Maximal dense ideal extensions of locally inverse semigroups

F. J. Pastijn and L. Oliveira*

Abstract

Every locally inverse semigroup has, within the class of all locally inverse semigroups, a maximal dense ideal extension.

1 Introduction

We shall use [3, 8, 9] as general references. The basics on locally inverse semigroups may be found in Section 6.1 of [3], and in [5, 6]. [7, 8] give a background on dense ideal extensions. We recall some of the definitions in what follows.

Let $\mathbf{C}$ be a class of semigroups. An **ideal extension** of a semigroup S is an injective homomorphism $\varphi : S \longrightarrow T$ where $S\varphi$ is an ideal of the semigroup T. We shall say that this ideal extension is **within** $\mathbf{C}$ if both S and T belong to $\mathbf{C}$. The ideal extensions $\varphi_1 : S \longrightarrow T_1$ and $\varphi_2 : S \longrightarrow T_2$ are said to be **equivalent** if there exists an isomorphism $\psi : T_1 \longrightarrow T_2$ which extends $\varphi_1^{-1}\varphi_2$. If $\varphi : S \longrightarrow T$ is an ideal extension within $\mathbf{C}$ then we shall say that this is a **dense ideal extension within** $\mathbf{C}$ if whenever $\psi : T \longrightarrow U$ is a homomorphism such that $\varphi\psi : S \longrightarrow U$ is an ideal extension within $\mathbf{C}$, then ψ is injective. If $\varphi : S \longrightarrow T$ is a dense ideal extension within $\mathbf{C}$ then we call $\varphi : S \longrightarrow T$ a **maximal dense ideal extension within** $\mathbf{C}$ if whenever $\psi : T \longrightarrow U$ is a homomorphism such that $\varphi\psi : S \longrightarrow U$ is a dense ideal extension within $\mathbf{C}$, then ψ is an isomorphism of T onto U. If $\varphi_1 : S \longrightarrow T_1$ and $\varphi_2 : S \longrightarrow T_2$ are maximal dense ideal extensions within $\mathbf{C}$, then these ideal extensions are equivalent.

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Dense ideal extensions play an important role in the construction of ideal extensions in general, and as we are often only interested in constructing ideal extensions within a given class **C**, we are led into considering dense ideal extensions within **C**. A considerable simplification in this construction occurs if every semigroup of **C** has a maximal dense ideal extension within **C**. Unfortunately it seems that there are only a few naturally defined classes **C** for which this is the case. Not every semigroup has a maximal dense ideal extension within the class of all semigroups [2, 14]: a semigroup has a maximal dense ideal extension within the class of all semigroups if and only if it is weakly reductive. And, though every regular semigroup is weakly reductive, not every regular semigroup has a maximal dense ideal extension within the class of all regular semigroups.

The class of all inverse semigroups behaves nicely however, for it so happens that if S is an inverse semigroup and $\varphi : S \longrightarrow T$ a maximal dense ideal extension within the class of all semigroups, then T needs to be an inverse semigroup [11]. Also, if **NBG** denotes the class of all normal bands of groups, then for every $S \in$ **NBG** there exists a maximal dense ideal extension $\varphi : S \longrightarrow T$ within **NBG**, but $\varphi : S \longrightarrow T$ need not be a maximal dense ideal extension within the class of all semigroups [10]. Recall that a regular semigroup S is a locally inverse semigroup if eSe is an inverse semigroup for every idempotent e of S. Every inverse semigroup and every normal band of groups is locally inverse. An ideal extension within the class **LI** of all locally inverse semigroups will also be called an ideal extension of locally inverse semigroups. Given the complicated nature of the maximal dense ideal extensions of completely 0-simple semigroups within the class of all semigroups (see Section 5 of [7] and Section V.3 of [8]) the theorem announced in the abstract should be curious and unexpected.

Dense ideal extensions are usually realized by using the concept of the **translational hull**, as introduced in [2]. In this paper we leave translational hulls out of our discussion. We follow instead the more natural approach initiated by Schein in [12] for inverse semigroups which fully exploits the fact that a partial order, compatible with the multiplication, is available. This line of investigation was also followed by Petrich in [10]. The idea will be to consider a canonical embedding τ_S of the locally inverse semigroup S into a sufficiently large semigroup $O(S)$ and then to take the regular part $T(S)$ of the idealizer $I(S)$ of $S\tau_S$ in $O(S)$, therewith to produce the desired maximal dense ideal extension.

To keep our exposition short we resisted the temptation to derive some results of [10] and [12] from the work done in the present paper. We avoided to be side-tracked by examples and special cases (completely 0-simple semigroups, strict locally inverse semigroups, straight locally inverse semigroups). However, for future reference we formulate and prove some results in greater generality than is strictly necessary for proving our main Theorem 4.7.

It remains to recall some commonly used notation and terminology. For a regular semigroup S we let $E(S)$ be the set of idempotents of S. On $E(S)$ define ω^l, ω^r and ω by: for $e, f \in E(S)$,

$$\begin{aligned} e\,\omega^l f &\iff ef = e, \\ e\,\omega^r f &\iff fe = e, \\ e\,\omega f &\iff ef = e = fe. \end{aligned}$$

Then ω^l and ω^r are quasi-orders and ω is a partial order. For $f \in E(S)$ we put $\omega^l(f) = \{e \in E(S) \mid e\,\omega^l f\}$ and we define $\omega^r(f)$ and $\omega(f)$ similarly.

In a regular semigroup S we put $a \leq b$ if $a = eb = bf$ for some idempotents e and f. The relation $\leq$ is a partial order, usually called the **natural partial order** on S. The partial order $\leq$ was introduced by Nambooripad in [5]. In Corollary 2.3 we remind the reader of Nambooripad's result which states that a regular semigroup is a locally inverse semigroup if and only if $\leq$ is compatible with the given multiplication. We shall assume that the reader is familiar with the other equivalent characterizations of $\leq$ listed in Section 6.1 of [3].

If S is a regular semigroup, $a \in S$ and $e \in E(S)$, then $a \leq e$ implies that $a \in E(S)$. For $e, f \in E(S)$, $e\,\omega f$ if and only if $e \leq f$. We shall most often use the notation $\leq$ instead of ω, and in keeping with the notation which will be introduced in the next section we shall prefer to use the notation $(e]$ instead of $\omega(e)$ for $e \in E(S)$.

If S is a regular semigroup and $a \in S$, then $V(a)$ denotes the set of inverses of a. For $e, f \in E(S)$, the set $S(e, f) = V(ef) \cap fSe$ consists of idempotents and will be called the **sandwich set** of e and f. For $a, b \in S$, idempotents e and f in L_a and R_b respectively, and $h \in S(e, f)$, we have that

$$ab\,\mathcal{R}\,ah\,\mathcal{L}\,h\,\mathcal{R}\,hb\,\mathcal{L}\,ab,$$

where $ab = ahb = (ah)(hb)$, $ah \leq a$ and $bh \leq b$.

2 Order ideals

Let S be a regular semigroup and $P(S)$ the set of all nonempty subsets of S. For $A, B \in P(S)$ we define AB by

$$AB = \{ab \mid a \in A,\ b \in B\}.$$

Then $P(S)$ becomes a semigroup which we shall call the **global semigroup** of S. For $a \in S$ and $B \in P(S)$ we shall also write $\{a\}B$ as aB and $B\{a\}$ as Ba.

