On a progress in the Kontorovich-Lebedev transform theory and related integral operators

SEMYON B. YAKUBOVICH [1]

Department of Pure Mathematics, Faculty of Science,
University of Porto, Campo Alegre str., 687,
4169-007 Porto, Portugal

This survey represents mainly recent results of the author, which were obtained during the last decade and concern the theory of the Kontorovich-Lebedev transformation and some related integral operators of the convolution and non-convolution type. In particular, we discuss their boundedness in the Lebesgue, Hardy and Sobolev type spaces and in some spaces of distributions. Special attention is given to the L_2-case and Plancherel's theory for these operators and to the harmonic analysis of the so-called Lebedev- Stieltjes integrals. Moreover, we concentrate on asymptotic, summation formulas and uncertainty principles for the Kontorovich - Lebedev transform.

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1. Introduction

The Kontorovich-Lebedev transform was introduced for the first time in [1], [2] to solve certain boundary- value problems of mathematical physics. It arises naturally when the method of separation of variables is used to solve boundary -value problems formulated in terms of cylindrical coordinate systems. The mathematical L_1, L_2- theories, inversion formulas and applications were given later by Lebedev, see refs. [3], [4], [5, Vol. 2], [6], Sneddon [7], Lowndes [8], Jones [9] and certain generalizations of the Kontorovich- Lebedev transformation were

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investigated in some early papers and books of the author (see, for instance, in [10, 11, 12, 13]).

So we will define the operator of the Kontorovich-Lebedev transform in the form of the Lebedev integral

$$K_{i\tau}[f] = \int_0^\infty K_{i\tau}(x)f(x)dx,\ \tau \in \mathbb{R}_+, \tag{1}$$

and its adjoint operator correspondingly

$$(KLf)(x) = \int_0^\infty K_{i\tau}(x)f(\tau)d\tau,\ x \in \mathbb{R}_+. \tag{2}$$

Operators (1), (2) are associated with the modified Bessel function $K_{i\tau}(x)$ [5, Vol. 2] of the pure imaginary index or subscript $i\tau$. The principal difference between operators (1) and (2) that the integration process in the adjoint operator (2) is realized with respect to the index of the modified Bessel function. This circumstance drives to some difficulties to estimate the kernel function in (2) in order to get the norm estimates of the operator in certain functional spaces.

The modified Bessel function can be represented by the Fourier integral [5, Vol. 2]

$$K_{i\tau}(x) = \int_0^\infty e^{-x\cosh u} \cos \tau u du,\ x > 0. \tag{3}$$

Therefore for $x > 0, \tau \in \mathbb{R}$ it is real-valued and even function with respect to the index $i\tau$. Furthermore, via the analytic properties of the integrand in (3) the latter integral can be extended to the strip $\delta \in [0, \pi/2)$ in the upper half-plane, i.e.

$$K_{i\tau}(x) = \frac{1}{2}\int_{i\delta-\infty}^{i\delta+\infty} e^{-x\cosh\beta + i\tau\beta} d\beta. \tag{4}$$

This gives us for each $x > 0$ an immediate uniform estimate

$$|K_{i\tau}(x)| \leq e^{-\delta|\tau|}K_0(x\cos\delta),\ 0 \leq \delta \leq \delta_0 < \frac{\pi}{2}. \tag{5}$$

In particular, we have the inequality $|K_{i\tau}(x)| \leq K_0(x)$. We note that generally the modified Bessel function $K_\mu(z)$ satisfies the differential equation

$$z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} - (z^2 + \mu^2)u = 0.$$

It has the asymptotic behavior [5, Vol. 2]

$$K_\mu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}[1 + O(1/z)], \qquad z \to \infty, \tag{6}$$

and near the origin

$$K_\mu(z) = O(z^{-|\mathrm{Re}\mu|}),\ z \to 0,\ \mu \neq 0, \tag{7}$$

$$K_0(z) = -\log z + O(1),\ z \to 0. \tag{8}$$

When $|\tau| \to \infty$ and $x > 0$ is fixed it has, see refs. [4], [12] the following behavior

$$K_{i\tau}(x) = O\left(\frac{e^{-\pi|\tau|/2}}{\sqrt{|\tau|}}\right). \tag{9}$$

By using relation (2.16.51.8) in [14] we obtain the useful formula

$$\int_0^\infty \tau \sinh((\pi-\delta)\tau) K_{i\tau}(x) K_{i\tau}(y) d\tau$$

$$= \frac{\pi}{2} xy \sin\delta \frac{K_1((x^2+y^2-2xy\cos\delta)^{1/2})}{(x^2+y^2-2xy\cos\delta)^{1/2}},\ x,y>0,\ 0<\delta\leq\pi. \tag{10}$$

Returning to the definition (1) of the Kontorovich-Lebedev transform and taking into account (5) we have the estimate

$$|K_{i\tau}[f]| \leq \int_0^\infty K_0(x)|f(x)|dx, \tag{11}$$

which gives the natural domain of definition of the Kontorovich-Lebedev transform, i.e. $f \in L^0 \equiv L_1(\mathbb{R}_+; K_0(x)dx)$

$$L^0 := \left\{ f : \int_0^\infty K_0(x)|f(x)|dx < \infty \right\}. \tag{12}$$

In particular it contains all spaces $L^{(\alpha,\beta)} \equiv L_1(\mathbb{R}_+; K_\alpha(\beta x)dx),\ \alpha \in \mathbb{R},\ 0 < \beta \leq 1$ and $L_p(\mathbb{R}_+; xdx),\ 2 < p \leq \infty$ with the norms

$$||f||_{L^{(\alpha,\beta)}} = \int_0^\infty K_\alpha(\beta x)|f(x)|dx < \infty, \tag{13}$$

$$||f||_{L_p(\mathbb{R}_+;xdx)} = \left(\int_0^\infty |f(x)|^p x dx\right)^{1/p} < \infty, \tag{14}$$

$$||f||_{L_\infty(\mathbb{R}_+;xdx)\nu,\infty} = \text{ess sup}_{x\geq 0}|f(x)| < \infty. \tag{15}$$

Thus simple properties arise for the operator (1):

1. $K_{i\tau}[f]$ is bounded, namely, $|KLf(\tau)| \leq ||f||_{L^0}$ for any $\tau \in \mathbb{R}_+, f \in L^0$ and the integral (1) exists as a Lebesgue integral;

2. If a sequence $\{f_k\}_1^\infty$ converges in the L^0-norm to f, then $K_{i\tau}[f_k]$ converges uniformly to $K_{i\tau}[f]$;

3. $K_{i\tau}[f]$ is uniformly continuous in $\mathbb{R}_+$;

4. $K_{i\tau}[f]$ tends to 0 as $\tau \to \infty$ (an analog of the Riemann-Lebesgue lemma).

The points $x \in \mathbb{R}_+$ at which

$$\int_{x-\delta}^{x+\delta} |f(y) - f(x)| dy = o(\delta), \delta \to 0$$

are called Lebesgue points of f.

We have the following inversion L_1-theorem.

Theorem 1.1 [11]. *Let $f \in L^{(0,\beta)}$ for some $0 < \beta < 1$. Then at each Lebesgue point of f*

$$f(x) = \lim_{\eta \to \pi - 0} \frac{2}{x\pi^2} \int_0^\infty \tau \sinh \eta\tau K_{i\tau}(x) K_{i\tau}[f] d\tau. \tag{16}$$

The right-hand side of formula (16) can be written in the form of the Poisson type integral. Indeed, by using (10) we change the order of integration via Fubini's theorem and we find

$$u(x,\eta) = \frac{2}{x\pi^2} \int_0^\infty \tau \sinh \eta\tau K_{i\tau}(x) K_{i\tau}[f] d\tau$$

$$= \frac{\sin \eta}{\pi} \int_0^\infty \frac{K_1\left(\sqrt{x^2+y^2+2xy\cos\eta}\right)}{\sqrt{x^2+y^2+2xy\cos\eta}} y f(y) dy. \tag{17}$$

Theorem 1.2. (the uniqueness theorem). *If two functions of L^0 have the same transform (1), then they coincide almost everywhere on $\mathbb{R}_+$.*

Theorem 1.3. *Let $f \in L^{(0,\beta)}$ for some $0 < \beta < 1$. If also $K_{i\tau}[f] \in L_1(\mathbb{R}_+; \tau \cosh(\pi\tau/2) d\tau)$ then the inversion formula is valid*

$$f(x) = \frac{2}{x\pi^2} \int_0^\infty \tau \sinh \pi\tau K_{i\tau}(x) K_{i\tau}[f] d\tau. \tag{18}$$

2. Asymptotic and summation formulas

Asymptotic expansions of the Kontorovich-Lebedev transform (1) and its adjoint (2) were studied by Naylor (see refs. [15, 16, 17, 18, 19, 20] and Wong [21]. Numerical methods for computing this transform and its inverse were applied in [22], [23]. In this section following ref. [24], we will exhibit analogs of the Watson lemma and the Poisson summation formulas for the Kontorovich-Lebedev transformation.

