Ideal extensions of locally inverse semigroups

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Abstract

The translational hull of a locally inverse semigroup has a largest locally inverse subsemigroup containing the inner part. A construction is given for ideal extensions within the class of all locally inverse semigroups.

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1 Introduction

We refer the reader to [3, 10] for the basic results and all notions which are left undefined in this paper. A regular semigroup S is said to be a **locally inverse semigroup** if eSe is an inverse semigroup for every idempotent e of S. The class **LI** of all locally inverse semigroups has been studied widely for more than two decades and includes remarkable semigroups such as inverse semigroups, completely 0-simple semigroups and normal bands of groups. The locally inverse semigroups are precisely the regular semigroups for which the **natural partial order** $\leq$ is compatible with the multiplication [4]. Equivalently, a regular semigroup S is locally inverse if and only if for all idempotents e, f in the set $E(S)$ of idempotents of S there exists a unique inverse $f \wedge e \in E(S)$ of ef in S which belongs to fSe [5]. The binary algebra $(E(S), \wedge)$ is called the **pseudosemilattice** (of idempotents) **of** S. Such pseudosemilattices have been characterized abstractly [5].

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Let S be a locally inverse semigroup. An injective homomorphism $\varphi : S \longrightarrow T$ is said to be an **ideal extension** [**within LI**] if $S\varphi$ is an ideal of the [locally inverse] semigroup T. If this is the case then $K = T/S\varphi$ is a [locally inverse] semigroup with zero and we say that T is an **ideal extension of** S **by** K. It is our aim to construct, for given locally inverse semigroups S and K, where K has a zero, every locally inverse semigroup T which is an extension of S by K. As explained in [9, 10], the concept of a dense ideal extension plays a crucial role in this construction. Recall that ideal extensions $\varphi_1 : S \longrightarrow T_1$ and $\varphi_2 : S \longrightarrow T_2$ are **equivalent** if there exists an isomorphism $\psi : T_1 \longrightarrow T_2$ which extends $\varphi_1^{-1}\varphi_2$. The ideal extension $\varphi : S \longrightarrow T$ is **dense** if whenever $\psi : T \longrightarrow U$ is a homomorphism such that $\varphi\psi : S \longrightarrow U$ is an ideal extension, then ψ is injective. A dense ideal extension $\varphi : S \longrightarrow T$ [within **LI**] is called a **maximal** dense ideal extension [within **LI**] if whenever $\psi : T \longrightarrow U$ is a homomorphism such that $\varphi\psi : S \longrightarrow U$ is a dense ideal extension [within **LI**], then ψ is an isomorphism. For the given locally inverse semigroup S such maximal dense ideal extensions exist and this facilitates finding all dense ideal extensions [within **LI**] of S.

For a locally inverse semigroup S the (nonempty) order ideals of $(S, \leq)$ form an ideal $O(S)$ of the power semigroup of S (Proposition 2.1 of [6]). For $a \in S$, let $(a]$ be the principal order ideal of $(S, \leq)$ generated by a. Then the mapping $\tau_S : S \longrightarrow O(S),\ a \longrightarrow (a]$ is an injective homomorphism and the regular part of the idealizer of $S\tau_S$ in $O(S)$ is a locally inverse semigroup $T(S)$ (Proposition 2.4 and Theorem 3.9 of [6]). Then $\tau_S : S \longrightarrow T(S)$ is a maximal dense ideal extension of S within **LI**; if $V \in$ **LI** and $S\tau_S \subseteq V \subseteq T(S)$ then $\tau_S : S \longrightarrow V$ is a dense ideal extension within **LI** and every dense ideal extension of S within **LI** is equivalent to an ideal extension obtained in this way (Theorems 4.6 and 4.7 of [6]). This paradigm was used before for constructing dense ideal extensions of some special locally inverse semigroups [12, 14].

2 The translational hull

Let S be a locally inverse semigroup. Let λ and ρ be, respectively, a left and right translation of S. Then λ and ρ are said to be **linked** if $(a\rho)b = a(\lambda b)$ for all $a, b \in S$. The **translational hull** $\Omega(S)$ of S consists of all such linked pairs (λ, ρ) and is a subsemigroup of $\Lambda(S) \times P(S)$, where $\Lambda(S)$ $[P(S)]$ is the semigroup of all left [right] translations of S. For $\omega = (\lambda, \rho) \in \Omega(S)$ and $a \in S$ we write $a\omega = a\rho$ and $\omega a = \lambda a$. For $a \in S$, λ_a [ρ_a] is the inner left [right] translation associated with a. Then $\pi_a = (\lambda_a, \rho_a) \in \Omega(S)$ and $\Pi(S) = \{\pi_a \mid a \in S\}$ is called the **inner part** of $\Omega(S)$. It follows from [2] (see also [9, 10]) that $\pi_S : S \longrightarrow \Omega(S),\ a \longrightarrow \pi_a$ is a maximal dense ideal extension (within the class of all semigroups) of the locally inverse semigroup S.

Lemma 2.1. *Let S be a locally inverse semigroup. For $H \in T(S)$ define λ_H and ρ_H by: if $s \in S$, then*

$$\lambda_H s = s_1 \qquad \text{if} \quad H(s] = (s_1],$$
$$s\rho_H = s_2 \qquad \text{if} \quad (s]H = (s_2].$$

Then $\omega_H = (\lambda_H, \rho_H) \in \Omega(S)$ and

$$\psi : T(S) \longrightarrow \Omega(S), \quad H \longrightarrow \omega_H$$

is the unique homomorphism of $T(S)$ into $\Omega(S)$ which extends

$$\tau_S^{-1}\pi_S : S\tau_S \longrightarrow S\pi_S,\ (a] \longrightarrow \pi_a.$$

Moreover, ψ is injective.