A nonempty subset H of the regular semigroup S will be called an order ideal of S if the following is satisfied:

$$b \in H,\ a \leq b \text{ in } S \quad \Rightarrow \quad a \in H.$$

The set of all order ideals of S will be denoted by $O(S)$. In particular, $O(S) \subseteq P(S)$. We emphasize the fact that an order ideal of S need not be a subsemigroup of S. For $a \in S$, we use the notation

$$(a] = \{b \in S \mid b \leq a \text{ in } S\},$$

and we call $(a]$ the **principal order ideal** generated by a.

Proposition 2.1. *If S is a regular semigroup then $O(S)$ is an ideal of $P(S)$.*

Proof. Let $H \in O(S)$, $a \in S$ and $b \in H$. Assume that $c \leq ab$, thus $c \in Sab$ and $c = abe$ for some idempotent $e \in L_c$. Then $be \leq b$ and so $be \in H$ since H is an order ideal. It follows that $c = abe \in aH$. We proved that $aH \in O(S)$. Thus, for any $A \in P(S)$, $AH = \cup_{a \in A} aH \in O(S)$. In a dual way we can show that $HA \in O(S)$. Therefore $O(S)$ is an ideal of $P(S)$. ■

We shall give a characterization of locally inverse semigroups in terms of order ideals. Recall that the natural partial order $\leq$ on the regular semigroup S is said to be **right compatible** if for all $a, b, c \in S$ with $a \leq b$ in S we have that $ac \leq bc$. **Left compatibility** is defined in a dual way, and we say that $\leq$ is **compatible** if it is both left and right compatible. It may be of interest to characterize the regular semigroups for which $\leq$ is right compatible.

Recall that a regular semigroup S is called **$\mathcal{L}$-unipotent** if every $\mathcal{L}$-class of S contains a unique idempotent, and S is called **locally $\mathcal{L}$-unipotent** if for every $e \in E(S)$, eSe is $\mathcal{L}$-unipotent. L_2^1 will denote the two-element left zero semigroup L_2 with an identity adjoined.

Proposition 2.2. *For a regular semigroup S the following are equivalent:*

(i) $\leq$ *is right compatible,*

(ii) S *does not contain a copy of* L_2^1 *as a subsemigroup,*

(iii) S *is locally* $\mathcal{L}$*-unipotent,*

(iv) *for every* $e \in E(S)$, $\omega^r(e)$ *forms a right regular subband of* S,

(v) *for all* $e, f \in E(S)$, $S(e, f)$ *forms a right zero semigroup,*

(vi) *for all* $e, f \in E(S)$, $|S(e, f)f| = 1$,

(vii) *for all* $a, b \in S$ *with* $L_a \leq L_b$, *there exists a unique* $c \in L_a$ *such that* $c \leq b$,

(viii) $(a]b = (ab]$ *for all* $a, b \in S$,

(ix) $aH = (a]H$ *for all* $a \in S$, $H \in O(S)$,

(x) $a(b] = (a](b]$ *for all* $a, b \in S$.

Proof. The proof of the equivalence of the statements (i)-(vii) follows from [1] and the results of Section 6.1 of [3]. We concentrate here on what is new.

$(i) \Rightarrow (viii)$. Let $a, b \in S$. We know from Proposition 2.1 that $(a]b \in O(S)$, and so $(ab] \subseteq (a]b$ since $ab \in (a]b$. If $c \leq a$, then $cb \leq ab$ since $\leq$ is right compatible. Therefore the equality $(ab] = (a]b$ prevails.

$(viii) \Rightarrow (ix)$. Let $a \in S$ and $H \in O(S)$. Clearly $aH \subseteq (a]H$. Let $c \leq a$ in S and $b \in H$. Then $cb \in (a]b = (ab]$. Using Proposition 2.1 we have that $aH \in O(S)$, thus since $ab \in aH$, so $cb \in aH$. We conclude that $aH = (a]H$.

$(ix) \Rightarrow (x)$ is obvious.

$(x) \Rightarrow (i)$. Let $a, b, c \in S$ such that $c \leq a$ in S. We need to show that $cb \leq ab$. Since $cb \in (a](b] = a(b]$, there exists $d \leq b$ such that $cb = ad$. Thus $d \in bS$ and so $cb = ad \in abS$. There exists an idempotent f in R_c such that $c = fa$. We choose an idempotent $g \in R_{cb}$. Then since $R_g = R_{cb} \leq R_c = R_f$ we have that $g\,\omega^r f$, whence gf is an idempotent in R_{cb}. Moreover $gfab = gcb = cb$. From $cb \in abS$ and $gfab = cb$ we have that $cb \leq ab$, as required. ∎

Clearly a regular semigroup is a locally inverse semigroup if and only if it is both a locally $\mathcal{L}$-unipotent and a locally $\mathcal{R}$-unipotent semigroup. Therefore

Corollary 2.3. *A regular semigroup is a locally inverse semigroup if and only if it satisfies both any of the conditions* $(i) - (x)$ *and the dual of any of the conditions* $(i) - (x)$ *of Proposition 2.2.*

The following variant of Corollary 2.3 will turn out to be useful.

Proposition 2.4. *For a regular semigroup* S *the following are equivalent:*

(i) *S is a locally inverse semigroup,*

(ii) $(a]b = a(b] = (a](b] = (ab]$ *for all* $a, b \in S$,

(iii) $aH = (a]H$ *and* $Ha = H(a]$ *for all* $a \in S$, $H \in O(S)$,

(iv) *the mapping*

$$\tau_S : S \longrightarrow O(S), \quad a \longrightarrow (a]$$

is an injective homomorphism.

Proof. The equivalence of (i), (ii) and (iii) follows from Proposition 2.2 and Corollary 2.3. That (ii) implies (iv) is also immediate.

Assume that (iv) holds, hence for all $a, b \in S$, $(ab] = (a](b]$. Further $(a]b \subseteq (a](b]$ obviously holds since $b \in (b]$, whereas $(ab] \subseteq (a]b$ by Proposition 2.1. Therefore $(a]b = (a](b] = (ab]$, and by duality (ii) holds. We proved that (iv) implies (ii). ■

Throughout this paper, S will be a locally inverse semigroup and the mapping $\tau_S : S \longrightarrow O(S)$ of Proposition 2.4 will be called the **canonical embedding** of S into $O(S)$.

Using the condition (v) of Proposition 2.2 we have from Corollary 2.3 that a regular semigroup S is a locally inverse semigroup if and only if for all $e, f \in E(S)$, $|S(e, f)| = 1$. Therefore, if S is a locally inverse semigroup, then we can introduce a binary operation $\wedge$ on $E(S)$ by putting $S(e, f) = \{f \wedge e\}$ for $e, f \in E(S)$. The binary algebra $(E(S), \wedge)$ is called the **pseudosemilattice** of S. Pseudosemilattices were characterized abstractly by Nambooripad in [6] (see also [13]). From [6] we collect or easily derive the following

Result 2.5. *Let* S *be a locally inverse semigroup. Then*

(i) *for* $e, f \in E(S)$,

$$e\,\omega^l f \Leftrightarrow e = e\wedge f, \quad e\,\omega^r f \Leftrightarrow e = f\wedge e, \quad e\,\omega f \Leftrightarrow e\wedge f = e = f\wedge e,$$
$$\omega^r(e) \cap \omega^l(f) = \omega(e \wedge f),$$

(*ii*) *for* $e \in E(S)$, $\omega^l(e)$ *is a left normal subband of* S, *and if* $g, h \in \omega^l(f)$, *then* $gh = g \wedge h$,

(*iii*) *for* $e \in E(S)$, $\omega(e)$ *is a subsemilattice of* S *and if* $g, h \in \omega(e)$, *then* $gh = g \wedge h$,

(*iv*) *for* $e, f \in E(S)$, $ef = fe$ *if and only if* $e \wedge f = f \wedge e$, *and if this is the case, then* $e \wedge f = ef$.

