The dynamics of a conservative Hénon map

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Abstract

We construct an area-preserving homeomorphism of the plane that conjugates a conservative Hénon map to a translation.

1 Introduction and statement of the results

In this paper we consider an area-preserving map derived from the Hénon map defined by $H(x, y) = (a - x^2 - y, x)$, where $a < -1$. The main result is the following:

Theorem 1 *There exists an area-preserving C^0-conjugacy of the conservative Hénon map and $T(x, y) = (x + 1, y)$.*

To prove Theorem 1 we start constructing a fundamental domain of H, $\mathcal{D}$, whose saturate fills $\mathbb{R}^2$ (Lemma 3.1). We consider a fundamental domain $\mathcal{D}_c$ for the translation T and split $\mathcal{D}$ and $\mathcal{D}_c$ into countable bounded pieces. Then we construct local diffeomorphisms, whose domains cover $\mathcal{D}$, which conjugate the restrictions of H and T to each piece and such that each one preserves the area of its domain. Next, using Dacorogna-Moser Theorem (Theorem 4.1), we get an area-preserving homeomorphism from $\mathcal{D}$ to $\mathcal{D}_c$. Finally, using Lemma 3.1 and the fact that H is area-preserving, we extend these maps conservatively to $\mathbb{R}^2$ to get the area-preserving homeomorphism Ψ.

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This conservative conjugacy Ψ trivializes H which is very useful if one aims to build perturbations and remain in the area-preserving setting.

We observe that the arguments we developed to prove Lemma 3.1 may be used to prove that H has only a fundamental region (Definition 5.1). Hence by using the fact, proved by Andrea ([1]), that maps with only one fundamental region are conjugated to T, we obtain another prove that H and T are conjugated. However this does not guarantee that there exists an area-preserving conjugacy.

2 Some preliminary lemmas

Definition 2.1 *We consider the map $H_{a,b}$ with parameters $a, b \in \mathbb{R}$:*

$$\begin{array}{rccc} H_{a,b}: & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & (x,y) & \longrightarrow & (a - x^2 - by, x) \end{array}$$

*This is the so-called Hénon map. When we consider $|b| = 1$, then for all $p \in \mathbb{R}^2$ we have $|det(DH_{a,1})_p| = 1$, so the map preserves the area and that is why we call it **conservative Hénon map**. If $b = 1$, then, since $det(DH_{a,1}) = 1$, $H_{a,1}$ preserves also the orientation. Depending on the parameter $a \in \mathbb{R}$, $H_{a,1}$ may have, or not, fixed points. In fact, it is easy to see that $Fix(H_{a,1}) = \emptyset$ if and only if $a < -1$.*

In what follows we deduce some properties of $H_{a,1}$, which will be denoted by H and we consider $a < -1$. It is clear that H is a homeomorphism of the plane with inverse defined by $H_{a,1}^{-1}((x,y)) = (y, a - y^2 - x)$.

Remark 2.1 *Let $S((x,y)) = (y,x)$, hence*

$$S(H^{-1}(S(x,y))) = S(H^{-1}(y,x)) = S(x, a-x^2-y) = (a-x^2-y, x) = H(x,y).$$

Therefore, H is topologically conjugate to its inverse by using the symmetry on the line $y = x$, say $H \circ S = S \circ H^{-1}$. In this case the conjugacy is equal to its inverse, say $S = S^{-1}$. We conclude that the action of the map H^{-1} may be seen as the reflection on the line $y = x$ of the action of H itself.

We will deduce some elementary properties of the dynamics of the conservative Hénon map. Let us define

$I = \{(x, y) : x, y \geq 0\}$, $II = \{(x, y) : x \leq 0, y \geq 0\}$, $III = \{(x, y) : x, y \leq 0\}$ and $IV = \{(x, y) : x \geq 0, y \leq 0\}$.
The subset of III defined by $\{(x, y) \in \mathbb{R}^2 : x \leq y\}$ will be denoted by D^+, finally denote $S(D^+)$ by D^-.

Lemma 2.1 . *One has*

1) $H(D^+) \subseteq D^+$ *and* $H^n(D^+) \subseteq int(D^+)$, $\forall n \geq 2$.
2) $H^{-1}(D^-) \subseteq D^-$ *and* $H^{-n}(D^-) \subseteq int(D^-)$, $\forall n \geq 2$.

