A convolution related to the inverse Kontorovich-Lebedev transform

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Abstract

We establish the mapping properties in the space $L_2(\mathbb{R}_+; \frac{dt}{t\sinh\nu t})$, $0 < \nu \leq \pi$ for a convolution related to the transformation

$$F(x) = \int_0^\infty f(t) K_{it}(x) dt, \quad x \in \mathbb{R}_+$$

involving the modified Bessel function $K_{it}(x)$ as a kernel. As a consequence, we get the multiplication theorem for two modified Bessel functions of different subscripts. Further applications to the corresponding class of convolution integral equations are obtained.

Keywords: *Kontorovich-Lebedev transform, Modified Bessel function, Convolution integral equations, Plancherel theorem, Multiplication theorem*

AMS subject classification: 44A15, 44A05, 44A35, 33C10, 45A05

1 Introduction

In this paper we consider the following Kontorovich-Lebedev transformation [1, 2]

$$F(x) = \underset{N\to\infty}{\text{l.i.m.}} \int_0^N f(t) K_{it}(x) dt, \quad x \in \mathbb{R}_+, \tag{1}$$

where the integration is realized in the mean convergence with respect to an index of the modified Bessel function of the second kind $K_{it}(x)$ [3], which is real-valued and even with respect to t. A function $f(t)$ is supposed to be from the space $L_2^\nu \equiv L_2(\mathbb{R}_+; \frac{dt}{t\sinh\nu t})$, $0 < \nu \leq \pi$, i.e.

$$L_2^\nu := \left\{ f : \int_0^\infty |f(t)|^2 \frac{dt}{t\sinh\nu t} = ||f||^2_{L_2^\nu} < \infty \right\}.$$

For $0 < \nu_1 \leq \nu_2 \leq \pi$ it is easily seen the embeddings

$$L_2^{\nu_1} \subseteq L_2^{\nu_2} \subseteq L_2^\pi = L_2\left(\mathbb{R}_+; \frac{dt}{t\sinh\pi t}\right).$$

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As it is known [1], the operator (1) maps the space L_2^π onto the space $L_2\left(\mathbb{R}_+;\frac{dx}{x}\right)$ and its inverse is described by

$$f(t)=\frac{2}{\pi^2}\underset{N\to\infty}{\text{l.i.m.}}\, t\sinh\pi t\int\limits_{1/N}^{N}F(x)K_{it}(x)\frac{dx}{x} \tag{2}$$

in the mean convergence sense with respect to the norm in L_2^π. Moreover, the Parseval equality

$$\int\limits_0^\infty |f(t)|^2\frac{dt}{t\sinh\pi t}=\frac{2}{\pi^2}\int\limits_0^\infty |F(x)|^2\frac{dx}{x} \tag{3}$$

is valid.

Using the Macdonald formula [1, 4] for the product of the modified Bessel functions of different arguments

$$K_{it}(x)K_{it}(y)=\frac{1}{2}\int\limits_0^\infty e^{-\frac{1}{2}\left(u\frac{x^2+y^2}{xy}+\frac{xy}{u}\right)}K_{it}(u)\frac{du}{u}, \tag{4}$$

one can introduce a convolution for the transform (2) defined by the following double integral

$$(F\overset{\gamma}{*}G)(x)=\frac{1}{2}\int\limits_0^\infty\int\limits_0^\infty F(y)G(u)e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy}+\frac{uy}{x}\right)}\frac{dydu}{yu}, \tag{5}$$

where $F(x)$ and $G(x)$ are two functions from a suitable functional space, $\gamma=\frac{\pi^2}{2t\sinh t}$ is the weight function. This operator was first introduced formally in Kakichev [5] as an example of integral nonstandard convolution. Later this operator was considered in detail in [1, 2, 6, 7, 8, 9].

The main goal of this paper is to establish mapping properties of the following convolution

$$\left(f\overset{\alpha}{*}g\right)(t)=\frac{2}{\pi^2}t\sinh\pi t\int\limits_0^\infty\int\limits_0^\infty f(\tau)g(\theta)\Omega_\alpha(t,\tau,\theta)d\tau d\theta, \tag{6}$$

where

$$\Omega_\alpha(t,\tau,\theta)=\int\limits_0^\infty x^{\alpha-1}K_{i\tau}(x)K_{i\theta}(x)K_{it}(x)dx,\quad \alpha,\ t,\ \tau,\ \theta\in\mathbb{R}_+. \tag{7}$$

Concerning (6), it was first introduced in Yakubovich [1], p. 142 and announced later in [10]. The kernel (7) can be calculated employing relation 2.16.46.6 from [4] and we have

$$\begin{aligned}\Omega_\alpha(t,\tau,\theta) &= 2^{\alpha-3}\,\mathrm{Re}\Big[\Gamma(it)\Gamma(\alpha-it)\\ &\times\ B\left(\frac{\alpha}{2}+i\frac{\tau+\theta-t}{2},\frac{\alpha}{2}-i\frac{\tau+\theta+t}{2}\right)B\left(\frac{\alpha}{2}-i\frac{\tau-\theta+t}{2},\frac{\alpha}{2}+i\frac{\tau-\theta-t}{2}\right)\\ &\times\ {}_4F_3\left(\frac{\alpha}{2}+i\frac{\tau+\theta-t}{2},\frac{\alpha}{2}-i\frac{\tau+\theta+t}{2},\frac{\alpha}{2}-i\frac{\tau-\theta+t}{2},\frac{\alpha}{2}+i\frac{\tau-\theta-t}{2};\right.\\ &\left.1-it,\frac{\alpha-it}{2},\frac{1+\alpha-it}{2};\frac{1}{4}\right)\Big].\end{aligned} \tag{8}$$

It contains the generalized hypergeometric function ${}_4F_3(a_1,a_2,a_3,a_4;b_1,b_2,b_3;z)$ at the point $z=1/4$ and as usual $\Gamma(z)$, $B(x,y)$ stand for Euler's gamma- and beta- functions [3].

In the sequel we will give some applications of the convolution (6). In particular, we prove an analog of the multiplication theorem for two modified Bessel functions with different subscripts. Finally, we will exhibit certain convolution integral equations related to (6).

2 Mapping properties of the convolution $\left(f \overset{\alpha}{*} g\right)(t)$

Let us prove an auxiliary lemma which gives a boundedness and the norm estimation for the Kontorovich-Lebedev transform (1) as an operator from $L_2^\nu,\ 0<\nu<\pi$ into the space $C(\mathbb{R}_+)$ of bounded continuous functions on $\mathbb{R}_+$.