Let $z \in \mathbb{C}$ be a complex number and let f be a complex -valued measurable function on $\mathbb{R}_+$. We generalize operator (1) considering the Kontorovich- Lebedev transform

$$F(z) \equiv K_z[f] = \int_0^\infty K_z(x) f(x) dx, \tag{19}$$

of general complex index z. Let us define first the product of the modified Bessel functions of different arguments by the known Macdonald formula [14, relation (2.16.9.1)]

$$K_\nu(x)K_\nu(y) = \frac{1}{2}\int_0^\infty e^{-\frac{1}{2}\left(u\frac{x^2+y^2}{xy}+\frac{xy}{u}\right)} K_\nu(u)\frac{du}{u}. \tag{20}$$

This formula generates the following convolution operator

$$(f * g)(x) = \frac{1}{2x}\int_0^\infty \int_0^\infty e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy}+\frac{yu}{x}\right)} f(u)g(y)dudy,\ x > 0, \tag{21}$$

satisfying under some conditions (see refs. [11, 12]) the factorization property in terms of the integral (19). Precisely, we get

$$K_z[f * g] = K_z[f]K_z[g]. \tag{22}$$

We suppose that $f(x)$ admits the series representation $f(x) = \sum_{n=0}^\infty \frac{a_n}{n!}x^n$ as an entire function of the exponential type with $\limsup_{n\to\infty}|a_n|^{1/n} = \sigma$, where σ is a type of this function. In [24] the Watson lemma is proved for integrals (19), which gives the asymptotic behavior of $F(z)$ when $z \to \infty$. Indeed, we have

Lemma 2.1. *Let f be an entire function of the exponential type with $\sigma < 1$. Then $F(z)$ admits in the strip $|\mathrm{Re}\ z| < 1$ when $z \to \infty$ the following asymptotic expansion*

$$F(z) \sim \frac{\pi}{2\cos(\pi z/2)} f(iz). \tag{23}$$

Further, taking the known Poisson formula for the cosine Fourier transform [25]

$$\sqrt{\beta}\left[\frac{1}{2}F_c(0) + \sum_{n=1}^\infty F_c(n\beta)\right] = \sqrt{\alpha}\left[\frac{1}{2}f(0) + \sum_{n=1}^\infty f(n\alpha)\right], \tag{24}$$

where $\alpha\beta = 2\pi, \alpha > 0$ and

$$F_c(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos xt dt, \tag{25}$$

we apply integral representation (3) for the modified Bessel function and we arrive at the identity

$$K_0(x) + 2\sum_{n=1}^\infty K_{in\beta}(x) = \alpha\left[\frac{e^{-x}}{2} + \sum_{n=1}^\infty e^{-x\cosh n\alpha}\right],\ x > 0,\ \alpha\beta = 2\pi,\ \alpha > 0. \tag{26}$$

This is a key identity, which allows to prove an analog of the Poisson summation formula for the Kontorovich- Lebedev transform (1). Precisely, we obtain

Theorem 2.1. *Let $f \in L^{(\mu,\xi)}$, where $|\mu| > 1/2,\ 0 < \xi < 1$. Then the Poisson type formula is true*

$$K_0[f] + 2\sum_{n=1}^{\infty} K_{in\beta}[f] = \alpha \left[\frac{1}{\pi}\int_0^{\infty} K_{i\tau}[f]d\tau + \sum_{n=1}^{\infty}(Lf)(\cosh n\alpha) \right],\ \alpha\beta = 2\pi,\ \alpha > 0, \tag{27}$$

where $(Lf)(x)$ is the Laplace integral

$$(Lf)(x) = \int_0^{\infty} e^{-xt} f(t)dt. \tag{28}$$

Let us exhibit some interesting particular cases of the formula (27). Indeed, letting $f(x) \equiv 1,\ \beta = \frac{2\pi}{\alpha}$ we calculate the corresponding Kontorovich- Lebedev integral by the relation (2.16.2.1) in [14] and we derive the identity

$$\sum_{n=1}^{\infty} \left[\frac{\pi}{\cosh\left(\frac{n\pi^2}{\alpha}\right)} - \frac{\alpha}{\cosh n\alpha} \right] = \frac{\alpha - \pi}{2},\ \alpha > 0. \tag{29}$$

If $f(x) = x^{\gamma-1},\ \gamma > 0$ then we appeal to the relation (2.16.2.2) in [14] to obtain the formula

$$2^{\gamma-1}\left[\Gamma^2\left(\frac{\gamma}{2}\right) + 2\sum_{n=1}^{\infty}\left|\Gamma\left(\frac{\gamma}{2} + \frac{\pi i n}{\alpha}\right)\right|^2 \right] = \alpha\Gamma(\gamma)\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{\cosh^{\gamma} n\alpha} \right],\ \alpha > 0. \tag{30}$$

The value $\gamma = 1$ leads again to (29). Let $f(x) = e^{-x}x^{\gamma-1},\ \gamma > 0$. Then by using relation (2.16.6.4) in [14] the identity (27) becomes ($\alpha > 0$)

$$\frac{2^{-\gamma}\sqrt{\pi}}{\Gamma(\gamma + 1/2)}\left[\Gamma^2(\gamma) + 2\sum_{n=1}^{\infty}\left|\Gamma\left(\gamma + \frac{2\pi i n}{\alpha}\right)\right|^2 \right] = \alpha\Gamma(\gamma)\left[2^{-\gamma-1} + \sum_{n=1}^{\infty}\frac{1}{(1+\cosh n\alpha)^{\gamma}} \right]. \tag{31}$$

Letting $\gamma = 1$ in (31) we invoke the reduction and supplement formulas for gamma-functions and we get the identity

$$1 + \frac{2\pi^2}{\alpha}\sum_{n=1}^{\infty}\frac{n}{\sinh\left(\frac{\pi^2 n}{\alpha}\right)\cosh\left(\frac{\pi^2 n}{\alpha}\right)} = \alpha\left[\frac{1}{4} + \sum_{n=1}^{\infty}\frac{1}{1+\cosh n\alpha} \right]. \tag{32}$$

In particular, when $\alpha = \pi$ we have

$$\sum_{n=1}^{\infty}\left[\frac{2n}{\sinh \pi n \cosh \pi n} - \frac{1}{1+\cosh \pi n} \right] = \frac{\pi - 4}{4\pi}. \tag{33}$$

As a consequence of (26) and differential equation (2) one can get another identity involving series of the modified Bessel functions with respect to an index

$$2\beta^2\sum_{n=1}^{\infty} n^2 K_{in\beta}(x) + \alpha x\sum_{n=1}^{\infty} e^{-x\cosh n\alpha}(x\cosh^2 n\alpha - \cosh n\alpha - x)$$

$$= \frac{\alpha x}{2} e^{-x}, \; x > 0, \; \alpha\beta = 2\pi, \alpha > 0. \tag{34}$$

Integrating through (34) over $\mathbb{R}_+$ and changing accordingly the order of integration and summation we come out with the equality

$$\frac{4\pi^3}{\alpha^2} \sum_{n=1}^{\infty} \frac{n^2}{\cosh\left(\frac{n\pi^2}{\alpha}\right)} + \alpha \sum_{n=1}^{\infty} \frac{\tanh^2 n\alpha}{\cosh n\alpha} = \frac{\alpha}{2}, \; \alpha > 0.$$

In particular, $\alpha = \pi$ gives the value of the series

$$\sum_{n=1}^{\infty} \frac{4n^2 + \tanh^2 \pi n}{\cosh \pi n} = \frac{1}{2}.$$

The Poisson type formula (27) can be extended involving convolution operator (21), which satisfies the following property.

Lemma 2.2. *Let $f, g \in L^{(\mu,\xi)}$, where $0 < \xi < 1$. Then the convolution* (21) $(f * g)(x)$ *exists almost for all $x > 0$, belongs to $L^{(\mu,\xi)}$ and satisfies the inequality*

$$||f * g||_{L_1(\mathbb{R}_+; K_\mu(\xi x)dx)} \leq ||f||_{L_1(\mathbb{R}_+; K_\mu(\xi x)dx)} ||g||_{L_1(\mathbb{R}_+; K_\mu(\xi x)dx)}. \tag{35}$$

Moreover, factorization property (22) *holds in the closed strip* $|\mathrm{Re} z| \leq |\mu|$.

This lemma drives us to the following extension of the Poisson formula (27).

Theorem 2.2. *Let $f, g \in L^{(\mu,\xi)}$, where $|\mu| > \frac{1}{4}$, $0 < \xi < 1$. Then the Poisson type formula is true*

$$K_0[f]K_0[g] + 2\sum_{n=1}^{\infty} K_{in\beta}[f]K_{in\beta}[g] = \alpha \left[\frac{1}{\pi} \int_0^{\infty} K_{i\tau}[f]K_{i\tau}[g] d\tau \right.$$

$$\left. + \int_0^{\infty} \int_0^{\infty} f(u)g(y) \sum_{n=1}^{\infty} K_0\left(\sqrt{u^2 + y^2 + 2uy\cosh n\alpha}\right) du dy \right], \; \alpha\beta = 2\pi, \; \alpha > 0. \tag{36}$$

In particular, we have the identity

$$|K_0[f]|^2 + 2\sum_{n=1}^{\infty} |K_{in\beta}[f]|^2 = \alpha \left[\frac{1}{\pi} \int_0^{\infty} |K_{i\tau}[f]|^2 d\tau \right.$$

$$\left. + \int_0^{\infty} \int_0^{\infty} f(u)\overline{f(y)} \sum_{n=1}^{\infty} K_0\left(\sqrt{u^2 + y^2 + 2uy\cosh n\alpha}\right) du dy \right], \; \alpha\beta = 2\pi, \; \alpha > 0. \tag{37}$$

An interesting example of the formula (37) can be done taking $f(x) \equiv 1$. Appealing to the corresponding values of the Lebedev integrals (see above) we come out with the identity

$$\frac{\pi^2}{2} \sum_{n=1}^{\infty} \frac{1}{\cosh^2(\pi n\beta/2)} - \alpha \sum_{n=1}^{\infty} \int_0^{\infty} \int_0^{\infty} K_0\left(\sqrt{u^2 + y^2 + 2uy\cosh n\alpha}\right) du dy$$

$$= \frac{2\alpha - \pi^2}{4}, \ \alpha\beta = 2\pi, \ \alpha > 0.$$

But

$$\int_0^\infty \int_0^\infty K_0\left(\sqrt{u^2+y^2+2uy\cosh n\alpha}\right) dudy$$

$$= \int_0^{\pi/2}\int_0^\infty rK_0\left(\sqrt{1+2\cos\varphi\sin\varphi\cosh n\alpha}\right) drd\varphi$$

$$= \int_0^{\pi/2} \frac{d\varphi}{1+2\cos\varphi\sin\varphi\cosh n\alpha} = \int_0^\infty \frac{du}{u^2+2u\cosh n\alpha+1}$$

$$= \frac{1}{2\sinh n\alpha}\log\frac{\cosh n\alpha+\sinh n\alpha}{\cosh n\alpha-\sinh n\alpha} = \frac{n\alpha}{\sinh n\alpha}.$$

Therefore we finally get

$$\sum_{n=1}^\infty \left[\frac{\pi^2}{1+\cosh \pi n\beta} - \frac{n\alpha^2}{\sinh n\alpha}\right] = \frac{2\alpha-\pi^2}{4}, \ \alpha\beta = 2\pi, \ \alpha > 0.$$

In particular, letting $\alpha = \pi$ we have

$$\sum_{n=1}^\infty \left[\frac{1}{1+\cosh 2\pi n} - \frac{n}{\sinh \pi n}\right] = \frac{2-\pi}{4\pi}.$$

3. Plancherel theorems and Parseval relations. An extension on L_p-case

In this section we will exhibit Plancherel type theorems and Parseval identities for the Kontorvich-Lebedev operators (1), (2) in the Hilbert spaces of the Lebesgue and Sobolev type. We will show that in these cases an interpretation of the integrals (1), (2) should be different. Such kind of theorems were proved in refs. [26, 27, 28, 29]. We will follow mainly our results from refs. [30, 31, 32, 33].