Proof: First, using Remark 2.1 and the definition of D^-, note that

$$H^{-1}(D^-) = H^{-1}S((D^+)) = S \circ H(D^+).$$

Therefore *2)* follows from *1)*.

Now fix $(x, y) \in D^+$ and let $(x_1, y_1) = H(x, y) = (a - x^2 - y, x)$. It is clear that $y_1 = x \leq 0$ and, since $a - x^2 - x \leq x$, $\forall x \in \mathbb{R}$, we get $x_1 = a - x^2 - y \leq a - x^2 - x \leq x = y_1$. Therefore $(x_1, y_1) \in D^+$, which implies that $H(D^+) \subseteq D^+$. Finally, to prove that $H^n(D^+) \subseteq int(D^+)$, $\forall n \geq 2$, we first note $(0, 0)$ is the unique point of D^+ whose image belongs to the boundary of D^+.

Since,

$$H^2((0, 0)) = H((a, 0)) = (a - a^2, a) \in int(D^+),$$

we conclude that $H^2(D^+) \subseteq int(D^+)$ and so,

$$H^n(D^+) = H^{n-2}(H^2(D^+)) \subseteq H^{n-2}(int(D^+)) \subseteq int(D^+), \forall n \geq 2.$$

□

Lemma 2.2 *Given* $(x, y) \in \mathbb{R}^2$ *we have:*

1) If $(x, y) \in I$, *then* $H((x, y)) \in II$.
2) If $(x, y) \in II$, *then* $H((x, y)) \in D^+$.
3) If $(x, y) \in IV$, *then* $H((x, y)) \in I$ *or* $H((x, y)) \in II$.
4) If $(x, y) \in III$, *then* $H((x, y)) \in III$ *or* $H((x, y)) \in IV$.
5) If $H^n((x, y)) \in III$, $\forall n \in \mathbb{N}_0$, *then* $\exists k \in \mathbb{N}_0$ *such that* $H^k((x, y)) \in D^+$.

Proof:

1) If $(x,y) \in I$ then, since $H((x,y)) = (a - x^2 - y, x)$, $a - x^2 - y \le a < 0$ and $x \ge 0$, we get $H((x,y)) \in II$.

2) If $(x,y) \in II$, then $x \le 0$ and $y \ge 0$. Since $H((x,y)) = (a - x^2 - y, x)$ it remains to prove that $a - x^2 - y \le x$. Since $a < -1$ for all $x \in \mathbb{R}$ it follows that $a - x^2 - 2x \le 0$, therefore $a - x^2 - x \le x$ and, since $-y \le -x$, we conclude that $a - x^2 - y \le a - x^2 - x \le x$. So $H((x,y)) \in D^+$.

3) If $(x,y) \in IV$, then since $H((x,y)) = (a - x^2 - y, x)$ and $x \ge 0$ it is clear that $H((x,y))$ belongs to I or to II.

4) If $(x,y) \in III$, then since $H((x,y)) = (a - x^2 - y, x)$, having $H((x,y))$ in III or in IV is equivalent to $x \le 0$, which is obvious.

5) Consider $(x_0, y_0) \in III$ such that $H^n((x_0,y_0)) \in III$, $\forall n \in \mathbb{N}$. Assume that $H^n((x_0,y_0)) \notin D^+$, $\forall n \in \mathbb{N}$. Denote $(x_n, y_n) = H^n((x_0,y_0))$ and we have $0 \ge x_n \ge y_n = x_{n-1}$, $\forall n \in \mathbb{N}$. So, $(x_n)_{n\in\mathbb{N}}$ is an increasing bounded sequence, therefore it converges to $p \in \mathbb{R}_0^+$. Since $y_n = x_{n-1}$ it follows that $(y_n)_{n\in\mathbb{N}} \underset{n\longrightarrow+\infty}{\longrightarrow} p$. So, $H^n((x_0,y_0)) \underset{n\longrightarrow+\infty}{\longrightarrow} (p,p)$ and so (p,p) is a fixed point for H, which is a contradiction. □

Lemmas 2.1 and 2.2 allow us to conclude that for all $(x,y) \in \mathbb{R}^2$ there exists $n_0 \in \mathbb{N}$, depending on (x,y), such that $H^n((x,y)) \in int(D^+)$, $\forall n \ge n_0$. Analogously we conclude that exists $n_1 \in \mathbb{N}$ such that $H^{-n}((x,y)) \in int(D^-)$, $\forall n \ge n_1$. It follows that we just need to know the dynamics of H inside D^+ (respectively the dynamics of H^{-1} inside D^-). Figure 1 illustrates the image by H of the four regions.