Proof. That $\omega_H \in \Omega(S)$ for every $H \in T(S)$, and that the above defined ψ is the unique homomorphism extending $\tau_S^{-1}\pi_S$ follows from Theorem III.1.12 of [10] since $S\tau_S$ is an ideal of $T(S)$ and both π_S and τ_S are injective homomorphisms. Since $\tau_S : S \longrightarrow T(S)$ is a dense ideal extension it follows from Corollary III.5.6 of [10] that ψ is injective. □

With the notation of Lemma 2.1 we denote the canonically constructed locally inverse subsemigroup $T(S)\psi$ of $\Omega(S)$ by $\Omega_{\mathbf{LI}}(S)$. Since ψ is an isomorphism of $T(S)$ onto $\Omega_{\mathbf{LI}}(S)$ which extends $\tau_S^{-1}\pi_S$, so $\pi_S : S \longrightarrow \Omega_{\mathbf{LI}}(S)$ and $\tau_S : S \longrightarrow T(S)$ are equivalent ideal extensions. Thus, since $\tau_S : S \longrightarrow T(S)$ is a maximal dense ideal extension within **LI**, we have

Theorem 2.2. *If $S \in$ **LI**, then $\pi_S : S \longrightarrow \Omega_{\mathbf{LI}}(S)$ is a maximal dense ideal extension of S within* **LI**.

We shall use the following lemma in the proof of Theorem 2.11.

Lemma 2.3. *Let $S \in \mathbf{LI}$, K an order ideal of S and $H \in T(S)$. Then $K\omega_H = KH$ and $\omega_H K = HK$.*

Proof. By duality we shall only prove $K\omega_H = KH$. By definition $K\omega_H = \{s \mid (k]H = (s] \text{ for some } k \in K\}$, and clearly $K\omega_H \subseteq KH$. Let now $k \in K$, $h \in H$. Then $k(hh' \wedge k'k) \in K$ for some inverses h' and k' of h and k respectively. Since $k(hh' \wedge k'k)h = kh$ and $R_{k(hh' \wedge k'k)h_1} \leq R_{k(hh' \wedge k'k)} = R_{kh}$ for any $h_1 \in H$ we conclude that $(k(hh' \wedge k'k)]H = (kh]$. We have shown $K\omega_H = KH$. □

We set out to locate $\Omega_{\mathbf{LI}}(S)$ within $\Omega(S)$ and find the inverse $\psi^{-1} : \Omega_{\mathbf{LI}}(S) \longrightarrow T(S)$ of the isomorphism $\psi : T(S) \longrightarrow \Omega_{\mathbf{LI}}(S)$.

Lemma 2.4. *Let S be a locally inverse semigroup and U a locally inverse subsemigroup of $\Omega(S)$ such that $\Pi(S) \subseteq U$. Then for $\omega \in U$,*

$$E(S)\omega \cap \omega E(S) = \{t \in S \mid \pi_t \leq \omega \text{ in } U\} \tag{1}$$

is an element of $T(S)$.

Proof. Since $\pi_S : S \longrightarrow U$ is an ideal extension of S within $\mathbf{LI}$ it follows from Corollary 4.5 of [6] that for every $\omega \in U$, $((\omega] \cap \Pi(S))\pi_S^{-1}$ is an element of $T(S)$. Obviously $((\omega] \cap \Pi(S))\pi_S^{-1}$ is the right hand side of (1).

Let $\omega \in U$ and $t \in S$ such that $\pi_t \leq \omega$ in U. Let t' be any inverse of t in S. Then $\pi_{tt'}$ is an idempotent of $\Pi(S)$ which is in the $\mathcal{R}$-class of π_t, and since $\pi_t \leq \omega$ it follows that $\pi_t = \pi_{tt'}\omega = \pi_{tt'\omega}$, whence $t = tt'\omega \in E(S)\omega$. By duality, $t \in E(S)\omega \cap \omega E(S)$.

Conversely, if $t \in E(S)\omega \cap \omega E(S)$ for $t \in S$ and $\omega \in U$, then $t = e\,\omega = \omega f$ for some $e, f \in E(S)$. Hence $\pi_t = \pi_{e\omega} = \pi_e\omega$ and $\pi_t = \pi_{\omega f} = \omega\pi_f$ where π_e and π_f are idempotents of U, so $\pi_t \leq \omega$ in U. We proved that the equality (1) holds true. □

Lemma 2.5. *Let S and U be as in Lemma 2.4 and for $\omega \in U$, let H_ω denote the set (1). Let ψ be as in Lemma 2.1. Then $\omega = H_\omega\psi$.*

Proof. Let $\omega \in U$. From Lemma 2.4 we know that $H_\omega \in T(S)$. Using duality and the definition of ψ we need only show that for any $s \in S$, $(\omega s] = H_\omega(s]$.

We choose an inverse ω' of ω in U, an inverse s' of s in S and we let $\varepsilon = \pi_{ss'} \wedge \omega'\omega$ in the pseudosemilattice of idempotents of U. Then $\varepsilon\pi_{ss'} \leq \pi_{ss'}$ and $\varepsilon\pi_{ss'} \in \Pi(S)$ since $\Pi(S)$ is an ideal of U. Also $\varepsilon\pi_{ss'} \,\mathcal{R}\, \varepsilon \,\mathcal{L}\, \omega\varepsilon$. It follows that $\omega\varepsilon \in \Pi(S)$, that is, $\omega\varepsilon = \pi_t$ for some $t \in S$. One verifies that $\omega\varepsilon \leq \omega$ in U, thus $\pi_t \leq \omega$ in U and $t \in H_\omega$. Using $\omega\pi_{ss'} = \omega\varepsilon\pi_{ss'}$ we then obtain

$$\omega s = \omega ss's = \omega\pi_{ss'}s = \omega\varepsilon\pi_{ss'}s = \omega\varepsilon s = \pi_t s = ts \in H_\omega s \subseteq H_\omega(s],$$

and so $(\omega s] \subseteq H_\omega(s]$ since $H_\omega(s]$ is an order ideal of $(S, \leq)$.