Let $f \in L_2(\mathbb{R}_+; xdx)$ (see (14)). Then integral (1) in general, does not exists in Lebesgue's sense. For instance, we take

$$f(x) = \begin{cases} \frac{1}{x\log x}, & \text{if } 0 < x \le \frac{1}{2}, \\ 0, & \text{if } x > \frac{1}{2}. \end{cases}$$

Then evidently, $f \in L_2(\mathbb{R}_+; xdx)$ but via the asymptotic formulas (6), (8) for the modified Bessel functions we observe that $f \notin L^0$. Correspondingly, taking $f(x) = e^{x/2}$ we have $f \in L^0$ but $f \notin L_2(\mathbb{R}_+; xdx)$. Thus transformation (1) will be defined as

$$K_{i\tau}[f] = \lim_{N\to\infty} \int_{1/N}^N K_{i\tau}(x)f(x)dx, \tag{38}$$

where the limit is taken in the mean square sense with respect to the norm of the space $L_2(\mathbb{R}_+; \tau \sinh \pi\tau d\tau)$. We note, that two definitions (1) and (38) are equivalent, if

$$f \in L_2(\mathbb{R}_+; xdx) \cap L^0.$$

We have

Theorem 3.1. *Let $f \in L_2(\mathbb{R}_+; xdx)$. Then the Kontorovich-Lebedev transform can be interpreted by formula* (38) *with the convergence in the mean with respect to the norm of the space $L_2(\mathbb{R}_+; \tau \sinh \pi\tau d\tau)$. Its inversion formula is given by the integral*

$$f(x) = \frac{2}{\pi^2} \lim_{N\to\infty} \int_0^N \tau \sinh \pi\tau \frac{K_{i\tau}(x)}{x} K_{i\tau}[f] d\tau, \tag{39}$$

with the convergence in the mean with respect to the norm of the space $f \in L_2(\mathbb{R}_+; xdx)$. The Kontorovich-Lebedev operator (38)

$$K_{i\tau} : L_2(\mathbb{R}_+; xdx) \leftrightarrow L_2(\mathbb{R}_+; \tau \sinh \pi\tau d\tau) \tag{40}$$

is a bounded operator and forms an isometric isomorphism between these Hilbert spaces. Moreover, the following Plancherel identity is true

$$\int_0^\infty \tau \sinh \pi\tau K_{i\tau}[f] \overline{K_{i\tau}[g]} d\tau = \frac{\pi^2}{2} \int_0^\infty x f(x) \overline{g(x)} dx. \tag{41}$$

In particular, the Parseval equality holds

$$\int_0^\infty \tau \sinh \pi\tau |K_{i\tau}[f]|^2 d\tau = \frac{\pi^2}{2} \int_0^\infty x |f(x)|^2 dx. \tag{42}$$

This theorem gives immediately the value of the norm of the operator (40), which is equal to $\frac{\pi}{\sqrt{2}}$.

Let A_x be the differential operator (see Section 1), which has eigenfunction $K_\mu(x)$ with eigenvalue $-\mu^2$ and can be written in the form

$$A_x u = x^2 u(x) - x \frac{d}{dx}\left[x \frac{du}{dx}\right], \quad A_x K_\nu = -\mu^2 K_\mu(x). \tag{43}$$

As usual we denote by A_x^k the k-th iterate of A_x, $A_x^0 u = u$. We will study the adjoint Kontorovich-Lebedev operator (2) KLf in the following Hilbert space of Sobolev type with the finite norm

$$||u||_{S_2^N(\mathbf{R}_+)} = \left(\sum_{k=0}^N \int_0^\infty |A_x^k u|^2 \frac{dx}{x} \right)^{1/2} < \infty. \tag{44}$$

The main result is the Plancherel theorem for the adjoint operator (2).

Theorem 3.2 [**31**]. *Let $f \in L_2(\mathbf{R}_+; \omega_N(\tau)d\tau)$, where the weighted function $\omega_N,\ N \in \mathbb{N}_0$ is defined by the relation*

$$\omega_N(\tau) = \frac{\pi^2}{2} \frac{1-\tau^{4(N+1)}}{(1-\tau^4)\tau \sinh \pi\tau}. \tag{45}$$

Then the integral

$$g_n(x) = \int_0^n K_{i\tau}(x) f(\tau) d\tau,\ x > 0$$

converges to $(KLf)(x)$ when $n \to \infty$ with respect to the norm (44) in the space $S_2^N(\mathbf{R}_+)$; and

$$f_n(\tau) = \frac{2}{\pi^2} \tau \sinh \pi\tau \int_{1/n}^n K_{i\tau}(x)(KLf)(x) \frac{dx}{x}$$

converges in the mean to $f(\tau)$ with respect to the norm in $L_2(\mathbf{R}_+; \omega_N(\tau)d\tau)$. Moreover, the following Plancherel identity is true

$$\sum_{k=0}^N \int_0^\infty A_x^k KLf\ \overline{A_x^k KLh} \frac{dx}{x} = \frac{\pi^2}{2} \int_0^\infty f(\tau)\overline{h(\tau)}\ \frac{1-\tau^{4(N+1)}}{1-\tau^4}\ \frac{d\tau}{\tau \sinh \pi\tau},$$

where $f, h \in L_2(\mathbf{R}_+; \omega_N(\tau)d\tau)$. In particular,

$$||KLf||^2_{S_2^N(\mathbf{R}_+)} = ||f||^2_{L_2(\mathbf{R}_+; \omega_N(\tau)d\tau)}$$

that is the Parseval equality takes place

$$\sum_{k=0}^N \int_0^\infty |A_x^k KLf|^2 \frac{dx}{x} = \frac{\pi^2}{2} \int_0^\infty |f(\tau)|^2\ \frac{1-\tau^{4(N+1)}}{1-\tau^4}\ \frac{d\tau}{\tau \sinh \pi\tau}. \tag{46}$$

Remark 3.1. The case $N = 0$ corresponds to the Plancherel theorem for the adjoint Kontorovich-Lebedev operator (2) as an isometric isomorphism (see refs. [29], [32])

$$KL : L_2\left(\mathbf{R}_+; \frac{\pi^2}{2} \frac{d\tau}{\tau \sinh \pi\tau}\right) \leftrightarrow L_2\left(\mathbb{R}_+; \frac{dx}{x}\right).$$

Parseval equality (46) in this case becomes formula (42) up to an elementary functional substitution.

We will extend our results on the L_p-case considering the Kontorovich-Lebedev operator (1) as a map

$$K_{i\tau} : L_p(\mathbb{R}_+; xdx) \leftrightarrow L_p(\mathbb{R}_+; \tau \sinh \pi\tau d\tau), \tag{47}$$

where $2 \leq p \leq \infty$. We will show that for $1 \leq p < 2$ this map in general, does not exist. However, when $2 < p \leq \infty$ and $f(x) \in L_p(\mathbb{R}_+; xdx)$ (see (14)) it is not difficult to show employing the Hölder inequality and asymptotic formulas (6)-(8) for the modified Bessel functions integral (1) is understood as a Lebesgue integral. The image of this function under the Kontorovich-Lebedev operator (1) will be in $L_p(\mathbb{R}_+; \tau \sinh \pi\tau d\tau)$. It fails certainly when $1 \le p < 2$. For instance, we take

$$f(x) = \begin{cases} 1, & \text{if } 0 \le x \le 1, \\ 0, & \text{if } x > 1. \end{cases}$$

Then f clearly belongs to $L_p(\mathbb{R}_+; xdx)$, $1 \le p < 2$. Nevertheless, we find that the corresponding value of the Kontorovich-Lebedev transform (1) is $\varphi(\tau) = \int_0^1 K_{i\tau}(x)dx$. The latter integral is a continuous function with respect to τ and behaves as $O(e^{-\pi\tau/2})$ when $\tau \to +\infty$ (see (9)). Consequently, integral

$$\int_0^\infty |\varphi(\tau)|^p \tau \sinh \pi\tau d\tau$$

is divergent for this function when $1 \le p < 2$.

The boundedness of the Kontorovich-Lebedev transform (1) as an operator

$$K_{i\tau} : L_\infty(\mathbb{R}_+; xdx) \to L_\infty(\mathbb{R}_+; \tau \sinh \pi\tau d\tau).$$

This result is given by

Theorem 3.3 **[30]**. *The Kontorovich-Lebedev transformation* (1) *is a bounded operator* $L_\infty(\mathbb{R}_+; xdx) \to L_\infty(\mathbb{R}_+; \tau \sinh \pi\tau d\tau)$ *and its norm is equal to* $\frac{\pi}{2}$.

Now by the Riesz-Thorin convexity theorem we can extend the boundedness properties of the Kontorovich-Lebedev operator as a map (47).