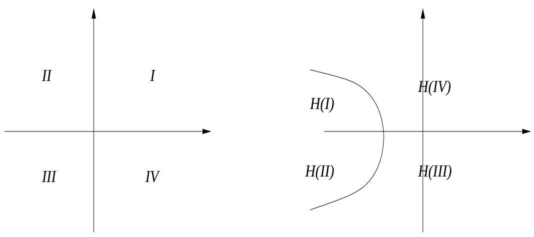

Figure 1: The dynamics of H on each region I, II, III and IV.

Lemma 2.3 *Let $(x_0,y_0) \in D^+$ and define $(x_n,y_n) = H^n((x_0,y_0))$. Then $\exists n_0 \in \mathbb{N}$, depending on (x_0,y_0), such that $\forall n \in \mathbb{N}$ with $n \ge n_0$ we have:*

1) $x_{n+1} \leq x_n + a$
2) $y_{n+1} \leq y_n$ *e* $y_{n+2} \leq y_n + a$

Proof: Given $(x_1, y_1) = (a - x_0^2 - y_0, x_0)$ we have that $y_1 = x_0 \leq y_0$ and $x_1 = a - x_0^2 - y_0 \leq a - x_0^2 - x_0 \leq x_0$, so the sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ are decreasing. Note that, $\forall (x, y) \in D^+$, there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} < -2$ because otherwise it implies the existence of a fixed point.
Since for $x < -2$ we have $x_{n+1} = a - x_n^2 - y_n \leq a - x_n^2 - x_n \leq x_n + a$, inequality *1)* is proved.

To prove *2)* we note that the first inequality follows from $(y_n)_{n \in \mathbb{N}}$ being decreasing and the second inequality follows directly from the fact that $y_{n+2} = x_{n+1} \leq x_n + a \leq y_n + a$. □

Lemma 2.3 tell us that under only two iterates and since $a < -1$, points move bellow a distance at least one and to the left a distance larger than two. So this result together with Lemmas 2.1 and 2.2 give us a very good description of how the orbit a point *tends to infinity.*

3 Construction of the conjugacy between the T and H

First we define the fundamental domain $\mathcal{D}$. Let $\lambda = \{(x, y) : y = x\}$ and $H(\lambda)$ its image by H. Let $\mathcal{D}$ be the connected, unbounded region of the plane delimited by λ and $H(\lambda)$. We will see first that the saturate of $\mathcal{D}$ fills the all plane.

Lemma 3.1 $\bigcup_{n \in \mathbb{Z}} H^n(\mathcal{D}) = \mathbb{R}^2$.

Proof: Note that it is sufficient to prove that, given (x, y) with $x \leq y$, there exists $n \in \mathbb{N}_0$ such that $H^{-n}((x, y)) \in \mathcal{D}$, because for the region $x \geq y$ we use the fact that $H^{-1} = S \circ H \circ S$. Since $H^2(\lambda)$ is contained in the region III it follows that, for $(x, y) \in II \setminus \mathcal{D}$, we have that $H^{-1}((x, y)) \in \mathcal{D}$.
It remains to consider points $(x, y) \in D^+ \setminus \mathcal{D}$. So, $H^{-1}((x, y))$ belongs to II or to III. Note that, if $H^{-1}((x, y)) \in II$ or more generally there exists $n \in \mathbb{N}$ such that $H^{-n}((x, y)) \in II$, then $H^{-1}((x, y)) \in \mathcal{D}$ or $H^{-2}((x, y)) \in \mathcal{D}$. Then, we just have to consider the case when $(x, y) \in D^+$ and $H^{-n}((x, y)) \in III$, $\forall n \in \mathbb{N}$. Suppose, by contradiction that $H^{-n}((x, y)) \notin \mathcal{D}$, $\forall n \in \mathbb{N}$; then, since $(x, y) \in D^+ \setminus \mathcal{D}$ we get $x < a - y^2 - y$ and so $H^{-1}((x, y)) \in D^+$.

