A note on an analog of Morgan's theorem for the Kontorovich-Lebedev transform

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Abstract

We establish an analog of the Morgan theorem for the Kontorovich-Lebedev operator of general complex index.

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Let $f(t)$ be a complex-valued measurable function defined on $t \in \mathbf{R}_+ = (0, \infty)$. We deal with the following Kontorovich-Lebedev operator of general complex index

$$K_{iz}[f] = \int_0^\infty K_{iz}(t) f(t) dt, \; z = x + iy \; \in \mathbb{C}, \tag{1}$$

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which is a modification of the Kontorovich-Lebedev transformation ($y = 0$) (cf. [5], [6]) with respect to an index of the Macdonald function $K_{iz}(t)$ [2]. As it is known, the function $K_\nu(t)$ satisfies the differential equation

$$t^2 \frac{d^2u}{dt^2} + t\frac{du}{dt} - (t^2 + \nu^2)u = 0, \tag{2}$$

for which it is the solution that remains bounded as t tends to infinity on the real line. The Macdonald function has the asymptotic behaviour [2]

$$K_\nu(t) = \left(\frac{\pi}{2t}\right)^{1/2} e^{-t}[1 + O(1/t)], \qquad t \to \infty, \tag{3}$$

and near the origin

$$t^{|\mathrm{Re}\nu|} K_\nu(t) = O(1), \ t \to 0, \tag{4}$$

$$K_0(t) = -\log t + O(1), \ t \to 0. \tag{5}$$

It can be defined by the following integral representations [5], [6]

$$K_\nu(t) = \int_0^\infty e^{-t\cosh u} \cosh \nu u du, \ t > 0, \tag{6}$$

$$K_\nu(t) = \frac{1}{2}\left(\frac{t}{2}\right)^\nu \int_0^\infty e^{-u-\frac{t^2}{4u}} u^{-\nu-1} du, \ t > 0. \tag{7}$$

Hence we easily find that $K_\nu(t)$ is a real-valued positive function when $\nu \in \mathbb{R}$ and an even function with respect to the index ν. Moreover, it satisfies the following inequality

$$|K_\nu(t)| \leq K_{\mathrm{Re}\nu}(t), \ t > 0. \tag{8}$$

Applying (8) to the kernel $K_{iz}(t)$ and using (6) we conclude, that this is an entire function with respect to z satisfying the inequality

$$|K_{iz}(t)| \leq K_y(t), \ t > 0, \ z = x + iy. \tag{9}$$

For the function $K_y(t)$ we will use the following estimate (see [7])

$$K_y(t) \leq \frac{\text{const.}}{\sqrt{t}} \exp\left[\frac{|y|^\alpha}{\alpha} + \frac{(2([\beta]+1))!}{4t}\right], \ t > 0, \ y \in \mathbb{R}, \tag{10}$$

where $\alpha, \beta > 1, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1$, $[\beta]$ is an integer part of β and the constant does not depend on $t, \ y$.

Let us assume that f in (1) satisfies the condition

$$f(t) \in L_1\left(\mathbb{R}_+; \exp\left[\frac{(2([\beta]+1))!}{4t}\right] \frac{dt}{\sqrt{t}}\right), \ \beta > 2, \tag{11}$$

i.e. f is integrable in the weighted Lebesgue space

$$\int_0^\infty |f(t)| \exp\left[\frac{(2([\beta]+1))!}{4t}\right] \frac{dt}{\sqrt{t}} < \infty,\ \beta > 2.$$

Hence taking (1), we estimate the absolute value of the Kontorovich-Lebedev transform. Indeed, invoking (10) and (11) we obtain the estimate

$$|K_{iz}[f]| \le \int_0^\infty |K_{iz}(t)f(t)|dt$$

$$\le \text{const.}e^{\frac{|y|^\alpha}{\alpha}} \int_0^\infty |f(t)| \exp\left[\frac{(2([\beta]+1))!}{4t}\right] \frac{dt}{\sqrt{t}} = \text{const.}e^{\frac{|y|^\alpha}{\alpha}},$$

which drives us to the inequality

$$|K_{i(x+iy)}[f]| \le \text{const.}e^{\frac{|y|^\alpha}{\alpha}} \tag{12}$$

for all $x, y \in \mathbb{R}$ and $\alpha = \frac{\beta}{\beta-1} \in]1,2[$. Therefore $K_{iz}[f]$ is entire on $\mathbb{C}$. Furthermore, assuming that its restriction on $\mathbb{R}$, which is the classical Kontorovich-Lebedev transform $K_{ix}[f]$, belongs to $L_p\left(\mathbb{R}; e^{\rho|x|^\alpha}dx\right)$, $p \in [1,\infty]$ with $\rho > \frac{1}{\alpha}\sin\frac{\pi}{2}(\alpha-1)$, we apply the Phragmen- Lindelöf type lemma from [1] in order to conclude that $K_{iz}[f] \equiv 0$.

On the other side condition (11) and integral representation (6) with Fubini's theorem guarantee the following composition equality

$$K_{iz}[f] = \frac{1}{2}\int_{-\infty}^\infty e^{ixu-yu}du \int_0^\infty e^{-t\cosh u} f(t)dt \tag{13}$$

in terms of the Fourier and Laplace transforms. Therefore the conclusion above $K_{iz}[f] \equiv 0$ yields

$$\int_0^\infty e^{-t\cosh u} f(t)dt = 0, \tag{14}$$

because

$$e^{yu}\int_0^\infty e^{-t\cosh u} f(t)dt \in L_1(\mathbb{R}; du)$$

for any $y \in \mathbb{R}$. In fact, we have (see (10))

$$\frac{1}{2}\int_{-\infty}^\infty e^{-yu}\left|\int_0^\infty e^{-t\cosh u} f(t)dt\right| du \le \frac{1}{2}\int_{-\infty}^\infty e^{-yu}du \int_0^\infty e^{-t\cosh u}|f(t)|dt$$

$$= \int_0^\infty K_y(t)|f(t)|dt < \infty.$$

This means that $f \in L_1(\mathbb{R}_+; K_0(t)dt)$. Consequently, from (14) and discussions in [7], Section 3 we get that $f = 0$ almost for all $t > 0$. Thus we have proved the following analog of the Morgan theorem [3] for the Kontorovich-Lebedev transform.

Theorem. *Let* $p \in [1, \infty]$ *and*

$$f(t) \in L_1 \left(\mathbb{R}_+; \exp\left[\frac{(2([\beta]+1))!}{4t} \right] \frac{dt}{\sqrt{t}} \right), \ \beta > 2.$$

Let the Kontorovich-Lebedev transform $K_{ix}[f]$, *which is a restriction of the general Kontorovich-Lebedev operator* (1) *on the real line, belong to the space*

$$L_p \left(\mathbb{R}; e^{\rho |x|^\alpha} dx \right), \alpha = \frac{\beta}{\beta - 1},$$

where $\rho > \frac{1}{\alpha} \sin \frac{\pi}{2}(\alpha - 1)$. *Then* $f(t)$ *is null almost for all* $t > 0$.

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