Theorem 3.4 **[30].** *Let* $2 \le p \le \infty$. *The Kontorovich-Lebedev transformation* (1) *is a bounded operator* $L_p(\mathbb{R}_+; xdx) \to L_p(\mathbb{R}_+; \tau \sinh \pi\tau d\tau)$. *Moreover, the following inequality*

$$\int_0^\infty \tau \sinh \pi\tau |K_{i\tau}[f]|^p d\tau \le \frac{\pi^p}{2^{p-1}} \int_0^\infty x|f(x)|^p dx, \tag{48}$$

holds true which is equivalent to

$$||K_{i\tau}[f]||_{L_p(\mathbb{R}_+;\tau \sinh \pi\tau d\tau)} \le \frac{\pi}{2^{1-\frac{1}{p}}} ||f||_{L_p(\mathbb{R}_+;xdx)}. \tag{49}$$

As a corollary we have that the norm of the operator (47) satisfies the inequality

$$||K_{i\tau}|| \le \frac{\pi}{2^{1-\frac{1}{p}}},\ 2 \le p \le \infty. \tag{50}$$

It gives an equality when $p = 2, \infty$ (see above). An open problem is to prove that the norm of the Kontorovich-Lebedev operator (47) is equal to $\frac{\pi}{2^{1-\frac{1}{p}}}$ for all $2 \le p \le \infty$.

Further we consider the Kontorovich-Lebedev transformation (1) in $L_{\nu,p}$-spaces with $\nu \in \mathbb{R}$, $1 \le p \le \infty$, which are related to the Mellin transform (see refs. [12], [31], [33]) and equipped with the norm

$$||f||_{\nu,p} = \left(\int_0^\infty x^{\nu p-1} |f(x)|^p dx \right)^{1/p} < \infty, \tag{51}$$

$$||f||_{\nu,\infty} = \text{ess sup}_{x\geq 0}|x^\nu f(x)| < \infty. \tag{52}$$

Lemma 3.1 [33]. *Let $f \in L_{\nu,p}(\mathbb{R}_+),\ \nu < 1,\ 1 \leq p \leq \infty$. Then the Poisson type representation* (17) *takes place.*

Theorem 3.5 [33]. *Let $f \in L_{\nu,p}(\mathbb{R}_+),\ 0 < \nu < 1,\ 1 \leq p < \infty$. Then*

$$f(x) = \lim_{\varepsilon\to 0+} \frac{\sin\varepsilon}{\pi}\int_0^\infty \frac{K_1\left(\sqrt{x^2+y^2-2xy\cos\varepsilon}\right)}{\sqrt{x^2+y^2-2xy\cos\varepsilon}} yf(y)dy, \tag{53}$$

where the limit is with respect to the norm (51)*. Besides, it exists for almost all $x > 0$.*

For the modified adjoint Kontorovich-Lebedev operator

$$(\mathcal{KL}f)(x) = \int_0^\infty K_{i\tau}(x)f(\tau)\tau d\tau,\ x > 0, \tag{54}$$

we have the following results (see ref. [34])

Theorem 3.6. (the Hausdorff-Young type theorem). *The transform* (54) *is a bounded operator*

$$\mathcal{KL} : L_p(\mathbb{R}_+; dx) \to L_{p'}(\mathbb{R}_+; x^{-1}dx),$$

where $1 \leq p \leq 2$ and $p^{-1} + {p'}^{-1} = 1$. It satisfies the norm inequality

$$||\mathcal{KL}f||_{L_{p'}(\mathbb{R}_+;x^{-1}dx)} \leq \left(\frac{\pi}{2}\right)^{\frac{1}{p'}} ||f||_{L_p(\mathbb{R}_+;dx)}$$

and its norm $||KL||_{p,p'} \leq \left(\frac{\pi}{2}\right)^{\frac{1}{p'}}$.

An inversion formula for this adjoint Kontorovich-Lebedev transformation is given by

Theorem 3.7. *Let $f \in L_p(\mathbb{R}_+; dx),\ 1 < p < \infty$. Then*

$$\lim_{\varepsilon\to 0+} \tau \int_0^\infty x^{\varepsilon-1}K_{i\tau}(x)(\mathcal{KL}f)(x)dx = \frac{\pi}{2}|\Gamma(1+i\tau)|^2 f(\tau), \tag{55}$$

where the limit is with respect to the norm in $L_p(\mathbb{R}_+; dx)$. Besides, the limit in (55) *exists almost for all $\tau > 0$.*

4. A distributional analog

An investigation of the Kontorovich-Lebedev transform on distributions started from the pioneer paper by Zemanian [35] as a continuation of his methods of generalized integral transformations, which were developed in [36] (see also ref. [37], [6]). Later some modifications and generalizations were done in refs. [38, 39, 40, 41, 42, 43]. We will follow our results from [44] considering the Kontorovich - Lebedev transformation on distributions for a general complex index

$$\mathcal{KL}[f](s) := \langle f, K_s(\cdot)\rangle, \ s \in \mathbb{C}, \tag{56}$$

which is defined in a special testing - function space $\mathcal{A}_{\nu,p}$ of complex-valued, smooth functions $\varphi(x)$ on $\mathbb{R}_+$ for which the following quantity

$$\alpha_{k,\nu,p}(\varphi) = \alpha_{0,\nu,p}(A_x^k\varphi) = \left(\int_0^\infty \left|A_x^k\varphi\right|^p x^{\nu p-1}dx\right)^{1/p}, \ p \geq 2 \tag{57}$$

is finite for each $k \in \mathbb{N}_0$. Here A_x^k, where $k = 0, 1, \ldots$, is k-th iterate of the differential operator (43) A_x. It is proved in [44] that the space $\mathcal{A}_{\nu,p}$ is a complete Frechet space.

Denoting by $\mathcal{D}(\mathbb{R}_+), \mathcal{E}(\mathbb{R}_+)$ customary spaces of testing functions encountered in distribution theory (see [36, 37]) it is easily seen that $\mathcal{D}(\mathbb{R}_+) \subset \mathcal{A}_{\nu,p} \subset \mathcal{E}(\mathbb{R}_+)$. Since $\mathcal{D}(\mathbb{R}_+)$ is dense in $\mathcal{E}(\mathbb{R}_+)$, we have that $\mathcal{A}_{\nu,p}$ is also dense in $\mathcal{E}(\mathbb{R}_+)$. As usual we denote by $\mathcal{A}'_{\nu,p}$ the dual of $\mathcal{A}_{\nu,p}$. It's equipped with the weak topology and represents a Hausdorff locally convex space of distributions. From the imbedding above we obtain that $\mathcal{E}'(\mathbb{R}_+) \subset \mathcal{A}'_{\nu,p}$. Since $\mathcal{A}_{\nu,p} \subset L_{\nu,p}(\mathbb{R}_+)$ we imbed the dual space $L_{1-\nu,q}(\mathbb{R}_+), \ q = \frac{p}{p-1}$ into $\mathcal{A}'_{\nu,p}$ as a subspace of regular distributions. They act upon elements φ from $\mathcal{A}_{\nu,p}$ according to

$$\langle f, \varphi\rangle := \int_0^\infty f(x)\varphi(x)dx. \tag{58}$$

The continuity of the linear functional (58) on $\mathcal{A}_{\nu,p}$ follows from the fact that if $\{\varphi_m\}_{m=1}^\infty$ converges in $\mathcal{A}_{\nu,p}$ to zero, then by the Hölder inequality

$$|\langle f, \varphi\rangle| \leq \alpha_{0,1-\nu,q}(f)\alpha_{0,\nu,p}(\varphi_m) \to 0, \ m \to \infty.$$

We note that this imbedding of $L_{1-\nu,q}(\mathbb{R}_+)$ into $\mathcal{A}'_{\nu,p}$ is one-to-one. Indeed, if two members f and g of $L_{1-\nu,q}(\mathbb{R}_+)$ become imbedded at the same element of $\mathcal{A}'_{\nu,p}$, then $\langle f, \varphi\rangle = \langle g, \varphi\rangle$ for every $\varphi \in \mathcal{D}(\mathbb{R}_+)$. But this will imply that $f = g$ almost everywhere on $\mathbb{R}_+$ (cf. in [36]). Finally from the general theory of continuous linear functionals on countably multinormed spaces follows that for each element $f \in \mathcal{A}'_{\nu,p}$ there exists a nonnegative integer r and a positive constant C such that

$$|\langle f, \varphi\rangle| \leq C\max_{0\leq k\leq r}\alpha_{k,\nu,p}(\varphi)$$

for every $\varphi \in \mathcal{A}_{\nu,p}$. Here r, C depends on f but not on φ.

The Kontorovich-Lebedev transform (56) has the following properties.

Theorem 4.1 [44]. *For each* $f \in \mathcal{A}'_{\nu,p}$ $\mathcal{KL}[f](s)$ *is analytic on the strip* $\Omega_\nu := \{s = \text{Re}s + i\tau, |\text{Re}s| < \nu\}$ *and its derivative*

$$F'(s) := \frac{d}{ds}\mathcal{KL}[f](s) = \left\langle f, \frac{\partial}{\partial s}K_s(\cdot)\right\rangle, \ s \in \Omega_\nu.$$

Furthermore, the following estimate is true

$$|\mathcal{KL}[f](s)| \leq C_{f,\delta,p,\nu} max\{1, |s|^{2r}\} e^{-(\pi/2-\delta)|\tau|}, \ s \in \Omega_\nu,$$

where $\delta \in \left(0, \frac{\pi}{2}\right], r \in \mathbb{N}$ *and* $C_{f,\delta,p,\nu} > 0$ *is a constant.*

Theorem 4.2 (inversion theorem). *Let* $f \in \mathcal{A}'_{\nu,p}$ *with* $0 < \nu < 1$. *Then*

$$f(x) = \lim_{\varepsilon \to 0+} \frac{i}{x\pi^2} \int_{\mu - i\infty}^{\mu + i\infty} s \sin(\pi - \varepsilon) s K_s(x) \mathcal{KL}[f](s) ds, \ |\mu| < \nu,$$

where the convergence is understood in $\mathcal{D}'(\mathbb{R}_+)$.

