# The heat kernel and Heisenberg inequalities related to the Kontorovich-Lebedev transform


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**Abstract**

We introduce a notion of the heat kernel related to the familiar Kontorovich-Lebedev transform. We study differential and semigroup properties of this kernel and construct fundamental solutions of a generalized diffusion equation. An integral transformation with the heat kernel is considered. By using the Plancherel $L_2$-theory for the Kontorovich-Lebedev transform and norm estimates for its convolution we establish analogs of the classical Heisenberg inequality and uncertainty principle for this transformation. The proof is also based on the norm inequalities for the Mellin transform of the heat kernel.



**Keywords**: *Kontorovich-Lebedev transform, the Mellin transform, convolution, Heisenberg inequality, heat kernel, uncertainty principle, Plancherel theory, modified Bessel function*

**AMS subject classification**: 44A15, 44A05, 44A35


## 1 Introduction and preliminary results

As it is known (see [6,7,8,9]), the Kontorovich-Lebedev transform

$$K_{i\tau} : L_2(\mathbb{R}_+; xdx) \leftrightarrow L_2(\mathbb{R}_+; \tau \sinh \pi\tau d\tau)$$


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given by the formula

$$K_{i\tau}[f] = \int_0^\infty K_{i\tau}(x)f(x)dx, \tag{1.1}$$

where the integral in (1.1) converges with respect to the norm in $L_2(\mathbb{R}_+; \tau \sinh \pi\tau d\tau)$, is an isometric isomorphism between these Hilbert spaces. The corresponding Parseval identity holds

$$\int_0^\infty \tau \sinh \pi\tau |K_{i\tau}[f]|^2 d\tau = \frac{\pi^2}{2} \int_0^\infty x|f(x)|^2 dx \tag{1.2}$$

as well as the inversion formula

$$f(x) = \frac{2}{x\pi^2} \int_0^\infty \tau \sinh \pi\tau K_{i\tau}(x) K_{i\tau}[f] d\tau, \tag{1.3}$$

where the convergence of the integral (1.3) is understood with respect to the norm of the space $L_2(\mathbb{R}_+; xdx)$.

In this paper we will study differential and semigroup properties of the heat kernel related to the Kontorovich-Lebedev transform, which we will introduce below. This kernel and its relationship with the Mellin transform will be applied to deduce analogs of the Heisenberg inequality and its direct consequence, which is called the uncertainty principle for the Kontorovich-Lebedev transform. Similar problems related to the Jacobi-Olevskii transform were considered recently in [2]. Other uncertainty principles for the transform (1.1) were proved by the author in [15, 17].

The kernel of the Kontorovich-Lebedev transform is the modified Bessel function $K_{i\tau}(x)$, which is an eigenfunction of the following second order differential operator

$$\mathcal{A}_x \equiv x^2 - x\frac{d}{dx}x\frac{d}{dx}, \tag{1.4}$$

i.e. we have

$$\mathcal{A}_x \ K_{i\tau}(x) = \tau^2 K_{i\tau}(x). \tag{1.5}$$

It has the asymptotic behaviour (cf. [1] relations (9.6.8), (9.6.9), (9.7.2))

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}[1 + O(1/z)], \qquad z \to \infty, \tag{1.6}$$

and near the origin

$$K_\nu(z) = O\left(z^{-|\mathrm{Re}\nu|}\right), \ z \to 0, \tag{1.7}$$

$$K_0(z) = -\log z + O(1), \ z \to 0. \tag{1.8}$$

When $|\tau| \to \infty$ and $x > 0, \ \gamma \in \mathbb{R}$ are fixed, the kernel $K_{\gamma+i\tau}(x)$ behaves as (cf. [8, Ch. 1])

$$K_{\gamma+i\tau}(x) = \frac{\sqrt{2\pi}\ e^{\gamma\pi i}}{|\tau|^{\gamma+1/2}} \left(\frac{x}{2}\right)^\gamma e^{-\pi|\tau|/2} \sin\left(\tau\left(\log\frac{2|\tau|}{x} - 1\right) + \left(\gamma + \frac{1}{2}\right)\frac{\pi}{2} + \frac{x^2}{4|\tau|}\right)$$

$$\times \left(1 + O\left(\frac{1}{|\tau|}\right)\right). \tag{1.9}$$

Moreover it can be defined by the following integral representations [6, (6-1-2)], [5], Vol. I, relation (2.4.18.4)

$$K_\nu(x) = \int_0^\infty e^{-x\cosh u} \cosh \nu u du, x > 0, \tag{1.10}$$

$$K_\nu(x) = \frac{1}{2}\left(\frac{x}{2}\right)^\nu \int_0^\infty e^{-t-\frac{x^2}{4t}} t^{-\nu-1} dt,\ x > 0. \tag{1.11}$$

Hence we easily find that $K_\nu(x)$ is a real-valued positive function when $\nu \in \mathbb{R}$ and an even function with respect to the index $\nu$. We note here the Mellin transform of the modified Bessel function [7]

$$\int_0^\infty K_{i\tau}(x)x^{s-1}dx = 2^{s-2}\Gamma\left(\frac{s}{2}+\frac{i\tau}{2}\right)\Gamma\left(\frac{s}{2}+\frac{i\tau}{2}\right),\ \mathrm{Re}\ s > 0, \tag{1.12}$$

where $\Gamma(z)$ is Euler's gamma-function.

The convolution operator for the Kontorovich-Lebedev transform is defined as follows [7, 8]

$$(f * h)(x) = \frac{1}{2x}\int_0^\infty\int_0^\infty e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy}+\frac{yu}{x}\right)} f(u)h(y)dudy,\ x > 0. \tag{1.13}$$

It is well defined in the Banach ring $L^\alpha(\mathbb{R}_+) \equiv L_1(\mathbb{R}_+; K_\alpha(x)dx), \alpha \in \mathbb{R}$, i.e. the space of all summable functions $f : \mathbb{R}_+ \to \mathbb{C}$ with respect to the measure $K_\alpha(x)dx$ for which

$$||f||_{L^\alpha(\mathbb{R}_+)} = \int_0^\infty |f(x)|K_\alpha(x)dx \tag{1.14}$$

is finite. The following embeddings take place

$$L^\alpha(\mathbb{R}_+) \equiv L^{-\alpha}(\mathbb{R}_+),\ L^\alpha(\mathbb{R}_+) \subseteq L^\beta(\mathbb{R}_+),\ |\alpha| \geq |\beta| \geq 0, \alpha, \beta \in \mathbb{R}, \tag{1.15}$$

$$L^\alpha(\mathbb{R}) \supset L_p(\mathbb{R}_+; xdx),\ 2 < p \leq \infty,\ |\alpha| < 1 - \frac{2}{p}, \tag{1.16}$$

where $L_p(\mathbb{R}_+; xdx)$ is a weighted Banach space with the norm

$$||f||_{L_p(\mathbb{R}_+;xdx)} = \left(\int_0^\infty |f(x)|^p x dx\right)^{1/p},\ 1 \leq p < \infty, \tag{1.17}$$

$$||f||_{L_\infty(\mathbb{R}_+;xdx)} = \mathrm{ess\ sup}_{x\in\mathbb{R}_+}|f(x)|. \tag{1.18}$$

The factorization property is true for the convolution (1.13) in terms of the Kontorovich-Lebedev transform (1.2) in the space $L^\alpha(\mathbb{R}_+)$, namely

$$K_{i\tau}[f * h] = K_{i\tau}[f]K_{i\tau}[h],\ \tau \in \mathbb{R}_+. \tag{1.19}$$