Lemma 1 *Let $f(t)\in L_2^\nu$ with $0<\nu<\pi$. Then transformation (1) exists as a Lebesque integral for all $x>0$ and represents a continuous function on $\mathbb{R}_+$. Moreover, it is a bounded operator $F: L_2^\nu \to C(\mathbb{R}_+)$, namely*

$$\sup_{x>0}|F(x)| \le C||f||_{L_2^\nu}, \tag{9}$$

with

$$C=\frac{\sqrt{\pi}}{2\cos^{1/2}\left(\frac{\nu}{2}\right)}.$$

Proof. In fact, appealing to Schwarz's inequality we obtain

$$\begin{aligned}|F(x)| &\le \int_0^\infty |f(t)||K_{it}(x)|dt \le \left(\int_0^\infty |f(t)|^2\frac{dt}{t\sinh\nu t}\right)^{1/2}\left(\int_0^\infty t\sinh\nu t K_{it}^2(x)dt\right)^{1/2}\\ &= ||f||_{L_2^\nu}\left(\int_0^\infty t\sinh\nu t K_{it}^2(x)dt\right)^{1/2}.\end{aligned} \tag{10}$$

The latter integral in (10) is calculated in terms of the modified Bessel function K_1 (see [4], relation 2.16.52.8), namely

$$\int_0^\infty t\sinh(\nu t)K_{it}^2(x)dt=\frac{\pi}{2}x\sin\frac{\nu}{2}K_1\left(2x\cos\frac{\nu}{2}\right),\quad 0<\nu<\pi. \tag{11}$$

Moreover, since $xK_1(x)\le 1$, when $x>0$, it yields (9)

$$\begin{aligned}\sup_{x>0}|F(x)| &\le \sup_{x>0}\int_0^\infty |f(t)||K_{it}(x)|dt\\ &\le \sup_{x>0}\left(\frac{\pi}{2}x\sin\frac{\nu}{2}K_1\left(2x\cos\frac{\nu}{2}\right)\right)^{1/2}||f||_{L_2^\nu}\le\frac{\sqrt{\pi}}{2\cos^{1/2}\left(\frac{\nu}{2}\right)}||f||_{L_2^\nu}<\infty.\end{aligned}$$

In order to prove that $F(x)$ is continuous on $\mathbb{R}_+$ we will establish the uniform convergence of the integral (1) with respect to $x\ge x_0>0$. To do this we use the following inequality for the modified Bessel function (cf. [1], formula (1.100))

$$|K_{it}(x)|dt\le e^{-\delta t}K_0(x\cos\delta),$$

where we choose $\frac{\nu}{2}<\delta<\frac{\pi}{2}$. Then for sufficiently big $A>0$ we have

$$\begin{aligned}\int_A^\infty |f(t)||K_{it}(x)|dt \le ||f||_{L_2^\nu}\left(\int_A^\infty t\sinh\nu t K_{it}^2(x)dt\right)^{1/2} &\le K_0^2(x_0\cos\delta)\\ \times\left(\int_A^\infty t\sinh\nu t e^{-2\delta t}dt\right)^{1/2} &\to 0,\ A\to\infty.\end{aligned}$$

Lemma 1 is proved.

Applying the Parseval equality (3) we get

Lemma 2 *The equality*

$$\frac{2}{\pi^2}\int_0^\infty t\sinh \pi t \left|\Omega_\alpha(t,\tau,\theta)\right|^2 dt = \int_0^\infty x^{2\alpha-1}K^2_{i\tau}(x)K^2_{i\theta}(x)dx. \tag{12}$$

holds for any $\alpha,\ \tau,\ \theta \in \mathbb{R}_+$.

Proof. Indeed by asymptotic properties of the modified Bessel functions [3] we observe that the right-hand side of (12) is finite. Therefore $x^\alpha K_{i\tau}(x)K_{i\theta}(x) \in L_2\left(\mathbb{R}_+, \frac{dx}{x}\right)$. Moreover, transformation (2) of this function exists as a Lebesgue integral and is equal to $\frac{2}{\pi^2}t\sinh \pi t \Omega_\alpha(t,\tau,\theta)$ (see (7)). Thus via Parseval equality (3) we arrive at (12).

Lemma 2 is proved.

As a corollary we immediately obtain an analogue of the multiplication theorem for the modified Bessel functions (cf. (4)) of different subscripts.

Corollary 1 *Let* $0<\alpha<2$ *and* $\tau,\ \theta \in \mathbb{R}_+$ *be fixed. Then for any* $x>0$ *the following equality is true*

$$\begin{aligned} x^\alpha K_{i\tau}(x)K_{i\theta}(x) &= \frac{2^{\alpha-3}}{\pi}\int_{-\infty}^{\infty}\frac{\Gamma(\alpha-it)}{\Gamma(-it)}B\left(\frac{\alpha}{2}+i\frac{\tau+\theta-t}{2}, \frac{\alpha}{2}-i\frac{\tau+\theta+t}{2}\right) \\ &\times\ B\left(\frac{\alpha}{2}-i\frac{\tau-\theta+t}{2}, \frac{\alpha}{2}+i\frac{\tau-\theta-t}{2}\right) \\ &\times\ {}_4F_3\left(\frac{\alpha}{2}+i\frac{\tau+\theta-t}{2}, \frac{\alpha}{2}-i\frac{\tau+\theta+t}{2}, \frac{\alpha}{2}-i\frac{\tau-\theta+t}{2}, \frac{\alpha}{2}+i\frac{\tau-\theta-t}{2};\right. \\ &\left. 1-it, \frac{\alpha-it}{2}, \frac{1+\alpha-it}{2}; \frac{1}{4}\right) K_{it}(x)\, dt. \end{aligned} \tag{13}$$

Proof. Owing to the definition of the generalized hypergeometric function [3] we write the right-hand side of (13) as

$$I_\alpha(x) = \int_{-\infty}^{\infty}\omega_\alpha(t,\tau,\theta)K_{it}(x)\, dt, \tag{14}$$

where

$$\begin{aligned} \omega_\alpha(t,\tau,\theta) &= \frac{2^{\alpha-3}t}{\pi i}\sum_{n=0}^{\infty}B\left(\frac{\alpha}{2}+n+i\frac{\tau+\theta-t}{2}, \frac{\alpha}{2}+n-i\frac{\tau+\theta+t}{2}\right) \\ &\times\ B\left(\frac{\alpha}{2}+n-i\frac{\tau-\theta+t}{2}, \frac{\alpha}{2}+n+i\frac{\tau-\theta-t}{2}\right)\frac{\Gamma(\alpha-it+2n)}{n!\Gamma(1-it+n)}. \end{aligned} \tag{15}$$

