Finite idempotent inverse monoid presentations

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ABSTRACT

Several decidability problems for finite idempotent presentations of inverse monoids are solved, giving also insight into their structure. Besides providing a new elementary solution for the problem, solutions are obtained for the following problems: computing the maximal subgroups, being combinatorial, being semisimple, being fundamental, having infinite $\mathcal{D}$-classes. The word problem for the least fundamental quotient is also solved.

1 Introduction

Inverse monoids arise naturally as monoids of injective transformations closed under inversion. Indeed, up to isomorphism, these are all the inverse monoids, as stated in the classical Vagner-Preston representation theorem. This makes inverse monoids ubiquitous in geometry, topology and other fields.

An abstract approach turns inverse monoids into a variety in the (2,1,0) signature, and so there exist free inverse monoids on an arbitrary set, a well-established fact in the fifties, the decade that boosted the systematic study of inverse monoids. However, the word problem remained unsolved until the early seventies, when Scheiblich [12] and Munn [9] independently provided solutions.

The beautiful solution by Munn, where the elements of the free inverse monoid are identified with finite tree inverse automata (known today as Munn trees), inspired Stephen to develop a general combinatorial theory of inverse monoid presentations [17]. The two cornerstones of Stephen's approach are the following facts:

- the full structure of the inverse monoid is determined by the strongly connected components of its Cayley graph (the Schützenberger graphs) with respect to the presentation considered;
- if the presentation is finite, each Schützenberger graph is the direct limit of a sequence of finite automata effectively constructible from an appropriate Munn tree, analogously to the Todd-Coxeter procedure.

This approximation of Schützenberger graphs is effective if the Schützenberger graphs are finite, but it cannot of course be effective in general. In the infinite case, success has been achieved mostly in two opposite directions:

- the group case (we note that a finitely presented group is also finitely presented as an inverse monoid);
- the idempotent-presented case.

An inverse monoid presentation is an idempotent presentation if all the relators involve idempotents. Let τ denote the congruence on the free inverse monoid FIM_A defined by the inverse monoid presentation $\langle A \mid R \rangle$ and let σ denote the least group congruence on FIM_A. Then

- FIM_A/τ is a group if and only if $\sigma \subseteq \tau$;
- $\langle A \mid R \rangle$ is an idempotent presentation if and only if $\tau \subseteq \sigma$.

Therefore the idempotent-presented case can be viewed as some sort of *anti-group case* (even though there exists one group which admits an idempotent presentation of the form $\langle A \mid R \rangle$: the free group on A).

Schützenberger graphs of idempotent presentations are trees, and this allows a favourable adaptation of Stephen's construction. These presentations were studied by Margolis and Meakin in a fruitful series of papers (see e.g. [2, 5, 6, 7, 8]). In [7], they obtained the first solution for the word problem. Linear solutions were recently obtained by Lohrey and Ondrusch [4], and subsequently by Diekert, Lohrey and Ondrusch [3].

The present paper aims at solving a few other algorithmic questions that arise naturally in the study of (inverse) monoids. Its structure is organized as follows.

In Section 2 we sum up the required preliminaries on free inverse monoid, and on Section 3 we start to consider (finite) idempotent presentations, including a new elementary solution for the word problem. Section 4 is devoted to maximal subgroups and combinatoriality, Section 5 to the existence of infinite $\mathcal{D}$-classes, Section 6 to semisimplicity and Section 7 to fundamentality. Finally, the word problem for the least fundamental quotient is solved in Section 8.

2 Free inverse monoids

For generalities on inverse monoids, see [10]. Throughout the paper, A denotes a finite alphabet. Let A^{-1} denote a set of formal inverses of A and write $\widetilde{A} = A \cup A^{-1}$. Let

$$R_A = \widetilde{A}^* \setminus (\bigcup_{a \in \widetilde{A}} \widetilde{A}^* a a^{-1} \widetilde{A}^*)$$

denote the set of all reduced words in the free monoid $\widetilde{A}^*$. Given $w \in \widetilde{A}^*$, we denote by $\overline{w}$ the reduced word obtained by successively erasing in w all the factors of the form aa^{-1}. Note that, whenever we write uv for $u, v \in \widetilde{A}^*$, we are considering uv as a word in the free monoid $\widetilde{A}^*$, not in the free group. We denote by C_A the set of all *cyclically reduced words* on $\widetilde{A}$.

We extend ${}^{-1} : A \to A^{-1} : a \mapsto a^{-1}$ to an involution on $\widetilde{A}^*$ through

$$(a^{-1})^{-1} = a, \quad (uv)^{-1} = v^{-1}u^{-1} \qquad (a \in A;\ u, v \in \widetilde{A}^*)\,.$$

The *free inverse monoid on* A is the quotient $FIM_A = \widetilde{A}^*/\rho$, where

$$\rho = (\{(ww^{-1}w, w) \mid w \in \widetilde{A}^*\} \cup \{(uu^{-1}vv^{-1}, vv^{-1}uu^{-1}) \mid u, v \in \widetilde{A}^*\})^\sharp$$

denotes the *Vagner congruence* on $\widetilde{A}^*$.

An *inverse automaton* over the alphabet $\widetilde{A}$ is a structure of the form $\mathcal{A} = (Q, i, t, E)$, where

- Q is the set of vertices,
- $i, t \in Q$ are the initial and terminal vertices, respectively,
- $E \subseteq Q \times \widetilde{A} \times Q$ is the set of edges,

satisfying the following properties:

- *deterministic*: $(p, a, q), (p, a, q') \in E \Rightarrow q = q'$;
- *involutive*: $(p, a, q) \in E \Leftrightarrow (q, a^{-1}, p) \in E$;
- *trim*: every vertex lies in some successful path.

We say that the inverse automaton $\mathcal{A}$ is a *tree inverse automaton* if no loop is labelled by a nonempty reduced word. An important example is given by the *Cayley graph* $\Gamma_A(FG_A)$ on the standard generators of the free group on A, when we fix the identity 1 as the initial vertex and g as the terminal vertex. This tree inverse automaton recognizes the set of all words on $\widetilde{A}^*$ equivalent to g in FG_A. Moreover, the underlying graph of a tree inverse automaton on $\widetilde{A}$ must be a connected subgraph of $\Gamma_A(FG_A)$.

Given $w \in \widetilde{A}$, let Pref(w) denote the set of all prefixes of w and write $T_0(w) = \overline{\mathrm{Pref}(w)}$. The *Munn tree* of w is the finite tree inverse automaton

$$\mathrm{MT}(w) = (T_0(w), 1, \overline{w}, E_0(w)),$$

where

$$E_0(w) = \{(p, a, q) \in T_0(w) \times \widetilde{A} \times T_0(w) \mid q = \overline{pa}\}.$$

The prefix-closed language $T_0(w)$ is the set of labels of *geodesics* connecting 1 to each vertex in MT(w).

W. D. Munn gave the following elegant solution for the word problem of FIM_A (see also [12] by Scheiblich):

Theorem 2.1 [9] *For all $u, v \in \widetilde{A}^*$, the following conditions are equivalent:*

(i) $u\rho = v\rho$;

(ii) $MT(u) \cong MT(v)$;

(iii) $T_0(u) = T_0(v)$ *and* $\overline{u} = \overline{v}$.

We can now characterize the idempotents of FIM_A: given $w \in \widetilde{A}^*$, we have

$$w\rho \in E(FIM_A) \Leftrightarrow \overline{w} = 1.$$

Recall that w is said to be a *Dyck word* if $\overline{w} = 1$. We denote the set of all Dyck words on the alphabet $\widetilde{A}$ by DW_A.

We note also that, for all $u, v \in \widetilde{A}^*$, we have

$$T_0(uv) = T_0(u) \cup \overline{uT_0(v)}, \qquad T_0(u^{-1}) = \overline{u^{-1}T_0(u)}.$$

Moreover, for all $e\rho, f\rho \in E(FIM_A)$, we have

$$e\rho \leq f\rho \Leftrightarrow T_0(e) \supseteq T_0(f).$$

Recall that $e\rho \prec f\rho$ if $e\rho < f\rho$ and there exists no $g\rho \in E(FIM_A)$ such that $e\rho < g\rho < f\rho$. It is easy to see that

$$e\rho \prec f\rho \;\;\Leftrightarrow\;\; (\; T_0(e) \supset T_0(f) \;\;\wedge\;\; |T_0(e)| = |T_0(f)| + 1 \;).$$

3 Idempotent presentations

In this paper, a (finite) inverse monoid presentation is a formal expression of the form $\mathcal{P} = \langle A \mid R \rangle$, where A is a (finite) alphabet and R is a (finite) subset of $\widetilde{A}^* \times \widetilde{A}^*$. Write $\tau = (\rho \cup R)^\sharp$. The quotient $M = FIM(A)/\tau$ is the inverse monoid defined by $\mathcal{P}$. We say that $\mathcal{P}$ is an *idempotent presentation* if, for every $(u, v) \in R$, we have $\overline{u} = \overline{v} = 1$.