For a locally inverse semigroup S and F a nonempty subset of $E(S)$, we shall say that F is an **order ideal** if $F \in O(S)$. We call F a **subpseudosemilattice** of $E(S)$ if $(F, \wedge)$ is a subalgebra of $(E(S), \wedge)$. If this is the case then F is called a **subsemilattice** of $E(S)$ if $(F, \wedge)$ is a semilattice. By Result 2.5 F is a subsemilattice of $E(S)$ if and only if F is a subsemigroup of S which is itself a semilattice: the multiplication in F coincides with the operation $\wedge$. In the next section we shall be interested in subsemilattices of $E(S)$ which are also order ideals. Such order ideals of S are necessarily idempotents of $O(S)$.

3 The locally inverse semigroup $T(S)$

Throughout the remainder of this paper S will be a locally inverse semigroup and $\tau_S : S \longrightarrow O(S)$ the canonical embedding mentioned in the statement of Proposition 2.4. We let $I(S)$ be the **idealizer** of $S\tau_S$ in $O(S)$, that is, the largest subsemigroup of $O(S)$ which contains $S\tau_S$ as an ideal. We let $T(S) = Reg\, I(S)$ be the set of all the regular elements of $I(S)$. Clearly $S\tau_S \subseteq T(S)$. There is no reason to believe that $I(S)$ is a regular semigroup, and at this stage we also do not know whether $T(S)$ is a subsemigroup of $I(S)$. In this section we shall show that $T(S)$ constitutes a locally inverse semigroup and in the next section we prove that $\tau_S : S \longrightarrow T(S)$ is a maximal dense ideal extension of S within the class of all locally inverse semigroups.

Our interest in the set $T(S)$ stems from the following

Proposition 3.1. *Let* S *and* T *be locally inverse semigroups where* S *is an ideal of* T. *Then the mapping*

$$\psi : T \longrightarrow O(S), \quad t \longrightarrow (t] \cap S$$

is a homomorphism which induces τ_S on S and $T\psi \subseteq T(S)$.

Proof. Let $t \in T$, $a \in S$ and choose an idempotent e in R_{ta}. Then $et \in R_{ta} = R_e \leq R_t$ and so $et \in (t] \cap S$ since S is an ideal of T. We conclude that $(t] \cap S$ is nonempty for every $t \in T$, and then clearly $(t] \cap S \in O(S)$.

Let $s, t \in T$. From Proposition 2.4 it follows that $(s](t] = (st]$, and thus also that $((s] \cap S)((t] \cap S) \subseteq (st] \cap S$. In order to show that ψ is a homomorphism, we need to prove that the equality holds. Therefore, let $a \in S$ such that $a \in (st]$. Let e and f be any idempotents in L_s and R_t respectively, so that $f \wedge e$ is the unique element of $S(e, f)$. Then

$$st = s(f \wedge e) \cdot (f \wedge e)t \,\mathcal{R}\, s(f \wedge e) \,\mathcal{L}\, f \wedge e \,\mathcal{R}\, (f \wedge e)t \,\mathcal{L}\, s(f \wedge e) \cdot (f \wedge e)t = st$$

in T. We use Proposition 2.2 and its dual several times. From $R_a \leq R_{st}$ we know that there exists a unique $s' \in R_a$ such that $s' \leq s(f \wedge e)$; since $L_{s'} \leq L_{s(f\wedge e)}$ there exists a unique $h \in L_{s'}$ such that $h \leq f \wedge e$; since $R_h \leq R_{f\wedge e}$ there exists a unique $t' \in R_h$ such that $t' \leq (f \wedge e)t$. From $s' \leq s(f \wedge e) \leq s$ and $t' \leq (f \wedge e)t \leq t$ it follows that $s't' \leq st$. From $s' \,\mathcal{L}\, h \,\mathcal{R}\, t'$ and since h is an idempotent it follows that $s't' \,\mathcal{R}\, s' \,\mathcal{R}\, a$. Thus a and $s't'$ are $\mathcal{R}$-related, $a \leq st$, $s't' \leq st$ and so by the dual of Proposition 2.2, $a = s't'$. Here $s' \in (s] \cap S$ and $t' \in (t] \cap S$ since S is an ideal of T. We conclude that $a \in ((s] \cap S)((t] \cap S)$.

The remaining statements are easily verified. ∎

In order to prove that $T(S)$ is a subsemigroup of $I(S)$ we shall focus our attention on the idempotents of $I(S)$.

Lemma 3.2. *Let S be a locally inverse semigroup. Then*

- (i) *if $H \in I(S)$, then no distinct elements of H are $\mathcal{L}$- or $\mathcal{R}$-related in S,*
- (ii) *if F is an idempotent of $O(S)$ such that no distinct elements of F are $\mathcal{L}$- or $\mathcal{R}$-related in S, then F is a subsemilattice and an order ideal of S.*

Proof. (i). Let $H \in I(S)$ and $a, b \in H$ such that $a \,\mathcal{L}\, b$ in S. We choose an idempotent e in $L_a = L_b$ and we find that $H(e] = (c]$ for some $c \in S$ since $S\tau_S$ is an ideal of $I(S)$. Hence $a, b \in (c]$ since $a = ae$, $b = be$. From $a \,\mathcal{L}\, b$, $a \leq c$, $b \leq c$ it follows that $a = b$ by Proposition 2.2. Thus (i) follows by symmetry.

(ii). Let F be an idempotent of $O(S)$ such that no distinct elements of F are $\mathcal{L}$- or $\mathcal{R}$-related in S. Since $F^2 = F$ we have that F is a subsemigroup of S.

Let $s \in F$. Since $F = F^2$, there exist $a, b \in F$ such that $s = ab$. By Proposition 2.2 and since $L_s \leq L_b$ there exists a unique $b' \in L_s$ such that $b' \leq b$. Since F is an order ideal and $b \in F$, so $b' \in F$. But then $s \mathcal{L} b'$ with $s, b' \in F$ and so $s = b'$, whence $s \leq b$. By duality $s \leq a$.

We choose idempotents e and f in L_a and R_b respectively and consider the unique element $f \wedge e \in S(e, f)$. Then $s = ab \mathcal{L} (f \wedge e)b$, $s \leq b$, $(f \wedge e)b \leq b$ and so by Proposition 2.2, $s = (f \wedge e)b$. Using a dual argument we find that $s = a(f \wedge e)$. Hence $s = ab = a(f \wedge e) \cdot (f \wedge e)b = s^2$ and F is a band. Since no distinct elements of F are $\mathcal{L}$- or $\mathcal{R}$-related in S we conclude that F is a subsemilattice of S. ∎

If E and F are nonempty subsets of the set $E(S)$ of idempotents of the locally inverse semigroup S, then we shall put

$$E \wedge F = \{e \wedge f \mid e \in E,\ f \in F\}.$$

For $e \in E(S)$ we put

$$e \wedge F = \{e \wedge f \mid f \in F\}$$

and

$$F \wedge e = \{f \wedge e \mid f \in F\}.$$

Lemma 3.3. *Let $E, F \in O(S)$ such that E and F are subsemilattices of S. Put $G = E \wedge F$. Then*

- (i) *$G \in O(S)$ and G is a subpseudosemilattice of $E(S)$,*
- (ii) *$G = \{g \in E(S) \mid e \mathcal{R} g \mathcal{L} f$ for some $e \in E,\ f \in F\}$,*
- (iii) *$EG = G = GF$.*

Proof. Let $g \in G$, hence $g = e \wedge f$ for some $e \in E$ and $f \in F$. Then $ge \leq e$ and $fg \leq f$, where $fg \mathcal{L} g \mathcal{R} ge$. Here $ge \in E$ and $fg \in F$ since E and F are order ideals. Therefore $G \subseteq \{g \in E(S) \mid e \mathcal{R} g \mathcal{L} f$ for some $e \in E,\ f \in F\}$. The reverse inclusion also holds because if $g \in E(S)$ such that $e \mathcal{R} g \mathcal{L} f$ for some $e \in E$ and $f \in F$, then $g = e \wedge f \in E \wedge F$. We conclude that (ii) holds.