Then by an elementary inequality for the beta -function

$$|B(s,t)| \le B(\mathrm{Re}s, \mathrm{Re}t), \quad \mathrm{Re}s>0,\ \mathrm{Re}t>0, \tag{16}$$

and the duplication formula for the gamma-function [3] we derive the following estimations

$$\begin{aligned}
\frac{\pi}{2^{\alpha-3}}\ |\omega_\alpha(t,\tau,\theta)| \ &\le \ \left|B\left(\frac{\alpha}{2}+i\frac{\tau+\theta-t}{2}, \frac{\alpha}{2}-i\frac{\tau+\theta+t}{2}\right)\right| \\
&\times \ \left|B\left(\frac{\alpha}{2}-i\frac{\tau-\theta+t}{2}, \frac{\alpha}{2}+i\frac{\tau-\theta-t}{2}\right)\right| \frac{|\Gamma(\alpha-it)|}{|\Gamma(-it)|} \\
&+ \ |t|\sum_{n=1}^{\infty} B^2\left(\frac{\alpha}{2}+n, \frac{\alpha}{2}+n\right)\frac{|\Gamma(\alpha-it+2n)|}{n!|\Gamma(1-it+n)|} \\
&\le \ \left|B\left(\frac{\alpha}{2}+i\frac{\tau+\theta-t}{2}, \frac{\alpha}{2}-i\frac{\tau+\theta+t}{2}\right)\right| \\
&\times \ \left|B\left(\frac{\alpha}{2}-i\frac{\tau-\theta+t}{2}, \frac{\alpha}{2}+i\frac{\tau-\theta-t}{2}\right)\right| \frac{|\Gamma(\alpha-it)|}{|\Gamma(-it)|} + \frac{2^{\alpha-1}|t|}{\sqrt{\pi}\left|\Gamma(1-\frac{\alpha+it}{2})\right|} \\
&\times \ \sum_{n=1}^{\infty}\frac{4^n}{n!}B^2\left(\frac{\alpha}{2}+n, \frac{\alpha}{2}+n\right)B\left(\frac{\alpha}{2}+n, 1-\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha+1}{2}+n\right) \\
&= \ \left|\Gamma\left(\frac{\alpha}{2}+i\frac{\tau+\theta-t}{2}\right)\Gamma\left(\frac{\alpha}{2}-i\frac{\tau+\theta+t}{2}\right)\right| \\
&\times \ \left|\Gamma\left(\frac{\alpha}{2}-i\frac{\tau-\theta+t}{2}\right)\Gamma\left(\frac{\alpha}{2}+i\frac{\tau-\theta-t}{2}\right)\right| \frac{1}{|\Gamma(\alpha-it)\Gamma(-it)|} \\
&+\frac{\Gamma(1-\frac{\alpha}{2})|t|}{2^{\alpha-1}\left|\Gamma(1-\frac{\alpha+it}{2})\right|}\sum_{n=1}^{\infty}\frac{\Gamma^3\left(\frac{\alpha}{2}+n\right)}{4^n(n!)^2\Gamma\left(\frac{\alpha+1}{2}+n\right)}, \ 0<\alpha<2.
\end{aligned}$$

Consequently via Stirling's asymptotic formulas for gamma-functions and factorials [3] the general term of the latter series is $O\left(n^{\alpha-5/2}4^{-n}\right),\ n\to\infty$. Therefore it converges and accordingly

$$|\omega_\alpha(t,\tau,\theta)| = O\left(|t|^{\frac{\alpha-1}{2}}e^{\pi|t|/4}\right), \quad |t|\to\infty. \tag{17}$$

Further appealing to the asymptotic formula for the modified Bessel function with respect to an index (see [1], formula (1.148)) we find

$$|K_{it}(x)| = \sqrt{\frac{2\pi}{|t|}}e^{-\pi|t|/2}\left(1+O\left(1/|t|\right)\right), \ |t|\to\infty. \tag{18}$$

Therefore $I_\alpha(x)$ exists as an absolutely convergent integral for all $x\in\mathbb{R}_+$ and $0<\alpha<2$.

On the other hand, by virtue of Lemma 2 and formulas (2), (7), (8), (14) we derive

$$\begin{aligned}
x^\alpha K_{i\tau}(x)K_{i\theta}(x) \ &= \ \frac{2}{\pi^2}\operatorname*{l.i.m.}_{N\to\infty}\int_0^N t\sinh\pi t\,\Omega_\alpha(t,\tau,\theta)K_{it}(x)dt \\
&= \ 2\operatorname*{l.i.m.}_{N\to\infty}\int_0^N K_{it}(x)\mathrm{Re}\,\omega_\alpha(t,\tau,\theta)dt \\
&= \ \operatorname*{l.i.m.}_{N\to\infty}\int_{-N}^N \omega_\alpha(t,\tau,\theta)K_{it}(x)dt.
\end{aligned} \tag{19}$$

This means that the latter limit in mean coincides with $I_\alpha(x)$. Thus we prove (13) and end the proof of Corollary 1.

The mapping properties for convolution (6) are given by

Theorem 1 *Let $f,\ g \in L_2^\nu$ with $0<\nu<\pi$. Then convolution $\left(f \overset{\alpha}{*} g\right)(t)$ exists as a Lebesque integral and is continuous on $\mathbb{R}_+$. Besides it belongs to L_2^π.*

Proof. Calling Schwarz's inequality for double integrals we deduce

$$\begin{aligned}\left|\left(f \overset{\alpha}{*} g\right)(t)\right| &\le \frac{2}{\pi^2} t\sinh \pi t \int_0^\infty\int_0^\infty |f(\tau)||g(\theta)||\Omega_\alpha(t,\tau,\theta)|d\tau d\theta \\ &\le \frac{2}{\pi^2} t\sinh \pi t \left(\int_0^\infty\int_0^\infty |f(\tau)|^2|g(\theta)|^2 \frac{d\tau d\theta}{\tau\theta\sinh\nu\tau\sinh\nu\theta}\right)^{1/2} J^{1/2}(t) \\ &= \frac{2}{\pi^2}||f||_{L_2^\nu}||g||_{L_2^\nu}\, t\sinh\pi t J^{1/2}(t),\end{aligned}$$

where we denoted by

$$J(t) = \int_0^\infty\int_0^\infty \tau\theta\sinh\nu\tau\sinh\nu\theta|\Omega_\alpha(t,\tau,\theta)|^2 d\tau d\theta. \tag{20}$$