Note that we may always assume in an idempotent presentation that $f\rho \prec e\rho$ for every $(e, f) \in R$. Indeed, we may replace (e, f) by the pair $(e, ef), (f, ef)$ in $\mathcal{P}$ to assume that $f\rho < e\rho$. If $f\rho = g_0\rho \prec \ldots \prec g_n\rho = e\rho$, we may replace (e, f) by all the (g_{i-1}, g_i), hence the claim holds. We call such presentations *normalized*. From now on, we fix $\mathcal{P} = \langle A \mid R \rangle$ to be a normalized idempotent presentation, fixing τ and M as well.

Solving the word problem and other decidability problems for idempotent presentations requires understanding their Schützenberger graphs, that is, the strongly connected components of the Cayley graph (on the standard generators). The *Schützenberger automaton* of $w \in \widetilde{A}^*$ has as vertex set the $\mathcal{R}$-class of $w\tau$ in M. It is a tree inverse automaton that can be approximated using a favourable variation of Stephen's construction:

Following [7], we build a sequence $(T_n(w))_n$ of finite prefix-closed subsets of R_A inductively. We have already defined $T_0(w)$ before, hence assume that $T_{n-1}(w)$ is defined. Then $T_n(w)$ is obtained by adding to $T_{n-1}(w)$, for all instances of $(e, f) \in R$ and $p \in T_{n-1}(w)$ such that $\overline{pT_0(e)} \subseteq T_{n-1}(w)$, the unique element of $\overline{p(T_0(f) \setminus T_0(e))}$.

A sequence of finite tree inverse automata is now defined through

$$\mathcal{T}_n(w) = (T_n(w), 1, \overline{w}, E_n(w)),$$

where

$$E_n(w) = \{(p, a, q) \in T_n(w) \times \widetilde{A} \times T_n(w) \mid q = \overline{pa}\}.$$

Note that $\mathcal{T}_0(w) = \mathrm{MT}(w)$. We call $(\mathcal{T}_n(w))_n$ the *Stephen's sequence* of w with respect to $\mathcal{P}$. From a combinatorial viewpoint, we consider in this inductive construction, for all

$(e,f) \in R$ and $p \in T_{n-1}(w)$, whether or not the Munn tree $\mathrm{MT}(e)$ embeds in $\mathcal{T}_{n-1}(w)$ at vertex p. If it does, we expand $\mathcal{T}_{n-1}(w)$ by gluing $\mathrm{MT}(f)$ at vertex p.

Let $T(w) = \cup_{n\geq 0} T_n(w)$, $E(w) = \cup_{n\geq 0} E_n(w)$ and $\mathcal{T}(w) = (T(w), 1, \overline{w}, E(w))$. It turns out that $\mathcal{T}(w)$ is the Schützenberger automaton of w with respect to $\mathcal{P}$. Moreover, $\mathcal{T}(w)$ is the *direct limit* of the sequence $(\mathcal{T}_n(w))_n$ (see [7, 17]). Note that $T(w)$ is a $\mathcal{P}$-*closed* prefix-closed subset of R_A in the sense that it cannot suffer any proper expansion: if $(e,f) \in R$ and $\mathrm{MT}(e)$ embeds in $\mathcal{T}(w)$ at vertex p, so does $\mathrm{MT}(f)$.

The underlying graph of $\mathcal{T}(w)$ is the *Schützenberger graph* of w and is denoted by $\Gamma S(w)$. The language recognized by $\mathcal{T}(w)$ is easily described in terms of the natural partial order of M:

Proposition 3.1 [7, 17] *For every* $w \in \widetilde{A}^*$,

$$L(\mathcal{T}(w)) = \{u \in \widetilde{A}^* \mid u\tau \geq w\tau\}.$$

The role played by the Schützenberger automaton in the solution of the word problem is evident in the following result:

Theorem 3.2 [7, 17] *For all* $u, v \in \widetilde{A}^*$*, the following conditions are equivalent:*

(i) $u\tau = v\tau$;

(ii) $\mathcal{T}(u) \cong \mathcal{T}(v)$;

(iii) $T(u) = T(v)$ *and* $\overline{u} = \overline{v}$;

(iv) $T_0(u) \subseteq T(v)$, $T_0(v) \subseteq T(u)$ *and* $\overline{u} = \overline{v}$.

Therefore the word problem is solvable if the membership problem for the sets $T(w)$ is decidable. Indeed, it turns out that:

Theorem 3.3 [7] *For every* $w \in \widetilde{A}^*$, $T(w)$ *is an effectively constructible rational language.*

The original proof, due to Margolis and Meakin, relies on an adaptation to free groups of Rabin's Tree Theorem. An alternative language-theoretic solution was subsequently proposed by the author [13]. Recently, a most efficient linear time solution was obtained by Lohrey and Ondrusch using tree automaton techniques [4]. A second linear time solution, of wider scope, was later obtained by Diekert, Lohrey and Ondrusch using rewriting systems [3]. We shall provide soon a new short combinatorial proof, although less efficient. We recall also the following result:

Theorem 3.4 [15] *Given* $L \subseteq R_A$ *rational, it is decidable whether or not:*

(i) $L = T(w)$ *for some idempotent presentation* $\langle A \mid R \rangle$ *and* $w \in \widetilde{A}^*$;

(ii) $L = T(w)$ *for some finite idempotent presentation* $\langle A \mid R \rangle$ *and* $w \in \widetilde{A}^*$.

Given $T \subseteq R_A$ prefix-closed and $p \in T$, let

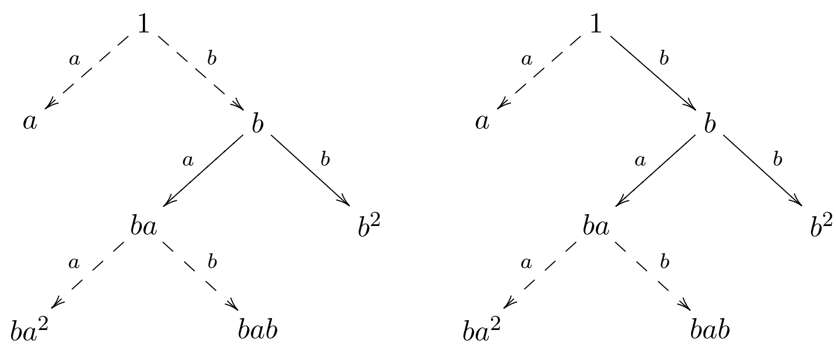
$$\mathrm{cone}(p, T) = \{w \in R_A \mid pw \in T\}.$$

For every $m \geq 0$, let also

$$\mathrm{cone}(p, T, m) = \{w \in \mathrm{cone}(p, T) : |w| \leq m\},$$

$$\text{nbh}(p, T, m) = \{w \in \overline{p^{-1}T} : |w| \leq m\}.$$

Clearly, $\text{nbh}(p, T, m)$ is the set of labels of geodesics leaving p in $\mathcal{T}(w)$, having length at most m. If we consider only geodesics oriented outwards the initial vertex, we get $\text{cone}(p, T, m)$.



$\text{cone}(b, T, 1) = \{1, a, b\}$, $\qquad$ $\text{nbh}(b, T, 1) = \{1, a, b, b^{-1}\}$

We may also use the terms *cone* and *neighbourhood* to refer to the vertices determined by the operators cone and nbh.

Finally, for every $w \in \widetilde{A}^*$, let

$$||w|| = \max\{|p| : p \in T_0(w)\},$$

$$||R|| = \max\{||f|| : (e, f) \in R\},$$

where we assume R normalized.

Lemma 3.5 *Let $w \in \widetilde{A}^*$ and $p, q \in T(w)$ be such that $\text{cone}(p, T(w), m) = \text{cone}(q, T(w), m)$ for $m = \max\{||w||, 2||R||\}$. Then $\text{cone}(p, T(w)) = \text{cone}(q, T(w))$.*

Proof. Since $m \geq ||w||$, all vertices of $\mathcal{T}(w)$ corresponding to elements of $\text{cone}(p, T(w)) \setminus \text{cone}(p, T(w), m)$ must have been obtained through expansions. Since $m \geq 2||R||$, all those vertices must have been successively obtained in Stephen's sequence through expansions which originated from (a subset of) $\text{cone}(p, T(w), m)$. $\square$

A new proof of Theorem 3.3. Fix $w \in \widetilde{A}^*$. Write $T = T(w)$ and $T_n = T_n(w)$. If we consider all the vertices to be terminal in $\mathcal{T}(w)$ and keep only the edges oriented outwards the initial vertex, we get a trim deterministic automaton recognizing T. By the standard minimization algorithm of rational languages (considering the Nerode equivalence) [1, Section I.4], $\{\text{cone}(p, T) \mid p \in T\}$ can be taken as the vertex set of the minimal automaton of T. By Lemma 3.5, this latter automaton is finite and so T is rational.

Given vertices $p, q \in T$, we say that p is *older* than q if it appeared in a previous iterate of the Stephen's sequence. Keeping $m = \max\{||w||, 2||R||\}$, suppose that there exist some $n, k \in \mathbb{N}$ satisfying the following conditions:

(R1) For every $p \in T_n$ of length k, there exists some shorter $p' \in T_n$ such that

- $\text{cone}(p, T_n, m) = \text{cone}(p', T_n, m)$;

- every vertex from $\mathrm{nbh}(p, T_n, m)$ is younger than the corresponding vertex from $\mathrm{nbh}(p', T_n, m)$;

(R2) $\mathrm{cone}(1, T_n, k) = \mathrm{cone}(1, T_{n+1}, k)$.

