}P_{\mathbb{X}}$, then also

$$\|P_{\mathbb{X}}Q_{\mathbb{Y}}\| < 1.$$

Hence from the proof of Corollary 1 and from Lemma 1 we obtain

Theorem 2 *Let (a_n), (b_m) be real vectors such that $b_j > -\frac{1}{2}$, $\quad j = 1, \cdots, m$, and $a_j < \frac{1}{2}$, $j = 1, \cdots, n$. Let $\mathbb{X}$ be a bounded set, and $|\mathbb{Y}^c| > 0$. If f is an $L_2(\mathbb{R}_+)$ function, that is bandlimited to the set $\mathbb{Y}$ (i.e. Gf vanishes off $\mathbb{Y}$), and r is the observation of f, corrupted by both noise $n \in L_2(\mathbb{R}_+)$ and unregistered values on $\mathbb{X}$, then $(I - P_{\mathbb{X}}Q_{\mathbb{Y}})^{-1}r$ stably recovers f.*

3 Paley-Wiener Theorem

Let $f \in L_2(R)$ and $\hat{f}$ be its Fourier transform. The Paley-Wiener theorem [10] asserts that $\hat{f} \in L_2(R)$ has compact support if and only if $f \in L_2(R)$ is analytically extendable into the complex plane as an entire function of exponential type. In [3] Bang proved another version of the Paley-Wiener theorem, namely

Theorem A. *$\hat{f}$ has compact support if and only if f is infinitely differentiable, $f^{(n)} \in L_2(R)$ for any n, and*

$$\lim_{n\to\infty} \|f^{(n)}\|_{L_2(R)}^{1/n} < \infty.$$

Theorem A can be considered as a real-valued version of the Paley-Wiener theorem since no complexification of f is involved. From Theorem A and the Paley-Wiener theorem one can describe the class of square integrable functions on R that are analytically extendable into entire functions of exponential type in the complex plane.
Bang based his proof on the Bernstein-Kolmogorov inequality. In [14] one finds a different proof of Theorem A and its generalization for the multidimensional Fourier transform, and other integral transforms. It is based on the following observation [15]

$$\lim_{n\to\infty} \|\lambda^n f(\lambda)\|^{1/n} = \sup_{\lambda\in \operatorname{supp} f} |\lambda|. \tag{3.1}$$

In signal analysis, f is called a signal, and $\hat{f}$ is the signal frequency content. In general, a linear integral transform is called a linear system, and the output of a signal through a system is called a frequency content. The support of the frequency content is the spectrum of the signal. A signal is band-limited, if its spectrum is bounded. Thus, within this contest Theorem A and the Paley-Wiener theorem described band-limited signals.

Let l be a positive integer. An l-th order differential operator is an operator of the form

$$Ly := p_0(x)\frac{d^l y}{dx^l} + p_1(x)\frac{d^{l-1}y}{dx^{l-1}} + \cdots p_l(x)y, \quad -\infty \le a < x < b \le \infty, \tag{3.2}$$

where the p_k are complex-valued functions with $n-k$ continuous derivatives on the open interval (a, b), with $p_0(t) \neq 0$ for any $a < t < b$.
Associated with the operator L is another operator, called the adjoint operator of L, which is denoted by L^* and given by

$$L^*y := (-1)^l \frac{d^l}{dx^l}(\bar{p}_0(x)y) + (-1)^{l-1}\frac{d^{l-1}}{dx^{l-1}}(\bar{p}_1(x)y) + \cdots \bar{p}_l(x)y, \quad -\infty \le a < x < b \le \infty.$$

Let L be formally self-adjoint, that is, L coincides with L^*. The operator L is singular if the interval is infinite or the coefficients in L have sufficiently singular behavior at one or both ends of the interval [4].