To prove the converse we take any element in $H_\omega(s]$. Since by Proposition 2.4 of [6] $H_\omega(s] = H_\omega s$, such an element is of the form qs with $q \in H_\omega$, that is, $\pi_q \leq \omega$ in U. We let q' be an inverse of q in S and let $\varepsilon' = \pi_{ss'} \wedge \pi_{q'q}$ in the pseudosemilattice of idempotents of U. Since $\pi_q \in U\omega$ it follows from elementary results concerning pseudosemilattices [5] that $\varepsilon' \leq \varepsilon$ in U, and thus also that $\pi_q\varepsilon' \leq \omega\varepsilon$, since $\leq$ is compatible with the multiplication in U. Since $\pi_{qs} = \pi_q\pi_s = \pi_q\varepsilon'\pi_s \leq \omega\varepsilon\pi_S = \omega\pi_s = \pi_{\omega s}$ in U we also have that $\pi_{qs} \leq \pi_{\omega s}$ in the ideal $\Pi(S)$ of U, and so $qs \leq \omega s$ in S since π_S is an isomorphism of S onto $\Pi(S)$. We proved that $qs \in (\omega s]$. We conclude that $(\omega s] = H_\omega(s]$, as required. □

Theorem 2.6. *If S is a locally inverse semigroup, then $\Omega_{\mathbf{LI}}(S)$ is the largest locally inverse subsemigroup of $\Omega(S)$ containing $\Pi(S)$. $\Omega_{\mathbf{LI}}(S) = \Omega(S)$ if and only if S is an inverse semigroup.*

Proof. The first statement follows immediately from Lemmas 2.4 and 2.5. Assume that $\Omega_{\mathbf{LI}}(S) = \Omega(S)$. Since $\Omega(S)$ has an identity element, $\Omega_{\mathbf{LI}}(S)$ is a locally inverse semigroup which has an identity element, hence $\Omega_{\mathbf{LI}}(S)$ is an inverse semigroup. Then $\Pi(S)$ is an ideal of the inverse semigroup $\Omega(S)$, whence $\Pi(S)$ and thus S itself are inverse semigroups.

If S is an inverse semigroup, then $\Omega(S)$ is an inverse semigroup by a result of Ponizovskiĭ [13] (see also [14] or Theorem V.4.6 of [10]), hence $\Omega(S) = \Omega_{\mathbf{LI}}(S)$ by the first statement. □

With the notation introduced before we have

Theorem 2.7. *Let S be a locally inverse semigroup. Then the mappings*

$$\psi : T(S) \longrightarrow \Omega_{\mathbf{LI}}(S), \quad H \longrightarrow \omega_H$$

and

$$\chi : \Omega_{\mathbf{LI}}(S) \longrightarrow T(S), \quad \omega \longrightarrow H_\omega$$

are pairwise inverse isomorphisms.

Proof. By Corollary 4.5 of [6] the mapping $\Omega_{\mathbf{LI}}(S) \longrightarrow T(S)$, $\omega \longrightarrow ((\omega] \cap \Pi(S))\pi_S^{-1}$ is the unique homomorphism which extends $\pi_S^{-1}\tau_S : \Pi(S) \longrightarrow S\tau_S$. Here obviously $((\omega] \cap \Pi(S))\pi_S^{-1} = H_\omega$ as in (1) and so the mentioned mapping is precisely the mapping χ of the statement of the theorem. That ψ is an isomorphism of $T(S)$ onto $\Omega_{\mathbf{LI}}(S)$ we know from Lemma 2.1. From Lemma 2.5 it follows that $\chi\psi$ is the identity transformation on $\Omega_{\mathbf{LI}}(S)$. We conclude that $\chi = \psi^{-1}$. □

The following gives more information about the idempotents and the local submonoids of $\Omega_{\mathbf{LI}}(S)$ and $T(S)$.

Theorem 2.8. *Let S be a locally inverse semigroup. An idempotent ε of $\Omega(S)$ belongs to $\Omega_{\mathbf{LI}}(S)$ if and only if $E = E(S)\varepsilon \cap \varepsilon E(S)$ is a subsemilattice of S. If this is the case, then E is the idempotent of $T(S)$ with $\varepsilon = E\psi$.*

Proof. If $e \in E(S)$ such that $\pi_e \leq \varepsilon = \varepsilon^2$ in $E(\Omega(S))$, then $\pi_e = \pi_e\varepsilon = \pi_{e\varepsilon}$, whence $e = e\varepsilon$, and dually also $e = \varepsilon e$. Therefore, if $e \in E(S)$ such that $\pi_e \leq \varepsilon$ then $e \in E(S)\varepsilon \cap \varepsilon E(S)$.

Assume that $E = E(S)\varepsilon \cap \varepsilon E(S)$ is a subsemilattice of S. Clearly $U = \Pi(S) \cup \{\varepsilon\}$ is a regular subsemigroup of $\Omega(S)$ which contains the locally inverse semigroup $\Pi(S)$ as an ideal. Therefore $\varepsilon U\varepsilon$ is a regular subsemigroup of U. If π_e and π_f are any idempotents of $\varepsilon U\varepsilon \cap \Pi(S)$, then by the above e and f belong to the subsemilattice E of S, hence π_e commutes with π_f. It follows that $\varepsilon U\varepsilon$ is an inverse semigroup and we may conclude that U is a locally inverse subsemigroup of $\Omega(S)$. By Theorem 2.6, $U \subseteq \Omega_{\mathbf{LI}}(S)$ and so $\varepsilon \in \Omega_{\mathbf{LI}}(S)$. By Theorem 2.7, $E = \varepsilon\psi^{-1}$.

Assume conversely that ε is an idempotent of $\Omega_{\mathbf{LI}}(S)$. Then by Theorem 2.7 $H_\varepsilon = E(S)\varepsilon \cap \varepsilon E(S) = E$ is an idempotent of $T(S)$. From Lemma 3.2 of [6] it follows that E is a subsemilattice of S. □

It will be useful to recall the following result of [6].