We obtain a sequence in D^+, $(x_{-n}, y_{-n}) = H^{-n}((x,y))$, $n \in \mathbb{N}$, such that $x_{-n} \geq x_{-n+1}$ and $y_{-n} \geq y_{-n+1}$, $\forall n \in \mathbb{N}$. Therefore $((x_{-n}, y_{-n}))_{n\in\mathbb{N}}$ converges to (x_0, y_0) which is a fixed point, and we get a contradiction. $\square$

Remark 3.1 *Clearly* $int(\mathcal{D}) \cap H^n(int(\mathcal{D})) = \emptyset$, $\forall n \in \mathbb{Z} \setminus \{0\}$.

Now we will construct the topological conjugacy, i.e. a homeomorphism $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $\Phi \circ H = T \circ \Phi$, where $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is the translation $T((x,y)) = (x+1, y)$. Let λ_c denote the line $x = 0$ and

$$\mathcal{D}_c = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1\},$$

the idea is to construct first a homeomorphism $\widetilde{\Phi} : \mathcal{D} \longrightarrow \mathcal{D}_c$ such that $\widetilde{\Phi}(\lambda) = \lambda_c$ and $\widetilde{\Phi}(H(x)) = T(\widetilde{\Phi}(x))$, $\forall x \in \lambda$, and then extend it to $\mathbb{R}^2$.

Denote by $[a, b]$ the oriented segment from the point a through the point b, and, for $x \neq 0$, let $\mathcal{A}(x)$ be the region bounded by the three segments $[(a - x^2 - x, x), (x, x)]$, $[(x, x), (0, 0)]$ and $[(0, 0), (a, 0)]$ and by the curve $H([(0, 0), (x, x)])$; for $x' \neq 0$ let $\mathcal{B}(x')$ be the region delimited by the quadrilateral with vertices $(0, 0)$, $(1, 0)$, $(1, x')$ and $(0, x')$.

We want a diffeomorphism $\tilde{\Phi}$ such that, for all $x \neq 0$:

- $\tilde{\Phi}([(0, 0), (x, x)]) = [(0, 0), (0, x')]$,
- $\tilde{\Phi} \circ H((x, x)) = T \circ \tilde{\Phi}((x, x))$, and
- $\tilde{\Phi}(\mathcal{A}(x)) = \mathcal{B}(x')$,

where x' depends on x and will be chosen such that $\mu(\mathcal{A}(x)) = \mu(\mathcal{B}(x'))$, where μ denotes the Lebesgue measure, see Figure 2.

Construction of $\tilde{\Phi}$:

We consider the parametrized segment:

$$(1 - t)(x, x) + t(a - x^2 - x, x), \ t \in [0, 1]$$

and send this segment, by the map $\widetilde{\Phi}$, into the segment:

$$(1 - t)(0, x') + t(1, x'), \ t \in [0, 1],$$

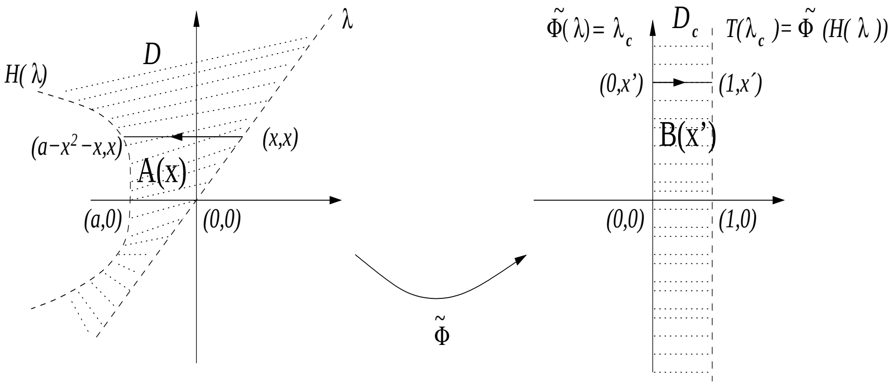

Figure 2: Construction of the conjugacy.

where x' will be defined bellow. We have that

$$\mu(\mathcal{A}(x)) = \int_0^{|x|} t - (a - t^2 - t)dt = -a|x| + \frac{|x|^3}{3} + x^2, \quad x \in \mathbb{R} \setminus \{0\}.$$

So we take

- $x' := -ax + \frac{x^3}{3} + x^2$, if $x > 0$, and
- $x' := -(-a|x| + \frac{|x|^3}{3} + x^2) = -ax + \frac{x^3}{3} - x^2$, otherwise.

