Integral and series transformations via Ramanujan's identities and Salem's type equivalences to the Riemann hypothesis

Semyon YAKUBOVICH

Abstract

We consider integral and series transformations, which are associated with Ramanujan's identities, involving arithmetic functions $a(n), \omega(n), \sigma_a(n),\ d(n),\ \mu(n),\ \lambda(n), \varphi(n)$ and a ratio of products of Riemann's zeta functions of different arguments. Reciprocal inversion formulas are proved in a Banach space of functions whose Mellin's transforms are integrable over the vertical line Re $s > 1$. Examples of new transformations like Widder-Lambert and Kontorovich-Lebedev type are exhibited. Particular cases include familiar Lambert and Möbius transformations. Finally a class of equivalences of the Salem type to the Riemann hypothesis is established.

Keywords: *Mellin transform, Riemann zeta-function, Kontorovich-Lebedev transform, modified Bessel functions, Lambert transform, Möbius transform, Ramanujan's formulas, arithmetic functions, Lambert series, the Riemann hypothesis*

AMS subject classification: 44A15, 33C05, 33C10, 33C15, 11M06, 11M36, 11N 37

1 Introduction and auxiliary results

Integral and series transformations, which will be derived in the sequel are based on remarkable Ramanujan's identities involving arithmetic and Riemann's zeta-functions [6], [10], namely

$$\frac{\zeta(s)\zeta(s-a)\zeta\,(s-b)\,\zeta\,(s-a-b)}{\zeta(2s-a-b)} = \sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s}, \tag{1.1}$$

where Re $s > \max\{1, \text{Re } a+1, \text{Re } b+1, \text{Re } (a+b)+1\}$,

$$\zeta\,(s)\,\zeta\,(s-a) = \sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^s}, \tag{1.2}$$

where Re$s > \max\{1, \text{Re } a+1\}$,

$$\frac{\zeta^2(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s}, \quad \text{Re}s > 1, \tag{1.3}$$

$$\zeta^2(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}, \quad \text{Re}s > 1, \tag{1.4}$$

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \quad \text{Re}s > 1, \tag{1.5}$$

$$\frac{\zeta(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s}, \quad \text{Re}s > 1, \tag{1.6}$$

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}, \quad \text{Re}s > 1, \tag{1.7}$$

$$\frac{\zeta^3(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{d\left(n^2\right)}{n^s}, \quad \text{Re}s > 1, \tag{1.8}$$

$$\frac{\zeta^4(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{d^2(n)}{n^s}, \quad \text{Re}s > 1, \tag{1.9}$$

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s}, \quad \text{Re}s > 2, \tag{1.10}$$

$$\frac{1-2^{1-s}}{1-2^{-s}}\zeta(s-1) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}, \quad \text{Re}s > 2. \tag{1.11}$$

Here $\zeta(s)$ is the Riemann zeta-function [10], which satisfies the familiar functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s), \tag{1.12}$$

where $\Gamma(z)$ is Euler's gamma-function, and in the half-plane $\text{Re}s = c_0 > 1$ it is represented by the absolutely and uniformly convergent series with respect to $t \in \mathbb{R},\ s = c_0 + it$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1.13}$$

and by the uniformly convergent series

$$(1-2^{1-s})\zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \ \text{Re}s > 0. \tag{1.14}$$

Further, $a(n)$ in (1.11) denotes the greatest odd divisor of n, $\sigma_a(n)$ in (1.1), (1.2) is the sum of a-th powers of the divisors of $n \in \mathbb{N}$. In particular, for pure imaginary $a = i\tau$ $|\sigma_{i\tau}(n)| \le d(n)$, where $d(n)$ is the Dirichlet divisor function, i.e. the number of divisors of n, including 1 and n itself. It has the estimate [10] $d(n) = O(n^{\varepsilon}),\ n \to \infty,\ \varepsilon > 0$. The Möbius function is denoted by $\mu(n)$ and $|\mu(n)| \le 1$. The symbol $\omega(n)$ in (1.3) represents the number of distinct prime factors of n and it behaves as $\omega(n) = O(\log\log n),\ n \to \infty$ (see in [8]). By $\varphi(n)$ Euler's totient function is denoted and its asymptotic behavior satisfies [cf. [8]) $\varphi(n) = O\left(n[\log\log n]^{-1}\right),\ n \to \infty$. Finally, $\lambda(n)$ in (1.7) is the Liouville function, $|\lambda(n)| \le 1$.

Acknowledgments

The present investigation was supported, in part, by the "Centro de Matemática" of the University of Porto.

References

1. A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions*, Vols. I and II, McGraw-Hill, New York, London and Toronto (1953).

2. R.R. Goldberg, Inversions of generalized Lambert transforms, *Duke Math. J.*, **25** (1958), 459- 476.

3. I.I. Hirschman and D.V. Widder, *The Convolution Transform*, Princeton University Press, Princeton, New Jersey (1955).

4. W.B. Pennington, Widder's inversion formula for the Lambert transform, *Duke Math. J.*, **27** (1960), 561-568.

5. A.P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, *Integrals and Series: Vol. 2: Special Functions*, Gordon and Breach, New York (1986); *Vol. 3: More Special Functions*, Gordon and Breach, New York (1990).

6. S. Ramanujan, Some formulas in the analytic theory of numbers, *Messenger of Math.*, **45** (1916), 81-84.

7. R. Salem, Sur une proposition equivalente a l'hipothese de Riemann. *Comptes Rendus Math.*, **236** (1953), 1127-1128.

8. J. Sandor, D.S. Mitrinovich, B. Crstici, *Handbook of Number Theory*, Vol. I, Springer, Dordrecht (2006).

9. I.N. Sneddon, *The Use of Integral Transforms*, McGray Hill, New York (1972).

10. E.C. Titchmarsh, *The Theory of The Riemann Zeta- Function*, The Clarendon Press, Oxford, Second edition (1986).

11. E.C. Titchmarsh, *An Introduction to the Theory of Fourier Integrals*, Clarendon Press, Oxford (1937).

12. Vu Kim Tuan, O.I. Marichev and S.B. Yakubovich, Composition structure of integral transformations, *J. Soviet Math.*, **33** (1986), 166-169.

13. D. V. Widder, An inversion of the Lambert transform, *Math. Mag.*, **23** (1950), 171-182.

14. S. B. Yakubovich and Yu. F. Luchko, *The Hypergeometric Approach to Integral Transforms and Convolutions*. Mathematics and its Applications, 287. Kluwer Academic Publishers Group, Dordrecht (1994).

15. S.B. Yakubovich and B. Fisher, A class of index transforms with general kernels, *Math. Nachr.*, **200** (1999), 165-182.

16. S.B. Yakubovich, *Index Transforms*, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong (1996).

S.Yakubovich
Department of Mathematics,
Faculty of Sciences,
University of Porto,
Campo Alegre st., 687
4169-007 Porto

Portugal
E-Mail: syakubov@fc.up.pt