This property is based on the Macdonald formula [1]

$$K_\nu(x)K_\nu(y) = \frac{1}{2}\int_0^\infty e^{-\frac{1}{2}\left(t\frac{x^2+y^2}{xy}+\frac{xy}{t}\right)} K_\nu(t)\frac{dt}{t}. \tag{1.20}$$

It is also proved (see [7], [8]) that the Kontorovich-Lebedev transform is a bounded operator from $L^\alpha(\mathbb{R}_+)$ into the space of bounded continuous functions on $\mathbb{R}_+$ vanishing at infinity. Furthermore, the convolution (1.13) of two functions $f, h \in L^\alpha(\mathbb{R}_+)$ exists as a Lebesgue integral and belongs to $L^\alpha(\mathbb{R}_+)$. It satisfies the Young -type inequality

$$||f * h||_{L^\alpha(\mathbb{R}_+)} \le ||f||_{L^\alpha(\mathbb{R}_+)}||h||_{L^\alpha(\mathbb{R}_+)}. \tag{1.21}$$

Another type of a sharp Young inequality for convolution (1.13) was established in [12]. Precisely, we have

**Theorem 1**. *Let* $1 < p \le \infty,\ f \in L_p(\mathbb{R}_+; xdx)$ *and* $h \in L^{\frac{p-2}{p-1}}(\mathbb{R}_+)$. *Then convolution* (1.13) *exists as a Lebesgue integral for all* $x > 0$ *and belongs to the space* $L_p(\mathbb{R}_+; xdx)$. *Moreover, it satisfies the following inequality*

$$||f * h||_{L_p(\mathbb{R}_+;xdx)} \le ||f||_{L_p(\mathbb{R}_+;xdx)}||h||_{L^{\frac{p-2}{p-1}}(\mathbb{R}_+)}. \tag{1.22}$$

In particular, for $p = 2$ we get (see [10, 11])

$$||f * h||_{L_2(\mathbb{R}_+;xdx)} \le ||f||_{L_2(\mathbb{R}_+;xdx)}||h||_{L^0(\mathbb{R}_+)}. \tag{1.23}$$

## 2 The heat kernel and its properties

We begin with

**Definition 1**. Let $t > 0,\ (x, y) \in \mathbb{R}_+$. The following integral

$$h(t, x, y) \equiv h_t(x, y) = \frac{2}{x\pi^2}\int_0^\infty e^{-t\tau^2}\tau \sinh \pi\tau\ K_{i\tau}(x)K_{i\tau}(y)d\tau, \tag{2.1}$$

is called the heat kernel for the Kontorovich-Lebedev transform.

**Lemma 1**. *The function* $h(t, x, y)$ *is infinitely differentiable of* $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$ *and* $t > 0$ *satisfying the estimate*

$$\left|\frac{\partial^m h(t, x, y)}{\partial t^m}\right| \le \frac{\Gamma^{1/4}\left(4m + \frac{3}{2}\right)}{2^{m+1}\pi^{7/8}}\ \frac{e^{\frac{\pi^2}{4t}}}{x\ t^{m+1/2}}\ K_0^{1/2}\left(2\sqrt{x^2 + y^2}\right). \tag{2.2}$$

*Moreover, the Kontorovich-Lebedev transform* (1.1) *by* $x$ *of the heat kernel is equal to*

$$K_{i\tau}[h] = \int_0^\infty K_{i\tau}(x)h(t, x, y)dx = e^{-t\tau^2}K_{i\tau}(y), \tag{2.3}$$

*and by $y$ correspondingly,*

$$K_{i\tau}\left[\frac{h}{y}\right] = \int_0^\infty K_{i\tau}(y)h(t,x,y)\frac{dy}{y} = e^{-t\tau^2}\frac{K_{i\tau}(x)}{x}. \tag{2.4}$$

*Finally $h(t,x,y)$ is a solution of generalized diffusion equations ($u = u(t,x,y)$)*

$$\frac{\partial u}{\partial t} = x^2\frac{\partial^2 u}{\partial x^2} + 3x\frac{\partial u}{\partial x} - (x^2-1)u \tag{2.5}$$

*for each fixed $y \in \mathbb{R}_+$,*

$$\frac{\partial u}{\partial t} = y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} - y^2 u \tag{2.6}$$

*for each fixed $x \in \mathbb{R}_+$ under the initial condition in the sense of distributions*

$$\lim_{t\to 0} h(t,x,y) = \delta(x-y), \tag{2.7}$$

*where $\delta$ is Dirac's delta-function.*

**Proof.** Appealing to the following inequality (see [16]) for derivatives of the modified Bessel function with respect to $x$

$$\left|\frac{\partial^m K_{i\tau}(x)}{\partial x^m}\right| \le e^{-\delta\tau}K_m(x\cos\delta),\ x>0,\ \tau>0,\ \delta\in\left[0;\frac{\pi}{2}\right),\ m=0,1,\dots$$

it is not difficult to verify that for $t > 0$ integral (2.1) and its derivatives of any order with respect to $x$ and $y$ converge absolutely and uniformly by $x \ge x_0 > 0$, and $y \ge y_0 > 0$. Therefore the heat kernel (2.1) is infinitely differentiable of $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+$. Similar motivation can be done for the derivatives of the order $m \in \mathbb{N}_0$ with respect to $t > 0$ and it gives the expression

$$\frac{\partial^m h(t,x,y)}{\partial t^m} = \frac{2(-1)^m}{x\pi^2}\int_0^\infty e^{-t\tau^2}\tau^{2m+1}\sinh\pi\tau\ K_{i\tau}(x)K_{i\tau}(y)d\tau,\ m=0,1,\dots\ . \tag{2.8}$$

Hence the Schwarz inequality yields

$$\left|\frac{\partial^m h(t,x,y)}{\partial t^m}\right| \le \frac{2}{x\pi^2}\left(\int_0^\infty e^{-2t\tau^2}\tau^{4m+1}\sinh\pi\tau d\tau\right)^{1/2}$$

$$\times\left(\int_0^\infty \tau\sinh\pi\tau\left|K_{i\tau}(x)K_{i\tau}(y)\right|^2 d\tau\right)^{1/2}. \tag{2.9}$$

Meanwhile, using (1.1), (1.2), (1.20) and relation (2.3.16.1) in [5], Vol.1 we find

$$\int_0^\infty \tau\sinh\pi\tau\left|K_{i\tau}(x)K_{i\tau}(y)\right|^2 d\tau = \frac{\pi^2}{8}\int_0^\infty e^{-\left(u\frac{x^2+y^2}{xy}+\frac{xy}{u}\right)}\frac{du}{u} = \frac{\pi^2}{4}K_0\left(2\sqrt{x^2+y^2}\right).$$

Further,

$$\int_0^\infty e^{-2t\tau^2}\tau^{4m+1}\sinh\pi\tau d\tau \le \frac{1}{2}\left(\int_{-\infty}^\infty e^{2(\pi\tau-t\tau^2)}d\tau\right)^{1/2}\left(\int_0^\infty e^{-2t\tau^2}\tau^{8m+1}d\tau^2\right)^{1/2}$$

$$=\frac{\pi^{1/4}}{2}\Gamma^{1/2}\left(4m+\frac{3}{2}\right)e^{\pi^2/(2t)}(2t)^{-(2m+1)}.$$

Therefore, substituting these values into (2.9) we deduce

$$\left|\frac{\partial^m h(t,x,y)}{\partial t^m}\right| \le \frac{\Gamma^{1/4}\left(4m+\frac{3}{2}\right)}{2^{m+1}\pi^{7/8}}\ \frac{e^{\frac{\pi^2}{4t}}}{x\ t^{m+1/2}}\ K_0^{1/2}\left(2\sqrt{x^2+y^2}\right),$$

which drives us to the estimate (2.2). As it follows from (1.5) and absolute and uniform convergence by $x$ and $y$ of the corresponding integrals on any compact set of $\mathbb{R}_+\times\mathbb{R}_+$, formula (2.8) can be written as the following differential equations

$$\frac{\partial^m h(t,x,y)}{\partial t^m} = (-1)^m\mathcal{A}_y^m h(t,x,y), \tag{2.10}$$

$$\frac{\partial^m h(t,x,y)}{\partial t^m} = \frac{(-1)^m}{x}\mathcal{A}_x^m\left[x\ h(t,x,y)\right], \tag{2.11}$$

where $m=0,1,\dots$ and $A_x^m$, $A_y^m$ are $m$-th iterates of the operator (1.4). In particular, letting $m=1$ we obtain that the heat kernel satisfies the generalized diffusion equations (2.5), (2.6).