Therefore, if $x \geq 0$, we define:

$$\widetilde{\Phi}(((1-t)x + t(a - x^2 - x), x)) = (t, x') = (t, -ax + \frac{x^3}{3} + x^2),$$

and if $x \leq 0$, we define:

$$\widetilde{\Phi}(((1-t)x + t(a - x^2 - x), x)) = (t, x') = (t, -ax + \frac{x^3}{3} - x^2),$$

Now, let $(x, y) \in int(\mathcal{A}(x))$ and assume that $y \geq 0$; this point belongs to $[(y, y), (a - y^2 - y, y)]$, so there exists one and only one $t \in [0, 1]$ such that:

$$(x, y) = (1-t)(y, y) + t(a - y^2 - y, y)$$

hence we obtain:

$$x = (1-t)y + t(a - y^2 - y) \Leftrightarrow x - y = t(a - y^2 - 2y) \Leftrightarrow t = \frac{x - y}{a - y^2 - 2y}$$

and since $\widetilde{\Phi}((y,y)) = (0,y')$ and $\widetilde{\Phi}((a-y^2-y,y)) = (1,y')$ we can define:

$$\widetilde{\Phi}((x,y)) = (1-t)(0,y') + t(1,y') = (t,y') = (\frac{x-y}{a-y^2-2y}, y').$$

If $y \leq 0$ we can argue exactly in the same way and define:

$$\widetilde{\Phi}((x,y)) = (1-t)(0,y') + t(1,y') = (t,y') = (\frac{x-y}{a-y^2-2y}, y').$$

Using the previous definition of y' we conclude that the homeomorphism $\widetilde{\Phi} : \mathcal{D} \longrightarrow \mathcal{D}_(c)$ is defined by:

$$\widetilde{\Phi}((x,y)) = (\frac{x-y}{a-y^2-2y}, -ay + \frac{y^3}{3} + y^2), \text{ if } y \geq 0, \text{ or}$$

$$\widetilde{\Phi}((x,y)) = (\frac{x-y}{a-y^2-2y}, -ay + \frac{y^3}{3} - y^2), \text{ if } y \leq 0.$$

Clearly $\mu(\mathcal{A}(y)) = \mu(\mathcal{B}(y'))$. Note that, by construction, if $p \in \lambda$, then $\tilde{\Phi} \circ H(p) = T \circ \tilde{\Phi}(p)$.

Construction of Φ:

In order to obtain the desired conjugacy Φ we extend $\widetilde{\Phi}$ to the plane in the standard way. Lemma 3.1 assures that, for each $p \in \mathbb{R}^2$, there exists $n_p \in \mathbb{Z}$ such that $H^{n_p}(p) \in \mathcal{D}$, therefore we define $\Phi(p) = T^{-n_p} \circ \widetilde{\Phi} \circ H^{n_p}(p)$. It is easy to verify that the map Φ is well defined, that it is a homeomorphism and that $\Phi \circ H = T \circ \Phi$.

4 Obtaining the area-preserving conjugacy

Write $\mathcal{D} = \cup_{i \in \mathbb{Z}} C_i$, where, for each $i \in \mathbb{Z}$, $C_i = \mathcal{A}(i) \setminus \mathcal{A}(i-1)$. In order to construct the area-preserving conjugacy Ψ we first modify $\tilde{\Phi}$ in the interior of each bounded region C_i in such way that the resulting map, say $\tilde{\Psi}_i$ is area-preserving and coincides with $\tilde{\Phi}$ on ∂C_i. As the maps $\tilde{\Psi}_i$ and $\tilde{\Psi}_{i+1}$ coincide on $[(i,i),(a-i^2-i,i)]$, for all $i \in \mathbb{Z}$, these maps define an area-preserving map on $\mathcal{D}$, say $\tilde{\Psi}$, such that $\tilde{\Psi}(\mathcal{D}) = \mathcal{D}_c$; as the maps $\tilde{\Psi}_i$

coincide with $\tilde{\Phi}$ on λ and $H(\lambda)$ one has that $\tilde{\Psi} \circ H(p) = T \circ \tilde{\Psi}(p)$, for all $p \in \lambda$. Finally the area-preserving conjugacy is obtained extending the map $\tilde{\Psi}$ to the plane as before.