Result 2.9. (Lemma 3.6 and Theorem 3.9 of [6].) *Let S be a locally inverse semigroup. Then E is an idempotent of $T(S)$ if and only if E is a subsemilattice and an order ideal of S, and for any $f \in E(S)$ there exist $k, l \in E(S)$ such that*

$$\{e \wedge f \mid e \in E\} = (k], \quad \{f \wedge e \mid e \in E\} = (l].$$

The $\wedge$-operation of the pseudosemilattice of $T(S)$ is given by: if E and F are idempotents of $T(S)$, then

$$E \wedge F = \{e \wedge f \mid e \in E,\, f \in F\}.$$

Since for a semilattice the operation $\wedge$ coincides with the multiplication it follows from Result 2.9 that for an inverse semigroup V the semilattice $E(T(V))$ coincides with $T(E(V))$, which is the $\cap$-semilattice of **retract ideals** of the semilattice $E(V)$ (see Exercise V.4.7.2 and Lemma V.6.1 of [10], or Lemmas 2.6 and 2.8 of [14]).

Proposition 2.10. *Let S be a locally inverse semigroup.*

(*i*) *If $H \in O(S)$ then $H \in T(S)$ if and only if there exists an inverse H' of H in $O(S)$ such that HH' and $H'H$ are idempotents of $T(S)$.*

(*ii*) *$\Omega_{\mathbf{LI}}(S)$ consist of the $\omega \in \Omega(S)$ which are $\mathcal{D}$-related in $\Omega(S)$ to some idempotent of $\Omega_{\mathbf{LI}}(S)$.*

Proof. (*i*). The direct part is obvious since $T(S)$ was defined to be the regular part of the idealizer of $S\tau_S$ in $O(S)$. To prove the converse, let H and H' be pairwise inverse elements of $O(S)$ such that HH' and $H'H$ are idempotents of $T(S)$. By duality it suffices to show that for any $a \in S$, $H(a] \in S\tau_S$.

Since $H'H \in T(S)$ there exists $b \in S$ such that $H'H(a] = (b]$. Since $H'H(a] = H'Ha$ by Proposition 2.4 of [6], there exists $f \in H'H$ such that $b = fa$. From Result 2.9 we know that $H'H$ is a subsemilattice and an order ideal of S and so in particular f is an idempotent. Let e be the unique idempotent in R_b such that $e \leq f$. Then $ea = fa = b$ and $e \in H'H$. By Lemma 3.11 of [6] there exist unique $t \in H$ and $t' \in H'$ such that t' is an inverse of t and $e = t't$. Hence $tb = ta$ and so $(tb] \subseteq H(a]$ since $H(a] \in O(S)$. We shall show that the reverse inclusion holds true.

Since $H(a] = Ha$ by Proposition 2.4 of [6], every element of $H(a]$ is of the form sa for some $s \in H$. Using Lemma 3.11 of [6] we may take the unique inverse s' of s in H' and then $s's$ belongs to the semilattice $H'H$. Then $s'sa \in H'H(a] = (b]$, and thus $s'sa \leq b$ in S. Since $b = ea\,\mathcal{R}\,e = t't$ we have that $s'sa = t'ts'sa$. Since $t't$ and $s's$ belong to the semilattice $H'H$, so $st't\,\mathcal{L}\,s'st't = t'ts's\,\mathcal{L}\,ts's$, where $st't, ts's \in HH'H = H$. By Lemma 3.11 of [6], H cannot contain distinct $\mathcal{L}$-related elements, hence $st't = ts's$. Therefore $sa = ss'sa = st'ts'sa = ts'sa \leq tb$. We conclude that $H(a] = (tb]$, as required.

(ii). Let δ be any idempotent of $\Omega(S)$ and ε an idempotent of $\Omega_{\mathbf{LI}}(S)$ such that $\delta\,\mathcal{D}\,\varepsilon$ in $\Omega(S)$. There exist pairwise inverse elements ω and ω' of $\Omega(S)$ such that $\varepsilon = \omega\omega'$ and $\delta = \omega'\omega$. Then $\varepsilon\Omega(S)\varepsilon \longrightarrow \delta\Omega(S)\delta$, $\omega\alpha\omega' \longrightarrow \omega'\omega\alpha\omega'\omega$ and $\delta\Omega(S)\delta \longrightarrow \varepsilon\Omega(S)\varepsilon$, $\omega'\alpha\omega \longrightarrow \omega\omega'\alpha\omega\omega'$ are pairwise inverse isomorphisms which induce isomorphisms between $\delta\Pi(S)\delta$ and $\varepsilon\Pi(S)\varepsilon$. Here $\varepsilon\Pi(S)\varepsilon$ is an inverse semigroup since it is an ideal of the inverse semigroup $\varepsilon\Omega_{\mathbf{LI}}(S)\varepsilon$. Therefore $\Pi(S) \cup \{\delta\}$ is a locally inverse subsemigroup of $\Omega(S)$ and $\delta \in \Omega_{\mathbf{LI}}(S)$ by Theorem 2.6.