In order to establish equalities (2.3), (2.4) we easily observe that for fixed positive $t,x,y$ the corresponding right-hand sides belong to the space $L_2(\mathbb{R}_+;\tau\sinh\pi\tau d\tau)$. Therefore the validity of (2.3), (2.4) follows immediately from reciprocities (1.1), (1.3) with the convergence of integrals in $L_2$-sense. Nevertheless, via the estimate (2.2) and asymptotic formulas (1.6), (1.7), (1.8) for the modified Bessel functions we get that integral (2.4) converges absolutely and clearly to the same limit. The absolute convergence of the integral (2.3) is based on (2.2) and on the asymptotic behavior of the heat kernel $h(t,x,y)$ when $x\to 0$. We deduce the mentioned asymptotic expression writing the heat kernel (2.1) in the equivalent form. In fact, appealing to the relation formulas for the modified Bessel functions [1] we find

$$h(t,x,y)=\frac{1}{x\pi i}\int_{-\infty}^\infty e^{-t\tau^2}\tau\ I_{-i\tau}(x)K_{i\tau}(y)d\tau, \tag{2.12}$$

where $I_\nu(z)$ is the modified Bessel function of the first kind represented by the series [1]

$$I_\nu(z)=\sum_{k=0}^\infty\frac{(z/2)^{2k+\nu}}{k!\Gamma(k+\nu+1)}.$$

Hence substituting this series into (2.12) separating its first term we have for $x \to 0$

$$h(t,x,y) = \frac{1}{x\pi i}\int_{-\infty}^{\infty} e^{-t\tau^2}\tau K_{i\tau}(y) e^{-i\tau\log(x/2)}\frac{d\tau}{\Gamma(1-i\tau)}$$

$$+\frac{1}{\pi i}\int_{-\infty}^{\infty} e^{-t\tau^2}\tau K_{i\tau}(y)\sum_{k=1}^{\infty}\frac{(x/2)^{2k-i\tau-1}}{k!\Gamma(k+1-i\tau)}d\tau. \tag{2.13}$$

By straightforward estimates we see that the second term in (2.13) is $O(x),\ x \to 0$ for any fixed $t, y > 0$. The first term can be treated carrying out the integration by parts and eliminating the corresponding boundary terms. Hence making use this procedure $n$ times we derive the estimate

$$\frac{1}{x\pi i}\int_{-\infty}^{\infty} e^{-t\tau^2}\tau K_{i\tau}(y) e^{-i\tau\log(x/2)}\frac{d\tau}{\Gamma(1-i\tau)} = O\left(\frac{1}{x\log^n x}\right),\ x \to 0,\ n \in \mathbb{N}.$$

Consequently, via the inequality $|K_{i\tau}(x)| \le K_0(x)$ and asymptotic formula (1.8) we take $n = 3, 4, \ldots$ and establish the absolute convergence of the integral (2.3) to the same limit.

Finally we prove (2.7). Indeed, for any $\varphi$ from the testing space $\mathcal{D}(\mathbb{R}_+)$ we understand the limit (2.7) as follows (cf. [7], [18])

$$\lim_{t\to 0}\langle h(t,x,\ \cdot),\ \varphi(\cdot)\rangle\ = \lim_{t\to 0}\frac{2}{x\pi^2}\int_0^{\infty} e^{-t\tau^2}\tau\sinh\pi\tau\ K_{i\tau}(x)K_{i\tau}[\varphi]\,d\tau. \tag{2.14}$$

Meanwhile, employing relation (2.16.14.1) in [5], Vol. II and Parseval's equality for the cosine Fourier transform we deduce

$$\frac{2}{x\pi^2}\int_0^{\infty} e^{-t\tau^2}\tau\sinh\pi\tau\ K_{i\tau}(x)K_{i\tau}[\varphi]\,d\tau = \frac{\sqrt{2}}{x\pi\sqrt{\pi}}\int_0^{\infty} e^{-t\tau^2}\tau\sinh\left(\frac{\pi\tau}{2}\right)\ K_{i\tau}(x)$$

$$\times\int_0^{\infty}\cos\tau u\ F_c[\varphi;\sinh u]\,du\ d\tau, \tag{2.15}$$

where by $F_c[\varphi; v]$ we denote the cosine Fourier transform of $\varphi \in \mathcal{D}(\mathbb{R}_+)$

$$F_c[\varphi; v] = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\varphi(y)\ \cos yv\ dy.$$

The right-hand side of (2.15) can be treated, in turn, with the use of the differentiation under the integral sign, integration by parts, convolution properties and Parseval's equality for the Fourier transform. So we have

$$\frac{\sqrt{2}}{x\pi\sqrt{\pi}}\int_0^{\infty} e^{-t\tau^2}\tau\sinh\left(\frac{\pi\tau}{2}\right)\ K_{i\tau}(x)$$

$$\times \int_0^\infty \cos \tau u \ F_c\left[\varphi; \sinh u\right] \ du \ d\tau = -\frac{\sqrt{2}}{x\pi\sqrt{\pi}} \int_0^\infty e^{-t\tau^2} \sinh\left(\frac{\pi\tau}{2}\right) \ K_{i\tau}(x)$$

$$\times \int_0^\infty \sin \tau u \ \frac{d}{du} \left[F_c\left[\varphi; \sinh u\right] \ \right] du \ d\tau$$

$$= - \frac{1}{i2x\pi\sqrt{2\pi}} \int_{-\infty}^\infty e^{-t\tau^2} \sinh\left(\frac{\pi\tau}{2}\right) \ K_{i\tau}(x) \int_{-\infty}^\infty e^{i\tau u} \ \frac{d}{du} \left[F_c\left[\varphi; \sinh u\right] \ \right] du \ d\tau. \tag{2.16}$$

In the meantime, due to relations (2.5.36.1) and (2.5.54.6) in [5], Vol. I the product $e^{-t\tau^2} \sinh\left(\frac{\pi\tau}{2}\right) K_{i\tau}(x)$ can be represented as the Fourier transform of a convolution, namely

$$e^{-t\tau^2} \sinh\left(\frac{\pi\tau}{2}\right) K_{i\tau}(x) = \frac{i}{4t\sqrt{\pi t}} \int_{-\infty}^\infty y e^{-y^2/(4t)} \ e^{iy\tau} dy \int_{-\infty}^\infty e^{iy\tau} dy \int_y^\infty \sin(x \ \sinh v) \ dv$$

$$= \frac{i}{4t\sqrt{\pi t}} \int_{-\infty}^\infty e^{iy\tau} dy \int_{-\infty}^\infty u \ e^{-u^2/(4t)} \ \int_{y-u}^\infty \sin(x \ \sinh v) \ dv \ du$$

$$= \frac{i}{2\sqrt{\pi t}} \int_{-\infty}^\infty e^{iy\tau} dy \int_{-\infty}^\infty \ e^{-u^2/(4t)} \ \sin(x \ \sinh(y-u)) \ du.$$