We claim that

$$\mathrm{cone}(1, T, k+m) = \mathrm{cone}(1, T_n, k+m). \tag{1}$$

Suppose not. Let $q \in \mathrm{cone}(1, T, k+m) \setminus \mathrm{cone}(1, T_n, k+m)$ correspond to the oldest extra vertex. In view of condition (R2), we must have $|q| > k$, hence there exists some $p \in T_n$ of length k such that $q = pu$ and $u \in \mathrm{cone}(p, T, m) \setminus \mathrm{cone}(p, T_n, m)$. Clearly, this vertex is the result of some sequence of expansions in the cone of p. Now, by condition (R1), a similar sequence of expansions should have occurred earlier to produce $p'u \in \mathrm{cone}(1, T, k+m)$. Now $u \notin \mathrm{cone}(p, T_n, m) = \mathrm{cone}(p', T_n, m)$ together with $|p'| < |p| = k$ yields $p'u \notin \mathrm{cone}(1, T_n, k+m)$. Hence $p'u \in \mathrm{cone}(1, T, k+m) \setminus \mathrm{cone}(1, T_n, k+m)$, thus contradicting the seniority of q. Thus (1) holds.

It follows that $\mathrm{cone}(p, T, m) = \mathrm{cone}(p, T_n, m)$ for every $p \in \mathrm{cone}(1, T_n, k)$. We have all the relevant edges and application of the standard minimization algorithm yields the minimal automaton of T.

Clearly, we can check at any step of Stephen's construction if conditions (R1)-(R2) hold, colouring the vertices by their age to make the verification easier. Therefore it suffices to show that conditions (R1)-(R2) must eventually occur in Stephen's construction.

Indeed, for a fixed k, it is clear that $\mathrm{cone}(1, T_n, k)$ eventually stabilizes when $n \to +\infty$, and the claim follows from having only finitely many neighbourhood types of fixed size m. $\square$

We consider next Green's relations on M. Note that $T(w^{-1}) = \overline{w^{-1}T(w)}$ for every $w \in \widetilde{A}^*$ [7].

Proposition 3.6 [7] *Let $u, v \in \widetilde{A}^*$. Then:*

(i) $(u\tau)\,\mathcal{R}\,(v\tau) \quad \Leftrightarrow \quad T(u) = T(v)$;

(ii) $(u\tau)\,\mathcal{L}\,(v\tau) \quad \Leftrightarrow \quad \overline{u^{-1}T(u)} = \overline{v^{-1}T(v)}$;

(iii) $(u\tau)\,\mathcal{D}\,(v\tau) \quad \Leftrightarrow \quad T(v) = \overline{wT(u)}$ *for some* $w \in R_A \quad \Leftrightarrow \quad \Gamma S(u) \cong \Gamma S(v)$;

(iv) $(u\tau)\,\mathcal{H}\,((uu^{-1})\tau) \quad \Leftrightarrow \quad T(u) = \overline{uT(u)}$.

Let $K, L \subseteq R_A$ be rational. It was proved in [16, Theorem 5.1] that $\{u \in R_A \mid \overline{Ku} \subseteq L\}$ is an effectively constructible rational language. Since

$$\overline{uK} = L \qquad \Leftrightarrow \qquad (\overline{L^{-1}u} \subseteq K^{-1} \text{ and } \overline{K^{-1}u^{-1}} \subseteq L^{-1}),$$

and rational languages are closed for inversion (being closed for reversion and automorphisms), it follows also that

$$\{u \in R_A \mid \overline{uK} = L\} \text{ is an effectively constructible rational language.} \tag{2}$$

Proposition 3.7 *It is decidable, for $\mathcal{K} \in \{\mathcal{R}, \mathcal{L}, \mathcal{H}, \mathcal{D}, \mathcal{J}\}$ and arbitrary $u, v \in \widetilde{A}^*$, whether or not $(u\tau)\,\mathcal{K}\,(v\tau)$.*

Proof. The cases $\mathcal{R}, \mathcal{L}, \mathcal{H}$ follow immediately from Theorem 3.3 and Proposition 3.6. The case $\mathcal{D}$ follows from the same results and (2). We consider now the case $\mathcal{J}$. By Proposition 3.1, $u\tau \in M(v\tau)M$ if and only if v labels some path in $\mathcal{T}(u)$, i.e., if and only if MT(v) embeds in $\mathcal{T}(u)$. We show that this can be decided.

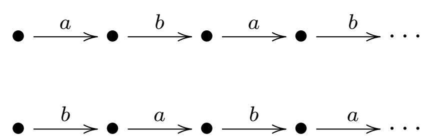
Indeed, if MT(v) embeds in $\mathcal{T}(u)$, such an embedding involves a unique vertex p of minimum depth in $\mathcal{T}(u)$ (relative to the initial vertex 1). Changing the initial vertex in MT(v) corresponds to moving inside the $\mathcal{L}$-class, so we must decide if some $v'\tau \in L_{v\tau}$ satisfies $T_0(v') \subseteq \text{cone}(p, T(u))$ for some $p \in T(u)$. Now $T_0(v')$ must be of the form $\overline{w^{-1}T_0(v)}$ for some $w \in T_0(v)$. If (Q, q_0, Q, E) is the minimum automaton of $T(u)$, then the sets cone$(p, T(u))$ correspond to rational languages of the form $L(Q, q, Q, E)$ for $q \in Q$. Hence $u\tau \in M(v\tau)M$ if and only if

$$\overline{w^{-1}T_0(v)} \subseteq L(Q, q, Q, E) \text{ for some } w \in T_0(v) \text{ and } q \in Q$$

and is therefore decidable. □

We show next that $\mathcal{D}$ may be strictly contained in $\mathcal{J}$:

Example 3.8 *Let M be defined by $\langle a, b \mid a^{-1}a = bb^{-1}, b^{-1}b = aa^{-1}\rangle$. Then $a\tau$ and $b\tau$ are $\mathcal{J}$-related but not $\mathcal{D}$-related, having*



as Schützenberger graphs.

Other known decidabiity results for finite idempotent presentations involve a solution of the *generalized word problem* by Lohrey and Ondrusch [4] and a solution of the *conjugacy problem* in restricted cases by the author [14].

4 Maximal subgroups

It follows easily from Proposition 3.6(iv) that the maximal subgroups of M, that is, its group $\mathcal{H}$-classes, are isomorphic to the automorphism group of the Schützenberger graph of the corresponding $\mathcal{D}$-class [17]. Since the Schützenberger graphs are trees in our case, it follows that the maximal subgroups of M are necessarily free groups. What about the rank? Recall that a monoid is said to be *combinatorial* if it has no nontrivial subgroups.

The free group FG_A acts on its Cayley graph $\Gamma_A(FG_A)$ by left translations. Given $w \in \widetilde{A}^*$, an automorphism of the tree $\mathcal{T}(w)$ can be seen as the restriction of an automorphism of $\Gamma_A(FG_A)$ which fixes $\mathcal{T}(w)$. It is then determined by the image of the vertex 1, that is, some word $u \in R_A$. Hence Aut($\mathcal{T}(w)$) is isomorphic to

$$\text{Stab}(T(w)) = \{u \in R_A \mid \overline{uT(w)} = T(w)\},$$

viewed as a subgroup of FG_A.

Lemma 4.1 *For every $w \in \widetilde{A}^*$, Stab($T(w)$) is an effectively computable finitely generated subgroup of FG_A.*

Proof. It follows from (2) and Benois' Theorem [1, Proposition III.2.8] that $\mathrm{Stab}(T(w))$ is an effectively constructible rational subgroup of FG_A. By Anisimov and Seifert's Theorem [11, Prop. II.6.2], $\mathrm{Stab}(T(w))$ is finitely generated as a subgroup of FG_A. □

We need now a technical lemma:

Lemma 4.2 *Suppose that $g \in Stab(T(w))$ is a cyclically reduced word ending in $a \in \widetilde{A}$. If $T_0(u) = cone(1, T(w), ||R||)$, then*

$$T(w) \setminus a^{-1}R_A \subseteq T(u) \setminus a^{-1}R_A.$$

Proof. Write $T = T(w)$ and $T' = T \setminus a^{-1}R_A$.

Replacing g by some positive power g^n if necessary, we may assume that $|g| > ||w||$. We show that, for every $v \in R_A$,

$$v \in T' \Leftrightarrow gv \in T'. \tag{3}$$

Indeed, $v \in T'$ implies that $gv \in R_A$ and so $\overline{gT} = T$ yields $gv = \overline{gv} \in T$ and therefore $gv \in T'$ since g is cyclically reduced.

Conversely, assume that $gv \in T'$. Then $\overline{g^{-1}T} = T$ yields $v \in T$. Since $gv \in R_A$, we get $v \in T'$ and so (3) holds.