Let Ω denote the union of the regular $\mathcal{D}$-classes of $\Omega(S)$ which contain an idempotent of $\Omega_{\mathbf{LI}}(S)$. We must show that $\Omega = \Omega_{\mathbf{LI}}(S)$. Clearly $\Omega_{\mathbf{LI}}(S) \subseteq \Omega$. For ω_1, $\omega_2 \in \Omega$ we may find idempotents $\varepsilon \in L_{\omega_1}$ and $\delta \in R_{\omega_2}$ and by the foregoing we have $\varepsilon, \delta \in \Omega_{\mathbf{LI}}(S)$. We take $\delta \wedge \varepsilon$ in the pseudosemilattice of idempotents of $\Omega_{\mathbf{LI}}(S)$ and we know that $\delta \wedge \varepsilon$ and $\varepsilon\delta$ are pairwise inverse elements. Since then $\delta\wedge\varepsilon\,\mathcal{D}\,\varepsilon\delta\,\mathcal{L}\,\omega_1\delta\,\mathcal{R}\,\omega_1\omega_2$ in $\Omega(S)$ and $\delta\wedge\varepsilon$ is an idempotent of $\Omega_{\mathbf{LI}}(S)$, so $\omega_1\omega_2 \in \Omega$. Therefore Ω is a regular subsemigroup of $\Omega(S)$, and obviously $\Pi(S) \subseteq \Omega$. If ε is an idempotent of Ω, then $\varepsilon\Omega\varepsilon$ is a regular semigroup in which the set of idempotents coincides with the set of idempotents of $\varepsilon\Omega_{\mathbf{LI}}(S)\varepsilon$. Since the latter constitutes a semilattice it follows that Ω is locally inverse. By Theorem 2.6 we may now conclude that $\Omega = \Omega_{\mathbf{LI}}(S)$. □

In the following we calculate the local submonoids of $\Omega_{\mathbf{LI}}(S)$ for any given locally inverse semigroup S.

Theorem 2.11. *Let S be a locally inverse semigroup, ε an idempotent of $\Omega_{\mathbf{LI}}(S)$ and E the idempotent $E(S)\varepsilon \cap \varepsilon E(S)$ of $T(S)$. Then*

$$\varepsilon\Omega(S)\varepsilon = \varepsilon\Omega_{\mathbf{LI}}(S)\varepsilon \cong ET(S)E = T(V) \cong \Omega(V)$$

where V is the largest inverse subsemigroup of S having E as its semilattice of idempotents.

Proof. Recall from Theorem 2.8 that E is the idempotent of $T(S)$ with $\varepsilon = E\psi$. Since ψ is an isomorphism of $T(S)$ onto $\Omega_{\mathbf{LI}}(S)$, it follows that $\varepsilon\Omega_{\mathbf{LI}}(S)\varepsilon \cong ET(S)E$. For any inverse semigroup V, $T(V) \cong \Omega_{\mathbf{LI}}(V) = \Omega(V)$ by Theorem 2.6 and $\chi_V : \Omega(V) \longrightarrow T(V)$, $\omega \longrightarrow E(V)\omega \cap \omega E(V)$ is an isomorphism by Theorem 2.7. It suffices to prove the equalities now.

Since E is an idempotent of $T(S)$, it is an order ideal and a subsemilattice of S which satisfies the additional conditions as stipulated in Result 2.9. Since E is a subsemilattice and an order ideal of S, ESE is the largest inverse subsemigroup of S having E as its pseudosemilattice of idempotents, and thus $V = ESE = \varepsilon S \varepsilon$ by Lemma 2.3. By Theorem 1 of [11] the mapping

$$\varphi : \varepsilon\Omega(S)\varepsilon \longrightarrow \Omega(V), \quad \omega \longrightarrow \omega|_V$$

is an isomorphism, and so by Theorem 2.6 $\varepsilon\Omega(S)\varepsilon$ is a locally inverse semigroup. Thus $\varepsilon\Omega(S)\varepsilon \cup \Pi(S)$ is a regular subsemigroup of $\Omega(S)$ since $\Pi(S)$ is an ideal of $\Omega(S)$. Since $\{\omega\} \cup \Pi(S)$ is a locally inverse semigroup for any idempotent ω of $\varepsilon\Omega(S)\varepsilon$, we have that $\omega \in \Omega_{\mathbf{LI}}(S)$ again by Theorem 2.6. Therefore $\varepsilon\Omega(S)\varepsilon \cup \Pi(S)$ is a locally inverse semigroup, and consequently $\varepsilon\Omega(S)\varepsilon \subseteq \Omega_{\mathbf{LI}}(S)$. We can now conclude that $\varepsilon\Omega(S)\varepsilon = \varepsilon\Omega_{\mathbf{LI}}(S)\varepsilon$, and thus

$$\psi\varphi\chi_V : ET(S)E \longrightarrow T(V), \quad EHE \longrightarrow E\,(\varepsilon\omega_H\varepsilon)|_V \cap (\varepsilon\omega_H\varepsilon)|_V\,E$$

is an isomorphism. By Lemma 2.3 we have

$$E\,(\varepsilon\omega_H\varepsilon)|_V = E\omega_{EHE} = EHE,$$

and by symmetry $(\varepsilon\omega_H\varepsilon)|_V\,E = EHE$. Hence $\psi\varphi\chi_V$ is the identity mapping and $ET(S)E = T(V)$. □

From the above theorem it follows that for a locally inverse semigroup S, every local submonoid $\varepsilon\Omega_{\mathbf{LI}}(S)\varepsilon$ of the locally inverse semigroup $\Omega_{\mathbf{LI}}(S)$ is isomorphic to the translational hull $\Omega(V)$ of a suitable inverse subsemigroup V of S, where V is also an order ideal of S. One cannot expect however that for any inverse subsemigroup V of S which is also an order ideal of S one can find a copy of $\Omega(V)$ in $\Omega_{\mathbf{LI}}(S)$: we return to this issue in Section 4.

3 Ideal extensions within LI

The following theorem describes how to construct, for given locally inverse semigroups S and K, where K has a zero, a locally inverse semigroup T which is an ideal extension of S by K. Our theorem follows the general pattern of the existing theory concerning ideal extensions (see in particular Section III.2 of [10]), the special features now being our use of $T(S)$ (or equivalently $\Omega_{\mathbf{LI}}(S)$), and our insistence on constructing an ideal extension where the resulting semigroup is again locally inverse.

Theorem 3.1. *Let S ba a locally inverse semigroup and K a locally inverse semigroup with zero 0. Let U be a locally inverse subsemigroup of $T(S)$ which contains $S\tau_S$ and $\beta : K \setminus \{0\} \longrightarrow U$ a mapping such that* *(i)* $(k_1k_2)\beta = (k_1\beta)(k_2\beta)$ *if* $k_1k_2 \neq 0$ *in* K,

(ii) $(k_1\beta)(k_2\beta) \in S\tau_S$ *if* $k_1k_2 = 0$ *in* K,

(iii) $U = (K \setminus \{0\})\beta \cup S\tau_S$.