Let $g \in G$ and $h \leq g$ in S. Then $g = e \wedge f$ for some $e \in E$, $f \in F$, whence $R_h \leq R_e$ and $L_h \leq L_f$, so $eh \,\mathcal{R}\, h \,\mathcal{L}\, hf$ with $eh \leq e$ and $hf \leq f$. Since E and F are order ideals, we have $eh \in E$ and $hf \in F$, thus $h \in G$ by (ii). We conclude that G is an order ideal of S.

Let $g, h \in G$. By (ii) there exist $e \in E$ and $f \in F$ such that $e \,\mathcal{R}\, g$ and $h \,\mathcal{L}\, f$. Thus $g \wedge h = e \wedge f \in G$. We proved (i).

If $g \in G$, then by (ii) there exists $e \in E$ such that $e \,\mathcal{R}\, g$. Therefore $g = eg \in EG$, and we conclude that $G \subseteq EG$. If $e_1 \in E$, then $e_1 g = e_1(g \wedge e_1)g = e_1(e \wedge e_1)g$. Since E is a subsemilattice of S, so $e_1(e \wedge e_1) \leq e$ by Result 2.5(iv), whence $e_1 g \leq eg = g$. By (i) G is an order ideal of S, so $e_1 g \in G$. We proved that $EG = G$, and in a dual way we can prove that $GF = F$. ∎

Lemma 3.4. *If E and F are idempotents of $I(S)$ then $E \wedge F$ is a subsemilattice and an order ideal of S.*

Proof. We shall put $G = E \wedge F$. From Lemmas 3.2 and 3.3 we know that G is a subpseudosemilattice of $E(S)$ and an order ideal of S. Using duality and the final remark of Section 2 it suffices to show that G does not contain distinct $\mathcal{R}$-related idempotents.

Let $g, h \in G$ such that $g \,\mathcal{R}\, h$ in S. By Lemma 3.3 there exist $f_1, f_2 \in F$ such that $f_1 \,\mathcal{L}\, g$ and $f_2 \,\mathcal{L}\, h$. Since $F \in I(S)$ we have $F(g] = (a]$ for some $a \in S$. Then $a \in Sg$ and $f_1 = f_1 g \in F(g] = (a] \subseteq Sa$, so $a \,\mathcal{L}\, f_1 \,\mathcal{L}\, g$. But then $a \,\mathcal{L}\, f_1$ and $f_1 \leq a$, hence $a = f_1$ by Proposition 2.2. We conclude that $F(g] = (f_1]$. It follows that $f_2 g \in F(g] = (f_1]$, where $f_2 g \in L_g = L_{f_1}$ since $f_2 \,\mathcal{L}\, h \,\mathcal{R}\, g$. Hence $f_2 g \in L_{f_1}$ and $f_2 g \leq f_1$, so $f_2 g = f_1$. Also $f_2 \,\mathcal{R}\, f_2 g$ since $f_2 \,\mathcal{L}\, h \,\mathcal{R}\, g$, whence $f_1 \,\mathcal{R}\, f_2$. It follows that $f_1 = f_2$ since F is a semilattice. From $f_1 \,\mathcal{L}\, g \,\mathcal{R}\, h \,\mathcal{L}\, f_2 = f_1$ it follows that $g = h$. ∎

Lemma 3.5. *Let $F \in O(S)$ be a subsemilattice of S. Then for every $e \in E(S)$, $e \wedge F = (e] \wedge F$.*

Proof. Let $g \in (e] \wedge F$. By Lemma 3.3 we have that $e' \,\mathcal{R}\, g \,\mathcal{L}\, f$ for some $e' \in (e]$ and $f \in F$, and then $g = e' \wedge f \in \omega^r(e) \cap \omega^l(f) = \omega(e \wedge f)$. On the other hand, $e \wedge f \,\omega^l f \mathcal{L} g$. From $e \wedge f \,\omega^l g \,\omega\, e \wedge f$ we have $e \wedge f = (e \wedge f)g = g$, whence $g \in e \wedge F$. ∎

Lemma 3.6. *Let F be a nonempty subset of S. Then F is an idempotent of $I(S)$ if and only if $F \in O(S)$, F is a subsemilattice of S, and for every $e \in E(S)$, there exist $k, l \in E(S)$ such that*

$$F \wedge e = (k], \quad e \wedge F = (l].$$

Proof. Let F be an idempotent of $I(S)$. By Lemma 3.2 F is a subsemilattice and an order ideal of S. Let $e \in E(S)$. Since $F \in I(S)$ we have from Proposition 2.4 that $Fe = F(e] = (p]$ for some $p \in S$. Thus $p = fe$ for some $f \in F$. We shall put $l = e \wedge f$ and we shall prove that $e \wedge F = (l]$. From Lemmas 3.4 and 3.5 we know that $e \wedge F = (e] \wedge F$ is a subsemilattice and an order ideal of S, thus in particular $(l] \subseteq e \wedge F$.

Let $e \wedge g \in e \wedge F$ for some $g \in F$. Since $g(e \wedge g)\,\mathcal{R}\,ge \in Fe = (p]$ and $f(e \wedge f)\,\mathcal{R}\,fe = p$, so $g(e \wedge g)\,\omega^r f(e \wedge f)$. Since $g(e \wedge g) \leq g \in F$, $f(e \wedge f) \leq f \in F$, we have that $g(e \wedge g), f(e \wedge f) \in F$. Since F is a semilattice, $g(e \wedge g)\,\omega^r f(e \wedge f)$ implies that $g(e \wedge g) \leq f(e \wedge f)$. Since $e \wedge g\,\mathcal{L}\,g(e \wedge g)$ and $e \wedge f\,\mathcal{L}\,f(e \wedge f)$ we then need to have $e \wedge g\,\omega^l e \wedge f$. Since $e \wedge F$ is a semilattice this entails $e \wedge g \leq e \wedge f = l$. We conclude that $e \wedge F = (l]$. In a dual way we can show that $F \wedge e = (k]$ for some $k \in E(S)$.

We set out to prove the converse. Therefore let F be a subsemilattice and an order ideal of S such that for every $e \in E(S)$ both $e \wedge F$ and $F \wedge e$ are principal order ideals. Let $a \in S$ and e an idempotent in R_a. Let $l \in E(S)$ such that $e \wedge F = (l]$. Thus $l = e \wedge f$ for some $f \in F$. By Proposition 2.4 we clearly have that $(fa] = (f](a] \subseteq F(a]$. We shall prove that the equality prevails. Let $g \in F$. Then $e \wedge g \leq l = e \wedge f$. Since $e \wedge g\,\mathcal{L}\,g(e \wedge g) \leq g \in F$ and $e \wedge f\,\mathcal{L}\,f(e \wedge f) \leq f \in F$ we have that $g(e \wedge g)\,\omega^l f(e \wedge f)$, where both $g(e \wedge g)$ and $f(e \wedge f)$ belong to the semilattice F. Therefore $g(e \wedge g) \leq f(e \wedge f)$. Hence

$$ga = g(e \wedge g)a \leq f(e \wedge f)a = fa$$

and it follows that $Fa \subseteq (fa]$. Using Proposition 2.4 we may now conclude that $F(a] = (fa]$. Using duality we see that F is an idempotent of $I(S)$. ■

Lemma 3.7. *If E and F are idempotents of $I(S)$ then $E \wedge F$ is an idempotent of $I(S)$.*

Proof. Let E and F be idempotents of $I(S)$ and put $G = E \wedge F$. From Lemma 3.4 we know that G is a subsemilattice and an order ideal of S. Using Lemma 3.6 and duality it suffices to show that if $k \in E(S)$ then there exists $j \in E(S)$ such that $k \wedge G = (j]$.