Substituting the latter expression into (2.16), taking into account (2.15) and the Parseval equality for the Fourier transform we derive the representation

$$\frac{2}{x\pi^2} \int_0^\infty e^{-t\tau^2} \tau \sinh \pi\tau \ K_{i\tau}(x) K_{i\tau}\left[\varphi\right] d\tau$$

$$= - \frac{1}{2x\pi\sqrt{2t}} \int_{-\infty}^\infty \frac{d}{dy} \left[F_c\left[\varphi; \ \sinh y\right] \ \right] dy \int_{-\infty}^\infty \ e^{-u^2/(4t)} \ \sin(x \ \sinh(y-u)) \ du$$

$$= \frac{1}{\pi \ 2\sqrt{2t}} \int_{-\infty}^\infty \ F_c\left[\varphi; \ \sinh y\right] \ dy \int_{-\infty}^\infty \ e^{-u^2/(4t)} \ \cos(x \ \sinh(y-u)) \ \cosh(y-u) \ du$$

$$= \frac{1}{\pi\sqrt{2}} \int_{-\infty}^\infty \ F_c\left[\varphi; \ \sinh y\right] \ dy \int_{-\infty}^\infty \ e^{-u^2} \ \cos\left(x \ \sinh\left(y - 2u\sqrt{t}\right)\right)$$

$$\times \ \cosh\left(y - 2u\sqrt{t}\right) \ du. \tag{2.17}$$

Hence it is not difficult to verify that the passage to the limit when $t \to 0$ under integral signs in the latter iterated integral of (2.17) is allowed via the dominated convergence theorem. Then by elementary substitutions and straightforward calculations appealing to the asymptotic and inversion properties of the cosine Fourier transform of test functions, we return to (2.14) and finally derive

$$\lim_{t\to 0} \ \langle h(t,x, \ \cdot), \ \varphi(\cdot)\rangle \ = \ \frac{1}{\pi\sqrt{2}} \int_{-\infty}^\infty \ F_c\left[\varphi; \ \sinh y\right] \ dy$$

$$\times \int_{-\infty}^{\infty} e^{-u^2} \cos(x \sinh y) \cosh y \, du = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \cos(x\lambda) \, F_c[\varphi; \lambda] \, d\lambda = \varphi(x).$$

This proves (2.7) and completes the proof of Lemma 1.

Following to [13] (see also in [14]) we define the space $S_2(\mathbb{R}_+)$, which we are going to use below.

**Definition 2.** A function $f : \mathbb{R}_+ \to \mathbb{C}$ is said to be in $S_2(\mathbb{R}_+)$ if $f \in L_2\left(\mathbb{R}_+; \frac{dx}{x}\right)$ and $\mathcal{A}_x f \in L_2\left(\mathbb{R}_+; \frac{dx}{x}\right)$, where the operator $\mathcal{A}_x$ is defined by (1.4).

Recall from [13, 16] that the $k$-th iterate of the operator $\mathcal{A}_x$ $\mathcal{A}_x^k f \in L_2\left(\mathbb{R}_+; \frac{dx}{x}\right)$, $k \in \mathbb{N}_0$ means that there exists a function $v(x)$ in $L_2\left(\mathbb{R}_+; \frac{dx}{x}\right)$ denoted by $\mathcal{A}_x^k f$ such that for all $\varphi \in \mathcal{D}(\mathbb{R}_+)$

$$\int_0^{\infty} f(x) \mathcal{A}_x^k \varphi \frac{dx}{x} = \int_0^{\infty} v(x)\varphi(x)\frac{dx}{x}.$$

It is proved that $S_2(\mathbb{R}_+)$ is a Banach space which can be endowed with the norm

$$||f||_{S_2(\mathbb{R}_+)} = \left( \int_0^{\infty} |f(x)|^2 \frac{dx}{x} + \int_0^{\infty} |\mathcal{A}_x f|^2 \frac{dx}{x} \right)^{1/2}.$$

A characterization of $S_2(\mathbb{R}_+)$ can be given in terms of the Kontorovich- Lebedev transform (1.1) (cf. [13]). First we observe that $f \in L_2\left(\mathbb{R}_+; \frac{dx}{x}\right)$ means $\frac{f(x)}{x} \in L_2\left(\mathbb{R}_+; x dx\right)$. We have

**Theorem 1. [13]**. *Let $f \in L_2\left(\mathbb{R}_+; \frac{dx}{x}\right)$ with the Kontorovich-Lebedev transform $K_{i\tau}\left[\frac{f(x)}{x}\right]$. Then $f \in S_2(\mathbb{R}_+)$ (i.e. $\mathcal{A}_x f \in L_2\left(\mathbb{R}_+; \frac{dx}{x}\right)$) if and only if the function $\tau \to \tau^2 K_{i\tau}\left[\frac{f(x)}{x}\right]$ is in $L_2(\mathbb{R}_+; \tau \sinh \pi\tau \, d\tau)$. Moreover, $K_{i\tau}\left[\frac{\mathcal{A}_x f}{x}\right] = \tau^2 K_{i\tau}\left[\frac{f(x)}{x}\right]$ and therefore*

$$||f||^2_{S_2(\mathbb{R}_+)} = \frac{2}{\pi^2} \int_0^{\infty} \tau \sinh \pi\tau \left| K_{i\tau}\left[\frac{f(x)}{x}\right] \right|^2 (1 + \tau^4) d\tau. \tag{2.18}$$

We define here the Weierstrass type integral transformation in $S_2(\mathbb{R}_+)$ as the action of the heat kernel (2.1) on functions $f(y)$

$$(g_t f)(x) = \int_0^{\infty} h(t, y, x) f(y) dy. \tag{2.19}$$

This integral is absolutely convergent via Schwarz's inequality, i.e. it exists as a Lebesgue integral. Moreover, by Lemma 1 (see (2.4)) and Theorem 1 one can prove that

$$K_{i\tau}\left[\frac{(g_t f)(x)}{x}\right] = e^{-t\tau^2} K_{i\tau}\left[\frac{f(x)}{x}\right] \tag{2.20}$$

and therefore we can denote the heat kernel (2.1) by $e^{-t\mathcal{A}}(x, y)$ and $(g_t f)(x) = e^{-t\mathcal{A}} f$. So the action of $\mathcal{A}$ on the Kontorovich-Lebedev transforms is multiplication by $\tau^2$ while the heat kernel is multiplication by $e^{-t\tau^2}$. Furthermore, from Lemma 1 and the Schwarz inequality it follows that $(g_t f)(x)$ is an infinitely differentiable function of $x, t > 0$. It satisfies the generalized diffusion equation (see (2.10))

$$\frac{\partial (g_t f)(x)}{\partial t} = -\mathcal{A}_x g_t f$$

classically for $t > 0$ with the initial condition

$$\lim_{t\to 0} (g_t f)(x) = f(x)$$

as a strong limit in $f \in L_2\left(\mathbb{R}_+; \frac{dx}{x}\right)$.