The paper [1] generalizes Theorem A to characterize band-limited signal passing through linear systems arising from some singular self-adjoint differential operators of the form (3.2) of any order l with the limitation that the equation $(L-\lambda)f = 0$ has no nontrivial solution in $L_2(a, b)$ for $\operatorname{Im}\lambda \neq 0$. If $l = 2$, $p_0(x) = -1$, and $p_2(x) = q(x)$ is real-valued, the operator L becomes the Sturm-Liouville operator

$$Ly = q(x)y - y'', \quad x \in (a, b),$$

and the condition is equivalent to the operator L being in the limit-point case at both ends of the interval (a,b).

In this section we will generalize Theorem A to a special case of the G-transform. Let $n=0$. The corresponding G-transform (1.2) takes the form

$$g(x)=\int_0^{\infty} G^{m,0}_{0,2m}\left(xy\,\middle|\,\begin{matrix} - \quad , \quad - \\ (b_m),-(\bar{b}_m)\end{matrix}\right) f(y)\,dy. \tag{3.3}$$

The kernel $G^{m,0}_{0,2m}\left(xy\,\middle|\,\begin{matrix} - \quad , \quad - \\ (b_m),-(\bar{b}_m)\end{matrix}\right)$ satisfies the following differential equation [9]

$$\frac{1}{x}\prod_{j=1}^{m}\left(b_j-x\frac{d}{dx}\right)\left(\bar{b}_j+x\frac{d}{dx}\right)G^{m,0}_{0,2m}\left(xy\,\middle|\,\begin{matrix} - \quad , \quad - \\ (b_m),-(\bar{b}_m)\end{matrix}\right)$$

$$=y\,G^{m,0}_{0,2m}\left(xy\,\middle|\,\begin{matrix} - \quad , \quad - \\ (b_m),-(\bar{b}_m)\end{matrix}\right). \tag{3.4}$$

The operator $\frac{1}{x}\prod_{j=1}^{m}\left(b_j-x\frac{d}{dx}\right)\left(\bar{b}_j+x\frac{d}{dx}\right)$ is in general, non-self-adjoint, and the G-transform (3.3), generally speaking, does not arise from eigenfunctions expansion of a self-adjoint operator.

Theorem 3 *f is infinitely differentiable and has a compact support if, and only if, its G-transform* (3.3) *g satisfies*
a) g is infinitely differentiable.
b) $\left(x\frac{d}{dx}\right)^k g(x)\in L_2(\mathbb{R}_+)$ for any $k=0,1,\cdots$
c) $\left[\frac{1}{x}\prod_{j=1}^{m}\left(b_j-x\frac{d}{dx}\right)\left(\bar{b}_j+x\frac{d}{dx}\right)\right]^k g(x)\in L_2(\mathbb{R}_+)$ for any $k=1,2,\cdots$
d) The following limit exists

$$\lim_{k\to\infty}\left\|\left[\frac{1}{x}\prod_{j=1}^{m}\left(b_j-x\frac{d}{dx}\right)\left(\bar{b}_j+x\frac{d}{dx}\right)\right]^k g(x)\right\|^{\frac{1}{k}}_{L_2(\mathbb{R}_+)}<\infty.$$

Moreover, the limit equals to the supremum of the support of f.

Proof. We start the proof by recalling some well-known facts for the Mellin transform (1.4) [9, 13]
1) $f(x), xf'(x)\in L_2(\mathbb{R}_+)$ if, and only if, $f^*(s), sf^*(s)\in L_2(\sigma)$. Moreover, $\left(xf'(x)\right)^*(s)=-sf^*(s)$.
2) $x^\alpha f(x), x^\beta f(x)\in L_2(\mathbb{R}_+)$ if, and only if $f^*(s)$ is analytic in the strip $\alpha+\frac{1}{2}<\Re(s)<\beta+\frac{1}{2}$, and $f^*(s)\in L_2(\gamma-i\infty,\gamma+i\infty)$ for $\gamma=\alpha+\frac{1}{2}$ and $\gamma=\beta+\frac{1}{2}$.
Let f be infinitely differentiable and have compact support. Then g is infinitely differentiable, and $y^k f(y)\in L_2(\mathbb{R}_+)$ for any $k=1,2,\cdots$. Thus, we have via (3.4)