Corollary 4.1 (uniqueness property). *If* $\mathcal{KL}[f](s) = F(s)$ *and* $\mathcal{KL}[g](s) = G(s), \ s \in \Omega_\nu, \ 0 < \nu < 1$ *and if* $F(s) = G(s), s \in \Omega_\nu$ *then* $f = g$ *in the sense of equality in* $\mathcal{D}'(\mathbb{R}_+)$.

5. A convolution and related integral operators

In this section we will consider mapping properties of the convolution operator (21), which is associated with the Kontorovich-Lebedev transformation (1) by factorization equality (22). We will exhibit its properties in Lebesgue spaces with norms (13), (14), (15) (see, for example, Lemma 2.2) and will study integral operators of the convolution type (cf. refs. [11, 12, 30, 45, 46, 47, 48, 49]). The convolution (21) on spaces of distributions was considered in [42]. In the sequel, following [50] we will construct a convolution for adjoint operator (2).

Recently in ref. [30] it was proved a sharp inequality in L_p- spaces for the convolution (21) as a generalization of the corresponding inequality in ref. [47] for the L_2-case. We have

Theorem 5.1 [30]. *Let* $1 < p \leq \infty, \ f \in L_p(\mathbb{R}_+; xdx)$ *and* $g \in L^{\left(\frac{p-2}{p-1}, 1\right)}(\mathbb{R}_+)$. *Then convolution* (21) *exists as a Lebesgue integral for all* $x > 0$ *and belongs to the space* $L_p(\mathbb{R}_+; xdx)$. *Moreover, it satisfies the following inequality*

$$||f * g||_{L_p(\mathbb{R}_+;xdx)} \leq ||f||_{L_p(\mathbb{R}_+;xdx)} ||g||_{L^{\left(\frac{p-2}{p-1},1\right)}(\mathbb{R}_+)}. \tag{59}$$

In particular, for $p = 2$ we get (see ref. [47])

$$||f * g||_{L_2(\mathbb{R}_+;xdx)} \leq ||f||_{L_2(\mathbb{R}_+;xdx)} ||g||_{L^0(\mathbb{R}_+)}. \tag{60}$$

The Hausdorff-Young type theorem for convolution (21) when f, g belong to the conjugate spaces (14) is given by

Theorem 5.2 [30]. *Let* $1 < p < \infty, \ g \in L_p(\mathbb{R}_+; xdx), \ f \in L_q(\mathbb{R}_+; xdx), \ q = p/(p-1)$. *Then convolution* (21) *exists as a Lebesgue integral for all* $x > 0$ *and belongs to the space* $L_r(\mathbb{R}_+; x^r dx)$ *with* $1 \leq r < \frac{pq}{2|p-q|}$. *Furthermore it satisfies the inequality*

$$||f * g||_{L_r(\mathbb{R}_+;x^r dx)} \leq C_{r,p,q} ||f||_{L_q(\mathbb{R}_+;xdx)} ||g||_{L_p(\mathbb{R}_+;xdx)}, \tag{61}$$

where

$$C_{r,p,q} = \left(\int_0^\infty \left[K^{1/q}_{1-q/p}(x)K^{1/p}_{1-p/q}(x)\right]^r dx\right)^{1/r}. \tag{62}$$

For a fixed function g we denote by

$$S_g(x,u) = \frac{1}{2x}\int_0^\infty e^{-\frac{1}{2}\left(x\frac{u^2+y^2}{uy}+\frac{yu}{x}\right)} g(y)dy \tag{63}$$

the kernel of the following convolution operator

$$(S_g f)(x) = \int_0^\infty S_g(x,u)f(u)du, \ x > 0. \tag{64}$$

As a consequence of Theorem 5.1 we establish boundedness properties of the operator (64) in the space $L_p(\mathbb{R}_+; xdx)$, $1 < p \leq \infty$. Thus we have

Theorem 5.3 [30]. *Let* $1 < p \leq \infty$, *and* $g \in L^{\left(\frac{p-2}{p-1},1\right)}(\mathbb{R}_+)$. *Then integral operator* (64) *is bounded in the space* $L_p(\mathbb{R}_+; xdx)$ *and its norm* $||S_g|| \leq ||g||_{L^{\left(\frac{p-2}{p-1},1\right)}(\mathbb{R}_+)}$. *If, in turn,* $g(x)$ *is a positive function on* $\mathbb{R}_+$ *and* $p \in \left(1+\frac{1}{\sqrt{3}}, \infty\right)$, *then* $||S_g|| = ||g||_{L^{\left(\frac{p-2}{p-1},1\right)}(\mathbb{R}_+)}$.

Let us consider two examples of convolution operator (64) (cf. [11], [12]) with the corresponding kernels (63), which are calculated for concrete functions g. If, for instance, we put $g(x) \equiv 1$ then we calculate integral (63) and we arrive at the following integral operator

$$(\mathcal{K}f)(x) = \int_0^\infty \frac{K_1(\sqrt{x^2+u^2})}{\sqrt{x^2+u^2}} uf(u)du, \ x > 0. \tag{65}$$

It is easily seen from (6), (7), (13) that $g(x) \equiv 1$ belongs to the space $L^{\left(\frac{p-2}{p-1},1\right)}(\mathbb{R}_+)$ if and only if $p \in \left(\frac{3}{2}, \infty\right)$. Consequently, appealing to Theorem 5.3 via relation (2.16.2.1) from [14] we find that (65) is a bounded operator in $L_p(\mathbb{R}_+; xdx)$, $p \in \left(\frac{3}{2}, \infty\right)$ and has a least value of its norm when $p \in \left(1+\frac{1}{\sqrt{3}}, \infty\right)$, namely

$$||\mathcal{K}|| = \frac{\pi}{2\cosh\left(\frac{\pi}{2}\frac{p-2}{p-1}\right)}, \ p \in \left(1+\frac{1}{\sqrt{3}}, \infty\right).$$

When, in turn, we put $g(x) = \frac{e^{-x}}{\sqrt{x}}$ then calculating the corresponding integral (63) and taking into account, that the modified Bessel function of the index $\frac{1}{2}$ reduces to

$$K_{1/2}(z) = e^{-z}\sqrt{\frac{\pi}{2z}}$$

(see [5, Vol. 2]), we arrive at the Lebedev convolution operator (cf. [11], [12])

$$(Lef)(x) = \sqrt{\frac{\pi}{2x}}\int_0^\infty \frac{e^{-x-u}}{x+u}\sqrt{u}f(u)du, \ x > 0. \tag{66}$$

In this case $g(x)=\frac{e^{-x}}{\sqrt{x}}$ belongs to the space $L^{\left(\frac{p-2}{p-1},\,1\right)}(\mathbb{R}_+)$ if and only if $p\in(\frac{5}{3},3)$. Moreover, using relation (2.16.6.4) from [14] and Theorem 5.3 we obtain that the Lebedev operator (66) is bounded in $L_p(\mathbb{R}_+;xdx),\ p\in(\frac{5}{3},3)$ and we have

$$||Le||=\pi\sqrt{\frac{\pi}{2}}\frac{1}{\cosh\left(\pi\frac{p-2}{p-1}\right)},\ p\in\left(\frac{5}{3},3\right).$$

Further, according to [50] we construct a convolution for the adjoint operator (2). It has the form

$$\left(f\overset{\alpha}{*}g\right)(t)=\frac{2}{\pi^2}t\sinh\pi t\int_0^\infty\int_0^\infty f(\tau)g(\theta)\Omega_\alpha(t,\tau,\theta)d\tau d\theta, \tag{67}$$

where

$$\Omega_\alpha(t,\tau,\theta)=\int_0^\infty x^{\alpha-1}K_{i\tau}(x)K_{i\theta}(x)K_{it}(x)dx,\quad \alpha,\,t,\,\tau,\,\theta\in\mathbb{R}_+. \tag{68}$$

The kernel (68) can be calculated in terms of the hypergeometric function ${}_4F_3$ employing relation (2.16.46.6) from [14]. Thus we obtain

$$\begin{aligned}\Omega_\alpha(t,\tau,\theta) &= 2^{\alpha-3}\,\mathrm{Re}\Bigg[\Gamma(it)\Gamma(\alpha-it)\\ &\times\ B\left(\frac{\alpha}{2}+i\frac{\tau+\theta-t}{2},\frac{\alpha}{2}-i\frac{\tau+\theta+t}{2}\right)B\left(\frac{\alpha}{2}-i\frac{\tau-\theta+t}{2},\frac{\alpha}{2}+i\frac{\tau-\theta-t}{2}\right)\\ &\times\ {}_4F_3\left(\frac{\alpha}{2}+i\frac{\tau+\theta-t}{2},\frac{\alpha}{2}-i\frac{\tau+\theta+t}{2},\frac{\alpha}{2}-i\frac{\tau-\theta+t}{2},\frac{\alpha}{2}+i\frac{\tau-\theta-t}{2};\right.\\ &\left.\left.1-it,\frac{\alpha-it}{2},\frac{1+\alpha-it}{2};\frac{1}{4}\right)\right].\end{aligned}$$

The main result is given by

Theorem 5.4 **[50]**. *Let $\alpha>0,\ f,g\in L_2(\mathbb{R}_+;[\tau\sinh\nu\tau]^{-1}d\tau)$ with $0<\nu<\pi$. Then convolution (67) exists as a Lebesgue integral and is continuous on $\mathbb{R}_+$. Moreover, it belongs to $L_2(\mathbb{R}_+;[\tau\sinh\pi\tau]^{-1}d\tau)$ and the following factorization equality is valid*

$$x^\alpha(KLf)(x)(KLg)(x)=\underset{N\to\infty}{\mathrm{l.i.m.}}\int_0^N\left(f\overset{\alpha}{*}g\right)(t)K_{it}(x)dt,$$

where the limit is with respect to the norm in $L_2(\mathbb{R}_+;x^{-1}dx)$. Besides, the Parseval equality of type

$$\int_0^\infty\left|\left(f\overset{\alpha}{*}g\right)(t)\right|^2\frac{dt}{t\sinh\pi t}=\frac{2}{\pi^2}\int_0^\infty x^{2\alpha-1}|(KLf)(x)(KLg)(x)|^2dx$$

holds true.