Let us recall the following theorem due to Dacorogna and Moser (Theorem 5, [2]) that will be used to obtain a conservative local change of coordinates that is crucial to get the maps $\tilde{\Psi}_i$. We point out that this version of Dacorogna-Moser theorem consider only regularity in the interior of a set Ω, whereas Theorem 1 of [2] requires a smooth boundary and as a consequence they obtain a smooth change of coordinates. Since we are only interested in homeomorphisms for the change of coordinates the following Theorem is sufficient.

Theorem 4.1 *(Dacorogna-Moser) Let $\Omega \subset \mathbb{R}^2$ be an open, bounded and connected set. Let $f, g : \overline{\Omega} \to \mathbb{R}$ be positive functions of class C^s for $s \in \mathbb{N}$ and such that $\int_\Omega f(y)dy = \int_\Omega g(y)dy$. Then there exists $\varphi : \overline{\Omega} \to \overline{\Omega}$ of class C^s with $\varphi(x) = x$ for $x \in \partial\Omega$ and such that $\int_U f(y)dy = \int_{\varphi(U)} g(y)dy$ for every open set $U \subset \Omega$.*

By a straightforward use of change of variables we obtain:

$$\int_{\varphi(U)} g(y)dy = \int_U g(\varphi(y))\det D\varphi_y dy,$$

but by Theorem 4.1 we know that $\int_\Omega f(y)dy = \int_\Omega g(y)dy$ so

$$\int_U f(y)dy = \int_U g(\varphi(y))\det D\varphi_y dy$$

and since this works for every open set U we obtain:

$$\det D\varphi_y g(\varphi(y)) = f(y), \tag{1}$$

Now fix $x = 1$ and consider the associated $x' = \frac{4}{3} - a$ as in the previous section. Let $int(\mathcal{B}(x')) = \Omega$ and we consider the functions $f, g : \overline{\Omega} \to \mathbb{R}$, defined by $f(y) = |\det D\tilde{\Phi}_y^{-1}|$ and $g = 1$. We note that $f \in C^0(\overline{\Omega})$.

Claim 4.1 *$\int_\Omega f(y)dy = \int_\Omega g(y)dy$.*

To see this we observe that,

$$\frac{\int_\Omega g(y)dy}{\int_\Omega f(y)dy} = \frac{\mu(\Omega)}{\int_\Omega |\det D\tilde{\Phi}_y^{-1}|dy} = \frac{\mu(\Omega)}{\mu(\tilde{\Phi}^{-1}(\Omega))} = \frac{\mu(\mathcal{B}(x'))}{\mu(\mathcal{A}(x))} = 1.$$

Last equality follows by construction and the claim is proved. Hence we can apply Theorem 4.1 and we obtain a homeomorphism $\varphi : \overline{\Omega} \to \overline{\Omega}$ verifying (1). Now we define,

$$\begin{array}{rccc} \tilde{\Psi}_1 : & \mathcal{A}(1) & \longrightarrow & \Omega \\ & p & \longrightarrow & \varphi \circ \tilde{\Phi}(p) \end{array}$$

To see that $\tilde{\Psi}_1$ is area-preserving we compute the determinant of the Jacobian,

$$\det D\tilde{\Psi} = \det D\varphi . \det D\tilde{\Phi}_1 = \det D\varphi . f(y)^{-1}.$$

Now the partial differential equation (1) says that $\det D\varphi . f(y)^{-1} = 1$, therefore we obtain that $\det D\tilde{\Psi}_1 = 1$, that is $\tilde{\Psi}_1$ is area-preserving.

In the same way we obtain $\tilde{\Psi}_{-1}$.

The maps $\tilde{\Psi}_i : \mathcal{A}(i) \setminus \mathcal{A}(i-1) \to \mathcal{B}(x'_i) \setminus \mathcal{B}(x'_{i-1})$, $i \in \mathbb{Z}$, are obtained by applying recursively the previous arguments to the points $x_i = i$ and the corresponding x'_i, and to the region $\Omega = int(\mathcal{B}(x'_i) \setminus \mathcal{B}(x'_{i-1}))$, thus ending the proof of the Theorem 1.

5 Existence of a conjugacy via fundamental regions

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a free map of the plane, say a fixed point free homeomorphism of the plane. We define an equivalence relation by saying that x is related with y, denoted by $x \sim y$, if there exists a compact arc γ from x through y such that given any compact $K \subseteq \mathbb{R}^2$, there exists $n_0 \in \mathbb{N}$ such that for any $n > n_0$ and $n < -n_0$ we have $H^n(\gamma) \cap K = \emptyset$.