$$\left[\frac{1}{x}\prod_{j=1}^{m}\left(b_j-x\frac{d}{dx}\right)\left(\bar{b}_j+x\frac{d}{dx}\right)\right]^k g(x)=\int_0^{\infty} G^{m,0}_{0,2m}\left(xy\,\middle|\,\begin{matrix} - \quad , \quad - \\ (b_m),-(\bar{b}_m)\end{matrix}\right) y^k f(y)\,dy,$$

where the integral on the right-hand side is in fact over a finite interval. Hence,

$$\left[\frac{1}{x}\prod_{j=1}^{m}\left(b_j - x\frac{d}{dx}\right)\left(\bar{b}_j + x\frac{d}{dx}\right)\right]^k g(x),$$

as the G-transform (3.3) of $y^k f(y)$, belongs to $L_2(\mathbb{R}_+)$. Consequently, Parseval's formula for the G-transform (1.3) and formula (3.1) yield

$$\lim_{k\to\infty}\left\|\left[\frac{1}{x}\prod_{j=1}^{m}\left(b_j - x\frac{d}{dx}\right)\left(\bar{b}_j + x\frac{d}{dx}\right)\right]^k g(x)\right\|_{L_2(\mathbb{R}_+)}^{\frac{1}{k}} = \lim_{k\to\infty}\left\|y^k f(y)\right\|_{L_2(\mathbb{R}_+)}^{\frac{1}{k}} = \sup_{y\in \operatorname{supp} f} y < \infty.$$

On the other hand, the Parseval formula for the Mellin convolution [9, 13] (see (1.5), (1.6)) yields

$$g(x) = \frac{1}{2\pi i}\int_\sigma \frac{\prod_{k=1}^{m}\Gamma\left(b_k + s\right)}{\prod_{k=1}^{m}\Gamma\left(1+\bar{b}_k - s\right)} f^*(1-s)\, x^{-s}\, ds.$$

Since $\left(x\frac{d}{dx}\right)^k f(x) \in L_2(\mathbb{R}_+)$ for any k, then $s^k f^*(s) \in L_2(\sigma)$ for any k. As

$$\left|\frac{\prod_{k=1}^{m}\Gamma\left(b_k + s\right)}{\prod_{k=1}^{m}\Gamma\left(1+\bar{b}_k - s\right)}\right| = 1 \qquad s\in\sigma,$$

then

$$s^k\frac{\prod_{k=1}^{m}\Gamma\left(b_k + s\right)}{\prod_{k=1}^{m}\Gamma\left(1+\bar{b}_k - s\right)} f^*(1-s) \in L_2(\sigma)$$

for any k. In other words, $s^k g^*(s) \in L_2(\sigma)$ for any k. Hence $\left(x\frac{d}{dx}\right)^k g(x) \in L_2(\mathbb{R}_+)$ for any $k = 0, 1, \cdots$

Now let g satisfy all conditions a)-d). Since $\left(x\frac{d}{dx}\right)^k g(x) \in L_2(\mathbb{R}_+)$ for any k, then $s^k g^*(s) \in L_2(\sigma)$ for any k. As

$$\left|\frac{\prod_{k=1}^{m}\Gamma\left(b_k + s\right)}{\prod_{k=1}^{m}\Gamma\left(1+\bar{b}_k - s\right)}\right| = 1 \qquad s\in\sigma,$$

then if we define

$$f^*(s) = \frac{\prod_{k=1}^{m}\Gamma\left(\bar{b}_k + s\right)}{\prod_{k=1}^{m}\Gamma\left(1+b_k - s\right)} g^*(1-s),$$

then $s^k f^*(s) \in L_2(\sigma)$ for any k. Hence, $\left(x\frac{d}{dx}\right)^k f(x) \in L_2(\mathbb{R}_+)$ for any k. In particular, f is infinitely differentiable.
From the definition of $f^*(s)$ it follows

$$g^*(s) = \frac{\prod\limits_{k=1}^{m} \Gamma\left(b_k + s\right)}{\prod\limits_{k=1}^{m} \Gamma\left(1 + \overline{b}_k - s\right)} f^*(1-s).$$