Let $gx_1, gx_2, \ldots$ be an enumeration of the vertices in $g \cdot \mathrm{cone}(g, T)$ as they appear when we apply Stephen's construction to $\mathrm{MT}(w)$ (performing one expansion at the time). In view of (3), $T' = \mathrm{cone}(g, T) = \{x_1, x_2, \ldots\}$ We claim that $x_1, x_2, \ldots$ can be successively obtained from $\mathrm{MT}(u)$ through Stephen's construction, therefore proving the lemma.

Indeed, assume that this holds for all $j < i$ and consider x_i. If $|x_i| \leq ||R||$, then $x_i \in T_0(u)$ and there is nothing else to prove. Hence we may assume that $|x_i| > ||R||$.

Now gx_i must be the extra element provided by some expansion from $\mathrm{MT}(e)$ to $\mathrm{MT}(f)$ in the Stephen's construction of $\mathcal{T}(w)$, at some vertex gp, for some $(e, f) \in R$. Since $|x_i| > ||e||$, $\mathrm{MT}(e)$ is fully embedded in the subtree of $\mathcal{T}_{i-1}(w)$ induced by the vertices $gx_1, \ldots, gx_{i-1}$ and possibly some older vertices from $\mathrm{nbh}(g, T, ||R||) = \overline{gT_0(u)}$ (since $g \in \mathrm{Stab}(T)$). By the induction hypothesis, $\mathrm{MT}(e)$ is fully embedded at p in the subtree of $\mathcal{T}_{i-1}(u)$ induced by $T_0(u)$ and the vertices $x_1, \ldots, x_{i-1}$. Hence x_i can also be obtained through an expansion, featuring as a vertex in $\mathcal{T}_i(u)$. This proves our claim and therefore the lemma. □

We can now prove the following result:

Theorem 4.3 *The following conditions are equivalent:*

(i) M is combinatorial;

(ii) $Stab(T(w))$ is trivial whenever $||w|| \leq ||R||$.

Proof. (i) ⇒ (ii): Immediate from the previous discussion.

(ii) ⇒ (i): Assume that there exists some $w \in \widetilde{A}^*$ such that $\mathrm{Stab}(T(w))$ is nontrivial. Since $T(w) = T(ww^{-1})$, we may assume that $\overline{w} = 1$. Let $g \in \mathrm{Stab}(T(w))$ be nonempty and write $g = xhx^{-1}$ with h cyclically reduced. Then there exists a path $1 \xrightarrow{x} q$ in $\mathcal{T}(w)$ for some q, and it is immediate that $\mathcal{T}(x^{-1}wx)$ is obtained from $\mathcal{T}(w)$ by taking q as the new initial/terminal vertex. Hence $\mathcal{T}(x^{-1}wx)$ also has a nontrivial automorphism, and $\overline{hT(x^{-1}wx)} = T(x^{-1}wx)$. Therefore we may assume that g is cyclically reduced.

Write $T = T(w)$. Since $\overline{gT} = T$, we have $\overline{g^nT} = T$ for every $n \in \mathbb{Z}$. We now apply Lemma 4.2 to both g and g^{-1}, taking $u \in \widetilde{A}^*$ such that $T_0(u) = \mathrm{cone}(1, T, ||R||)$. Assume that a is the first letter of g and b its last letter. By Lemma 4.2, we have

$$T \setminus b^{-1}R_A \subseteq T(u) \setminus b^{-1}R_A, \quad T \setminus aR_A \subseteq T(u) \setminus aR_A.$$

Since g is cyclically reduced, we get

$$T = (T \setminus b^{-1}R_A) \cup (T \setminus aR_A) \subseteq T(u).$$

Since $T_0(u) \subseteq T$ and T is $\mathcal{P}$-closed, the inclusion $T(u) \subseteq T$ holds. Therefore $T = T(u)$ and so $\overline{gT(u)} = T(u)$. Since $||u|| \leq ||R||$ and $g \neq 1$, the theorem follows. □

Corollary 4.4 *It is decidable whether or not M is combinatorial.*

Proof. Indeed, there are only finitely many Munn trees $\mathrm{MT}(u)$ such that $||u|| \leq ||R||$. By Theorem 3.3, we can build $T(u)$ for each one of them and compute $\mathrm{Stab}(T(u))$ by Lemma 4.1. □

Corollary 4.5 *If D is a $\mathcal{D}$-class of M with nontrivial subgroups, then $D = D_{u\tau}$ for some $u \in \widetilde{A}^*$ such that $||u|| \leq ||R||$. Furthermore, the maximal subgroups of these $\mathcal{D}$-classes can be effectively computed.*

Proof. If $D = D_{w\tau}$ is such a $\mathcal{D}$-class, it follows from the proof of Theorem 4.3 that $T(w) = T(u)$ for some $u \in \widetilde{A}^*$ such that $||u|| \leq ||R||$. Hence $D = D_{u\tau}$. The maximal subgroups of these $\mathcal{D}$-classes can be effectively computed by Lemma 4.1.□

5 Finite $\mathcal{D}$-classes

For all $w \in \widetilde{A}^*$ and $u \in T(w)$, we fix $\delta_w(u) \in \mathrm{DW}_A$ satisfying

$$T_0(\delta_w(u)) = \mathrm{nbh}(u, T(w), ||R||) = \{v \in T(u^{-1}ww^{-1}u) : |v| \leq ||R||\}.$$

We can take $\delta_w(u)$ as the minimum such word for the lexicographic order. Since the right hand side is a finite nonempty prefix-closed subset of R_A, then $\delta_w(u)$ is well defined. The next result shows that every tree $T(w)$ can be decomposed as a union of finitely many tree types.

Lemma 5.1 *If $w \in \widetilde{A}^*$, then $T(w) = \cup_{u \in T_0(w)} \overline{uT(\delta_w(u))}$.*

Proof. ($\subseteq$): Given a word $v \in T(w)$, we factor it as $v = (v\alpha)(v\beta)$, where $v\alpha$ denotes the longest prefix of v in $T_0(w)$. We prove that

$$u\alpha^{-1} \subseteq \overline{uT(\delta_w(u))} \tag{4}$$

holds for every $u \in T_0(w)$. Since $T(w) = \cup_{u \in T_0(w)} u\alpha^{-1}$, this proves the direct inclusion.

Fix $u \in T_0(w)$. Clearly. $u\alpha^{-1} \cap T_0(w) = \{u\}$, so we only have to care about $T(w) \setminus T_0(w)$. We assume that expansions are performed one at the time, so the elements of

$T(w) \setminus T_0(w)$ can be written as a sequence $w_1, w_2, \ldots$ in which each w_n can be obtained from its predecessors and $T_0(w)$ by performing a certain expansion. We can now delete from this sequence the elements that are not in $u\alpha^{-1}$. We prove that $w_n \in \overline{uT(\delta_w(u))}$ by induction on n.

Assume that $w_i \in \overline{uT(\delta_w(u))}$ for every $i < n$. If $|w_n\beta| \leq ||R||$, then $w_n\beta \in T_0(\delta_w(u))$ and so $w_n = u(w_n\beta) \in \overline{uT(\delta_w(u))}$, hence we may assume that $|w_n\beta| > ||R||$. Now w_n was obtained by performing some expansion from $\mathrm{MT}(e)$ to $\mathrm{MT}(f)$ at some vertex q. Since $||f|| \leq ||R||$, it follows that $q \in u\alpha^{-1} \setminus \{u\}$. Moreover, $\overline{qT_0(e)} \subseteq u\alpha^{-1} \cup \overline{uT_0(\delta_w(u))}$ and so all its elements are in $\{w_1, \ldots, w_{n-1}\} \cup \overline{uT_0(\delta_w(u))}$. Using the induction hypothesis, we obtain $\overline{qT_0(e)} \subseteq \overline{uT(\delta_w(u))}$. Hence $\overline{u^{-1}qT_0(e)} \subseteq T(\delta_w(u))$ and so $\overline{u^{-1}qT_0(f)} \subseteq T(\delta_w(u))$ since $T(\delta_w(u))$ is $\mathcal{P}$-closed. In particular, $u^{-1}w_n \in T(\delta_w(u))$ and so $w_n \in \overline{uT(\delta_w(u))}$, completing the induction. Thus (4) holds as required.

($\supseteq$): Let $u \in T_0(w)$. Since $T_0(\delta_w(u)) = \mathrm{nbh}(u, T(w), ||R||)$, we have $\overline{uT_0(\delta_w(u))} \subseteq T(w)$ and so $\overline{uT(\delta_w(u))} \subseteq T(w)$ since $T(w)$ is $\mathcal{P}$-closed. $\square$

We fix $e_0 \in \mathrm{DW}_A$ such that $T_0(e_0) = \{u \in R_A : |u| \leq ||R||\}$, say the minimum for the lexicographic order. We can now prove the main result of this section:

Theorem 5.2 *It is decidable:*

(i) whether or not all the Schützenberger graphs of M are finite;

(ii) whether or not all the $\mathcal{D}$-classes of M are finite.

Proof. (i) It follows from Lemma 5.1 that M has an infinite Schützenberger graph if and only if $T(f)$ is infinite for some $f \in \mathrm{DW}_A$ satisfying $||f|| \leq ||R||$. Since $T(f) \subseteq T(e_0)$ and $T(e_0)$ is an effectively constructible rational language by Theorem 3.3, the problem is decidable.