Let $k \in E(S)$. By Lemma 3.6 there exist $l, m \in E(S)$ such that $k \wedge F = (l]$ and $E \wedge l = (m]$. Then $m\,\omega^l l$ and so $m\,\mathcal{L}\,lm \leq l$. We want to show that $k \wedge G = (lm]$.

From the details of the proof of Lemma 3.6 there exists $f \in F$ such that $l = k \wedge f$ and $(fk] = Fk = F(k]$. From $m\,\omega^l l\,\omega^l f$ we have that $m\,\mathcal{L}\,fm \leq$

$f \in F$, and so $fm \in F$ since F is an order ideal. By Lemmas 3.3 and 3.6 and the dual of Lemma 3.5 there exists $e \in E$ such that $e\,\mathcal{R}\,m$. Thus $m = e \wedge fm \in G$ since $e\,\mathcal{R}\,m\,\mathcal{L}\,fm$. Since $lm \leq l = k \wedge f$ we have that $lm\,\omega^r k$, thus $lmk \in E(S)$ with $lm\,\mathcal{R}\,lmk \leq k$. Using Lemma 3.5 it then follows that $lm = lmk \wedge m \in k \wedge G$, and since $k \wedge G \in O(S)$ by Lemmas 3.3, 3.5, so $(lm] \subseteq k \wedge G$.

Let $g \in k \wedge G$. By Lemmas 3.3, 3.5 there exist idempotents $v, h \in E(S)$ such that $v \leq k$, $h \in G$ and $v\,\mathcal{R}\,g\,\mathcal{L}\,h$. Again by Lemma 3.3 there exist $u \in E$ and $w \in F$ such that $u\,\mathcal{R}\,h\,\mathcal{L}\,w$. Hence $g = v \wedge w \in k \wedge F = (l]$. Since $u\,\mathcal{R}\,h\,\mathcal{L}\,g$ we have that $h = u \wedge g \in E \wedge (l] = E \wedge l = (m]$ by Lemma 3.6 and the dual of Lemma 3.5. From $h\,\mathcal{L}\,g \leq l$ we have that $g = lh$ by Proposition 2.2. Hence $g = lh \leq lm$ and we see that $k \wedge G \subseteq (lm]$. We conclude that the equality $k \wedge G = (lm]$ prevails. ∎

Lemma 3.8. *If E and F are idempotents of $I(S)$, then FE and $E \wedge F$ are pairwise inverse elements of $I(S)$.*

Proof. We shall put $G = E \wedge F$. Clearly $FE \in I(S)$ and by Lemma 3.7 also $G \in I(S)$. By Lemmas 3.2 and 3.3, $GFEG = G^2 = G$ and $FEGFE = FGE$. It remains to show that $FE = FGE$. For $f \in F$ and $e \in E$ we have that $fe = f(e \wedge f)e \in FGE$, whence $FE \subseteq FGE$. It remains to show the reverse inclusion. It suffices to show that $GE \subseteq E$.

Let $ge \in GE$ for some $g \in G$ and $e \in E$. We let $h = e \wedge g$. By Lemma 3.3 there exists $f \in F$ such that $g\,\mathcal{L}\,f$, thus $h = e \wedge f \in G$. Since G is a semilattice by Lemma 3.6, $g, h \in G$ and $h\,\omega^l g$, so $h \leq g$. Therefore $ge = ghe = he \leq e$ and so $ge \in E$ since E is an order ideal. We proved that $GE \subseteq E$, as required. ∎

Theorem 3.9. *Let S be a locally inverse semigroup. Then $T(S)$ is a locally inverse semigroup. The $\wedge$-operation of the pseudosemilattice of $T(S)$ is given by: if E and F are idempotents of $T(S)$, then*

$$E \wedge F = \{e \wedge f \mid e \in E,\ f \in F\}.$$

Proof. Let $H, K \in T(S)$ and let H' and K' be inverses of H and K respectively within $T(S)$. We put $F = H'H$, $E = KK'$ and $G = E \wedge F$. Then by Lemma 3.7 E, F and G are idempotents of $I(S)$ and by Lemmas 3.3 and 3.6 $EG = G = GF$. It follows that HGK and $K'GH'$ are pairwise inverse elements of $T(S)$. Further

$$HGK = HH'HEGFKK'K = HFEGFEK = HFEK = HK$$

by Lemma 3.8. We proved that $T(S)$ is a regular semigroup.

Let E be an idempotent of $T(S)$ and F and G idempotents in $ET(S)E$. By Lemma 3.6 F and G are order ideals and thus also ideals of the semilattice E. Therefore $FG = F \cap G = GF$. It follows that $ET(S)E$ is an inverse semigroup. We conclude that $T(S)$ is a locally inverse semigroup.

Let E and F be any idempotents of $T(S)$, and put $G = \{e \wedge f \mid e \in E,\ f \in F\}$. By Lemma 3.3 and 3.8, G is an inverse of FE in $T(S)$ and $G \in ET(S)F$. Hence since $T(S)$ is a locally inverse semigroup, G is the unique element in the sandwich set $S(F, E)$. We proved that the $\wedge$-operation of Lemma 3.7 coincides with the $\wedge$-operation of the pseudosemilattice of $T(S)$. ■

It will be useful to give some details about the structure of $T(S)$.

Lemma 3.10. *Let $E, F \in O(S)$ such that E and F are subsemilattices of S. Then $EF = FE = E$ if and only if $E \subseteq F$.*

Proof. If $E \subseteq F$ then E is an order ideal, and thus also an ideal of F, hence $EF = FE = E$. Assume that conversely $EF = FE = E$ and let $e \in E$. Then $e = e_1f_1 = f_2e_2$ for some $e_1, e_2 \in E$ and $f_1, f_2 \in F$, and so $e \in \omega^l(f_1) \cap \omega^r(f_2) = \omega(f_2 \wedge f_1)$. By Result 2.5 $e \leq f_2 \wedge f_1 = f_2f_1 \in F$, and since $F \in O(S)$, $e \in F$. We proved that $E \subseteq F$. ■

Lemma 3.11. *Let H and H' be pairwise inverse elements of $O(S)$ such that both $E = HH'$ and $F = H'H$ are subsemilattices of S. Then for every $a \in H$ there exists a unique inverse a' of a in S which belongs to H', and the mapping*

$$H \longrightarrow H', \quad a \longrightarrow a'$$

is an order isomorphism. Every $e \in E$ can be written uniquely as $e = hh'$ for some $h \in H$ and $h' \in H'$, with $h \,\mathcal{R}\, e \,\mathcal{L}\, h'$ in S, and then h and h' are pairwise inverse elements of S.

