# A MUNN TREE TYPE REPRESENTATION FOR THE ELEMENTS OF THE BIFREE LOCALLY INVERSE SEMIGROUP

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Abstract. The Munn tree representation for the elements of the free inverse monoid is an elegant and useful tool in the theory of inverse semigroups. It has been the starting point for many of the subsequent developments in this theory. In the present paper we generalize this representation for the elements of the bifree locally inverse semigroup. We will represent each element of the bifree locally inverse semigroup as an undirected tree whose vertices, called blocks, are special vertex-labeled graphs themselves. Another distinctive characteristic of these graphs is that they have different types of edges.

## 1. Introduction

A semigroup $S$ is called *regular* if for every $x \in S$ there exists an $x' \in S$ such that $xx'x = x$ and $x'xx' = x'$. The element $x'$ is called an *inverse* of $x$. In a regular semigroup, each element $x$ can have many distinct inverses. If the regular semigroup $S$ has the property that each element has a unique inverse, then $S$ is called an *inverse semigroup*. Inverse semigroups are usually seen as unary semigroups where the unary operation is the inverse operation. As unary semigroups, the class of all inverse semigroups constitutes a variety of algebras.

There have been two usual approaches when studying free inverse semigroups: Scheiblich's approach and Munn's approach. Scheiblich's approach [16, 17] is based on a notion of reduced words and the elements of the free inverse monoid are represented as pairs $(A, w)$ where $A$ is a special set of reduced words containing $w$. Munn's approach [10] is based on a description of the elements of the free inverse monoid as edge-labeled directed graphs with a pair of distinguished vertices. Both approaches have been influential in the development of the theory of inverse semigroups, but also in the development of the general theory of semigroups. For a taste of the importance of Munn's approach to the development of inverse semigroup theory see, for example, [5, 7, 8, 9, 18].

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If $x'$ is an inverse of $x$, then $xx'$ and $x'x$ are idempotents in $S$. As usual, we denote by $E(S)$ the set of idempotents of $S$. For $f \in E(S)$, let

$$(f]_{\mathscr{R}} = \{e \in E(S) \,:\, e = fe\} \quad \text{and} \quad (f]_{\mathscr{L}} = \{e \in E(S) \,:\, e = ef\}\,;$$

and set $(f]_{\leq} = (f]_{\mathscr{R}} \cap (f]_{\mathscr{L}} = \{e \in E(S) \,:\, e = ef = fe\}$. A regular semigroup is called *locally inverse* if for every $f, g \in E(S)$, there exists (a unique) $e \in E(S)$ such that $(f]_{\mathscr{R}} \cap (g]_{\mathscr{L}} = (e]_{\leq}$; we will denote this idempotent $e$ by $f \wedge g$. Often, locally inverse semigroups are seen as binary semigroups with the second binary operation $\wedge$ defined by $a \wedge b = (aa') \wedge (bb')$ for inverses $a'$ and $b'$ of $a$ and $b$, respectively. We will call this second binary operation on a locally inverse semigroup its *pseudosemilattice* operation. The terminology "locally inverse" comes from another equivalent description for these semigroups: they are the regular semigroups whose local submonoids (the submonoids $eSe$ for $e \in E(S)$) are inverse semigroups. In fact, there are many equivalent descriptions for the locally inverse semigroups (see [15, Corollary 2.3]).

The class of all locally inverse semigroups constitutes an *e-variety* of regular semigroups [4, 6]: a class of regular semigroups closed for homomorphic images, direct products and regular subsemigroups. Unlike the concept of variety of algebras that has free objects, in general there is no object in an e-variety with a similar universal property [19]. However, that is not the case for e-varieties of locally inverse semigroups. Every e-variety of locally inverse semigroups has a *bifree object* on every non-empty set $X$ [19].

Given a non-empty set $X$, let $x \to x'$ be a one-to-one mapping from $X$ onto a disjoint copy $X'$ of $X$. A mapping $\theta : X \cup X' \to S$, where $S$ is a regular semigroup, is called *matched* if $x'\theta$ is an inverse of $x\theta$ in $S$. Let $\mathbf{V}$ be an e-variety of locally inverse semigroups. A bifree object on $X$ in $\mathbf{V}$ is a semigroup $S \in \mathbf{V}$, together with a matched mapping $\iota : X \cup X' \to S$, such that for any other matched mapping $\theta : X \cup X' \to T$ with $T \in \mathbf{V}$, there exists a (unique) homomorphism $\varphi : S \to T$ satisfying $\theta = \iota \circ \varphi$.

We denote the e-variety of all locally inverse semigroups by $\mathbf{LI}$ and the bifree object on $X$ in $\mathbf{LI}$ by $\mathrm{BFLI}(X)$. The semigroup $\mathrm{BFLI}(X)$ is called the *bifree locally inverse semigroup* on the set $X$. A model for the bifree locally inverse semigroup has been presented in [2] where the elements are represented as complex sets of canonical forms for a specific type of words. We can look at this model as the natural generalization, for the bifree locally inverse semigroup case, of Scheiblich's approach to the free inverse monoid. We will recall it in Section 2 since it will be useful to us.

In the present paper we will construct another model for $\mathrm{BFLI}(X)$. In our model, the elements of $\mathrm{BFLI}(X)$ are represented as special vertex-labeled trees. Thus, our approach is similar to Munn's approach to the free inverse monoid. In our opinion, this graph representation of the elements of $\mathrm{BFLI}(X)$ captures better the structure of this semigroup. In particular, it contains information to give an easier description for the Green's relations and for the natural partial order on $\mathrm{BFLI}(X)$.