(ii) This follows from (i) since the cardinal of a $\mathcal{D}$-class is the square of the cardinal of the vertex set of the respective Schützenberger graph [17]. $\square$

6 Semisimplicity

We consider now semisimplicity. A monoid N is said to be semisimple if N has no bicyclic subsemigroup. This is known to be equivalent to the condition

$$\forall x, y \in N \ \forall e \in E(N) \qquad (xy = e \wedge xe = ex = x \wedge ye = ey = y) \quad \Rightarrow \quad yx = e.$$

It is immediate that in an inverse monoid, the left hand side conditions imply $y = x^{-1}$, hence our condition can be simplified to

$$\forall x \in N \qquad x^2x^{-1} = x \quad \Rightarrow \quad x^{-1}x \geq xx^{-1}.$$

Back to our finite idempotent presentation $\mathcal{P}$, defining the inverse monoid M, we can prove the following result:

Lemma 6.1 *M is semisimple if and only if every endomorphism of a Schützenberger graph is necessarily an automorphism.*

Proof. We can consider only Schützenberger graphs of Dyck words. First we note that any endomorphism of a tree must necessarily be a monomorphism. Moreover, if $\Gamma S(e)$ is the Schützenberger graph of e, then the endomorphisms of $\Gamma S(e)$ are characterized by the image of the vertex 1 and therefore correspond to the reduced words u satisfying $\overline{uT(e)} \subseteq T(e)$. If this inclusion is actually an equality, that is, if $T(e) \subseteq \overline{uT(e)}$, we have an automorphism.

Assume that M is semisimple and let φ be a monomorphism of $\Gamma S(e)$ for some $e \in \mathrm{DW}_A$. Then φ is determined by $u = 1\varphi$. Now $((eu)^2(eu)^{-1})\tau = (eu)\tau$ follows easily from the fact that $(eu)^2$ can be read off vertex 1 in $\Gamma S(e)$ (due to φ being a monomorphism), hence $(u^{-1}eu)\tau \geq (euu^{-1})\tau = e\tau$ by semisimplicity. Hence $\overline{u^{-1}T_0(e)} \subseteq T(e)$, that is, $\mathrm{MT}(e)$ embeds in $\Gamma S(e)$ at vertex u^{-1}. Since $T(e)$ is $\mathcal{P}$-closed, we get $\overline{u^{-1}T(e)} \subseteq T(e)$ and so $T(e) \subseteq \overline{uT(e)}$. Thus φ is an automorphism.

Conversely, assume that every endomorphism of a Schützenberger graph is an automorphism. Let $w \in \widetilde{A}^*$ and suppose that $(w^2w^{-1})\tau = w\tau$. Take $e = ww^{-1}$. Then $\overline{wT_0(e)} \subseteq T(w) = T(e)$ and so $\overline{wT(e)} \subseteq T(e)$ since the latter is $\mathcal{P}$-closed. Hence $\overline{w}$ defines an endomorphism of $\Gamma S(e)$. Since this endomorphism is actually an automorphism, we get $\overline{w^{-1}T(e)} = T(e)$ and so $T_0(w^{-1}) = \overline{w^{-1}T_0(e)} \subseteq T(e)$. Thus $(w^{-1}w)\tau \geq e\tau = (ww^{-1})\tau$ and M is semisimple. $\square$

Theorem 6.2 *It is decidable whether or not M is semisimple.*

Proof. We show that M is semisimple if and only if $\mathrm{End}\Gamma S(f) = \mathrm{Aut}\Gamma S(f)$ for every $f \in \mathrm{DW}_A$ with $||f|| \leq ||R||$. Then we can certainly decide $\mathrm{End}\Gamma S(f) \neq \mathrm{Aut}\Gamma S(f)$ for some given f, since this amounts to decide if there exists some $u \in R_A$ such that $\overline{uT(f)} \subset T(f)$. The set of solutions of such an equation can be proved to be rational and effectively constructible using similar arguments to the proof of (2).

In view of Lemma 6.1, it suffices to show that $\mathrm{End}\Gamma S(e) \neq \mathrm{Aut}\Gamma S(e)$ for some $e \in \mathrm{DW}_A$ implies $\mathrm{End}\Gamma S(f) \neq \mathrm{Aut}\Gamma S(f)$ for some $f \in \mathrm{DW}_A$ with $||f|| \leq ||R||$.

Assume that φ is an endomorphism (and therefore a monomorphism) of $\Gamma S(e)$ which is not an automorphism. Then φ is determined by $u = 1\varphi \neq 1$ and so $\overline{uT(e)} \subset T(e)$. Writing $u = xcx^{-1}$ with c cyclically reduced, it follows that c determines a monomorphism of $\Gamma S(x^{-1}ex) \cong \Gamma S(e)$ (we just compute the geodesics at a different vertex) which is not an automorphism either. Thus we may assume that u is cyclically reduced. Since $\overline{uT(e)} \subset T(e)$ and u is cyclically reduced, we have $u^n \in T(e)$ for every $n \geq 0$. Let $v = u^p u_1$ denote the longest prefix of the infinite word u^ω in $T_0(e)$, where u_1 is a proper prefix of u, and write $u = u_1u_2$. Let $q > 0$ be such that $|u^q| > ||e||$. We claim that

$$\overline{(u_2u_1)^{p+q+1}T(\delta_e(v))} \subset T(\delta_e(v)). \tag{5}$$

Indeed, $\overline{uT(e)} \subset T(e)$ yields $\overline{u^{p+q+1}T_0(e)} \subseteq T(e)$. Since $|u^q| > ||e||$, we get $\overline{u^{p+q+1}T_0(e)} \subseteq u^{p+1}R_A$ and so $\overline{u^{p+q+1}T_0(e)} \subseteq v\alpha^{-1}$.

By (4), it follows that $\overline{u^{p+q+1}T_0(e)} \subseteq \overline{vT(\delta_e(v))}$ and so

$$\overline{u_2u^qT_0(e)} \subseteq T(\delta_e(v)). \tag{6}$$

Since $T(\delta_e(v))$ is $\mathcal{P}$-closed, we get $\overline{u_2u^qT(e)} \subseteq T(\delta_e(v))$. It follows from Lemma 5.1 that $\overline{u_2u^qvT(\delta_e(v))} \subseteq T(\delta_e(v))$, i.e.,

$$\overline{(u_2u_1)^{p+q+1}T(\delta_e(v))} \subseteq T(\delta_e(v)).$$

It suffices to prove that this inclusion is strict.

Suppose that $\overline{(u_2u_1)^{p+q+1}T(\delta_e(v))} = T(\delta_e(v))$. By (6), we get

$$\overline{(u_1^{-1}u_2^{-1})^{2p+2q+2}u_2u^qT_0(e)} \subseteq T(\delta_e(v))$$

and so $\overline{u_1^{-1}u^{-2p-q-1}T_0(e)} \subseteq T(\delta_e(v))$. Thus $\overline{u_1^{-1}u^{-2p-q-1}T(e)} \subseteq T(\delta_e(v))$ by $\mathcal{P}$-closedness and so

$$\overline{u^{-p-q-1}T(e)} = \overline{vu_1^{-1}u^{-2p-q-1}T(e)} \subseteq \overline{vT(\delta_e(v))} \subseteq T(e)$$

by Lemma 5.1. Now $\overline{uT(e)} \subseteq T(e)$ because u determines the endomorphism of $T(e)$. It follows easily that $\overline{u^{p+q+1}T(e)} = T(e)$. Hence

$$\overline{u^{-1}T(e)} = \overline{u^{p+q}T(e)} \subseteq T(e)$$

and so $\overline{uT(e)} = T(e)$, a contradiction. Thus $\overline{(u_2u_1)^{p+q+1}T(\delta_e(v))} \subset T(\delta_e(v))$ and so (5) holds.

Therefore $\mathrm{End}\Gamma S(\delta_e(v)) \neq \mathrm{Aut}\Gamma S(\delta_e(v))$. Since $||\delta_e(v)|| \leq ||R||$, the proof is now complete. □

7 Fundamentality

We turn now our attention to fundamental inverse monoids. Recall that, given an inverse monoid S, the *maximum idempotent-separating* congruence on S is the largest congruence contained in $\mathcal{H}$. Alternatively, it can be defined by

$$x\mu y \qquad \text{if} \qquad xex^{-1} = yey^{-1} \text{ for every } e \in E(S).$$

If μ is the identity congruence, S is said to be *fundamental*.

Back to our finite idempotent presentation $\mathcal{P}$, we prove:

Lemma 7.1 *Let $x \in \widetilde{A}^*$ and $uvw \in T(x)$ satisfy*

$$uh \in T_0(x) \Rightarrow |h| \leq ||R|| \tag{7}$$

for every $h \in R_A$. Then $vw \in T(\delta_x(u))$.