Meanwhile (see (7))

$$\begin{aligned}|\Omega_\alpha(t,\tau,\theta)|^2 &\le \left[\int_0^\infty x^{\alpha-1}|K_{i\tau}(x)K_{i\theta}(x)K_{it}(x)|dx\right]^2 \\ &\le \int_0^\infty x^{\alpha-1}K_{i\tau}^2(x)K_{i\theta}^2(x)dx \int_0^\infty y^{\alpha-1}K_{it}^2(y)dy.\end{aligned}$$

Hence plainly

$$\begin{aligned}J(t) &\le \int_0^\infty y^{\alpha-1}K_{it}^2(y)dy\int_0^\infty\int_0^\infty \tau\theta\sinh\nu\tau\sinh\nu\theta\int_0^\infty x^{\alpha-1}K_{i\tau}^2(x)K_{i\theta}^2(x)dxd\tau d\theta dt \\ &= \int_0^\infty y^{\alpha-1}K_{it}^2(y)dy\int_0^\infty x^{\alpha-1}\left[\int_0^\infty \tau\sinh\nu\tau K_{i\tau}^2(x)d\tau\right]^2 dx.\end{aligned} \tag{21}$$

Calculating the inner integral with respect to τ by (11) and integrals with respect to y and x via [4], relation 2.16.33.2

$$\int_0^\infty y^{\alpha-1}K_\mu^2(y)dy = 2^{\alpha-3}B\left(\frac{\alpha}{2}+\mu, \frac{\alpha}{2}-\mu\right)\Gamma^2\left(\frac{\alpha}{2}\right), \quad \alpha > 2|\mathrm{Re}\mu| \tag{22}$$

we arrive at the estimate

$$J^{1/2}(t) \le \frac{C_\alpha}{\cos^{\alpha/2+1}\frac{\nu}{2}}\left|\Gamma(\frac{\alpha}{2}+it)\right|,\ 0<\nu<\pi, \tag{23}$$

where $C_\alpha > 0$ is a constant depending only on α. Therefore,

$$\left|\left(f \overset{\alpha}{*} g\right)(t)\right| \le \text{const.}||f||_{L_2^\nu}||g||_{L_2^\nu}\frac{|\Gamma(\frac{\alpha}{2}+it)|}{|\Gamma(it)|^2} = O\left(t^{\frac{\alpha+1}{2}}e^{\pi t/2}\right), t\to+\infty. \tag{24}$$

Thus we obtain that under conditions of the theorem convolution $\left(f \overset{\alpha}{*} g\right)(t)$ exists as a Lebesgue integral for all $t > 0$. To establish its continuity on $\mathbb{R}_+$ we will do it in a similar way as in the proof of Lemma 1. Indeed, choosing a big $A > 0, t \in [0,T]$ and $\delta \in (\nu/2, \pi/2)$ we invoke the inequality [1] $|K_{it}(x)| \le K_0(x)$ and we derive (see (21))

$$\frac{2}{\pi^2} t \sinh \pi t \int_A^\infty \int_A^\infty |f(\tau)||g(\theta)||\Omega_\alpha(t,\tau,\theta)| d\tau d\theta \le \frac{2}{\pi^2} t \sinh \pi t ||f||_{L_2^\nu} ||g||_{L_2^\nu}$$

$$\times \left(\int_A^\infty \int_A^\infty \tau\theta \sinh \nu\tau \sinh \nu\theta |\Omega_\alpha(t,\tau,\theta)|^2 d\tau d\theta \right)^{1/2} \le \frac{2}{\pi^2} T \sinh \pi T ||f||_{L_2^\nu} ||g||_{L_2^\nu}$$

$$\times \left(\int_0^\infty y^{\alpha-1} K_0^2(y) dy \int_0^\infty x^{\alpha-1} K_0^2(x \cos\delta) dx \right)^{1/2} \int_A^\infty \tau \sinh \nu\tau e^{-2\delta\tau} d\tau$$

$$= \text{const.} \int_A^\infty \tau \sinh \nu\tau e^{-2\delta\tau} d\tau \to 0, \ A \to \infty.$$

Therefore integral (6) converges uniformly by $t \in [0,T]$ for any $T > 0$.

Now we are ready to prove that $\left(f \overset{\alpha}{*} g\right)(t) \in L_2^\pi$. In fact

$$\begin{aligned} \left| \left(f \overset{\alpha}{*} g\right)(t) \right|^2 &\le \frac{4}{\pi^4} t^2 J(t) \sinh^2 \pi t \int_0^\infty \int_0^\infty |f(\tau)|^2 |g(\theta)|^2 \frac{d\tau d\theta}{\tau\theta \sinh \nu\tau \sinh \nu\theta} \\ &= \frac{4}{\pi^4} t^2 \sinh^2 \pi t ||f||^2_{L_2^\nu} ||g||^2_{L_2^\nu} J(t), \end{aligned}$$

where $J(t)$ is defined by (20). Then

$$\begin{aligned} ||f \overset{\alpha}{*} g||_{L_2^\pi} &\le \frac{2}{\pi^2} ||f||_{L_2^\nu} ||g||_{L_2^\nu} \left(\int_0^\infty t \sinh \pi t J(t) dt \right)^{1/2} \\ &= \frac{2}{\pi^2} ||f||_{L_2^\nu} ||g||_{L_2^\nu} \\ &\times \left(\int_0^\infty \int_0^\infty \tau\theta \sinh \nu\tau \sinh \nu\theta \int_0^\infty t \sinh \pi t |\Omega_\alpha(t,\tau,\theta)|^2 dt d\tau d\theta \right)^{1/2}. \end{aligned}$$