Next we exhibit a recently obtained pair of integral transformations (see refs. [51], [52]), which is associated with the product of the modified Bessel functions of different arguments and related to the class of the Kontorovich-Lebedev type transforms (cf. [53]). Namely, we will consider the following reciprocal formulas

$$F(x,y)=\frac{2}{\sqrt{\pi}}\int_0^\infty \tau K_{i\tau}\left(\sqrt{x^2+y^2}-y\right)K_{i\tau}\left(\sqrt{x^2+y^2}+y\right)f(\tau)d\tau, \tag{69}$$

$$f(\tau)=\frac{8\sinh 2\pi\tau}{\pi^3\sqrt{\pi}}\int_0^\infty\int_0^\infty K_{i\tau}\left(\sqrt{x^2+y^2}-y\right)K_{i\tau}\left(\sqrt{x^2+y^2}+y\right)F(x,y)\frac{dx\,dy}{x}. \tag{70}$$

It is shown, that (69) is one-to-one transformation between two Lebesgue spaces

$$F: L_2\left(\mathbb{R}_+;\frac{\tau d\tau}{\sinh 2\pi\tau}\right)\leftrightarrow L_2\left(\mathbb{R}_+\times\mathbb{R}_+;\frac{dx\,dy}{x}\right),$$

and has the inversion (70). The convergence of the integrals (69), (70) is in the mean square sense. This result is summarized by the corresponding Plancherel theorem.

Theorem 5.5 [52]. *Let* $f\in L_2(\mathbb{R}_+;\tau[\sinh 2\pi\tau]^{-1}d\tau)$. *Then, as* $N\to\infty$, *the integral*

$$F_N(x,y)=\frac{2}{\sqrt{\pi}}\int_{\frac{1}{N}}^N \tau K_{i\tau}\left(\sqrt{x^2+y^2}-y\right)K_{i\tau}\left(\sqrt{x^2+y^2}+y\right)f(\tau)d\tau$$

converges in mean to $F(x,y)$ *with respect to the norm of* $L_2\left(\mathbb{R}_+\times\mathbb{R}_+;\frac{dx\,dy}{x}\right)$, *and*

$$f_N(\tau)=\frac{8}{\pi^3\sqrt{\pi}}\sinh 2\pi\tau\int_{\frac{1}{N}}^N\int_{\frac{1}{N}}^N K_{i\tau}\left(\sqrt{x^2+y^2}-y\right)K_{i\tau}\left(\sqrt{x^2+y^2}+y\right)F(x,y)\frac{dx\,dy}{x}$$

converges in mean to $f(\tau)$ *with respect to the norm of* $L_2(\mathbb{R}_+;\tau[\sinh 2\pi\tau]^{-1}d\tau)$. *Moreover, almost for all* $\tau\in\mathbb{R}_+$ *and* $(x,y)\in\mathbb{R}_+\times\mathbb{R}_+$, *respectively, the following reciprocal formulas take place:*

$$f(\tau)=\frac{8}{\pi^3\sqrt{\pi}}\frac{\sinh 2\pi\tau}{\tau}\frac{d}{d\tau}\int_0^\infty\int_0^\infty\int_0^\tau \xi K_{i\xi}\left(\sqrt{x^2+y^2}-y\right)$$
$$\times K_{i\xi}\left(\sqrt{x^2+y^2}+y\right)F(x,y)\,d\xi\frac{dx\,dy}{x},$$

$$F(x,y)=\frac{2}{\sqrt{\pi}}\frac{\partial}{\partial x\partial y}\int_0^\infty\int_0^x\int_0^y \tau K_{i\tau}\left(\sqrt{u^2+v^2}-v\right)$$
$$\times K_{i\tau}\left(\sqrt{u^2+v^2}+v\right)f(\tau)\,du\,dv\,d\tau.$$

Finally, for all $f_1,f_2\in L_2\left(\mathbb{R}_+;\frac{\tau d\tau}{\sinh 2\pi\tau}\right)$ *and the corresponding* $F_1,F_2\in L_2\left(\mathbb{R}_+\times\mathbb{R}_+;\frac{dx\,dy}{x}\right)$ *the Plancherel identity*

$$\int_0^\infty\int_0^\infty F_1(x,y)\overline{F_2(x,y)}\frac{dx\,dy}{x} = \frac{\pi^3}{4}\int_0^\infty \frac{\tau}{\sinh 2\pi\tau} f_1(\tau)\overline{f_2(\tau)}d\tau.$$

holds. In particular for $f_1 \equiv f_2$ it takes the form of Parseval's equality

$$\int_0^\infty\int_0^\infty |F(x,y)|^2\frac{dx\,dy}{x} = \frac{\pi^3}{4}\int_0^\infty \frac{\tau}{\sinh 2\pi\tau} |f(\tau)|^2 d\tau.$$

Another pair of reciprocal transformations, which is related to the Kontorovich-Lebedev transform contains a product of Euler's gamma-functions (see refs. [32], [54]) (the so-called Gamma -product transform). Precisely, we introduce the reciprocities

$$[\mathcal{G}f](x) = \text{l.i.m.}_{N\to\infty}\text{P.V.}\int_0^N \Gamma\left(i(x+\tau)\right)\Gamma\left(i(x-\tau)\right)f(\tau)d\tau,\ x\in\mathbb{R}, \tag{71}$$

$$f(\tau) = \text{l.i.m.}_{N\to\infty}\text{P.V.}\frac{1}{4\pi^2|\Gamma(2i\tau)|^2}\int_{-N}^N \Gamma\left(-i(x+\tau)\right)\Gamma\left(i(\tau-x)\right)[\mathcal{G}f](x)dx. \tag{72}$$

Integrals under the limit sign in (71), (72) are understood in the Cauchy principal value sense. So we have the following Plancherel theorem for the pair (71), (72).

Theorem 5.6 [54]. *Let $f(\tau) \in L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$. The Gamma-product transform* (71), *where the integral converges in mean with respect to the norm in $L_2(\mathbb{R}; dx)$ forms the isomorphism*

$$\mathcal{G} : L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau) \leftrightarrow L_2(\mathbb{R}; dx),$$

where the reciprocal inverse operator is given by formula (72) *with the convergence with respect to the norm in $L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$. If*

$$f, g \in L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$$

then the Plancherel identity holds, i.e.,

$$\int_{-\infty}^\infty [\mathcal{G}f](x)\overline{[\mathcal{G}g](x)}dx = 4\pi^2\int_0^\infty f(\tau)\overline{g(\tau)}|\Gamma(2i\tau)|^2 d\tau. \tag{73}$$

In particular, we have the Parseval equality

$$\int_{-\infty}^\infty |[\mathcal{G}f](x)|^2\,dx = 4\pi^2\int_0^\infty |f(\tau)|^2|\Gamma(2i\tau)|^2 d\tau. \tag{74}$$

Let us introduce the function

$$\Phi_f(z) = \int_0^\infty \Gamma\left(z+i\tau\right)\Gamma\left(z-i\tau\right)f(\tau)d\tau, \tag{75}$$

where $z = \alpha + ix$, $\alpha > 0$, $x \in \mathbb{R}$. In [54] it was described a class of functions $f \in L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$ having the corresponding Gamma-product transform (71) $[\mathcal{G}f](x) \in L_2(\mathbb{R}; dx)$

as a limit almost everywhere as $\alpha \to 0+$ of the function $\Phi_f(z)$ from the Hardy space $\mathbb{H}_2$. This means that Φ_f is analytic in the right half-plane $\alpha > 0$ and satisfies the condition

$$\sup_{\alpha>0} \int_{-\infty}^{\infty} |\Phi_f(\alpha + ix)|^2 dx < \infty.$$

Theorem 5.7 [54]. *Let $f \in L_2(\mathbb{R}_+; |\Gamma(2i\tau)|^2 d\tau)$. The Gamma-product transform* (71) *is the limit of $\Phi_f(\alpha + ix) \in \mathbb{H}_2$ almost everywhere as $\alpha \to 0+$ if and only if the modified adjoint Kontorovich- -Lebedev transformation*

$$(KL_m)(x) = l.i.m._{N\to\infty} \int_0^N K_{2i\tau}(2\sqrt{x}) f(\tau) d\tau, \quad x > 0,$$

is equal to zero for almost all $x > 1$.

6. Uncertainty principles

The main aim of this section is to establish the so-called uncertainty principles for the Kontorovich-Lebedev operator (1), which say that a nonzero original and its image under transformation (1) cannot be simultaneously too small in the pointwise or integrable decay. This comes as a generalization of the classical Heisenberg uncertainty principle. It was extended to the Fourier transform in refs. [56, 57, 58, 59]. The corresponding principles have been proved also for the Y-transform [60], the Dunkl transform [61] and recently for the Hankel transform [62]. So we will state here the Hardy, Beurling, Cowling-Price, Gelfand-Shilov and Donoho-Stark uncertainty principles for the Kontorovich-Lebedev transformation (1).

As t is known, Hardy's classical theorem for the Fourier transform [58], [63] says that if $|f(t)| \le Ce^{-at^2}$ and $|F_c(x)| \le Ce^{-\frac{x^2}{4a}}$, $a > 0$, then $f(t)$ is a multiple of e^{-at^2}. Here $C > 0$ is a universal constant, which is different in distinct places and $F_c(x)$ is the cosine Fourier transform (25). Let us suppose that transformation (1) admits the following series expansion with respect to an index of the modified Bessel functions

$$K_{i\tau}[f] = \frac{C}{\cosh(\pi\tau/2)} \sum_{n=0}^{\infty} \alpha_n \left[K_{i(\frac{\tau}{2}+n)}\left(\frac{a}{2}\right) + K_{i(\frac{\tau}{2}-n)}\left(\frac{a}{2}\right) \right], \ a > 0, \tag{76}$$

where $\sum_{n=0}^{\infty} |\alpha_n| < \infty$. We have

Theorem 6.1 [55]. *Let $K_{i\tau}[f]$ satisfy* (76) *and $|f(x)| \le Ce^{-\frac{x^2}{4a}}$. Then $f(x)$ is a multiple of $e^{-\frac{x^2}{4a}}$.*

Corollary 6.1. *Under conditions of Theorem 6.1*

$$K_{i\tau}[f] = C \ \mathrm{sech}(\pi\tau/2) K_{i\tau/2}\left(\frac{a}{2}\right) = O(e^{-\frac{3\pi}{4}\tau}), \quad \tau \to +\infty.$$

As a consequence we are ready to state an analog of the Hardy uncertainty principle for the Kontorovich-Lebedev transformation (1).