Proof. Let $N = \mathrm{nbh}(u, T(x), ||R||)$. We assume that when we apply Stephen's construction to $\mathrm{MT}(x)$, we consider expansions one by one, so the new vertices come out as a totally ordered set $x_1, x_2, \ldots$. We consider the subsequence $uy_1, uy_2, \ldots$ of all vertices having u as a prefix. In view of (7), every word of $T(x)$ having u as a prefix must appear in our subsequence or in $\overline{uN}$, in particular uvw. We show that $y_n \in T(\delta_x(u))$ by induction on n. Let $n \geq 1$ and assume that the claim holds for all $i < n$. We may assume that $y_n \notin T_0(\delta_x(u)) = \mathrm{nbh}(u, T(x), ||R||)$. On the other hand, since $uy_n \in T(x)$, we have $y_n \in \mathrm{cone}(u, T(x))$. Thus $|y_n| > ||R||$.

By (7), uy_n was obtained by applying some expansion from $\mathrm{MT}(e)$ to $\mathrm{MT}(f)$, and it follows easily that the corresponding embedding of $\mathrm{MT}(e)$ must take place inside the subtree of $T(x)$ defined by $\{uy_1, \ldots, uy_{n-1}\} \cup \overline{uN}$ at some vertex p of $\mathcal{T}(x)$. Indeed, $|y_n| > ||R||$

implies that u is a prefix of p, and (7) implies that all the vertices involved are either in $\overline{uN}$ or were obtained through previous expansions.

Hence y_n can be obtained from the tree defined by $T_0(\delta_x(u)) \cup \{y_1, \ldots, y_{n-1}\}$ through an expansion and by the induction hypothesis this tree is a subtree of $T(\delta_x(u))$. Thus $y_n \in T(\delta_x(u))$ and our claim holds. Since we have already remarked that uvw must be among the uy_i, the lemma is proved. $\square$

Recall e_0 from Section 5.

Theorem 7.2 *The following conditions are equivalent for M:*

(i) M is fundamental;

(ii) $T(e_0) \neq R_A$;

(iii) $E(M)$ has no zero;

(iv) M has no group as a minimal ideal.

Proof. (i) $\Rightarrow$ (ii). Suppose that $T(e_0) = R_A$. Then $\overline{uT(e_0)} = T(e_0)$ for some $u \in R_A \setminus \{1\}$ and so $((e_0u)\tau)\,\mathcal{H}\,(e_0\tau)$. We claim that $((e_0u)\tau)\,\mu\,(e_0\tau)$. Since $\overline{e_0u} = u \neq 1 = \overline{e_0}$, it will follow that $(e_0u)\tau \neq e_0\tau$ and so M is not fundamental. Let $e \in \mathrm{DW}_A$. We must show that $(e_0ueu^{-1})\tau = (e_0e)\tau$. Since $T(e_0) = R_A$, it is immediate that $T(e_0ueu^{-1}) = T(e_0e) = R_A$. Together with $\overline{e_0ueu^{-1}} = 1 = \overline{e_0e}$, this implies $(e_0ueu^{-1})\tau = (e_0e)\tau$ and so M is not fundamental.

(ii) $\Rightarrow$ (i). Assume that M is not fundamental. Then we claim that $(e_0\tau)\,\mu\,(u\tau)$ for some nonidempotent $u\tau$. Indeed, we may assume that $(x\tau)\,\mu\,(y\tau)$ for some distinct $x\tau, y\tau \in M$. Since $\mu \subseteq \mathcal{H} \subseteq \mathcal{R}$, we have $T(x) = T(y)$ and so Theorem 3.2 yields $\overline{x} \neq \overline{y}$. Hence $((xx^{-1})\tau)\,\mu\,((yx^{-1})\tau)$ with $(yx^{-1})\tau$ nonidempotent.

By Corollary 4.5, we have $((xx^{-1})\tau)\,\mathcal{D}\,(e\tau)$ for some $e \in \mathrm{DW}_A$ such that $||e|| \leq ||R||$. Hence $e\tau = (zxx^{-1}z^{-1})\tau$ for some $z \in \widetilde{A}^*$ and so

$$e_0\tau = (e_0e)\tau = ((e_0zxx^{-1}z^{-1})\tau)\,\mu\,((e_0zyx^{-1}z^{-1})\tau).$$

Now $\overline{x} \neq \overline{y}$ yields $\overline{e_0zyx^{-1}z^{-1}} \neq 1$ and so our claim holds for $u = e_0zyx^{-1}z^{-1}$.

Thus $(u\tau)\,\mathcal{R}\,(e_0\tau)$, and we may write $u\tau = (e_0w)\tau$ for some $w \in R_A$ nonempty. Note that

$$(e_0\tau)\,\mu\,((e_0p)\tau) \Leftrightarrow (e_0\tau)\,\mu\,((e_0p^{-1})\tau) \tag{8}$$

holds for every $p \in R_A$. Indeed, $(e_0\tau)\,\mu\,((e_0p)\tau)$ yields $((e_0p^{-1})\tau)\,\mu\,((e_0pp^{-1})\tau) = e_0\tau$ since μ is a congruence and $(e_0\tau)\,\mathcal{R}\,((e_0p)\tau)$. The converse implication follows by symmetry.

Let $v \in R_A$. Write $w = dcd^{-1}$ with c cyclically reduced. In view of (8), we may replace w by w^{-1} if necessary to assume that $cv \in R_A$. Moreover, since $(e_0\tau)\,\mu\,((e_0dc^nd^{-1})\tau)$ for every $n > 0$, we may assume that $|c| > ||R||, |v|$. Now $(e_0\tau)\,\mu\,((e_0dcd^{-1})\tau)$ yields $((e_0dv)\tau)\,\mu\,((e_0dcd^{-1}dv)\tau)$ and therefore $((e_0dv)\tau)\,\mathcal{R}\,((e_0dcd^{-1}dv)\tau)$. It follows that $dcv = \overline{e_0dcd^{-1}dv} \in T(e_0dv)$. Note that $dc \notin T_0(e_0dv)$ since $|c| > ||R||, |v|$. Using the factorization $dc \cdot 1 \cdot v$, it follows from Lemma 7.1 that $v \in T(\delta_{e_0dv}(dc))$. Since $||\delta_{e_0dv}(dc)|| \leq ||R||$, it follows that $T(\delta_{e_0dv}(dc)) \subseteq T(e_0)$. Thus $v \in T(e_0)$ and so $T(e_0) = R_A$.

(ii) $\Rightarrow$ (iii). Assume that $f\tau$ is the zero of $E(M)$ for some $f \in \mathrm{DW}_A$. Then $(fww^{-1})\tau = f\tau$ for every $w \in R_A$ and so $T(f) = R_A$. Let $a \in \widetilde{A}$. Then $\overline{aT(f)} = \overline{a^{-1}T(f)} = T(f)$ and it follows from Lemma 4.2 that $T(f) \setminus a^{-1}R_A \subseteq T(e_0) \setminus a^{-1}R_A$ and $T(f) \setminus aR_A \subseteq T(e_0) \setminus aR_A$. Thus

$$T(f) = (T(f) \setminus a^{-1}R_A) \cup (T(f) \setminus aR_A) \subseteq T(e_0)$$

and so $T(e_0) = R_A$.

(iii) $\Rightarrow$ (iv). Suppose that I is the minimal ideal of M and is a group. Let e be the idempotent in I. For every $f \in E(M)$, $ef \in I \cap E(M)$, hence $ef = e$ and so e is the zero of $E(M)$.

(iv) $\Rightarrow$ (ii). Assume that $T(e_0) = R_A$. Then $M(e_0\tau)M = \{u\tau \mid T(u) = R_A\}$ is clearly the minimal ideal of M, and a free group of rank $|A|$. $\square$

Corollary 7.3 *It is decidable whether or not M is fundamental.*

Proof. It follows from Theorems 3.3 and 7.2. $\square$

We remark that fundamental does not imply combinatorial for M:

Example 7.4 *Let M be defined by the presentation $\langle a, b \mid aa^{-1} = a^{-1}a = 1\rangle$. Clearly, M is not combinatorial since it is the free product of a free group of rank 1 by a free monogenic inverse monoid.*

However, M is fundamental by Theorem 7.2: we have $e_0 = aa^{-2}abb^{-2}b$ and $T(e_0) = a^* \cup (a^{-1})^* \cup \{b, b^{-1}\}$.

8 The least fundamental quotient

We consider next the word problem for M/μ, the least fundamental quotient of M. We start with some technical lemmas.