Put

$$T = \{(k\beta, k),\ (s\tau_s, 0) \mid 0 \neq k \in K,\ s \in S\}.$$

Then T *is a locally inverse semigroup which is a subdirect product of* U *and* K, *and* $\varphi_1 : S \longrightarrow T$, $s \longrightarrow (s\tau_S, 0)$ *is an ideal extension which is equivalent to an ideal extension of* S *by* K.

Conversely, every ideal extension of S *by* K *within* **LI** *is equivalent to an ideal extension constructed in this way.*

Proof. It is routine to verify the direct part and we shall proceed to prove the converse. Therefore we let $\varphi : S \longrightarrow V$ be an ideal extension of S by K within **LI**. By Corollary 4.5 of [6],

$$\beta : V \longrightarrow T(S), \quad v \longrightarrow ((v] \cap S\varphi)\varphi^{-1}$$

is the unique homomorphism of V into $T(S)$ which extends $\varphi^{-1}\tau_S$. Then $U = V\beta$ is a locally inverse semigroup since the class of all locally inverse semigroups is closed for the taking of homomorphic images, and $S\tau_S \subseteq U$. The mapping

$$\psi : V \longrightarrow U \times K, \quad v \longrightarrow (v\beta, v/S\varphi)$$

is a homomorphism onto a locally inverse subsemigroup T of $U \times K$. If $v_1 \neq v_2$ in V and $v_1, v_2 \notin S\varphi$, $v_1/S\varphi \neq v_2/S\varphi$. If $v_1 \neq v_2$ and $v_1, v_2 \in S\varphi$, then $v_1\varphi^{-1} \neq v_2\varphi^{-1}$ in S and thus also $v_1\beta \neq v_2\beta$. If $v_1 \in S\varphi$ and $v_2 \notin S\varphi$, then $v_1/S\varphi \neq v_2/S\varphi$. It follows that ψ is an isomorphism from V onto T.

The subdirect product T of U and K consists precisely of the pairs

$$(u, k), \quad u \in U,\ 0 \neq k \in K,\ k\beta = u$$

and

$$(s\tau_S, 0), \quad s \in S.$$

The ideal extension $\varphi_1 : S \longrightarrow T,\ s \longrightarrow (s\tau_S, 0)$ is obtained as in the direct part of the statement of the theorem, and the ideal extensions $\varphi : S \longrightarrow V$ and $\varphi_1 : S \longrightarrow T$ are equivalent via the isomorphism $\psi : V \longrightarrow T$. □

If $\varphi : S \longrightarrow V$ is an ideal extension within **LI** then we say that this ideal extension is **strict** if the homomorphism $\beta : V \longrightarrow T(S),\ v \longrightarrow ((v] \cap S\varphi)\varphi^{-1}$ considered in the converse part of the proof (see also Corollary 4.5 of [6]) maps V into $S\tau_S$. Note that in this case $V\beta = S\tau_S$ since β extends $\varphi^{-1}\tau_S$. It follows that

Corollary 3.2. *Let S be a locally inverse semigroup and K be a locally inverse semigroup with zero 0. Let $\beta : K \setminus \{0\} \longrightarrow S\tau_S$ be a mapping such that $(k_1k_2)\beta = (k_1\beta)(k_2\beta)$ whenever $k_1k_2 \neq 0$ in K. Put*

$$T = \{(k\beta, k),\ (s\tau_s, 0) \mid 0 \neq k \in K,\ s \in S\}.$$

Then T is a locally inverse semigroup which is a subdirect product of $S\tau_S$ and K and $S \longrightarrow T,\ s \longrightarrow (s\tau_s, 0)$ is a strict ideal extension of S by K within **LI**. *Every strict ideal extension of S by K within* **LI** *is equivalent to an ideal extension thus obtained.*

Strict ideal extensions within **LI** are interesting, not only because our construction in Theorem 3.1 greatly simplifies (the conditions (*ii*) and (*iii*) are trivially satisfied) but also because $\varphi : S \longrightarrow V$ is a strict ideal extension within **LI** if and only if $S\varphi$ is a retract ideal of the locally inverse semigroup V, that is, $S\varphi$ is an ideal of V which is the image of V under an idempotent endomorphism of V (see Proposition III.4.4 of [10]). This leads us to the question: find the locally inverse semigroups S such that every ideal extension $\varphi : S \longrightarrow V$ within **LI** is strict. The following follows from general principles [10].

Theorem 3.3. *For $S \in$* **LI** *the following are equivalent:*

(*i*) *every ideal extension $\varphi : S \longrightarrow V$ within* **LI** *is strict,*

(*ii*) $\tau_S : S \longrightarrow T(S)$ $[\pi_S : S \longrightarrow \Omega_{\mathbf{LI}}(S)]$ *is a strict ideal extension,*

(*iii*) $T(S) = S\tau_S$ $[\Omega_{\mathbf{LI}}(S) = \Pi(S)]$,

(*iv*) *if ε is an idempotent of $\Omega_{\mathbf{LI}}(S)$ then $E = E(S)\varepsilon \cap \varepsilon E(S)$ is a subsemilattice of S which has an identity element.*

It follows that the following is a sufficient condition for a locally inverse semigroup to satisfy the equivalent conditions of Theorem 3.3. We give an example in Section 4 to show that the condition is not a necessary condition.