Corollary 6.2 [55]. *Under conditions of Theorem 6.1 let* $|f(x)| \le Ce^{-bx^2},\ b > \frac{1}{4a}$. *Then* $f(x) = 0$.

This principle can be formulated in terms of the composition $F_c \circ (\cosh(\pi\tau/2)K_{i\tau}[f])(x)$. Precisely, we have

Corollary 6.3. *One cannot have both* $|f(x)| \le Ce^{-ax^2},\ a > 0$ *and*

$$F_c \circ (\cosh(\pi\tau/2)K_{i\tau}[f])(x) \le Ce^{-b\sinh^2 x},\ b > 0,$$

where $ab > \frac{1}{4}$ *unless* $f(x) = 0$.

As a consequence of Theorem 6.1 and Corollary 6.1 we get

Corollary 6.4. *Let* $|f(x)| \le Ce^{-ax^2},\ a > 0$ *and*

$$|F_c \circ (\cosh(\pi\tau/2)K_{i\tau}[f])(x)| \le Ce^{-b\sinh^2 x},\ b > 0,$$

where $0 < ab \le \frac{1}{4}$. *If* $|K_{i\tau}[f]| \le Ce^{-c\tau},\ \tau > 0, c > \frac{3\pi}{4}$, *then* $f(x) = 0$.

The Beurling condition related to the cosine Fourier transform $F_c(x)$ (25) of f says (cf. [59]), that if

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} |f(y)F_c(x)|e^{xy}dxdy < \infty, \tag{77}$$

then $f = 0$. An analog of the Beurling theorem for the Kontorovich-Lebedev transformation (1) is the following statement.

Theorem 6.2 [55]. *Let*

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} |f(x)K_{i\tau}[f]|K_\tau(x)d\tau dx < \infty. \tag{78}$$

Then $f = 0$.

The Gelfand-Shilov uncertainty principle [57] has the form: if

$$\int_{\mathbb{R}_+} |f(y)|e^{(ay)^p/p}dy < \infty, \quad \int_{\mathbb{R}_+} |F_c(x)|e^{(bx)^q/q}dx < \infty, \tag{79}$$

with $1 < p, q < \infty,\ p^{-1} + q^{-1} = 1$ and $ab > 1/4$, then $f = 0$.

We have accordingly for the Kontorovich-Lebedev transform

Theorem 6.3 [55]. *Let* $1 < p, q < \infty,\ p^{-1} + q^{-1} = 1,\ [q]$ *be an integer part of* q *and*

$$\int_{\mathbb{R}_+} |f(x^2)|e^{\frac{(2([q]+1))!}{4x^2}}dx < \infty, \quad \int_{\mathbb{R}_+} |K_{i\tau}[f]|e^{\tau^p/p}d\tau < \infty. \tag{80}$$

Then $f = 0$.

Next we exhibit the Cowling -Price theorem for the Kontorovich-Lebedev transform (1). This will be an analog of the following result for the Fourier transform (25) (cf. [56]): if $1 \leq p, q < \infty$ and

$$\left|\left|e^{ax^2} f(x)\right|\right|_{L_p(\mathbb{R}_+;dx)} + \left|\left|e^{b\lambda^2} F_c(\lambda)\right|\right|_{L_q(\mathbb{R}_+;dx)} < \infty$$

with $ab > 1/4$, then $f = 0$.

We have

Theorem 6.4 [55]. *If*

$$\left|\left|e^{a\tau^2} K_{i\tau}[f]\right|\right|_{L_p(\mathbb{R}_+;d\tau)} < \infty, \quad \left|\left|e^{6b^2/x^2} f(x^2)\right|\right|_{L_1(\mathbb{R}_+;dx)} < \infty, \tag{81}$$

where $p \in [1, \infty)$ *and* $ab > 1/4$, *then* $f = 0$.

As it is known in Section 3, when $f \in L_2(\mathbb{R}_+; xdx)$, then $K_{i\tau}[f] \in L_2(\mathbb{R}_+; \tau \sinh \pi\tau d\tau)$ and vice versa. Moreover, by virtue of (42) $||K_{i\tau}[f]||_{L_2(\mathbb{R}_+;\tau \sinh \pi\tau d\tau)} = \frac{\pi}{\sqrt{2}} ||f||_{L_2(\mathbb{R}_+;xdx)}$ and the Kontorovich-Lebedev reciprocal integrals can be interpreted in the mean convergence sense by formulas (38), (39).

Let $\mathbb{X} = [0, X]$, $\mathbb{Y} = [1/Y, Y]$ the Lebesgue measurable sets, $|\mathbb{X}|$, $|\mathbb{Y}|$ be their Lebesgue measures and $g(x) = K_{ix}[f]$. Denoting by $P_{\mathbb{X}}$ the operator

$$(P_{\mathbb{X}} g)(x) = \begin{cases} g(x), & \text{if } x \in \mathbb{X}, \\ 0, & \text{if } x \notin \mathbb{X}, \end{cases}$$

we have

$$||g - P_{\mathbb{X}} g||_{L_2(\mathbb{R}_+;x \sinh \pi x dx)} \leq \varepsilon_{\mathbb{X}},$$

and this means that g is $\varepsilon_{\mathbb{X}}$-concentrated on the set $\mathbb{X}$. Plainly $||P_{\mathbb{X}}|| = 1$. Another auxiliary operator is given by the formula

$$(Q_{\mathbb{Y}} g)(x) = \int_{\mathbb{Y}} K_{ix}(y) f(y) dy,$$

where f is the reciprocal inverse Kontorovich-Lebedev transform (39). If $h = Q_{\mathbb{Y}} g$ the transform (39) $\hat{h}(y)$ is equal to

$$\hat{h}(y) = \begin{cases} f(y), & \text{if } y \in \mathbb{Y}, \\ 0, & \text{if } y \notin \mathbb{Y}. \end{cases}$$

Meanwhile by Parseval's equality (42) we find

$$\left|\left|f - \hat{h}\right|\right|_{L_2(\mathbb{R}_+;ydy)} = \frac{\sqrt{2}}{\pi} ||g - Q_{\mathbb{Y}} g||_{L_2(\mathbb{R}_+;x \sinh \pi x dx)}, \tag{82}$$

and f is ε-concentrated on $\mathbb{Y}$ if, and only if, $||g - Q_{\mathbb{Y}} g||_{L_2(\mathbb{R}_+;x \sinh \pi x dx)} \leq \varepsilon$. Moreover, one can show that $||Q_{\mathbb{Y}}|| = 1$.

Now we are ready to formulate an analog of the Donoho-Stark uncertainty principle (cf. [64])

Theorem 6.5 [55]. *Let g is $\varepsilon_{\mathbb{X}}$-concentrated on $\mathbb{X} = [0, X]$ and its Kontorovich-Lebedev reciprocity f is $\varepsilon_{\mathbb{Y}}$-concentrated on $\mathbb{Y} = [1/Y, Y]$. Then*

$$|\mathbb{X}|^{3/2}\ |\mathbb{Y}| \geq \frac{\pi^{7/4}\sqrt{24}}{\Gamma^2(1/4)}(1 - (\varepsilon_{\mathbb{X}}^2 + \varepsilon_{\mathbb{Y}}^2)^{1/2})^2.$$

7. Lebedev- Stieltjes integrals

The aim of this final section is to expand the Bochner technique [25] given for the Fourier-Stieltjes integrals on the following integral

$$F(x) = \int_{\mathbb{R}} \frac{K_{i\tau}(e^x)}{|\Gamma(i\tau)|}\, dV(\tau),\ x \in \mathbb{R}. \tag{83}$$

Here $V(\tau)$ is a distribution function in the Bochner sense [25], which is bounded and monotone increasing on $\mathbb{R}$, and satisfies everywhere the following equality

$$V(\tau) = \frac{1}{2}[V(\tau + 0) + V(\tau - 0)].$$

Integral (83) is clearly related to the adjoint Kontorovich-Lebedev operator (2). Indeed, putting $V(\tau) = \int_{-\infty}^{\tau} f(t)dt$, where f is nonnegative and belongs to $L_1(\mathbb{R}; dt)$, $y = e^x$, $\psi(y) = F(\log y)$ and taking into account the value of the modulus of the gamma- function $|\Gamma(i\tau)| = \sqrt{\frac{\pi}{\tau \sinh \pi\tau}}$, integral (83) becomes

$$\psi(y) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \sqrt{\tau \sinh \pi\tau} K_{i\tau}(y) f(\tau) d\tau,\ y > 0,$$

which corresponds to the Kontorovich-Lebedev transformation (2). We note here two inequalities for the modified Bessel functions (cf. refs. [3], [65]), which we will use below. Precisely, for all $x > 0$ and $\tau \in \mathbb{R}$ it satisfies

$$\left| \frac{K_{i\tau}(x)}{\Gamma(i\tau)} \right| \leq e^x, \tag{84}$$

$$\left| \frac{K_{i\tau}(x)}{\Gamma(i\tau)} \right| \leq C \frac{\sqrt{|\tau|}}{x^{1/4}}, \tag{85}$$

where $C > 0$ is an absolute constant. In fact, appealing to the inequality (84) we immediately obtain the estimates

$$\left| \int_{-\infty}^{a} \frac{K_{i\tau}(e^x)}{|\Gamma(i\tau)|}\, dV(\tau) \right| \leq e^{e^x} \left[V(a) - V(-\infty) \right], \tag{86}$$

$$\left|\int_b^\infty \frac{K_{i\tau}(e^x)}{|\Gamma(i\tau)|}\, dV(\tau)\right| \leq e^{e^x}\left[V(\infty) - V(b)\right], \tag{87}$$

for any $b \in \mathbb{R}$. This means that for each distribution function $V(\tau)$ integral (83) exists for all $x \in \mathbb{R}$. Moreover, we have

$$|F(x)| \leq e^{e^x}\left[V(\infty) - V(-\infty)\right]. \tag{88}$$

Consequently, $F(x)$ is a real bounded function on any bounded set of $\mathbb{R}$. If we put

$$F_n(x) = \int_{-n}^n \frac{K_{i\tau}(e^x)}{|\Gamma(i\tau)|}\, dV(\tau)$$

then (see (86), (87))

$$|F(x) - F_n(x)| \leq e^{e^x}\left[V(\infty) - V(n) + V(-n) - V(-\infty)\right],$$

which tends to 0 when $n \to \infty$ on any compact set of $\mathbb{R}$. Thus the sequence $\{F_n(x)\}_{n=1}^\infty$ converges uniformly to $F(x)$ on any compact set and represents there a continuous functions.