Using (12) we have

$$\begin{aligned} ||f \overset{\alpha}{*} g||_{L_2^\pi} &\le \frac{\sqrt{2}}{\pi}||f||_{L_2^\nu}||g||_{L_2^\nu} \\ &\times \left(\int_0^\infty\int_0^\infty \tau\theta \sinh\nu\tau \sinh\nu\theta \int_0^\infty x^{2\alpha-1} K^2_{i\tau}(x)K^2_{i\theta}(x)dx d\tau d\theta\right)^{1/2} \\ &= \frac{\sqrt{2}}{\pi}||f||_{L_2^\nu}||g||_{L_2^\nu} \\ &\times \left(\int_0^\infty x^{2\alpha-1}\int_0^\infty \tau\sinh\nu\tau K^2_{i\tau}(x)d\tau \int_0^\infty \theta\sinh\nu\theta K^2_{i\theta}(x)d\theta dx\right)^{1/2} \\ &= \frac{\sqrt{2}}{\pi}||f||_{L_2^\nu}||g||_{L_2^\nu}\left(\int_0^\infty x^{2\alpha-1}\left[\int_0^\infty \tau\sinh\nu\tau K^2_{i\tau}(x)d\tau\right]^2 dx\right)^{1/2}. \end{aligned}$$

The latter iterated integral can be easily calculated explicitly invoking again (11) and (22). So the final inequality for the L_2^π-norm of the convolution (6) can be written as

$$||f \overset{\alpha}{*} g||_{L_2^\pi} \le C_{\alpha,\nu}||f||_{L_2^\nu}||g||_{L_2^\nu}, \tag{25}$$

where $C_{\alpha,\nu} > 0$ is a constant.

Theorem 1 is proved.

Remark 1. It is clear via Fubini's theorem that convolution (6) is a commutative operation, i.e. $f \overset{\alpha}{*} g = g \overset{\alpha}{*} f$.

Theorem 2 *Under conditions of Theorem 1 the following factorization equality for convolution (6) is valid*

$$x^\alpha F(x)G(x) = \underset{N\to\infty}{\text{l.i.m.}}\int_0^N \left(f \overset{\alpha}{*} g\right)(t)K_{it}(x)dt, \tag{26}$$

where F, G are transforms (1) of f and g, respectively. Besides, the Parseval equality of type

$$\int_0^\infty \left|\left(f \overset{\alpha}{*} g\right)(t)\right|^2 \frac{dt}{t\sinh\pi t} = \frac{2}{\pi^2}\int_0^\infty x^{2\alpha-1}|F(x)G(x)|^2 dx \tag{27}$$

holds true.

Proof. Indeed, appealing to Lemma 1 and Fubini's theorem we derive the chain of equalities

$$\begin{aligned} \left(f \overset{\alpha}{*} g\right)(t) &= \frac{2}{\pi^2}t\sinh\pi t\int_0^\infty\int_0^\infty f(\tau)g(\theta)\Omega_\alpha(t,\tau,\theta)d\tau d\theta \\ &= \frac{2}{\pi^2}t\sinh\pi t\int_0^\infty\int_0^\infty f(\tau)g(\theta)\int_0^\infty x^{\alpha-1}K_{i\tau}(x)K_{i\theta}(x)K_{it}(x)dx d\tau d\theta \\ &= \frac{2}{\pi^2}t\sinh\pi t\int_0^\infty x^{\alpha-1}\left(\int_0^\infty f(\tau)K_{i\tau}(x)d\tau\right)\left(\int_0^\infty g(\theta)K_{i\theta}(x)d\theta\right)K_{it}(x)dx \\ &= \frac{2}{\pi^2}t\sinh\pi t\int_0^\infty x^{\alpha-1}F(x)G(x)K_{it}(x)dx \end{aligned}$$

and all inner integrals are absolutely convergent. But at the same time since $\left(f \overset{\alpha}{*} g\right)(t) \in L_2^{\pi}$ we have $x^{\alpha}F(x)G(x) \in L_2\left(\mathbf{R}_+, \frac{dx}{x}\right)$. Therefore, equalities (26), (27) are direct consequences of the reciprocities (1), (2) and the Parseval identity (3) for the Kontorovich-Lebedev transformation.

Theorem 2 is proved.

This result can be extended if one of the functions under convolution (6) belongs to L_2^{π}. Precisely, we prove

Theorem 3 *Let $f \in L_2^{\pi}$ and $g \in L_2^{\nu}$, $0 < \nu < \pi$. Then convolution $\left(f \overset{\alpha}{*} g\right)(t)$ exists as a Lebesque integral and still is continuous on $\mathbb{R}_+$. Moreover, it belongs to L_2^{π}.*

Proof. Similar estimations as in Theorem 1 drive us at the chain of inequalities

$$\begin{aligned} \left|\left(f \overset{\alpha}{*} g\right)(t)\right| &\leq \frac{2}{\pi^2} t \sinh \pi t \int_0^{\infty}\int_0^{\infty} |f(\tau)||g(\theta)||\Omega_{\alpha}(t,\tau,\theta)| d\tau d\theta \\ &\leq \frac{2}{\pi^2} t \sinh \pi t \left(\int_0^{\infty}\int_0^{\infty} |f(\tau)|^2 |g(\theta)|^2 \frac{d\tau d\theta}{\tau\theta \sinh \pi\tau \sinh \nu\theta}\right)^{1/2} \Psi^{1/2}(t) \\ &= \frac{2}{\pi^2} ||f||_{L_2^{\pi}} ||g||_{L_2^{\nu}}\, t \sinh \pi t \Psi^{1/2}(t), \end{aligned} \tag{28}$$

with

$$\Psi(t) = \int_0^{\infty}\int_0^{\infty} \tau\theta \sinh \pi\tau \sinh \nu\theta |\Omega_{\alpha}(t,\tau,\theta)|^2 d\tau d\theta. \tag{29}$$

However, the inner integral with respect to τ in (29) can be expressed by using the Parseval equality (12). Hence employing (11), (22) we find

$$\begin{aligned} \Psi(t) &= \frac{\pi^2}{2} \int_0^{\infty}\int_0^{\infty} \theta \sinh \nu\theta x^{2\alpha-1} K_{i\theta}^2(x) K_{it}^2(x) dx d\tau \\ &= \frac{\pi^3}{4} \sin \frac{\nu}{2} \int_0^{\infty} x^{2\alpha} K_{it}^2(x) K_1\left(2x \cos \frac{\nu}{2}\right) dx \leq \text{const.} \int_0^{\infty} x^{2\alpha-1} K_{it}^2(x) dx \\ &= \text{const.} \left|\Gamma(\alpha + it)\right|^2. \end{aligned}$$