Corollary 3.4. *A locally inverse semigroup S satisfies the equivalent conditions of Theorem 3.3 if every [maximal] subsemilattice of S is dually well-ordered.*

Proof. If S is a locally inverse semigroup which satisfies the stated property, then according to Theorem 2.8, every idempotent of $T(S)$ belongs to $S\tau_S$, and thus according to Theorem 3.3, every ideal extension of S within **LI** is strict. $\square$

Corollary 3.5. *Every ideal extension within* **LI** *of a completely simple semigroup is strict.*

Proof. This follows from Corollary 3.4 since the maximal subsemilattices of a completely simple semigroup are trivial semilattices. $\square$

This result is of course the starting point for showing that a completely regular semigroup is in **LI** if and only if it is a strong semilattice of completely simple semigroups (Section IV.4 of [10], and [12]).

If S is a right zero semigroup and $\mathcal{T}(S)$ the full transformation semigroup on the set S, then there exists a canonical ideal extension $\varphi : S \longrightarrow \mathcal{T}(S)$ which associates with every $s \in S$ the constant transformation with value s. By Corollary V.3.12 of [10], $\varphi : S \longrightarrow \mathcal{T}(S)$ and $\pi : S \longrightarrow \Omega(S)$ are equivalent ideal extensions. By Theorem 3.3 and Corollary 3.4, $\Pi(S) = \Omega_{\mathbf{LI}}(S)$. It follows that the largest locally inverse subsemigroup of $\mathcal{T}(S)$ which contains the semigroup of all constant transformations on S is the semigroup of constant transformations on S. This fact may of course be verified directly: if γ is a nonconstant idempotent transformation of S then there exist distinct constant transformations α, β such that $\alpha \leq \gamma$ and $\beta \leq \gamma$, where obviously $\alpha \mathcal{R} \beta$.

For a more nontrivial example we may consider the four-spiral semigroup Sp_4 of [1]. In Sp_4 the maximal subsemilattices are dually well-ordered chains of order type ω^*. By Corollary 3.4 every ideal extension of Sp_4 within **LI** is strict, and $\Omega_{\mathbf{LI}}(Sp_4) \cong Sp_4$.

4 Completely 0-simple semigroups

In what follows S will be a completely 0-simple semigroup with zero 0. If $a, b \in S$ then $a \leq b$ if and only if $a = b$ or $a = 0$. Therefore $O(S)$ consists of all subsets of S containing 0.

It is easy enough to characterize the $E \in O(S)$ such that E is a subsemilattice of S: E is a subsemilattice and an order ideal of S if and only if E consists of idempotents only, $0 \in E$ and for any distinct $e, f \in E$, $R_e \cap L_f$ does not contain an idempotent. If E is a subsemilattice and an order ideal of S and $f \in E(S)$, then

$$\{e \wedge f \mid e \in E\} = \{0,\ g \in L_f \cap E(S) \mid g \,\mathcal{R}\, e \text{ for some } e \in E\}$$

and

$$\{f \wedge e \mid e \in E\} = \{0,\ g \in R_f \cap E(S) \mid g \,\mathcal{L}\, e \text{ for some } e \in E\}.$$

Therefore by Result 2.9 we have

Proposition 4.1. *Let S be a completely 0-simple semigroup. Then E is an idempotent of $T(S)$ if and only if E is a subsemilattice and an order ideal of S, and for any nonzero idempotent f of S, the sets $L_f \cap (\cup_{e \in E} R_e)$ and $R_f \cap (\cup_{e \in E} L_e)$ each contain at most one idempotent.*

Proposition 4.1 guarantees that for any idempotents E and F of $T(S)$ the set of idempotents in $(\cup_{e \in E} R_e) \cap (\cup_{f \in F} L_f)$ is a subsemilattice and an order ideal of S. In fact by Result 2.9 the $\wedge$-operation in the pseudosemilattice of $T(S)$ is given by:

Proposition 4.2. *Let S be a completely 0-simple semigroup. The $\wedge$-operation in the pseudosemilattice of idempotents of $T(S)$ is given by: for idempotents E and F of $T(S)$, $E \wedge F$ is the subsemilattice and order ideal of S consisting of the idempotents in $(\cup_{e \in E} R_e) \cap (\cup_{f \in F} L_f)$.*

Using Lemma 3.11 of [6] and Proposition 2.10 we can now construct every element of $T(S)$.

Proposition 4.3. *Let S be a completely 0-simple semigroup, E and F idempotents of $T(S)$ and $\pi : E \longrightarrow F$ a bijection such that $0\pi = 0$. For every $e \in E$ choose $a_e \in R_e \cap L_{e\pi}$ and let a'_e be the inverse of a_e in $L_e \cap R_{e\pi}$. Then $H = \{a_e \mid e \in E\}$ and $H' = \{a'_e \mid e \in E\}$ are pairwise inverse elements of $T(S)$ such that $HH' = E$ and $H'H = F$. Every $H \in T(S)$ can be so constructed.*

We shall now use Theorem 2.11 to determine the local submonoids of $T(S)$ for a given completely 0-simple semigroup S.

Proposition 4.4. *Let S be a completely 0-simple semigroup and E an idempotent of $T(S)$. Then*

$$V = ESE = \{a \in S \mid L_a \cap E \neq \emptyset \neq R_a \cap E\}$$

is a Brandt semigroup, and $ET(S)E \cong \Omega(V)$.

The translational hull of a Brandt semigroup is well-known and we refer to [7], or Section V.5 of [10], or 1.32 and 2.27 of [14] for the details. With the notation of Proposition 4.4, if $E \neq \{0\}$ and $I = E \setminus \{0\}$, then $\Omega(V)$ can be represented as a wreath product of the symmetric inverse semigroup $\mathcal{I}_I$ on the set I and a group G, where G is isomorphic to every nonzero maximal subgroup of S. It follows in particular that every maximal group of $T(S)$ [$\cong \Omega_{\mathbf{LI}}(S)$] is isomorphic to a wreath product of the symmetric group on I and the group G. If in particular G is trivial, that is, if $\mathcal{H}$ is the equality in S, then the local submonoids of $T(S)$ [$\cong \Omega_{\mathbf{LI}}(S)$] are each isomorphic to a symmetric inverse semigroup and the maximal subgroups of $T(S)$ [$\cong \Omega_{\mathbf{LI}}(S)$] are each isomorphic to a symmetric group. Note that in this case S is fundamental, and therefore also $T(S)$ and $\Omega_{\mathbf{LI}}(S)$ are fundamental since the ideal extensions $\tau_S : S \longrightarrow T(S)$ and $\pi_S : S \longrightarrow \Omega_{\mathbf{LI}}(S)$ are dense.