**Lemma 2**. *For each* $t > 0$ *the Weierstrass type transform* (2.19) *is a bounded operator in* $S_2(\mathbb{R}_+)$ *and the following norm estimate takes place*

$$||g_t f||_{S_2(\mathbb{R}_+)} \le ||f||_{S_2(\mathbb{R}_+)}. \tag{2.21}$$

**Proof.** From the generalized Parseval equality for the Kontorovich-Lebedev transform (see (1.2)) and (2.3) we have

$$(g_t f)(x) = \frac{2}{\pi^2} \int_0^\infty \tau \sinh \pi\tau K_{i\tau}\left[h(t,\cdot,x)\right] K_{i\tau}\left[\frac{f(\cdot)}{\cdot}\right] d\tau$$

$$= \frac{2}{\pi^2} \int_0^\infty \tau \sinh \pi\tau \ e^{-t\tau^2} K_{i\tau}(x) K_{i\tau}\left[\frac{f(\cdot)}{\cdot}\right] d\tau.$$

As in Lemma 1 we easily verify the absolute and uniform convergence by $x \ge x_0 > 0$ of the latter integral and its derivatives with respect to $x$. Therefore, via (1.5) we obtain

$$\mathcal{A}_x g_t f = \frac{2}{\pi^2} \int_0^\infty \tau^3 \sinh \pi\tau \ e^{-t\tau^2} K_{i\tau}(x) K_{i\tau}\left[\frac{f(\cdot)}{\cdot}\right] d\tau$$

and

$$||\mathcal{A}_x g_t f||^2_{L_2(\mathbb{R}_+;\frac{dx}{x})} = \frac{2}{\pi^2} \int_0^\infty \tau \sinh \pi\tau \ e^{-2t\tau^2} \left| K_{i\tau}\left[\frac{\mathcal{A}.f}{\cdot}\right]\right|^2 d\tau \le ||f||^2_{S_2(\mathbb{R}_+)} < \infty.$$

At the same time, plainly

$$||g_t f||^2_{L_2(\mathbb{R}_+;\frac{dx}{x})} = \frac{2}{\pi^2} \int_0^\infty \tau \sinh \pi\tau \ e^{-2t\tau^2} \left| K_{i\tau}\left[\frac{f(\cdot)}{\cdot}\right]\right|^2 d\tau \le ||f||^2_{S_2(\mathbb{R}_+)} < \infty.$$

Therefore taking into account (2.18) we get the inequality

$$||g_t f||^2_{S_2(\mathbb{R}_+;\frac{dx}{x})} = \frac{2}{\pi^2} \int_0^\infty \tau \sinh \pi\tau \ e^{-2t\tau^2} \left| K_{i\tau}\left[\frac{f(\cdot)}{\cdot}\right]\right|^2 (1+\tau^4) d\tau \le ||f||^2_{S_2(\mathbb{R}_+)},$$

which yields (2.21) and completes the proof of Lemma 2.

One can extend Theorem 1 for the iterates $\mathcal{A}^k,\ k \in \mathbb{N}_0$. Indeed, we have

**Theorem 2.** [**13**]. *The iterate* $\mathcal{A}_x^k f \in L_2\left(\mathbb{R}_+; \frac{dx}{x}\right),\ k \in \mathbb{N}_0$ *if and only if the function* $\tau \to \tau^{2k} K_{i\tau}\left[\frac{f(x)}{x}\right]$ *is in* $L_2(\mathbb{R}_+; \tau \sinh \pi\tau\ d\tau)$. *Moreover,*

$$K_{i\tau}\left[\frac{\mathcal{A}_x^k f}{x}\right] = \tau^{2k}\ K_{i\tau}\left[\frac{f(x)}{x}\right]. \tag{2.22}$$

Using this result we find a reciprocal inversion formula of the Weierstrass type transform (2.19). In fact, returning to (2.20) we derive

$$K_{i\tau}\left[\frac{f(x)}{x}\right] = e^{t\tau^2} K_{i\tau}\left[\frac{(g_t f)(x)}{x}\right],\ t > 0. \tag{2.23}$$

Since the left-hand side of (2.23) is from $L_2(\mathbb{R}_+; \tau \sinh \pi\tau\ d\tau)$ for any $f \in L_2\left(\mathbb{R}_+; \frac{dx}{x}\right)$ so the right-hand side possesses the same property. Consequently,

$$\tau^{2k} K_{i\tau}\left[\frac{(g_t f)(x)}{x}\right] \in L_2(\mathbb{R}_+; \tau \sinh \pi\tau\ d\tau) \tag{2.24}$$

for any $k \in \mathbb{N}_0$. Therefore, due to (2.22)

$$F_n(t,\tau) = \sum_{m=0}^{n} \frac{t^m}{m!} \tau^{2m} K_{i\tau}\left[\frac{(g_t f)(x)}{x}\right] = \sum_{m=0}^{n} \frac{t^m}{m!} K_{i\tau}\left[\frac{\mathcal{A}_x^m g_t f}{x}\right] = K_{i\tau}\left[\frac{P_n(t\mathcal{A}_x) g_t f}{x}\right],$$

where $P_n(z)$ is the $n$th Taylor's polynomial of the exponential function $e^z$. Hence from (1.3) we obtain

$$P_n(t\mathcal{A}_x) g_t f = \frac{2}{\pi^2} \int_0^\infty \tau \sinh \pi\tau K_{i\tau}(x) P_n(t\tau^2) K_{i\tau}\left[\frac{(g_t f)(x)}{x}\right] d\tau,\ n \in \mathbb{N}_0$$

where the latter integral converges absolutely since

$$e^{t\tau^2} K_{i\tau}\left[\frac{(g_t f)(x)}{x}\right] \in L_2(\mathbb{R}_+; \tau \sinh \pi\tau\ d\tau). \tag{2.25}$$

On the other hand we get from (2.23)

$$f(x) = \text{l.i.m}_{T\to\infty} \frac{2}{\pi^2} \int_0^T \tau \sinh \pi\tau K_{i\tau}(x) e^{t\tau^2} K_{i\tau}\left[\frac{(g_t f)(x)}{x}\right] d\tau$$

and the Parseval equality (1.2) yields

$$\int_0^\infty |f(x) - P_n(t\mathcal{A}_x) g_t f|^2 \frac{dx}{x} = \frac{2}{\pi^2} \int_0^\infty \tau \sinh \pi\tau \left|1 - P_n(t\tau^2) e^{-t\tau^2}\right|^2$$

$$\times e^{2t\tau^2}\left|K_{i\tau}\left[\frac{(g_t f)(\cdot)}{\cdot}\right]\right|^2 d\tau. \tag{2.26}$$

Since $1-P_n(t\tau^2)e^{-t\tau^2}<1$ and tends to zero when $n\to\infty$ for each $t>0$, by the Lebesgue dominated convergence theorem we obtain that the left-hand side of (2.26) tends to zero as well. Therefore with a convergence by the norm in $L_2\left(\mathbb{R}_+;\frac{dx}{x}\right)$ we arrive at the inversion formula of the transformation (2.19) which can be written in the symbolic form

$$f(x)=e^{t\mathcal{A}_x}g_t,\ t>0. \tag{2.27}$$

However, since (2.26) is true also for some subsequence $P_{n_k}$ when the convergence is pointwise, we have the equality (2.27) for almost all $x>0$. Similarly, we can get reciprocally (2.19) starting from (2.27). Finally, we summarize our results in

**Theorem 3.** *Let $t>0$. For any $f\in L_2\left(\mathbb{R}_+;\frac{dx}{x}\right)$, formula* (2.19) *defines for all $x>0$ an infinitely differentiable function $h(x)=(g_t f)(x)$ satisfying condition* (2.24). *Moreover, for almost all $x>0$ the reciprocal inversion formula* (2.27) *holds. Conversely, for any $h$, which satisfies* (2.24) *formula* (2.27) *with the convergence in mean square defines for almost all $x>0$ a function $f\in L_2\left(\mathbb{R}_+;\frac{dx}{x}\right)$ and the reciprocal formula (2.19) takes place.*

Let us prove now that $\mathcal{A}_x^2$ is the infinitesimal generator of the heat kernel (2.1).