Proof. Let $a \in H$. Since $H = HH'H$ there exist $a_1, a_2 \in H$ and $a'' \in H'$ such that $a = a_1a''a_2$. Using the fact that E and F are subsemilattices we have

$$\begin{aligned} aa''a &= a_1(a''a_2)(a''a_1)(a''a_2) \\ &= a_1(a''a_1)(a''a_2)(a''a_2) \\ &= a_1(a''a_1)(a''a_2) \\ &= (a_1a'')(a_1a'')a_2 \\ &= a_1a''a_2 = a \end{aligned}$$

and so $a' = a''aa'' \in H'HH' = H'$ is an inverse of a which belongs to H'. If $a_1', a_2' \in H' \cap V(a)$ then $aa_1' \mathcal{R} aa_2' \in E$, and since E is a semilattice, so $aa_1' = aa_2'$. Dually we also have that $a_1'a = a_2'a$, hence $a_1' = a_2'$. Thus for every $a \in H$ there exists a unique $a' \in V(a) \cap H'$. By symmetry we conclude that $H \longrightarrow H'$, $a \longrightarrow a'$ is a bijection.

Let $a_1, a_2 \in H$ such that $a_1 \leq a_2$. Let $a_1' \in V(a_1) \cap H'$, $a_2' \in V(a_2) \cap H'$, $e_1 = a_1a_1' \in E$ and $e_2 = a_2a_2' \in E$. Then $a_1 \leq a_2$ entails $e_1 \omega^r e_2$, and since E is a semilattice, so $e_1 \leq e_2$. Hence $L_{a_1'} \leq L_{a_2'}$ and by Proposition 2.2 there exists $a' \in L_{a_1'}$ such that $a' \leq a_2'$. Therefore $a' \in H'$ since $H' \in O(S)$, and so $a'a_1 \mathcal{L} a_1'a_1$, whence $a' = a_1'$, thus $a_1' \leq a_2'$. By symmetry we conclude that $H \longrightarrow H'$, $a \longrightarrow a'$ is an order isomorphism.

Every $e \in E$ can be written as $e = ab'$ for some $a \in H$ and $b' \in H'$. Then $e = (ea)(b'e)$ and it follows that ea and $b'e$ are pairwise inverse elements. From $e = ab'$ it follows that $ea \mathcal{R} e \mathcal{L} b'e$ with $ea \leq a$ and $b'e \leq b$. Since H and H' are order ideals, so $ea \in H$ and $b'e \in H'$. We proved that every $e \in E$ can be written in the form $e = hh'$ for some $h \in H$, $h' \in H'$, with $h \mathcal{R} e \mathcal{L} h'$, and if this is the case then h and h' are pairwise inverse elements. It remains to prove the uniqueness.

Assume that $e \in E$, $a_1, a_2 \in H$, $b_1, b_2 \in H'$ such that $e = a_1b_1 = a_2b_2$ with $a_1 \mathcal{R} a_2 \mathcal{R} e \mathcal{L} b_1 \mathcal{L} b_2$. By the foregoing a_1 and b_1 are pairwise inverse elements, whence $b_1a_1 \mathcal{R} b_1a_2$ in the semilattice F, whence $b_1a_1 = b_1a_2$ and it follows that $a_1 = a_1b_1a_1 = a_1b_1a_2 = ea_2 = a_2$. By symmetry, $b_1 = b_2$. ∎

Proposition 3.12. *Let S be a locally inverse semigroup and $H, K \in T(S)$. Then*

- (i) *$H \mathcal{R} K$ in $T(S)$ if and only if there exists a bijection $\varphi : H \longrightarrow K$ such that $h \mathcal{R} h\varphi$ for every $h \in H$,*
- (ii) *$H \leq K$ in $T(S)$ if and only if $H \subseteq K$.*

Proof. (i). It suffices to give a proof for the case where $K = E$ is an idempotent of $T(S)$. Let $H \mathcal{R} E$ in $T(S)$, and let H' be an inverse of H in $T(S)$ such that $HH' = E$. By Lemma 3.11 there exists for every $h \in H$ a unique $h' \in V(h) \cap H'$, hence $\varphi : H \longrightarrow E$, $h \longrightarrow hh'$ is a well-defined mapping and obviously $h \mathcal{R} h\varphi$ for every $h \in H$. By Lemma 3.2(i) the mapping φ is injective and by Lemma 3.11 φ is onto.

We now assume that $H, E \in T(S)$, E an idempotent of $T(S)$, and that there exists a bijection $\varphi : H \longrightarrow E$ such that $h \mathcal{R} h\varphi$ for every $h \in H$. We let H' be any inverse of H in $T(S)$ and put $F = HH'$. It suffices to prove that the idempotents E and F are $\mathcal{R}$-related. Using the result proved in the above paragraph we know that there exists a bijection $\psi : E \longrightarrow F$ such that $e \,\mathcal{R}\, e\psi$ for every $e \in E$.

Let $e \in E$ and $f \in F$. Then $f \wedge e \,\mathcal{R}\, (f \wedge e)f \leq f$ and so $(f \wedge e)f \in F$ since F is an order ideal. Let $e_1 \in E$ be such that $e_1\psi = (f \wedge e)f$. Then $e_1 \,\mathcal{R}\, f \wedge e \,\mathcal{L}\, e(f \wedge e) \leq e$ and thus $e(f \wedge e) \in E$ since E is an order ideal. The idempotents e_1 and $e(f \wedge e)$ commute since they belong to the semilattice E, whence $e_1 = f \wedge e = e(f \wedge e)$. It follows that $ef = e(f \wedge e)f = (f \wedge e)f \in F$. We proved that $EF \subseteq F$. Since every element of F is of the form $e\psi$ for some $e \in E$, and $e\psi = e(e\psi)$, so $EF = F$ holds. By symmetry we conclude that $E \,\mathcal{R}\, F$, as required.

(ii). Assume that $H \leq K$ in $T(S)$. Let F be any idempotent in the $\mathcal{L}$-class of K. From Proposition 2.2 we know that there exists a unique idempotent E in the $\mathcal{L}$-class of H such that $E \leq F$ in $T(S)$ and then $H = KE$. From Lemma 3.10 we have $E \subseteq F$ and therefore $H = KE \subseteq KF = K$.

To prove the converse, assume that $H \subseteq K$. Let K' be an inverse of K in $T(S)$ and let $a \in H$. By Lemma 3.11 there exists a unique $a' \in V(a) \cap K'$, thus $a = aa'a \in HK'K$. We proved that $H \subseteq HK'K$. Let $ab_1'b_2 \in HK'K$ for some $a \in H$, $b_1' \in K'$ and $b_2 \in K$. Then by Proposition 2.2 there exists a unique $a_1 \in R_{ab_1'b_2}$ such that $a_1 \leq a$. Here $a_1 \in H$ since H is an order ideal. Since $ab_1'b_2 \in KK'K = K$ we see that a_1 and $ab_1'b_2$ are $\mathcal{R}$-related elements of K and so $a_1 = ab_1'b_2$ by Lemma 3.2. Hence $ab_1'b_2 \in H$. We proved that $H = HK'K$. Since HK' is an order ideal of the semilattice KK' it follows that HK' is an idempotent of $T(S)$. In a dual way we show that $H = KK'H$ where $K'H$ is an idempotent of $T(S)$. Therefore $H \leq K$ in $T(S)$. ∎

4 Dense ideal extensions

So far we have seen that for every locally inverse semigroup S the mapping $\tau_S : S \longrightarrow T(S)$ is an ideal extension within the class of all locally inverse semigroups. We set out to prove that τ_S is, within this class, a maximal dense ideal extension. We shall need some auxiliary results. The following simple proof was communicated to us by P. R. Jones.