The Parseval formula yields

$$g(x) = \int_0^\infty G^{m,0}_{0,2m}\left(xy \,\middle|\, \begin{matrix} - \quad , \quad - \\ (b_m), -(\overline{b}_m) \end{matrix}\right) f(y)\, dy.$$

Denote $h(x) = \left[\prod_{j=1}^m \left(b_j - x\frac{d}{dx}\right)\left(\overline{b}_j + x\frac{d}{dx}\right)\right] g(x)$, then $h(x), \frac{1}{x}\, h(x) \in L_2(\mathbb{R}_+)$. Moreover,

$$h^*(s) = \prod_{j=1}^m \left(b_j + s\right)\left(\overline{b}_j - s\right)\, g^*(s)$$

$$= \prod_{j=1}^m \left(b_j + s\right)\left(\overline{b}_j - s\right) \frac{\prod\limits_{k=1}^{m} \Gamma\left(b_k + s\right)}{\prod\limits_{k=1}^{m} \Gamma\left(1 + \overline{b}_k - s\right)} f^*(1-s) = \frac{\prod\limits_{k=1}^{m} \Gamma\left(1 + b_k + s\right)}{\prod\limits_{k=1}^{m} \Gamma\left(\overline{b}_k - s\right)} f^*(1-s)$$

over σ. Since $h^*(s)$, therefore, $g^*(s)$ is analytic in the strip $-\frac{1}{2} < \Re(s) < \frac{1}{2}$, then $f^*(s)$ is analytic in the strip $\frac{1}{2} < \Re(s) < \frac{3}{2}$. As

$$h^*(s-1) = \frac{\prod\limits_{k=1}^{m} \Gamma\left(b_k + s\right)}{\prod\limits_{k=1}^{m} \Gamma\left(1 + \overline{b}_k - s\right)} f^*(2-s) \in L_2(\sigma),$$

then $f^*(s+1) \in L_2(\sigma)$. Hence, $y\, f(y) \in L_2(\mathbb{R})$, and

$$\frac{1}{x} h(x) = \left[\frac{1}{x}\prod_{j=1}^m \left(b_j - x\frac{d}{dx}\right)\left(\overline{b}_j + x\frac{d}{dx}\right)\right] g(x) = \int_0^\infty G^{m,0}_{0,2m}\left(xy \,\middle|\, \begin{matrix} - \quad , \quad - \\ (b_m), -(\overline{b}_m) \end{matrix}\right) yf(y)\, dy.$$

By induction one can show

$$\left[\frac{1}{x}\prod_{j=1}^m \left(b_j - x\frac{d}{dx}\right)\left(\overline{b}_j + x\frac{d}{dx}\right)\right]^k g(x) = \int_0^\infty G^{m,0}_{0,2m}\left(xy \,\middle|\, \begin{matrix} - \quad , \quad - \\ (b_m), -(\overline{b}_m) \end{matrix}\right) y^k f(y)\, dy.$$

The compactness of the support of f follows from

$$\infty > \lim_{k\to\infty} \left\| \left[\frac{1}{x}\prod_{j=1}^m \left(b_j - x\frac{d}{dx}\right)\left(\overline{b}_j + x\frac{d}{dx}\right)\right]^k g(x) \right\|_{L_2(\mathbb{R}_+)}^{\frac{1}{k}} = \lim_{k\to\infty} \left\| y^k f(y) \right\|_{L_2(\mathbb{R}_+)}^{\frac{1}{k}} = \sup_{y \in \operatorname{supp} f} y.$$

■

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