**Theorem 4**. *A function $f(x)$ is in $S_2(\mathbb{R}_+)$ if and only if it is in $L_2\left(\mathbb{R}_+;\frac{dx}{x}\right)$ and*

$$I^t(f)=\frac{1}{t}\left[\langle f,f\rangle\ -\ \langle f,\ e^{-t\mathcal{A}_\cdot^2}f\rangle\right]$$

*is uniformly bounded in $t$. (Here we mean $\langle\cdot,\ \cdot\rangle$ is the $L_2\left(\mathbb{R}_+;\frac{dx}{x}\right)$, not $S_2(\mathbb{R}_+)$, inner product. ) In that case*

$$\sup_{t>0}I^t(f)=\lim_{t\to 0}I^t(f)=||\mathcal{A}.f||^2_{L_2\left(\mathbb{R}_+;\frac{dx}{x}\right)}.$$

**Proof.** By Theorem 1 it is sufficient to show that $f(x)\in L_2\left(\mathbb{R}_+;\frac{dx}{x}\right)$ and $I^t(f)$ is uniformly bounded if and only if

$$\int_0^\infty \tau\sinh\pi\tau\left|K_{i\tau}\left[\frac{f(x)}{x}\right]\right|^2(1+\tau^4)d\tau<\infty.$$

First we note by the Parseval identity (1.2) and relation (2.22) for $k=2$

$$I^t(f)=\frac{2}{\pi^2 t}\int_0^\infty \tau\sinh\pi\tau\left[1-e^{-t\tau^4}\right]\left|K_{i\tau}\left[\frac{f(\cdot)}{\cdot}\right]\right|^2 d\tau. \tag{2.28}$$

Meanwhile, since $y^{-1}(1-e^{-y})$ is a decreasing function of $y>0$ and hence $1/t$ times the factor $1-e^{-t\tau^4}$ converges monotonically to $\tau^4$ as $t\to 0$. Thus if $f\in S_2(\mathbb{R}_+)$ (see (2.18)),

$I^t(f)$ is uniformly bounded. Conversely, if $I^t(f)$ is uniformly bounded, the monotone convergence theorem implies that

$$\sup_{t>0} I^t(f) = \lim_{t\to 0} I^t(f) = \frac{2}{\pi^2}\int_0^\infty \tau^5 \sinh \pi\tau \left| K_{i\tau}\left[\frac{f(\cdot)}{\cdot}\right]\right|^2 d\tau$$

$$= ||\mathcal{A}.f||^2_{L_2\left(\mathbb{R}_+;\frac{dx}{x}\right)} < \infty.$$

Theorem 4 is proved.

# 3 Heisenberg's type inequalities

The classical Heisenberg inequality for the Fourier transform states that for $f \in L_2(\mathbb{R}; dx)$

$$\int_{\mathbb{R}} x^2|f(x)|^2 dx \int_{\mathbb{R}} \xi^2 |f(\xi)|^2 d\xi \geq \frac{1}{4}||f||^4, \tag{3.1}$$

where

$$\hat{f}(\xi) = \frac{1}{\sqrt{2}\pi}\int_{\mathbb{R}} f(x) e^{-ixix} dx.$$

In this section we will establish some analogs of the Heisenberg inequality (3.1) for the Kontorovich-Lebedev transform (1.1). To proceed this we call the Parseval identity (1.2). We have

**Theorem 5.** *Let $f \in L_2(\mathbb{R}_+; xdx)$. Then*

$$\int_0^\infty |f(x)|^2 \, dx \int_0^\infty \tau^2 \sinh^2\left(\frac{\pi\tau}{2}\right) |K_{i\tau}[f]|^2 \, d\tau \geq \frac{\pi^2}{4}||f||^4_{L_2(\mathbb{R}_+;xdx)}, \tag{3.2}$$

$$\int_0^\infty |f(x)|^2 dx \left(\int_0^\infty \tau \sinh\left(\frac{\pi\tau}{2}\right) |K_{i\tau}[f]| \, d\tau\right)^2 \geq \frac{\pi^2}{4}||f||^4_{L_2(\mathbb{R}_+;xdx)}, \tag{3.3}$$

$$\int_0^\infty |f(x)|^p \, dx \int_0^\infty \tau^p \sinh^p\left(\frac{\pi\tau}{2}\right) |K_{i\tau}[f]|^p \, d\tau$$

$$\geq \frac{\pi^{1+\frac{p}{2}}}{2^p}\left[\frac{\Gamma\left(\frac{p-1}{2(2-p)}\right)}{\Gamma\left((2(2-p))^{-1}\right)}\right]^{p-2} ||f||^{2p}_{L_2(\mathbb{R}_+;xdx)}, \ 1 < p < 2. \tag{3.4}$$

**Proof.** In fact, writing (1.2) in the form

$$||f||^2_{L_2(\mathbb{R}_+;xdx)} = \frac{4}{\pi^2}\int_0^\infty \tau \sinh\left(\frac{\pi\tau}{2}\right) \cosh\left(\frac{\pi\tau}{2}\right) |K_{i\tau}[f]|^2 \, d\tau,$$

we appeal to the Schwarz inequality to get

$$||f||^2_{L_2(\mathbb{R}_+;xdx)} \le \frac{4}{\pi^2}\left(\int_0^\infty \tau^2 \sinh^2\left(\frac{\pi\tau}{2}\right)|K_{i\tau}[f]|^2 d\tau\right)^{1/2}$$

$$\left(\int_0^\infty \left|\cosh\left(\frac{\pi\tau}{2}\right) K_{i\tau}[f]\right|^2 d\tau\right)^{1/2}. \tag{3.5}$$

On the other hand by relation (2.16.14.1) from [5], Vol.2, the Parseval equality for the cosine Fourier transform and an elementary substitution we find in the case $f \in L_2(\mathbb{R}_+;xdx) \cap L_2(\mathbb{R}_+;dx)$

$$\cosh\left(\frac{\pi\tau}{2}\right) K_{i\tau}[f] = \sqrt{\frac{\pi}{2}} \int_0^\infty F_c[f;\sinh u]\cos\tau u \ du.$$

Substituting in (3.5) we obtain

$$||f||^2_{L_2(\mathbb{R}_+;xdx)} \le \frac{2}{\pi}\left(\int_0^\infty \tau^2 \sinh^2\left(\frac{\pi\tau}{2}\right)|K_{i\tau}[f]|^2 d\tau\right)^{1/2}\left(\int_0^\infty |F_c[F_c[f;\sinh u];\tau]|^2 d\tau\right)^{1/2}$$

$$= \frac{2}{\pi}\left(\int_0^\infty \tau^2 \sinh^2\left(\frac{\pi\tau}{2}\right)|K_{i\tau}[f]|^2 d\tau\right)^{1/2}\left(\int_0^\infty |F_c[f;\sinh u]|^2 du\right)^{1/2}$$

$$\le \frac{2}{\pi}\left(\int_0^\infty \tau^2 \sinh^2\left(\frac{\pi\tau}{2}\right)|K_{i\tau}[f]|^2 d\tau\right)^{1/2}\left(\int_0^\infty |F_c[f;u]|^2 du\right)^{1/2}$$