Thus combining with (28) and taking into account the asymptotic behavior of the gamma function at infinity we get that convolution (6) exists as a Lebesgue integral and satisfies the following estimate

$$\left|\left(f \overset{\alpha}{*} g\right)(t)\right| = O(e^{\pi t/2} t^{\alpha+1/2}),\ t \to +\infty.$$

In the same manner as in Theorem 1 we get its continuity. In fact,

$$\frac{2}{\pi^2} t\sinh \pi t \int_A^\infty \int_A^\infty |f(\tau)||g(\theta)||\Omega_\alpha(t,\tau,\theta)|d\tau d\theta \le \frac{2}{\pi^2} T\sinh \pi T ||f||_{L_2^\pi} ||g||_{L_2^\nu}$$

$$\times \left(\int_A^\infty \int_A^\infty \tau\theta \sinh\pi\tau \sinh\nu\theta |\Omega_\alpha(t,\tau,\theta)|^2 d\tau d\theta \right)^{1/2}$$

$$\le \text{const.} \left(\int_0^\infty x^{2\alpha-1} K_0^2(x) K_0^2(x\cos\delta) dx \right)^{1/2} \left(\int_A^\infty \theta \sinh\nu\theta e^{-2\delta\theta} d\theta \right)^{1/2}$$

$$= \text{const.} \int_A^\infty \theta\sinh\nu\theta e^{-2\delta\theta} d\theta \to 0, \ \delta > \frac{\nu}{2}, \ A \to \infty.$$

Therefore under conditions of the theorem integral (6) converges uniformly by $t \in [0, T]$ for any $T > 0$.

Further, let us estimate the norm of the convolution (6) in the space L_2^π and show that it is finite. First we observe that since integral (6) is a Lebesgue one we can write it for all $t > 0$ as a usual limit

$$\left(f \overset{\alpha}{*} g \right)(t) = \frac{2}{\pi^2} t\sinh\pi t \lim_{n\to\infty} \int_0^n \int_0^\infty f(\tau) g(\theta) \Omega_\alpha(t,\tau,\theta) d\theta d\tau.$$

Hence denoting by $f_n(\tau) = f(\tau), \ \tau \in [0, n]$, which vanishes outside of the interval $[0, n]$ we easily find that $f_n \in L_2^\nu$ for all $n \in \mathbb{N}$. Therefore appealing to the Parseval equality (27) via the Fatou lemma we obtain

$$\int_0^\infty \left| \left(f \overset{\alpha}{*} g \right)(t) \right|^2 \frac{dt}{t\sinh\pi t} \le \text{liminf}_{n\to\infty} \int_0^\infty \left| \left(f_n \overset{\alpha}{*} g \right)(t) \right|^2 \frac{dt}{t\sinh\pi t}$$

$$= \frac{2}{\pi^2} \text{liminf}_{n\to\infty} \int_0^\infty x^{2\alpha-1} |F_n(x) G(x)|^2 dx, \tag{30}$$

where $F_n(x)$ is the Kontorovich-Lebedev transformation (1) of the function f_n and the corresponding integral converges absolutely. Moreover, since $g \in L_2^\nu$ then making similar estimates as in the proof of Lemma 1 (see (10)) and invoking equality (11) we derive for all $\alpha > 0$

$$x^\alpha |G(x)| \le \text{const.} \sup_{x>0} \left[x^{\alpha+1/2} K_1^{1/2} \left(2x\cos\frac{\nu}{2} \right) \right] < \text{const.}$$

The latter estimate yields that $x^\alpha G(x)$ is bounded. Consequently, calling again equality (3) we return to (30) and it becomes

$$\int_0^\infty \left| \left(f \overset{\alpha}{*} g \right)(t) \right|^2 \frac{dt}{t\sinh\pi t} \le \text{const. liminf}_{n\to\infty} \int_0^\infty |F_n(x)|^2 \frac{dx}{x}$$

$$= \text{const. liminf}_{n\to\infty} \int_0^\infty |f_n(\tau)|^2 \frac{d\tau}{\tau\sinh\pi\tau} = \text{const.} ||f||^2_{L_2^\pi} < \infty.$$

Theorem 3 is proved.

Corollary 2. *Formulas (26), (27) keep true under conditions of Theorem 3.*
Proof. Similar to the proof of Theorem 2 we derive by straightforward calculations for all $t > 0$

$$\begin{aligned} \left(f \overset{\alpha}{*} g\right)(t) &= \frac{2}{\pi^2} t \sinh \pi t \lim_{n\to\infty} \int_0^n \int_0^\infty f(\tau) g(\theta) \Omega_\alpha(t,\tau,\theta) d\tau d\theta \\ &= \frac{2}{\pi^2} t \sinh \pi t \lim_{n\to\infty} \int_0^n \int_0^\infty f(\tau) g(\theta) \int_0^\infty x^{\alpha-1} K_{i\tau}(x) K_{i\theta}(x) K_{it}(x) dx d\tau d\theta \\ &= \frac{2}{\pi^2} t \sinh \pi t \lim_{n\to\infty} \int_0^\infty x^{\alpha-1} \left(\int_0^n f(\tau) K_{i\tau}(x) d\tau \right) \left(\int_0^\infty g(\theta) K_{i\theta}(x) d\theta \right) K_{it}(x) dx \\ &= \frac{2}{\pi^2} t \sinh \pi t \lim_{n\to\infty} \int_0^\infty x^{\alpha-1} F_n(x) G(x) K_{it}(x) dx \\ &= \frac{2}{\pi^2} t \sinh \pi t \int_0^\infty x^{\alpha-1} F(x) G(x) K_{it}(x) dx. \end{aligned}$$

The latter equality is because for all $t > 0$ we find

$$\int_0^\infty x^{\alpha-1} |F(x) - F_n(x)||G(x) K_{it}(x)| dx \le \left(\int_0^\infty |F(x) - F_n(x)|^2 \frac{dx}{x} \right)^{1/2}$$

$$\times \left(\int_0^\infty x^{2\alpha} |G(x) K_0(x)|^2 dx \right)^{1/2} = \text{const.} \left(\int_0^\infty |F(x) - F_n(x)|^2 \frac{dx}{x} \right)^{1/2} \to 0, \ n \to \infty.$$

Hence the statement of the corollary follows as in Theorem 2. Corollary 2 is proved.