Proposition 4.1. *Let S be a locally inverse semigroup and I an ideal of S. If ρ is a congruence relation on I and ι_S the equality on S, then $\rho \cup \iota_S$ is a congruence relation on S.*

Proof. We let $t \in S$ and $a, b \in I$ such that $a\rho b$. By duality it suffices to show that $ta\rho tb$. We let e be an idempotent in R_t and choose $(ta)' \in V(ta)$ and $(tb)' \in V(tb)$. Then

$$\begin{aligned} ta &= (ta(ta)'e)ta \;\rho\; (ta(ta)'e)tb \qquad \text{(since } ta(ta)'et \in I \text{ and } et = t) \\ &= (ta(ta)'e)(tb(tb)'e)tb \\ &= (tb(tb)'e)(ta(ta)'e)tb \\ &\qquad \text{(since } ta(ta)'e \text{ and } tb(tb)'e \text{ are commuting idempotents in } eSe). \end{aligned}$$

From this it follows that $(ta)\rho \leq (tb)\rho$ in I/ρ. By symmetry we conclude that $ta\,\rho\,tb$. ∎

The reader should not attempt to generalize Proposition 4.1 for regular semigroups S in general, even if the ideal I is itself a locally inverse semigroup. For this it suffices to contemplate the example mentioned in Section 4.5 of [12]: the full transformation semigroup on a set X contains the semigroup I of constant transformations as an ideal. If $|X| \geq 3$, then I is a locally inverse semigroup - in fact an $|X|$-element right zero semigroup - while the full transformation semigroup is regular. However, no proper nontrivial congruence on I can be extended to a congruence on the full transformation semigroup.

Lemma 4.2. *Let S and T be locally inverse semigroups and $\varphi : T \longrightarrow S$ a homomorphism. If $a, b \in T$ such that $a\varphi \leq b\varphi$ then there exists $c \in TaT$ such that $a\varphi = c\varphi$ and $c \leq b$ in T.*

Proof. We let e be an idempotent in L_b and f an idempotent in L_{ae}. Since $L_{ae} \leq L_b$ we know from Proposition 2.2 that bf is the unique element of L_{ae} such that $bf \leq b$. From $a\varphi \leq b\varphi \,\mathcal{L}\, e\varphi$ it follows that $a\varphi = (a\varphi)(e\varphi) = (ae)\varphi \,\mathcal{L}\, (bf)\varphi$. Since $(bf)\varphi \leq b\varphi$, $a\varphi \leq b\varphi$ and $a\varphi \,\mathcal{L}\, (bf)\varphi$ in the locally inverse semigroup S, we have $a\varphi = (bf)\varphi$ by Proposition 2.2. Since $bf \,\mathcal{L}\, ae$, so $bf \in TaT$. ∎

Theorem 4.3. *Let $\varphi : S_1 \longrightarrow T$ be an ideal extension of locally inverse semigroups and $\alpha : S_1 \longrightarrow S_2$ a surjective homomorphism. Then*

$$\psi : T \longrightarrow T(S_2), \quad t \longrightarrow ((t] \cap S_1\varphi)\varphi^{-1}\alpha \tag{1}$$

is a homomorphism. ψ is the unique homomorphism which extends $\varphi^{-1}\alpha\tau_{S_2} : S_1\varphi \longrightarrow T(S_2)$.

Proof. $\alpha\alpha^{-1}$ is the congruence relation induced on S_1 by α, and by Proposition 4.1 $\rho = \varphi^{-1}(\alpha\alpha^{-1})\varphi \cup \iota_T$ is a congruence relation on T. Therefore $\theta : S_2 \longrightarrow T/\rho,\ a \longrightarrow a\alpha^{-1}\varphi$ is an ideal extension of S_2. By Proposition 3.1 the mapping

$$\psi' : T \longrightarrow T(S_2),\ t \longrightarrow ((t\rho] \cap S_1\varphi/\varphi^{-1}\alpha\alpha^{-1}\varphi)\theta^{-1} \tag{2}$$

is a homomorphism. For any $s \in S_1$ we have that

$$((s\varphi)\rho] \cap S_1\varphi/\varphi^{-1}\alpha\alpha^{-1}\varphi = ((s\varphi)\rho] = (s\alpha\theta]$$

and thus by (2),

$$(s\varphi)\psi' = (s\alpha] = s\alpha\tau_{S_2} = s\varphi\varphi^{-1}\alpha\tau_{S_2}.$$

Therefore ψ' extends $\varphi^{-1}\alpha\tau_{S_2}$.

We claim that the mappings given by (1) and (2) coincide. For this we need to show that the equality

$$((t] \cap S_1\varphi)\varphi^{-1}\alpha = ((t\rho] \cap S_1\varphi/\varphi^{-1}\alpha\alpha^{-1}\varphi)\theta^{-1} \tag{3}$$

holds true for all $t \in T$.

Let $a \in ((t] \cap S_1\varphi)\varphi^{-1}\alpha$, thus there exists $s \in S_1$ such that $a = s\alpha$ with $s\varphi \leq t$. Then

$$a\theta = a\alpha^{-1}\varphi = s\alpha\alpha^{-1}\varphi = (s\varphi)\varphi^{-1}\alpha\alpha^{-1}\varphi = (s\varphi)\rho$$

and so $a \in ((t\rho] \cap S_1\varphi/\varphi^{-1}\alpha\alpha^{-1}\varphi)\theta^{-1}$.

Let $a \in ((t\rho] \cap S_1\varphi/\varphi^{-1}\alpha\alpha^{-1}\varphi)\theta^{-1}$, that is, $a = s\alpha$ for some $s \in S_1$ with

$$(s\varphi)\varphi^{-1}\alpha\alpha^{-1}\varphi = s\alpha\alpha^{-1}\varphi = a\theta \in (t\rho] \cap S_1\varphi/\varphi^{-1}\alpha\alpha^{-1}\varphi,$$

thus $(s\varphi)\rho \leq t\rho$ in T/ρ. By Lemma 4.2 there exists $c \in T(s\varphi)T$ such that $c \leq t$ and $c\rho = (s\varphi)\rho$. Since $S_1\varphi$ is an ideal of T we have $c \in S_1\varphi$, whence $c\,(\varphi^{-1}\alpha\alpha^{-1}\varphi)\,s\varphi$ since ρ induces $\varphi^{-1}\alpha\alpha^{-1}\varphi$ on $S_1\varphi$. Therefore $a = c\varphi^{-1}\alpha \in ((t] \cap S_1\varphi)\varphi^{-1}\alpha$. We proved the equality (3).

We set out to prove the uniqueness. We let $\chi : T \longrightarrow T(S_2)$ be any homomorphism which extends $\varphi^{-1}\alpha\tau_{S_2}$ and for $t \in T$ we put $t\chi = H$. If for some $s \in S_1$, $s\varphi \leq t$ in T, then $s\alpha\tau_{S_2} = s\varphi\varphi^{-1}\alpha\tau_{S_2} = s\varphi\chi \leq t\chi = H$ and thus $s\alpha\tau_{S_2} \subseteq H$ by Proposition 3.12, hence $s\alpha \in H$. Assume that conversely $s \in S_1$ is such that $s\alpha \in H$. Then $s\varphi\chi = s\varphi\varphi^{-1}\alpha\tau_{S_2} = (s\alpha] \subseteq H$ and thus again by Proposition 3.12, $s\varphi\chi \leq H = t\chi$ in $T(S_2)$. By Lemma 4.2 there exists $c \in T(s\varphi)T$ such that $c \leq t$ and $c\chi = s\varphi\chi$. Since $S_1\varphi$ is an ideal of T, so $c \in S_1\varphi$ and there exists $u \in S_1$ such that $c = u\varphi$. We then have $u\varphi \leq t$ and

$$(u\alpha] = u\alpha\tau_{S_2} = (u\varphi)\varphi^{-1}\alpha\tau_{S_2} = c\chi = s\varphi\chi = (s\alpha]$$

whence $u\alpha = s\alpha$. We conclude that

$$\begin{aligned} t\chi &= H = \{s\alpha \mid s \in S_1 \text{ and } s\varphi \leq t\} \\ &= ((t] \cap S_1\varphi)\varphi^{-1}\alpha = t\psi \end{aligned}$$

for all $t \in T$, and thus $\chi = \psi$. ■

There are two special cases of Theorem 4.3 which are of particular interest. The first one concerns the case where $\varphi = \tau_{S_1}$ and $T = T(S_1)$, and the second one where $S_1 = S_2$ and α is the identity transformation.