$$= \frac{2}{\pi}\left(\int_0^\infty \tau^2 \sinh^2\left(\frac{\pi\tau}{2}\right)|K_{i\tau}[f]|^2 d\tau\right)^{1/2}\left(\int_0^\infty |f(x)|^2 \ dx\right)^{1/2},$$

which implies (3.2). In order to prove (3.3) we use the limit case of the Hölder inequality to find

$$||f||^2_{L_2(\mathbb{R}_+;xdx)} \le \frac{4}{\pi^2}\int_0^\infty \tau \sinh\left(\frac{\pi\tau}{2}\right)|K_{i\tau}[f]|\, d\tau$$

$$\times \sup_{\tau>0}\left|\cosh\left(\frac{\pi\tau}{2}\right) K_{i\tau}[f]\right| \le \frac{2\sqrt{2}}{\pi\sqrt{\pi}}\int_0^\infty \tau \sinh\left(\frac{\pi\tau}{2}\right)|K_{i\tau}[f]|\, d\tau$$

$$\times \int_0^\infty |F_c[f;u]| \frac{du}{\sqrt{u^2+1}} \le \frac{2}{\pi}\int_0^\infty \tau \sinh\left(\frac{\pi\tau}{2}\right)|K_{i\tau}[f]|\, d\tau$$

$$\times \left(\int_0^\infty |F_c[f;u]|^2 du\right)^{1/2} = \frac{2}{\pi}\int_0^\infty \tau \sinh\left(\frac{\pi\tau}{2}\right)|K_{i\tau}[f]|\, d\tau \left(\int_0^\infty |f(x)|^2 dx\right)^{1/2}.$$

Thus we arrive at the inequality (3.3).

Generally, assuming $1 < p < 2$ we appeal to the Hölder inequality to deduce

$$||f||^2_{L_2(\mathbb{R}_+;xdx)} \le \frac{4}{\pi^2}\left(\int_0^\infty \tau^p \sinh^p\left(\frac{\pi\tau}{2}\right)|K_{i\tau}[f]|^p\, d\tau\right)^{1/p}$$

$$\times\left(\int_0^\infty \left|\cosh\left(\frac{\pi\tau}{2}\right)K_{i\tau}[f]\right|^q d\tau\right)^{1/q},\ q = \frac{p}{p-1}. \tag{3.6}$$

The latter $q$-norm can be estimated by the familiar Hausdorff-Young inequality [8], [12] together with the Hölder inequality. Thus we obtain

$$\left(\int_0^\infty \left|\cosh\left(\frac{\pi\tau}{2}\right)K_{i\tau}[f]\right|^q d\tau\right)^{1/q} \le \left(\frac{\pi}{2}\right)^{\frac{1}{q}+\frac{1}{2}}\left(\int_0^\infty |F_c[f;\sinh u]|^p du\right)^{1/p}$$

$$= \left(\frac{\pi}{2}\right)^{\frac{1}{q}+\frac{1}{2}}\left(\int_0^\infty |F_c[f;u]|^p \frac{du}{\sqrt{u^2+1}}\right)^{1/p} \le \left(\frac{\pi}{2}\right)^{\frac{1}{q}+\frac{1}{2}}\left(\int_0^\infty |F_c[f;u]|^q du\right)^{1/q}$$

$$\times\left(\int_0^\infty (u^2+1)^{-(2(2-p))^{-1}}\, du\right)^{\frac{2-p}{p}} = \pi\ 2^{-\left(\frac{1}{p}+\frac{1}{2}\right)}\left[\frac{\Gamma\left(\frac{p-1}{2(2-p)}\right)}{\Gamma\left((2(2-p))^{-1}\right)}\right]^{\frac{2-p}{p}}$$

$$\times\left(\int_0^\infty |F_c[f;u]|^q du\right)^{1/q} \le \frac{\pi^{\frac{3}{2}-\frac{1}{p}}}{2}\left[\frac{\Gamma\left(\frac{p-1}{2(2-p)}\right)}{\Gamma\left((2(2-p))^{-1}\right)}\right]^{\frac{2-p}{p}}\left(\int_0^\infty |f(x)|^p du\right)^{1/p}.$$

Hence combining with (3.6) we easily come out with final inequality (3.4). Theorem 5 is proved.

**Corollary 1.** *Letting $p \to 2-$ in* (3.4) *we get* (3.2). *Putting $p = \frac{3}{2}$ in* (3.4) *we come out immediately with the following Heisenberg inequality*

$$\int_0^\infty |f(x)|^{3/2}\, dx \int_0^\infty \tau^{3/2}\sinh^{3/2}\left(\frac{\pi\tau}{2}\right)|K_{i\tau}[f]|^{3/2}\, d\tau \ge \left(\frac{\pi}{2}\right)^{3/2}||f||^3_{L_2(\mathbb{R}_+;xdx)}.$$

Another type of Heisenberg inequalities for the Kontorovich-Lebedev transform are based on the Mellin transform of the heat kernel (2.1) by $y$. These inequalities will give estimates of the convolution (1.13) $x^{\mu-1} * f,\ \mu > 0$ in the space $L_2(\mathbb{R}_+; xdx)$. In fact, using the Mellin transform (1.13) of the modified Bessel function we apply it to the heat kernel (2.1) and we find

$$h_t^*(x,\mu) \equiv \int_0^\infty h(t,x,y)y^{\mu-1}dy$$

$$= \frac{2^{\mu-1}}{x\pi^2}\int_0^\infty e^{-tv^2}v\ \sinh\pi v\left|\Gamma\left(\frac{\mu}{2}+\frac{iv}{2}\right)\right|^2 K_{iv}(x)dv. \tag{3.7}$$

Hence via (1.1), (1.3) we derive

$$K_{i\tau}[h_t^*(\cdot,\mu)] = 2^{\mu-2}e^{-t\tau^2}\left|\Gamma\left(\frac{\mu}{2}+\frac{i\tau}{2}\right)\right|^2, \ t,\mu>0. \tag{3.8}$$

Function (3.7) has an independent interest and was employed in various applications (see, for instance, in [4]).

In order to proceed to our goals we will first estimate the norm $||h_t^*||_{L_2(\mathbb{R}_+;xdx)}$. To do this we call the Parseval equality (1.2). Hence by virtue of (3.8)

$$||h_t^*||^2_{L_2(\mathbb{R}_+;xdx)} = \frac{2^{2\mu-3}}{\pi^2}\int_0^\infty e^{-2t\tau^2}\tau\sinh\pi\tau\left|\Gamma\left(\frac{\mu}{2}+\frac{i\tau}{2}\right)\right|^4 d\tau. \tag{3.9}$$

In the meantime to estimate the modulus of the gamma-function we will appeal to the following inequality, which is a direct consequence of Stirling's formula

$$|\Gamma(x+iy)| \le \sqrt{2\pi}|z|^{x-1/2}e^{-\pi|y|/2}e^{1/(6|z|)}, \ z=x+iy, \ x>0.$$