Remark 2. Theorem 3 and Corollary 2 guarantee the associativity of the convolution (6). Namely, it has the property

$$\left(f \overset{\alpha}{*} g\right) \overset{\alpha}{*} h = f \overset{\alpha}{*} \left(g \overset{\alpha}{*} h\right) = g \overset{\alpha}{*} \left(f \overset{\alpha}{*} h\right),$$

for any $f, g, h \in L_2^\nu, \ 0 < \nu < \pi$.

3 Convolution integral equations

As applications we consider in this last section the L_2^π-solvability of the first and second kind integral equations related to convolution (6). Precisely, we exhibit the following integral equations

$$g(x) = (k \overset{\alpha}{*} f)(x), \quad x \in \mathbb{R}_+, \tag{31}$$

$$f(x) = h(x) + \lambda (k \overset{\alpha}{*} f)(x), \quad t \in \mathbb{R}_+, \ \lambda \in \mathbb{C}, \tag{32}$$

where $g, h \in L_2^\pi$, $k \in L_2^\nu, \ 0 < \nu < \pi$ are given functions and $f(x) \in L_2^\pi$ is to be determined. We will prove two theorems, which will guarantee the existence and uniqueness of L_2^π-solutions and give them in the closed form. Certain examples of the function k will be considered. Similar questions for the convolution integral equations related to (5) were investigated in [1], Ch. 4.

Denoting by

$$\mathcal{K}_\alpha(x,y) = \int_0^\infty k(\theta)\Omega_\alpha(x,y,\theta)d\theta \tag{33}$$

integral equation (31) of the first kind can be written in the form

$$g(x) = \frac{2}{\pi^2}\, x\sinh\pi x \int_0^\infty \mathcal{K}_\alpha(x,y)f(y)dy. \tag{34}$$

Theorem 4 *Let $\alpha > 0,\ g \in L_2^\pi$ and $k \in L_2^\nu,\ 0 < \nu < \pi$. Then for the existence of a L_2^π-solution of the equation (34) it is necessary and sufficient that $\frac{G(u)}{u^\alpha \hat{k}(u)} \in L_2\left(\mathbb{R}_+; \frac{du}{u}\right)$, where $G,\ \hat{k}$ are the Kontorovich-Lebedev transformations (1) of the functions g, k, respectively. Moreover the solution is unique and is given by the formula*

$$f(x) = \frac{2}{\pi^2} \operatorname*{l.i.m.}_{N\to\infty} x\sinh\pi x \int_{1/N}^N \frac{G(u)}{\hat{k}(u)} K_{ix}(u)\frac{du}{u^{\alpha+1}}, \tag{35}$$

where the convergence is with respect to the norm in L_2^π.

Proof. *Necessity.* Indeed, if we assume that g, k, f belong to the corresponding L-classes and the equation (34) is satisfied, then via Corollary 2 we have the equality

$$G(u) = u^\alpha \hat{k}(u)F(u),$$

where F is the transformation (1) of the function f. Hence since $F \in L_2\left(\mathbb{R}_+; \frac{du}{u}\right)$ we get that $\frac{G(u)}{u^\alpha \hat{k}(u)} \in L_2\left(\mathbb{R}_+; \frac{dx}{x}\right)$ and the L_2^π-solution is given reciprocally by formula (35).

Sufficiency. If conversely, $\frac{G(u)}{u^\alpha \hat{k}(u)} \in L_2\left(\mathbb{R}_+; \frac{du}{u}\right)$, then f being defined by (35) belongs to L_2^π and by virtue of Theorem 3 the right-hand side of (34) belongs to L_2^π. Therefore by Corollary 2 the Kontorovich-Lebedev transform (1) of the right-hand side of (34) is equal to

$$u^\alpha \hat{k}(u)F(u) = u^\alpha \hat{k}(u)\frac{G(u)}{u^\alpha \hat{k}(u)} = G(u).$$

By the reciprocity (2) we see that equation (34) is satisfied and (35) is a unique L_2^π-solution.

Theorem 4 is proved.

Let us consider the convolution integral equation of the second kind (32), which can be written accordingly

$$f(x) = h(x) + \frac{2\lambda}{\pi^2}\, x\sinh\pi x \int_0^\infty \mathcal{K}_\alpha(x,y)f(y)dy. \tag{36}$$

We have

Theorem 5 *Let $\alpha > 0, \lambda \in \mathbb{C}\backslash\{0\},\ \ h \in L_2^\pi,\ \ k \in L_2^\nu,\ 0 < \nu < \pi$ and $H(u)$ is the Kontorovich-Lebedev transformation (1) of the function h. Let also $\sup_{u>0} |u^\alpha \hat{k}(u)| < \frac{1}{|\lambda|}$. Then*

$$f(x) = \frac{2}{\pi^2} \underset{N\to\infty}{\text{l.i.m.}}\ x \sinh \pi x \int\limits_{1/N}^{N} \frac{H(u)}{1 - \lambda u^\alpha \hat{k}(u)} K_{ix}(u) \frac{du}{u}, \tag{37}$$

is a solution of (36) belonging to L_2^π and any another solution from L_2^π coincides with (37) almost for all $x \in \mathbb{R}^+$.

Proof. Since $[1 - \lambda u^\alpha \hat{k}(u)]^{-1}$ is bounded we have $\frac{H(u)}{1-\lambda u^\alpha \hat{k}(u)} \in L_2\left(\mathbb{R}_+; \frac{du}{u}\right)$. Consequently, (37) exists in the mean sense and defines a function f from L_2^π. At the same time the convolution in the right-hand side of (36) belongs to L_2^π via Theorem 3. The transformation (1) of the right-hand side of (36) gives

$$H(u) + \lambda u^\alpha \hat{k}(u) \frac{H(u)}{1 - \lambda u^\alpha \hat{k}(u)} = \frac{H(u)}{1 - \lambda u^\alpha \hat{k}(u)} = F(u).$$

Therefore f by formula (37) satisfies equation (36) almost for all $x \in \mathbb{R}_+$.

Conversely, if $f, h \in L_2^\pi,\ k \in L_2^\nu$ and equality (36) takes place then by Corollary 2 we have

$$F(u) = H(u) + \lambda u^\alpha \hat{k}(u) F(u),$$

which implies (37). Theorem 5 is proved.