Corollary 4.4. *Let $\alpha : S_1 \longrightarrow S_2$ be a surjective homomorphism of locally inverse semigroups. Then*

$$T(S_1) \longrightarrow T(S_2), \quad H \longrightarrow H\alpha$$

is the unique homomorphism of $T(S_1)$ into $T(S_2)$ which extends $\tau_{S_1}^{-1}\alpha\tau_{S_2}$.

Proof. In view of Theorem 4.3 one need only verify that for all $H \in T(S_1)$,

$$H = ((H] \cap S_1\tau_{S_1})\tau_{S_1}^{-1}.$$

This follows immediately from Proposition 3.12. ■

Corollary 4.5. *Let $\varphi : S \longrightarrow T$ be an ideal extension of locally inverse semigroups. Then*

$$\psi : T \longrightarrow T(S), \quad t \longrightarrow ((t] \cap S\varphi)\varphi^{-1}$$

is the unique homomorphism which extends the isomorphism

$$\varphi^{-1}\tau_S : S\varphi \longrightarrow S\tau_S, \quad s\varphi \longrightarrow (s].$$

Theorem 4.6. *Let S be a locally inverse semigroup and T a locally inverse semigroup such that $S\tau_S \subseteq T \subseteq T(S)$. Then $\tau_S : S \longrightarrow T$ is a dense ideal extension. Conversely, every dense ideal extension of S within the class of all locally inverse semigroups is equivalent to a unique ideal extension $\tau_S : S \longrightarrow T$ obtained in this way.*

Proof. Let T be a locally inverse semigroup such that $S\tau_S \subseteq T \subseteq T(S)$. Obviously $\tau_S : S \longrightarrow T$ is an ideal extension of locally inverse semigroups, and we need to prove that this ideal extension is dense. Our proof resembles in part the proof of Theorem 2.9$(iv)(g)$ of [12].

Let $\psi : T \longrightarrow U$ be a homomorphism such that $\tau_S\psi : S \longrightarrow U$ is an ideal extension. Assume that $H, K \in T$ such that $H\psi = K\psi$. Let H and H' be pairwise inverse elements of T and $h \in H$. By Lemma 3.11 there exists a unique $h' \in V(h) \cap H'$. Since H and K belong to $I(S)$, both $H(h']$ and $K(h']$ belong to $S\tau_S$ and using Lemma 3.11 again it is not difficult to see that $H(h'] = (hh']$. Since $\tau_S\psi$ is injective we derive from

$$\begin{aligned} (hh')\tau_S\psi &= (hh']\psi = (H(h'])\psi \\ &= (H\psi)((h']\psi) = (K\psi)((h']\psi) \\ &= (K(h'])\psi \end{aligned}$$

that $(hh'] = K(h']$. Therefore also

$$(h] = (hh'](h] = K(h'](h] \subseteq KH'H.$$

Therefore $H \subseteq KH'H$ and thus also $H \leq KH'H$ by Proposition 3.12. By Proposition 2.2 we then have $H = KH'H$ and by duality also $H = HH'K$, whence $H \leq K$. By symmetry we also find that $K \leq H$, thus $H = K$. We conclude that ψ is injective and therefore $\tau_S : S \longrightarrow T$ is a dense ideal extension.

Let $\varphi : S \longrightarrow T$ be a dense ideal extension of locally inverse semigroups and let $\psi : T \longrightarrow T(S)$ be the homomorphism of Corollary 4.5. Since φ is a dense ideal extension, $\varphi\psi = \tau_S : S \longrightarrow T(S)$ an ideal extension since ψ extends $\varphi^{-1}\tau_S$, so ψ is injective. It follows that the ideal extensions $\varphi : S \longrightarrow T$ and $\tau_S : S \longrightarrow T\psi$ are equivalent via the isomorphism $\psi : T \longrightarrow T\psi$. That the extension $\varphi : S \longrightarrow T$ is equivalent to a unique extension of the form $\tau_S : S \longrightarrow V$ for some $S\tau_S \subseteq V \subseteq T(S)$ follows from the uniqueness in the statement of Corollary 4.5. ∎

The technique of proof for the following theorem derives from the proof of Theorem 2.9$(iv)(g)$ of [12].

Theorem 4.7. *Let S be a locally inverse semigroup. Then $\tau_S : S \longrightarrow T(S)$ is a maximal dense ideal extension of S within the class of all locally inverse semigroups.*

Proof. From Theorem 4.6 we already know that $\tau_S : S \longrightarrow T(S)$ is a dense ideal extension. We let $\psi : T(S) \longrightarrow U$ be a homomorphism of locally inverse semigroups such that $\tau_S\psi : S \longrightarrow U$ is a dense ideal extension. We must show that ψ is an isomorphism of $T(S)$ onto U. Clearly ψ is injective since $\tau_S : S \longrightarrow T(S)$ is a dense ideal extension. It remains to show that ψ is onto.

By Corollary 4.5

$$\xi : U \longrightarrow T(S), \quad u \longrightarrow ((u] \cap S\tau_S\psi)(\tau_S\psi)^{-1}$$

is the unique homomorphism of U into $T(S)$ which extends $(\tau_S\psi)^{-1}\tau_S$. Moreover, ξ is injective since $\tau_S\psi : S \longrightarrow U$ is a dense ideal extension. Then, applying Proposition 3.12 and the fact that ψ is injective, we have for $H \in T(S)$ and $s \in S$,

$$\begin{aligned} s \in H &\Leftrightarrow (s] \subseteq H \\ &\Leftrightarrow (s] \leq H \text{ in } T(S) \\ &\Leftrightarrow s\tau_S\psi \leq H\psi \text{ in } T(S)\psi \\ &\Leftrightarrow s\tau_S\psi \leq H\psi \text{ in } U \\ &\Leftrightarrow s\tau_S\psi \in (H\psi] \cap S\tau_S\psi \\ &\Leftrightarrow s \in ((H\psi] \cap S\tau_S\psi)(\tau_S\psi)^{-1} \\ &\Leftrightarrow s \in H\psi\xi \end{aligned}$$

and thus $H = H\psi\xi$ for every $H \in T(S)$. In particular $\xi : U \longrightarrow T(S)$ is onto, and since ξ is also injective, it follows that ψ and ξ are pairwise inverse isomorphisms. ∎

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Luís Oliveira
Departamento de Matemática Pura, F.C.U.P.,
R. Campo Alegre 687,
4169-007 Porto, Portugal
loliveir@fc.up.pt

Francis J. Pastijn
Department of Mathematics, Statistics and Computer Science
Marquette University
Milwaukee, WI 53201-1881, U.S.A.
francisp@mscs.mu.edu