A new Kontorovich-Lebedev like transformation

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Abstract

A different application of the familiar integral representation for the modified Bessel function drives to a new Kontorovich-Lebedev like integral transformation of a general complex index. Mapping properties and inversion formula are established.

Keywords: *Kontorovich-Lebedev transform, modified Bessel functions, Mellin transform, Laplace transform*

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1 Introduction

As it is known [1], Vol. II, the modified Bessel function $K_z(2\sqrt{x})$ can be represented by the following integral

$$K_z(2\sqrt{x}) = \frac{x^{-z/2}}{2}\int_0^\infty e^{-t-\frac{x}{t}} t^{z-1}dt,\ x > 0, \tag{1.1}$$

where $z = \nu + i\tau$ is a complex number. As it is easily seen, integral (1.1) converges absolutely for any $x \in \mathbb{R}_+, z \in \mathbb{C}$ and represents an entire function by z. Formula (1.1) can be written with the use of the Parceval relation for the Mellin transform [5], which leads to the integral representation

$$2x^{z/2}K_z(2\sqrt{x}) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(s+z)\Gamma(s)x^{-s}ds,\ x > 0, \tag{1.2}$$

where $\Gamma(w)$ is Euler's gamma function [1], Vol. 1 and $\gamma > \max(0, -\text{Re}z)$. Reciprocally, we have the direct Mellin transform of the modified Bessel function, namely

$$\Gamma(s+z)\Gamma(s) = 2\int_0^\infty K_z(2\sqrt{x})x^{s+z/2-1}dx. \tag{1.3}$$

Let us consider the following integral transformation with respect to a complex index of the modified Bessel function

$$F(z) = 2\int_0^\infty x^{z/2}K_z(2\sqrt{x})f(x)dx. \tag{1.4}$$

This transformation looks like the Kontorovich-Lebedev transform [4], [7], [8]. However, it is a completely different operator and cannot be reduced to the Kontorovich-Lebedev integral by any change of variables and functions. As far as the author is aware, the transform (1.4) was not studied yet, taking into account his mapping properties and inversion formula in an appropriate class of functions.

Our goal is to do this involving a special class of functions related to the Mellin transform and its inversion, which was introduced in [6]. Indeed, we have

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