Lemma 8.1 *Let $u, v, w \in \widetilde{A}^*$ be such that $(v\tau)\,\mathcal{R}\,(u\tau)\,\mathcal{H}\,((uu^{-1})\tau)$ and $\overline{v} \in \overline{u^* \cup (u^{-1})^*}$. Then $T(v\overline{u}w) = \overline{uT(vw)}$.*

Proof. Since the proof of the opposite inclusion is absolutely similar, we prove just the direct inclusion. By $\mathcal{P}$-closedness, it suffices to show that $T_0(v\overline{u}w) \subseteq \overline{uT(vw)}$. We can write

$$T_0(v\overline{u}w) = T_0(v) \cup \overline{vT_0(\overline{u})} \cup \overline{vuT_0(w)}.$$

Successive application of Proposition 3.6 yields $T_0(v) \subseteq T(v) = T(u) = \overline{uT(u)} = \overline{uT(v)} \subseteq \overline{uT(vw)}$. Now $\overline{uT(u)} = T(u)$ yields $\overline{vT(u)} = T(u) = \overline{uT(v)}$ and so $\overline{vT_0(\overline{u})} \subseteq \overline{vT(u)} = \overline{uT(v)} \subseteq \overline{uT(vw)}$. Finally, $\overline{vuT_0(w)} \subseteq \overline{vuT(w)} = \overline{uvT(w)} \subseteq \overline{uT(vw)}$. $\square$

For every $u \in \widetilde{A}^*$, write

$$||u||_1 = \max\{||u||, ||u^{-1}||\}.$$

We define

$$\widehat{C_A} = \{u \in \widetilde{A}^* \mid \overline{u} \in C_A \setminus \{1\}\},$$

$$J = \{u \in \widehat{C_A} \mid (uvv^{-1}u^{-1})\tau = (uu^{-1}vv^{-1})\tau \text{ for every } v \in R_A \text{ with } |v| \leq ||u||_1 + 2||R||\}.$$

For every $u \in J$, let

$$J_u = \{(e,v) \in \mathrm{DW}_A \times R_A : ||e|| \leq ||R|| \text{ and } |v| = ||u||_1 + ||R||\}.$$

For every $(e,v) \in J_u$, let $K(u,e,v)$ consist of all $w \in R_A \setminus T(e)$ such that

$$vw \in R_A, \quad v \notin T_0(uvew), \quad \mathrm{nbh}(\overline{uv}, T(uvew), ||R||) = T_0(e). \tag{9}$$

Lemma 8.2 *Let $u \in \widehat{C_A}$. Then the following conditions are equivalent:*

(i) $(u\tau)\,\mu\,((uu^{-1})\tau)$;

(ii) $u \in J$ and $K(u,e,v) = \emptyset$ for every $(e,v) \in J_u$.

Proof. (i) $\Rightarrow$ (ii): It is immediate that condition (i) implies $u \in J$. Take $(e,v) \in J_u$. Suppose that $w \in R_A$ satisfies all the conditions in (9). By (i), we have $(uvew w^{-1}v^{-1}u^{-1})\tau = (uu^{-1}vew w^{-1}v^{-1})\tau$. In particular,

$$vw = \overline{uu^{-1}vew} \in T(uu^{-1}vew w^{-1}v^{-1}) = T(uvew w^{-1}v^{-1}u^{-1}) = T(uvew).$$

Since $v \notin T_0(uvew)$, we may apply Lemma 7.1 to the factorization $v \cdot 1 \cdot w$ and get $w \in T(\delta_{uvew}(v))$. Since (i) yields $(u\tau)\,\mu\,((u^{-1}u)\tau)$ and so $((v^{-1}u^{-1}uvew)\tau)\,\mu\,((v^{-1}uvew)\tau)$, we get $((v^{-1}u^{-1}uvew)\tau)\,\mathcal{R}\,((v^{-1}uvew)\tau)$ and thus $T(v^{-1}u^{-1}uvew) = T(v^{-1}uvew)$. It follows that

$$T_0(\delta_{uvew}(v)) = \mathrm{nbh}(v, T(uvew), ||R||) = \mathrm{nbh}(\overline{uv}, T(uvew, ||R||) = T_0(e),$$

and so $w \in T(\delta_{uvew}(v)) = T(e)$. Therefore $K(u,e,v) = \emptyset$ and so condition (ii) holds.

(ii) $\Rightarrow$ (i): Assume that $(u\tau, (uu^{-1})\tau) \notin \mu$ and $u \in J$. We must prove that $K(u,e,v) \neq \emptyset$ for some $(e,v) \in J_u$.

Since $(u\tau, (uu^{-1})\tau) \notin \mu$, we have $(ufu^{-1})\tau \neq (uu^{-1}f)\tau$ for some $f \in \mathrm{DW}_A$ and so $(uzz^{-1}u^{-1})\tau \neq (uu^{-1}zz^{-1})\tau$ for some $z \in R_A$, or equivalently, $T(uz) \neq T(uu^{-1}z)$. Let such a z have minimum length. Since $u \in J$, we have $|z| > ||u||_1 + 2||R||$. Write $z = vw$ with $|v| = ||u||_1 + ||R||$. We prove that

$$|\overline{v^{-1}uv}| > |v|. \tag{10}$$

For a start, we claim that $\overline{u}$ is not a prefix of z. Indeed, $u \in J$ implies $(u\tau)\,\mathcal{H}\,((uu^{-1})\tau)$: taking $v = u, u^{-1}$, we get easily $(uu^{-1})\tau = (u^{-1}u)\tau$. Thus, if $z = \overline{u}z'$, we may apply Lemma 8.1 to get

$$T(uz) = T(u\overline{u}z') = \overline{uT(uz')}, \quad T(uu^{-1}z) = T(uu^{-1}\overline{u}z') = \overline{uT(uu^{-1}z')},$$

hence $T(uz) \neq T(uu^{-1}z)$ yields $T(uz') \neq T(uu^{-1}z')$, contradicting the minimality of z.

A similar argument shows that $\overline{u}^{-1}$ is not a prefix of z either. Now note that $|\overline{u}| < |v|$. Since $u \in \widehat{C_A}$, one of the products $v^{-1}\overline{u}$ and $\overline{u}v$ must be reduced. On the other hand, since neither $\overline{u}$ nor $\overline{u}^{-1}$ is a prefix of v, $\overline{u}$ cannot cancel completely v (nor v^{-1}). Therefore (10) holds.

Let $e = \delta_{uvw}(\overline{uv})$. Then $(e,v) \in J_u$. We prove that $w \in K(u,e,v)$. Of course, $vw = z \in R_A$. Moreover, $\mathrm{MT}(e)$ embeds in $\mathcal{T}(uvw)$ at vertex $\overline{uv}$ and so $T(uvew) = T(uvw)$. Thus

$$\mathrm{nbh}(\overline{uv}, T(uvew), ||R||) = \mathrm{nbh}(\overline{uv}, T(uvw), ||R||) = T_0(\delta_{uvw}(\overline{uv})) = T_0(e).$$

We prove next that

$$v \notin T_0(uvew) = T_0(u) \cup \overline{uT_0(v)} \cup \overline{uvT_0(e)} \cup \overline{uveT_0(w)}.$$

Since $|v| > ||u||$, then $v \notin T_0(u)$. Suppose that $v \in \overline{uT_0(v)}$. Then $v = \overline{u^{-1}v}x$ for some $x \in R_A$ and so $|\overline{v^{-1}uv}| = |x| \leq |v|$, contradicting (10). Thus $v \notin \overline{uT_0(v)}$. A similar argument shows that $v \notin \overline{uvT_0(e)}$. Finally, $\overline{v^{-1}u^{-1}v}$ and v^{-1} start by the same first letter in view of (10) and $|\overline{u}| < |v|$. Since $vw \in R_A$, then also $\overline{v^{-1}uvw} \in R_A$ and so $v \notin \overline{uveT_0(w)}$. Therefore $v \notin T_0(uvew)$.

It remains to be proved that $w \notin T(e)$. Indeed, suppose that $w \in T(e)$. Then $e\tau = (eww^{-1})\tau$. Since $u \in J$ and $||ve|| \leq ||u||_1 + 2||R||$, we have $(uvev^{-1}u^{-1})\tau = (uu^{-1}vev^{-1})\tau$. Hence $T(uve) = T(uu^{-1}ve)$ and so

$$T(uvew) = T(uveww^{-1}) = T(uve) = T(uu^{-1}ve) = T(uu^{-1}veww^{-1}) = T(uu^{-1}vew).$$

By definition of e, we have $(uvew)\tau = (uvw)\tau = (uz)\tau$ and since $u \in J$ implies $(uu^{-1})\tau = (u^{-1}u)\tau$, also

$$(uu^{-1}vew)\tau = (u^{-1}uvew)\tau = (u^{-1}uvw)\tau = (uu^{-1}z)\tau.$$

Hence $T(uz) = T(uvew) = T(uu^{-1}vew) = T(uu^{-1}z)$, a contradiction.

Thus $w \notin T(e)$ and so $w \in K(u, e, v)$. Therefore condition (ii) fails as required. □

Theorem 8.3 *M/μ has decidable word problem.*

Proof. As a first step, we reduce the word problem of M/μ to deciding whether or not $(u\tau)\,\mu\,((uu^{-1})\tau)$ for $u \in \widehat{C_A}$. Indeed, as with any inverse monoid congruence, deciding $x\mu y$ van be reduced to deciding $z\mu(zz^{-1})$ and $e\mu f$ for $e, f \in E(M)$: this follows from the equivalence

$$x = y \qquad \Leftrightarrow \qquad (xx^{-1} = yy^{-1} \quad \wedge \quad x^{-1}x = y^{-1}y \quad \wedge \quad xy^{-1} = xy^{-1}yx^{-1}).$$

Since μ is idempotent-separating, deciding $e\mu f$ follows from the word problem, and so we only need to decide $(u\tau)\,\mu\,((uu^{-1})\tau)$. If $\overline{u} = vcv^{-1}$ with $c \in C_A$, it is easy to check that $(u\tau)\mu$ is idempotent if and only if $((v^{-1}uv)\tau)\mu$ is idempotent, hence we are reduced to the case $(u\tau)\,\mu\,((uu^{-1})\tau)$ for $u \in \widehat{C_A}$, which we discuss now, in the light of Lemma 8.2.