Let S be a completely regular locally inverse semigroup. Then $\mathcal{H}$ is a congruence on S and we shall denote the canonical homomorphism from S onto $S/\mathcal{H}$ by μ. By Corollary 4.4 of [6], the mapping $T(S) \longrightarrow T(S/\mathcal{H})$, $H \longrightarrow H\mu$ is the unique homomorphism which extends $\tau_S^{-1}\mu\tau_{S/\mathcal{H}}$. The reader may consult Propositions 4.1 and 4.3 to prove that this homomorphism is surjective and from what precedes it follows that this homomorphism induces the greatest idempotent separating congruence on $T(S)$.

Proposition 4.5. *Let S be a completely 0-simple semigroup. The following are equivalent:*

(*i*) *$T(S)$ [$\cong \Omega_{\mathbf{LI}}(S)$] is completely semisimple,*

(*ii*) *the local submonoids of $T(S/\mathcal{H})$ [$\cong \Omega_{\mathbf{LI}}(S/\mathcal{H})$] are finite,*

(*iii*) *if E is an idempotent of $T(S)$, then E is a finite subsemilattice of S.*

Proof. In the following we shall apply the remarks made before the statement of the proposition.

There exists an idempotent separating homomorphism of $T(S)$ onto $T(S/\mathcal{H})$ and therefore $T(S)$ is completely semisimple if and only if $T(S/\mathcal{H})$

is completely semisimple. Every local submonoid of $T(S/\mathcal{H})$ is isomorphic to a symmetric inverse semigroup $\mathcal{I}_I$, with $I = E \setminus \{0\}$ for some idempotent E of $T(S)$. Here $\mathcal{I}_I$ is completely semisimple if and only if I is finite, that is, if and only if $\mathcal{I}_I$ is finite, or equivalently, if and only if E is finite. $\square$

For a completely 0-simple semigroup S we may encounter two possible extreme situations, namely 1. every subsemilattice of S which contains 0 is an idempotent of $T(S)$, or 2. the only idempotents of $T(S)$ are the trivial semilattice $\{0\}$ and the two-element subsemilattices of S. The latter is the case if and only if $T(S) = S\tau_S$, that is, every ideal extension of S within **LI** is strict. The following give a characterization for these extremes.

Proposition 4.6. *Let S be a completely 0-simple semigroup. Then every nontrivial subsemilattice of S is an idempotent of $T(S)$ if and only if for any two commuting idempotents e and f of S, $R_e \cup R_f$ does not contain distinct $\mathcal{L}$-related idempotents and $L_e \cup L_f$ does not contain distinct $\mathcal{R}$-related idempotents.*

Proposition 4.7. *Let S be a completely 0-simple semigroup. Then every ideal extension of S within* **LI** *is strict if and only if for any two distinct commuting nonzero idempotents e and f of S, $R_e \cup R_f$ contains distinct $\mathcal{L}$-related idempotents or $L_e \cup L_f$ contains distinct $\mathcal{R}$-related idempotents.*

It may be instructive to illustrate the situation of Proposition 4.7 by a concrete and natural example. Let X be a set with $|X| \geq 2$. A binary relation on X is said to be **rectangular** if it is of the form $A \times B$ for some subsets A and B of X. The semigroup R_X of all rectangular binary relations on X is the least nontrivial ideal of the semigroup B_X of all binary relations on X and R_X constitutes a completely 0-simple semigroup, the zero being the empty relation on X. The rectangular binary relation $A \times B$ is an idempotent if and only if $A \cap B \neq \emptyset$. If $A \times B$ and $C \times D$ are distinct nonempty idempotents, then they commute if and only if $B \cap C = \emptyset = A \cap D$. If this is the case then $A \neq C$ and $B \neq D$, whence $A \times X$ and $C \times X$ are distinct $\mathcal{L}$-related idempotents in $R_{A\times B} \cup R_{C\times D}$, and $X \times B$ and $X \times D$ are distinct $\mathcal{R}$-related idempotents in $L_{A\times B} \cup L_{C\times D}$. Therefore, according to Proposition 4.7, every ideal extension of R_X within **LI** is strict. Hence $\Omega_{\mathbf{LI}}(R_X) = \Pi(R_X)$ and $T(R_X) = R_X\tau_{R_X}$. The ideal extensions $\pi_{R_X} : R_X \longrightarrow \Omega(R_X)$ and $\iota : R_X \longrightarrow B_X$ are equivalent (see [15] or Theorem 6 of [8]). Hence $\Omega_{\mathbf{LI}}(R_X) = \Pi(R_X)$ is equivalent to the fact that the largest locally inverse subsemigroup of B_X which contains R_X

is R_X itself. Or equivalently, if ε is any idempotent binary relation on X and $\varepsilon \notin R_X$, then there exist distinct idempotents $\alpha, \beta \in R_X$ such that $\alpha \leq \varepsilon$, $\beta \leq \varepsilon$ and $\alpha \,\mathcal{R}\, \beta$ or $\alpha \,\mathcal{L}\, \beta$.

The paper [7] (see also Section V.3 of [10]) gives a structural description of the translational hull of a completely 0-simple semigroup $S = \mathcal{M}^0(I, G, \Lambda; P)$ which is given as a Rees matrix semigroup. The isomorphic copy of $\Omega(S)$ given there is somewhat complicated and it would be interesting to see how our description of $T(S)$ translates into a convenient description of $\Omega_{\mathbf{LI}}(S)$ in terms of the structural data used in [7].

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