Modifying the inversion theorem for the Lebedev integrals from [34], we state the corresponding result for absolutely continuous distributions. This means that $V(\tau)$ can be represented by the indefinite integral of a nonnegative summable function $s(t)$, i.e.

$$V(\tau) = \int_a^\tau s(t)dt, \quad a \in \mathbb{R}. \tag{89}$$

Thus we have

Theorem 7.1 [66]. *For a class of distributions* (89) *the Lebedev integral* (83) *has the following inversion*

$$\frac{1}{2}\left[V(\tau) - V(-\tau) + V(-a)\right] = \lim_{\varepsilon \to 0+} \int_{-\infty}^\infty e^{\varepsilon x} \mathcal{K}(\tau, x) F(x) dx,$$

where

$$\mathcal{K}(\tau, x) = \frac{1}{\pi}\int_a^\tau \frac{K_{iy}(e^x)}{|\Gamma(iy)|} dy.$$

As it is known [25], the distribution function $V(\tau)$ has at most a countable set of discontinuous points, which we denote by $\lambda_0,\ \lambda_1,\ \lambda,\ \dots, \lambda_\nu,\ \dots$, and the corresponding jumps by a_ν. Hence as usual $a_\nu = V(\lambda_\nu + 0) - V(\lambda_\nu - 0)$ and $\sum_\nu a_\nu \leq V(\infty) - V(-\infty)$. Furthermore, let us assume that there exists an equivalent odd distribution function [25], which differs from $V(\tau)$ on a constant and which we will denote again as $V(\tau)$. Then integral (83) can be written as

$$F(x) = 2\int_{\mathbb{R}_+} \frac{K_{i\tau}(e^x)}{|\Gamma(i\tau)|}\, dV(\tau),\ x \in \mathbb{R}. \tag{90}$$

Meanwhile, $V(\tau)$ can be represented as a sum of two distribution functions, namely

$$V(\tau) = S(\tau) + D(\tau), \tag{91}$$

where $S(\tau)$ is continuous and $D(\tau)$ is a jump function of the distribution $V(\tau)$. $D(\tau)$ is defined as follows: at each point τ, where $V(\tau)$ is continuous the value of $D(\tau)$ is equal to the sum of jumps of $V(\tau)$ from the left of τ, i.e.

$$D(\tau) = \sum_{\lambda_\nu < \tau} a_\nu.$$

Hence we immediately obtain

$$D(\lambda_\nu + 0) - D(\lambda_\nu - 0) = V(\lambda_\nu + 0) - V(\lambda_\nu - 0).$$

When $V(\tau)$ is continuous, then $D(\tau) \equiv 0$. Another least case takes place when $S(\tau) = \text{const}$.

Denoting by $\varphi_\tau(x) = 2\frac{K_{i\tau}(e^x)}{|\Gamma(i\tau)|}$ we easily find that $\varphi_0(x) = 0$ for all $x \in \mathbb{R}$. Taking into account (90) we write the Lebedev-Stieltjes integral (83) as $F(x) = G(x) + h(x)$, where

$$G(x) = \int_{\mathbb{R}_+} \varphi_\tau(x)\ dD(\tau), \tag{92}$$

$$h(x) = \int_{\mathbb{R}_+} \varphi_\tau(x)\ dS(\tau). \tag{93}$$

Let us consider function (92). By the properties of the Stieltjes integral it can be written in terms of series

$$G(x) = \sum_\nu a_\nu \varphi_{\lambda_\nu}(x). \tag{94}$$

Conversely one can show that each series of type (94) with positive a_ν such that $\sum_\nu a_\nu$ is convergent, represents a function of type (92). Moreover, $D(\tau)$ should be defined as follows: if τ is different from λ_ν for any ν then (91) holds. Otherwise we have the equality

$$D(\tau) = \frac{1}{2}\left[D(\tau + 0) + D(\tau - 0)\right].$$

We will appeal now at the Bohr type mean value

$$\mathcal{M}\{f(x)\} = \lim_{\omega\to\infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} f(x)dx.$$

Lemma 7.1 [66]. *The uncountable system of functions* $\varphi_{\lambda_n}(x) = 2\frac{K_{i\lambda_n}(e^x)}{|\Gamma(i\lambda_n)|},\ \lambda_n \in \mathbb{R}_+,\ n \in \mathbb{N}_0$ *is orthonormal in a sense that*

$$\mathcal{M}\{\varphi_{\lambda_n}(x)\varphi_{\lambda_m}(x)\} = 0, \quad \lambda_n \neq \lambda_m, \tag{95}$$

$$\mathcal{M}\{\varphi^2_{\lambda_n}(x)\} = 1, \quad \lambda_n > 0, \tag{96}$$

hold true.

This lemma allows us to consider Fourier type series (94) with respect to the system $\varphi_{\lambda_n}(x) = 2\frac{K_{i\lambda_n}(e^x)}{|\Gamma(i\lambda_n)|}$ in the Bohr type pre-Hilbert space [25]) equipped with the inner product

$$\mathcal{M}\{f(x)g(x)\} = \lim_{\omega\to\infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} f(x)\overline{g(x)}dx.$$

As it is known [67], the corresponding Hilbert space contains the space of almost periodic functions. Here below we will establish in a similar manner the fundamental results for the so-called spectral decompositions of the Lebedev- Stieltjes integrals $F(x)$ in terms of the Fourier type series (94).

Theorem 7.2 [66]. *Let the sequence $\{\lambda_\nu\}_0^\infty$ be with distinct positive numbers and the series* (94) *be with non zero coefficients such that $\sum_\nu a_\nu < \infty$. Then this series is the Fourier series of its sum $G(x)$.*

Next we consider the following formula

$$\mathcal{M}\left\{\left|F(x) - \sum_{n=0}^{N} c_n \varphi_{\lambda_n}(x)\right|^2\right\} = \mathcal{M}\{|F(x)|^2\} - \sum_{n=0}^{N}[a(\lambda_n)]^2 + \sum_{n=0}^{N} |c_n - a(\lambda_n)|^2,$$

where $F(x)$ is defined by (90), $a(\lambda_n)$ are Fourier coefficients of $G(x)$

$$a_\nu \equiv a(\lambda_\nu) = \mathcal{M}\{G(x)\varphi_{\lambda_\nu}(x)\}, \quad \nu \in \mathbb{N}_0, \tag{97}$$

and c_n are arbitrary complex numbers. This formula can be easily proved by using Lemma 7.1, the properties of the inner product and the Bohr mean value. If the numbers $a(\lambda_n)$ are chosen for the constants c_n, the there follows the formula

$$\mathcal{M}\left\{\left|F(x) - \sum_{n=0}^{N} a(\lambda_n)\varphi_{\lambda_n}(x)\right|^2\right\} = \mathcal{M}\{|F(x)|^2\} - \sum_{n=0}^{N}[a(\lambda_n)]^2.$$

This immediately yields the Bessel type inequality

$$\sum_{n=0}^{N}[a(\lambda_n)]^2 \leq \mathcal{M}\{|F(x)|^2\}. \tag{98}$$

Assuming that the right-hand side of (98) is finite, we observe from (98) by similar discussions as for Fourier coefficients of the almost periodic functions (cf. [67]) that $a(\lambda)$ is zero for all $\lambda > 0$ with the exception of an at most enumerable set of values of positive λ, which we denote by $\lambda_0, \lambda_1, \lambda_2, \dots$. Thus the series in (98) when $N \to \infty$ is convergent and we have

$$\sum_{n=0}^{\infty}[a(\lambda_n)]^2 \leq \mathcal{M}\{|F(x)|^2\}. \tag{99}$$

It is proved in [66] that for the class of the Lebedev- Stieltjes integrals $F(x)$ the equality sign always holds in (99), i.e. we will establish the Parseval equality for this class of functions

$$\mathcal{M}\{|F(x)|^2\} = \sum_{n=0}^{\infty}[a(\lambda_n)]^2. \tag{100}$$

Indeed, we have

Lemma 7.2 [66]. *Under conditions of Theorem* 7.2 *each function* $F(x)$*, which is defined by the Lebedev- Stieltjes integral* (90) *satisfies the equality*

$$a(\lambda_\nu) = \mathcal{M}\{F(x)\varphi_{\lambda_\nu}(x)\}, \quad \nu \in \mathbb{N}_0,$$

where $a(\lambda_\nu)$ *are given by* (97).

Theorem 7.3 [66]. *A function* $F(x)$*, which is represented by the Lebedev- Stieltjes integral* (90) *can be decomposed in the Fourier type series in the Bohr type pre-Hilbert space*

$$F(x) = 2\sum_{n=0}^{\infty} a_n \frac{K_{i\lambda_n}(e^x)}{|\Gamma(i\lambda_n)|}, \ \lambda_n > 0, \ x \in \mathbb{R}, \tag{101}$$

with positive coefficients a_n *and the convergent sum* $\sum_n a_n$. *Series* (101) *converges in the mean to* $F(x)$ *and the Parseval equality*

$$\mathcal{M}\{|F(x)|^2\} = \sum_{n=0}^{\infty} a_n^2$$

holds.

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