Let $\mathcal{A} = (Q, q_0, Q, E)$ denote the minimum automaton of $T(e)$ (note that all states must be terminal since $T(e)$ is prefix-closed). Clearly, $u \in J$ is decidable, so we may assume that $u \in J$. Moreover, we only need to consider finitely many $(e, v) \in J_u$, hence if we can bound the length of a possible element of some $K(u, e, v)$, we are done: indeed, all the conditions in (9) are decidable for a given $w \in R_A \setminus T(e)$. Therefore it suffices to prove that, if $(u\tau, (uu^{-1})\tau) \notin \mu$, then

$$K(u, e, v) \text{ contains a word of length } \leq |Q| + 2||R|| + 2 \text{ for some } (e, v) \in J_u. \qquad (11)$$

Assume that $(u\tau, (uu^{-1})\tau) \notin \mu$. As in the proof of the converse implication of Lemma 8.2, we have $T(uz) \neq T(uu^{-1}z)$ for some $z \in R_A$. Let such a z have minimum length. Since $u \in J$, we have $|z| > ||u||_1 + 2||R||$. Writing $z = vz'$ with $|v| = ||u||_1 + ||R||$, it follows from the proof that $(e,v) \in J_u$ and $z' \in K(u,e,v)$ for $e = \delta_{uvz'}(\overline{uv})$. Hence $K(u,e,v) \neq \emptyset$. Let $w \in K(u,e,v)$ have minimum length. We only need to prove that

$$|w| \leq |Q| + 2||R|| + 2. \tag{12}$$

Suppose that $|w| > |Q| + 2||R|| + 2$. Write $w = w_1 a$ with $a \in \widetilde{A}$. We claim that $w_1 \in T(e)$. Indeed, $vw_1 \in R_A$, $v \notin MT(uvew_1)$ and $\mathrm{nbh}(\overline{uv}, T(uvew_1), ||R||) = T_0(e)$ are easily verified, and so $w_1 \in T(e)$ must hold by minimality of $|w|$. Write $w_1 = bw_2w_3w_4$ with $b \in \widetilde{A}$, $|w_2| = ||R||$ and $|w_4| = ||R|| + 1$. Then we have a path

$$q_0 \xrightarrow{bw_2} q_2 \xrightarrow{w_3} q_3 \xrightarrow{w_4} q_4$$

in $\mathcal{A}$. Let $q_2 \xrightarrow{w_3'} q_3$ be a path of minimal length and set $w' = bw_2w_3'w_4a$. We claim that $w' \in K(u,e,v)$. Since $|w'| \leq 1 + ||R|| + |Q| - 1 + ||R|| + 1 + 1 = |Q| + 2||R|| + 2 < |w|$, we reach the desired contradiction.

Since $vw = vbw_2w_3w_4a \in R_A$ and $bw_2w_3'w_4 \in L(\mathcal{A}) = T(e) \subseteq R_A$, we have also $vw' = vbw_2w_3'w_4a \in R_A$. On the other hand, since $\mathcal{A}$ is deterministic, $w' = bw_2w_3'w_4a \in T(e)$ would imply $w = bw_2w_3w_4a \in T(e)$, a contradiction, thus $w' \notin T(e)$.

On the other hand, $v \notin T_0(uvew)$ yields $v \notin T_0(uve)$. By (10), and since $|\overline{u}| < |v|$, $\overline{v^{-1}u^{-1}v}$ and v^{-1} start by the same first letter. Since $vw' \in R_A$, it follows that $v \notin \overline{uveT_0(w')}$ and so $v \notin T_0(uvew')$.

It remains to prove the inclusion

$$\mathrm{nbh}(\overline{uv}, T(uvew'), ||R||) \subseteq T_0(e), \tag{13}$$

the opposite inclusion holding trivially.

Let $C = \mathrm{cone}(\overline{uv}bw_2w_3, T(uvew))$. It suffices to show that

$$T(v^{-1}u^{-1}uvew') \subseteq T(v^{-1}u^{-1}uve) \cup bw_2w_3'C. \tag{14}$$

Indeed, since $|w_2| = ||R||$ and $T(v^{-1}u^{-1}uve) \cup \{bw_2\} \subseteq T(v^{-1}u^{-1}uvew)$, it follows that $\mathrm{nbh}(\overline{uv}, T(uvew'), ||R||) \subseteq \mathrm{nbh}(\overline{uv}, T(uvew), ||R||) \subseteq T_0(e)$.

Let $w_1' = bw_2w_3'w_4$ so that $w' = w_1'a$. Since $w_1 \in T(e)$, we have $w_1' \in T(e)$ as well since both label paths $q_0 \longrightarrow q_4$ in $\mathcal{A}$. Hence $T(uvew_1') = T(uve) \subseteq T(uvew)$ and the crucial question is understanding the effects of adjoining the edge $w_1' \xrightarrow{a} w'$ to $\mathcal{T}(v^{-1}u^{-1}uvew_1')$ and performing the due expansions. We claim that the expansion process never takes us outside the cone $bw_2w_3'C$. This follows from the following claim: no expansion can create an edge *lying closer* to the vertex 1. And since the added edge $w_1' \xrightarrow{a} w'$ lies at depth $||R|| + 1$ in the cone C, the only way of expanding outside the cone would be to produce new edges closer to the cone root.

We show next that

$$\mathrm{cone}(bw_2w_3', T(v^{-1}u^{-1}uve)) = \mathrm{cone}(bw_2w_3', T(e)). \tag{15}$$

Indeed, let $p \in \mathrm{cone}(bw_2w_3', T(v^{-1}u^{-1}uve))$. Then we consider $\overline{uv} \cdot b \cdot w_2w_3'p \in T(uve)$. Since $|v| > ||u||_1$ and $||e|| \leq ||R||$, then $\overline{uv}h \in T_0(uve)$ implies $h \in T_0(e)$ and therefore $|h| \leq ||R||$. It follows from Lemma 7.1 that $bw_2w_3'p \in T(\delta_{uve}(\overline{uv}))$. Since $e = \delta_{uvw}(\overline{uv})$, it follows easily that $\delta_{uve}(\overline{uv}) = e$ and so $bw_2w_3'p \in T(e)$. Thus $p \in \mathrm{cone}(bw_2w_3', T(e))$ and so (15) holds.

Thus, before adding the extra edge, we have

$$\begin{aligned}\mathrm{cone}(bw_2w_3', T(v^{-1}u^{-1}uvew_1')) &= \mathrm{cone}(bw_2w_3', T(v^{-1}u^{-1}uve)) \\ &= \mathrm{cone}(bw_2w_3', T(e)) = L(Q, q_3, Q, E) = \mathrm{cone}(bw_2w_3, T(e)) \\ &= \mathrm{cone}(bw_2w_3, T(v^{-1}u^{-1}uve)) = \mathrm{cone}(bw_2w_3, T(v^{-1}u^{-1}uvew_1)).\end{aligned}$$

Indeed, the first equality follows from $w_1' \in T(e)$, the second from (15), the third is obvious and the remaining similar.

Thus our claim is equivalent to prove that, after adjoining the edge $w_1 \xrightarrow{a} w$ to the tree $\mathcal{T}(v^{-1}u^{-1}uvew_1)$ and performing the due expansions, the expansion proccess never takes us outside the cone bw_2w_3C.

So suppose that the expansion process produces a new edge $r \xrightarrow{c} rc$ in the Stephen's sequence of $v^{-1}u^{-1}uvew$ with $|r| < |w_1|$. Assume that $r \xrightarrow{c} rc$ is the first such edge to appear. We claim that rc would be a shorter alternative to w as an element of $K(u, e, v)$.

Clearly, $rc \in R_A \setminus T(e)$ since it was not in $\mathcal{T}(v^{-1}u^{-1}uvew_1)$. Moreover, since we are taking the older new edge, we have $r \in T(e)$ and $vrc \in R_A$ since bw_2w_3' is a prefix of rc due to $|w_4| = ||R|| + 1$. On the other hand, $v \notin T_0(uverc) = T_0(uve) \cup \overline{uvT_0(rc)}$ follows from $v \notin T_0(uvew)$ and $\overline{v^{-1}u^{-1}v}$ starting with a different letter from $rc \in bR_A$ as observed before.

Finally, $\mathrm{nbh}(\overline{uv}, T(uverc), ||R||) \subseteq \mathrm{nbh}(\overline{uv}, T(uvew), ||R||) = T_0(e)$ since rc can be produced through expansions of $\mathrm{MT}(uvew)$. The opposite inclusion holds trivially, hence $\mathrm{nbh}(\overline{uv}, T(uverc), ||R||) = T_0(e)$ and so $rc \in K(u, e, v)$. This contradicts the minimality of $|w|$, hence the expansion proccess never takes us outside the cone bw_2w_3C and so (14) holds. Thus (13) holds and so $w' \in K(u, e, v)$, contradicting the minimality of $|w|$. Therefore (12) holds and the proof is complete. □

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