Definition 5.1 *Each element of* $\mathbb{R}^2/\sim$ *is called a* fundamental region.

Proposition 5.1 *The conservative Hénon map has only one fundamental region.*

Proof: Let $x, y \in \mathbb{R}^2$, γ a compact segment from x through y and $K \subseteq \mathbb{R}^2$ any compact set. Clearly $K \subseteq Q_K$ for some square Q_K centered in the origin. Lemmas 2.1 to 2.3 are sufficient to conclude that exists $n_k \in \mathbb{N}$ such that for all $n > n_k$, we have $H^n(\gamma) \cap Q_K = \emptyset$. In fact, if we fix $z \in \gamma$, there exists $n_z \in \mathbb{N}$ such that $H^{n_z}(z) \notin Q_K$, $H^{n_z}(z) \in int(D^+)$ and the first coordinate of $H^{n_z}(z)$ is smaller than -2. Therefore, for all $n > n_z$ we have $H^n(z) \notin Q_K$ and by continuity n_z it is true also for some open neighborhood of z, V_z, sufficiently small. Let $\{V_z\}_{z\in\gamma}$ be an open covering of γ, since γ is compact $\{V_z\}_{z\in\gamma}$ admits a finite subcovering $\{V_{z_1}, ..., V_{z_s}\}$ and, as above to each z_i we associate n_{z_i} such that $H^{n_{z_i}}(V_{z_i}) \cap Q_K = \emptyset$, $H^{n_{z_i}}(V_{z_i}) \subseteq D^+$ and the first coordinate of any point of $H^{n_{z_i}}(V_{z_i})$is less than -2. Now, we just choose $n_0 = max\{n_{z_1}, ..., n_{z_s}\}$ and we have that for $n > n_0$ that $H^n(\gamma) \cap K = \emptyset$. By an analog procedure we prove that $H^{-n}(\gamma) \cap K = \emptyset$, by using the map S in D^-. □

Consider the following theorem due to Andrea (see [1]).

Theorem 5.2 *A free map of the plane is equivalent to a translation if and only if it has just one fundamental region.*

Now using the Proposition 5.1 jointly with Theorem 5.2 we conclude that H is conjugate to a translation. We note that this argument does not assure the existence of an area-preserving conjugacy.

6 Final remarks

In the case of the conservative Hénon map with $a = -1$, our map is no longer a free map, because it has a fixed point. Therefore it is not conjugate to a translation. Nevertheless the dynamics of this new map seems to be very similar to the case when $a < -1$, see Figure 3. After the bifurcation, say for $a > -1$, the dynamics earn rich properties, namely horseshoes and elliptical islands.

For the reversing-orientation conservative Hénon map without fixed points, that is when $b = -1$ and $a < 0$ in Definition 2.1, it is straigthforward to verify that $H \circ R = R \circ H^{-1}$, where $H = H_{a,-1}$ and $R(x, y) = (-y, -x)$. Therefore, considering the region $\mathcal{D}$ bounded by the curves $\lambda = \{(x, y) \in \mathbb{R}^2 : y = -x\}$ and $H(\lambda)$, we believe that, applying the same arguments, it is possible to prove that $\mathcal{D}$ is a fundamental domain and that there exists an area-preserving conjugacy between H and the map $Z(x, y) = (x + 1, -y)$.

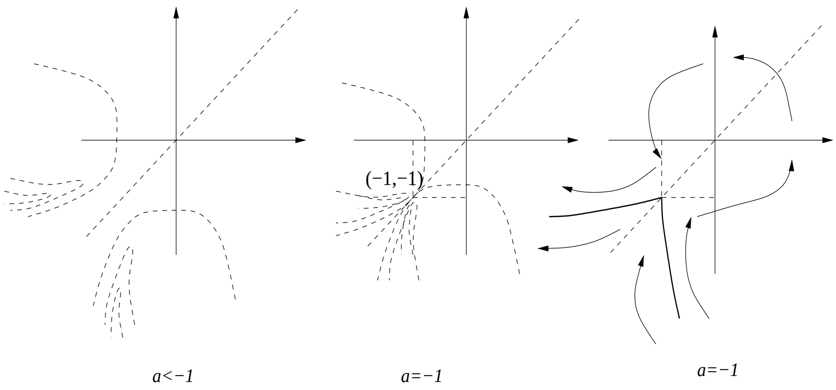

Figure 3: The bifurcation parameter $a = -1$.

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