Therefore, for $t>0$

$$||h_t^*||^2_{L_2(\mathbb{R}_+;xdx)} \le 2e^{4/(3\mu)}\int_0^\infty e^{-\pi\tau-2t\tau^2}\tau\sinh\pi\tau\,|\mu+i\tau|^{2(\mu-1)}\,d\tau$$

$$\le e^{4/(3\mu)}\int_0^\infty e^{-2ty}\left(\mu^2+y\right)^{\mu-1}dy$$

$$\le (2t)^{-1}\mu^{2(\mu-1)}e^{4/(3\mu)}, \quad \mu\in(0,1],$$

and

$$||h_t^*||^2_{L_2(\mathbb{R}_+;xdx)} \le e^{4/(3\mu)}\int_0^\infty e^{-2ty}\left(\mu^2+y\right)^{\mu-1}dy$$

$$= \mu^{2\mu}e^{4/(3\mu)}\left(\int_0^1+\int_1^\infty\right)e^{-2t\mu^2 y}\left(1+y\right)^{\mu-1}dy < 2^{\mu-2}\mu^{2(\mu-1)}e^{4/(3\mu)}t^{-1}$$

$$+2^{\mu-1}\mu^{2\mu}e^{4/(3\mu)}\int_0^\infty e^{-2t\mu^2 y}y^{\mu-1}dy$$

$$= \frac{e^{4/(3\mu)}}{2}\left[(2\mu^2)^{\mu-1}\,t^{-1}+\Gamma(\mu)t^{-\mu}\right], \ \mu>1.$$

Hence, it has the estimates

$$||h_t^*||_{L_2(\mathbb{R}_+;xdx)} \le (2t)^{-1/2}\mu^{\mu-1}e^{2/(3\mu)}, \quad t>0, \ \mu\in(0,1], \tag{3.10}$$

$$||h_t^*||_{L_2(\mathbb{R}_+;xdx)} \le \frac{e^{2/(3\mu)}}{\sqrt{2}}\left[(2\mu^2)^{\mu-1}\,t^{-1}+\Gamma(\mu)t^{-\mu}\right]^{1/2}, \quad t>0, \ \mu>1. \tag{3.11}$$

Further, from the Parseval equality (1.2) taking into account (1.12), (1.13), (1.19), (1.22), (3.8) and the inequality $|\Gamma(z)| \le \Gamma(\mathrm{Re} z),\ \mathrm{Re} z > 0$ for the Euler gamma-function we obtain

$$||x^{\mu-1} * f||_{L_2(\mathbb{R}_+;xdx)} = \frac{2^{\mu-3/2}}{\pi}\left(\int_0^\infty \tau\sinh\pi\tau\left|\Gamma\left(\frac{\mu}{2}+\frac{i\tau}{2}\right)\right|^4 |K_{i\tau}[f]|^2 d\tau\right)^{1/2}$$

$$= \frac{\sqrt{2}}{\pi}\left(\int_0^\infty \tau\sinh\pi\tau\left|K_{i\tau}\left[x^{\mu-1} * f\right]\right|^2 d\tau\right)^{1/2}$$

$$\le \frac{\sqrt{2}}{\pi}\left|\left|\left(1-e^{-t\tau^2}\right)K_{i\tau}[x^{\mu-1} * f]\right|\right|_{L_2(\mathbb{R}_+;\tau\sinh\pi\tau d\tau)} + \frac{\sqrt{2}}{\pi}\left|\left|e^{-t\tau^2}K_{i\tau}[x^{\mu-1} * f]\right|\right|_{L_2(\mathbb{R}_+;\tau\sinh\pi\tau d\tau)}$$

$$\le \frac{t\sqrt{2}}{\pi}\left|\left|\tau^2 K_{i\tau}[x^{\mu-1} * f]\right|\right|_{L_2(\mathbb{R}_+;\tau\sinh\pi\tau d\tau)} + \frac{\sqrt{2}}{\pi}||h_t^* * f||_{L_2(\mathbb{R}_+;xdx)}$$

$$\le \frac{2^{\mu-3/2}\, t}{\pi}\Gamma^2(\mu/2)\left|\left|\tau^2 K_{i\tau}[f]\right|\right|_{L_2(\mathbb{R}_+;\tau\sinh\pi\tau d\tau)} + \frac{\sqrt{2}}{\pi}||h_t^*||_{L_2(\mathbb{R}_+;xdx)}\,||f||_{L^0(\mathbb{R}_+)}. \qquad (3.12)$$

But,

$$||f||_{L^0(\mathbb{R}_+)} = \int_0^\infty K_0(x)|f(x)|dx \le ||f||_{L_2(\mathbb{R}_+;dx)}\left(\int_0^\infty K_0^2(x)dx\right)^{1/2} = \frac{\pi}{2}||f||_{L_2(\mathbb{R}_+;dx)}.$$

Hence using (3.10) we obtain from (3.12) for all $t > 0$ and $\mu \in (0,1]$ the inequality

$$||x^{\mu-1} * f||_{L_2(\mathbb{R}_+;xdx)} \le \frac{2^{\mu-3/2}\, t}{\pi}\Gamma^2(\mu/2)\left|\left|\tau^2 K_{i\tau}[f]\right|\right|_{L_2(\mathbb{R}_+;\tau\sinh\pi\tau d\tau)}$$

$$+\frac{\mu^{\mu-1}e^{2/(3\mu)}}{2\sqrt{t}}||f||_{L_2(\mathbb{R}_+;dx)}. \qquad (3.13)$$

However, when $\mu = 1$ we can get more sharp estimate. In fact, equality (3.9) becomes

$$||h_t^*||^2_{L_2(\mathbb{R}_+;xdx)} = \int_0^\infty e^{-2t\tau^2}\tau\tanh\left(\frac{\pi\tau}{2}\right)d\tau.$$

Hence

$$||h_t^*||^2_{L_2(\mathbb{R}_+;xdx)} \le \int_0^\infty e^{-2t\tau^2}\tau d\tau = \frac{1}{2t}.$$

Analogously, using this estimate in (3.12) we come out with the inequality

$$||1\ * f||_{L_2(\mathbb{R}_+;xdx)} \le \frac{t}{\sqrt{2}}\left|\left|\tau^2 K_{i\tau}[f]\right|\right|_{L_2(\mathbb{R}_+;\tau\sinh\pi\tau d\tau)} + \frac{1}{2\sqrt{t}}||f||_{L_2(\mathbb{R}_+;dx)}. \qquad (3.14)$$

Estimates (3.13), (3.14) are key relations to prove the Heisenberg inequalities for the Kontorovich-Lebedev transform. Letting in (3.14), for instance,

$$t = \left( \frac{||f||_{L_2(\mathbb{R}_+;dx)}}{\sqrt{2}\, ||\tau^2 K_{i\tau}[f]||_{L_2(\mathbb{R}_+;\tau \sinh \pi\tau d\tau)}} \right)^{2/3}$$

we arrive at the following inequality

$$\int_0^\infty |f(x)|^2 dx \left( \int_0^\infty \tau^5 \sinh \pi\tau |K_{i\tau}[f]|^2 d\tau \right)^{1/2} \geq \frac{1}{\sqrt{2}} ||1 * f||^3_{L_2(\mathbb{R}_+;xdx)}.$$

Generally, when

$$t = \left( \frac{||f||_{L_2(\mathbb{R}_+;dx)} \pi \sqrt{2} (\mu/2)^\mu e^{2/(3\mu)}}{||\tau^2 K_{i\tau}[f]||_{L_2(\mathbb{R}_+;\tau \sinh \pi\tau d\tau)}\, \mu\, \Gamma^2(\mu/2)} \right)^{2/3}$$

then from (3.13) we deduce

$$\int_0^\infty |f(x)|^2 dx \left( \int_0^\infty \tau^5 \sinh \pi\tau |K_{i\tau}[f]|^2 d\tau \right)^{1/2}$$

$$\geq \frac{2^{-\mu-5/2}\, \pi\, e^{-4/(3\mu)}}{\mu^{2(\mu-1)}\, \Gamma^2(\mu/2)}\, ||x^{\mu-1} * f||^3_{L_2(\mathbb{R}_+;xdx)}, \ \mu \in (0,1].$$

**Remark 1**. Analogously one can prove Heisenberg's inequalities employing the estimate (3.11).

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