Remark 3. If $u^\alpha \hat{k}(u) \in L_2\left(\mathbb{R}_+; \frac{du}{u}\right)$ then solution (37) can be written in terms of the resolvent. Indeed, denoting by

$$M(u) = \frac{u^\alpha \hat{k}(u)}{1 - \lambda u^\alpha \hat{k}(u)},$$

and by $m(\theta)$ the transform (2) of $M \in L_2^\nu$ we can write solution (37) in the form

$$f(x) = h(x) + \frac{2\lambda}{\pi^2}\, x \sinh \pi x \int\limits_0^\infty \mathcal{M}_\alpha(x, y) h(y) dy, \tag{38}$$

where

$$\mathcal{M}_\alpha(x, y) = \int\limits_0^\infty m(\theta) \Omega_\alpha(x, y, \theta) d\theta \tag{39}$$

Finally we give certain concrete examples of the kernel (33), the corresponding convolution integral equations (34), (36) and their L_2^π-solutions (35), (37).

Example 1. Let $k(\theta) = \theta \tanh \frac{\pi\theta}{2}$. It evidently belongs to L_2^ν for any $\nu > 0$. Calling relation 2.16.48.14 from [4] we calculate the inner integral in (33) and write the kernel in terms of the notation (7). Precisely we have $\mathcal{K}_\alpha(x, y) = \Omega_{\alpha+1}(x, y, 0)$. Thus we arrive at the following integral equations

$$g(x) = \frac{2}{\pi^2}\, x \sinh \pi x \int\limits_0^\infty \Omega_{\alpha+1}(x, y, 0) f(y) dy, \tag{40}$$

$$f(x)=h(x)+\frac{2\lambda}{\pi^2}\,x\sinh\pi x\int\limits_0^\infty\Omega_{\alpha+1}(x,y,0)f(y)dy. \tag{41}$$

Hence a unique L_2^π-solution of the equation (40) can be obtained via Theorem 4 by the formula

$$f(x)=\frac{2}{\pi^2}\,\underset{N\to\infty}{\text{l.i.m.}}\;x\sinh\pi x\int\limits_{1/N}^{N}\frac{G(u)}{K_0(u)}K_{ix}(u)\frac{du}{u^{\alpha+2}}$$

under condition $\frac{G(u)}{u^{\alpha+1}K_0(u)}\in L_2\left(\mathbb{R}_+;\frac{du}{u}\right)$. Moreover, if $|\lambda|<[\sup_{u>0}u^{\alpha+1}K_0(u)]^{-1}$ then (see Theorem 5) we have a unique L_2^π-solution of the equation (41) written in the form (38), where $\mathcal{M}_\alpha(x,y)$ is defined by (39) with

$$m(\theta)=\frac{2}{\pi^2}\,\theta\sinh\pi\theta\int\limits_0^\infty\frac{u^\alpha K_0(u)}{1-\lambda u^{\alpha+1}K_0(u)}K_{i\theta}(u)du.$$

Example 2. Let $k(t)=t\sinh\pi t\left|\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)\right|^4\in L_2^\nu$. By relation 2.16.49.2 from [4] we arrive at the following convolution integral equations

$$g(x)=2^{5/2}\sqrt{\pi}\,x\sinh\pi x\int\limits_0^\infty\Omega_{\alpha+1/2}(x,y,0)f(y)dy, \tag{42}$$

$$f(x)=h(x)+2^{5/2}\lambda\sqrt{\pi}\,x\sinh\pi x\int\limits_0^\infty\Omega_{\alpha+1/2}(x,y,0)f(y)dy. \tag{43}$$

Hence a unique L_2^π-solution of the equation (42) is represented by the formula

$$f(x)=\frac{1}{\sqrt{2\pi}\pi^4}\,\underset{N\to\infty}{\text{l.i.m.}}\;x\sinh\pi x\int\limits_{1/N}^{N}\frac{G(u)}{K_0(u)}K_{ix}(u)\frac{du}{u^{\alpha+3/2}}$$

under condition $\frac{G(u)}{u^{\alpha+1/2}K_0(u)}\in L_2\left(\mathbb{R}_+;\frac{du}{u}\right)$. Moreover, if $|\lambda|<[2\pi^2\sqrt{2\pi}\sup_{u>0}u^{\alpha+1/2}K_0(u)]^{-1}$ then we have a unique L_2^π-solution of the equation (43) written in the form (38), where $\mathcal{M}_\alpha(x,y)$ is defined by (39) with

$$m(\theta)=2^{5/2}\sqrt{\pi}\,\theta\sinh\pi\theta\int\limits_0^\infty\frac{u^{\alpha-1/2}K_0(u)}{1-2\pi^2\sqrt{2\pi}\lambda u^{\alpha+1/2}K_0(u)}K_{i\theta}(u)du.$$

Example 3. Let $k(t)=t\sin at,\ a\neq 0$. It is easily seen that $k(t)\in L_2^\nu$, when $|\mathrm{Ima}|<\frac{\nu}{2}$. Appealing [4], relation 2.16.48.19 we get the following equations

$$g(x)=\frac{\sinh a}{\pi}\,x\sinh\pi x\int\limits_0^\infty\mathcal{K}_\alpha(x,y)f(y)dy, \tag{44}$$

$$f(x) = h(x) + \frac{\lambda \sinh a}{\pi}\, x \sinh \pi x \int_0^\infty \mathcal{K}_\alpha(x,y) f(y) dy, \tag{45}$$

where

$$\mathcal{K}_\alpha(x,y) = \int_0^\infty u^\alpha e^{-u\cosh\alpha} K_{ix}(u) K_{iy}(u) du.$$

A unique L_2^π-solution of the equation (44) is given by the formula

$$f(x) = \frac{4}{\pi^3 \sinh a} \underset{N\to\infty}{\text{l.i.m.}}\, x \sinh \pi x \int_{1/N}^{N} e^{u\cosh\alpha} G(u) K_{ix}(u) \frac{du}{u^{\alpha+2}}$$

under condition $u^{-(\alpha+1)} e^{u\cosh\alpha} G(u) \in L_2\left(\mathbb{R}_+; \frac{du}{u}\right)$. Moreover, if $|\lambda| < \left[\frac{\pi \sinh a}{2} \sup_{u>0} u^{\alpha+1} e^{-u\cosh\alpha}\right]^{-1}$ then we have a unique L_2^π-solution of the equation (45) written in the form (38), where $\mathcal{M}_\alpha(x,y)$ is defined by (39) with

$$m(\theta) = \frac{2\sinh a}{\pi}\, \theta \sinh \pi\theta \int_0^\infty \frac{u^\alpha K_{i\theta}(u)}{2\, e^{u\cosh\alpha} - \pi\lambda \sinh a\, u^{\alpha+1}} du.$$

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