The trees we will consider have a few peculiar characteristics that distinguish them from the Munn trees. Firstly, our trees are undirected trees with labels on the vertices (and not on the edges as happens for the Munn trees). But the more distinctive facts about them are the following: the vertices are vertex-labeled graphs themselves and the edges can be of different types. We will call the vertices of our graphs *blocks* and we will use the term vertex to refer to the vertices of the blocks. A block will be just a "square" graph (four vertices with four edges making a cycle). The label of the block will induce a labeling for its vertices. As for the Munn tree representation, we will need to distinguish two vertices (and not two blocks) in our graphs.

In the next section we recall some results needed to prove that our construction gives a model for BFLI($X$). The notions of birooted bipartite block-trees and of reduced birooted bipartite block-trees are introduced in Section 3. We will denote by $\mathfrak{A}(X)$ the set of all reduced birooted bipartite block-trees. Two binary operations $\cdot$ and $\wedge$ on $\mathfrak{A}(X)$ will be introduced also in Section 3. In Section 4 we prove that $(\mathfrak{A}(X), \cdot)$ is a model for the bifree locally inverse semigroup on the set $X$, while in Section 5 we study the Green's relations and the natural partial order on $\mathfrak{A}(X)$. As a consequence, we conclude that the operation $\wedge$ introduced in Section 3 is the pseudosemilattice operation on the locally inverse semigroup $\mathfrak{A}(X)$.

The Munn tree representation gives us an easy description of the least group congruence $\sigma$ on the free inverse monoid: two Munn trees belong to the same $\sigma$-class if and only if they have the same simple path connecting their distinguished vertices. Thus, it is possible to modify Munn's product of trees in a natural way that allows us to regard the elements of the free group as Munn chains (Munn trees that are chains and have the distinguished vertices at the endpoints) with this modified product. Thus, the Munn tree representation reflects the straight bound between the free inverse monoid and the free group.

We know from [2] that there exists a similar straight bound between the bifree locally inverse semigroup and the bifree completely simple semigroup. In Section 6 we will analyze this relationship on the model we construct. If $\rho$ denotes the least completely simple congruence on BFLI($X$), we will see that two reduced birooted block-trees belong to the same $\rho$-class if and only if they have the same simple path connecting the distinguished vertices. We will then present a natural modified product for reduced birooted block-chains (the block-trees that are chains and have the distinguished vertices at blocks in the opposite endpoints of the chain) and find another model for the bifree completely simple semigroup.

## 2. Preliminaries

If $A$ is an "alphabet", let $A^+$ be the set of all words in the alphabet $A$ (that is, the free semigroup on $A$ with the concatenation as its operation) and let $A^* = A^+ \cup \{1\}$ (the free monoid on $A$) where 1 denotes the empty word. Let $\lambda u$ and $u\tau$ be the first and last letters of $u$, respectively. We will denote by 1 the empty word in any alphabet. Fix $X$ as a non-empty set for the entire paper and let $X'$ be a disjoint copy of $X$ such that $x \to x'$ is a bijection from $X$ onto $X'$. We will denote by $\overline{X}$ the set $X \cup X'$. If $a = x' \in X'$, then $a'$ will denote the element $x \in X$. Thus, we can look at $'$ as an involution on $\overline{X}$. We will call $a'$ the *inverse letter* of $a$.

In general, there is no natural way of defining a binary operation on the set of idempotents of a regular semigroup $S$. What is usually done is to consider $E(S)$ endowed with a natural partial multiplication. These partial algebras were introduced and studied by Nambooripad [11] who called them biordered sets. However, the case is different if $S$ is locally inverse: we can define a binary operation $\wedge$ on $E(S)$ by setting $e \wedge f$, for any $e, f \in E(S)$, as the unique idempotent $g$ such that $(e]_{\mathscr{R}} \cup (f]_{\mathscr{L}} = (g]_{\leq}$. The binary algebras $(E(S), \wedge)$ were characterized by Nambooripad [13]. They are called *pseudosemilattices* and constitute a variety of binary algebras.

Let $S$ be a locally inverse semigroup. Since for any $a, b \in S$, $aa' \wedge b'b$ gives always the same idempotent independently of the chosen inverses $a'$ of $a$ and $b'$ of $b$, we can extend the operation $\wedge$ to the whole semigroup $S$ by defining $a \wedge b = aa' \wedge b'b$. Thus, we can regard a locally inverse semigroup as either a semigroup $(S, \cdot)$ or a binary semigroup $(S, \cdot, \wedge)$. Nevertheless, the $\wedge$ operation is completely determined by the $\cdot$ operation, and any homomorphism between locally inverse semigroups always respects $\wedge$ [19]. Usually, the term pseudosemilattice operation refers to the restriction of $\wedge$ to the set of idempotents $E(S)$. However, for our convenience, we will adopt a more general setting in this paper and we will refer to the binary operation $\wedge$ on the whole locally inverse semigroup $S$ as its pseudosemilattice operation.

For the purpose of this paper, it is irrelevant if we look at a locally inverse semigroup as a type $\langle 2 \rangle$ algebra or a type $\langle 2, 2 \rangle$ algebra. We choose to look at them as type $\langle 2, 2 \rangle$ algebras. The following immediate observation will be used later: $a \wedge a' = aa'$ for any $a \in S$ and any inverse $a'$ of $a$ in $S$.

Let $F$ be the free semigroup on the set $\overline{X} \cup \{(a \wedge b) \,:\, a, b \in \overline{X}\}$. The strings $(a \wedge b)$ are considered as new letters in this context. Auinger introduced in [2] a notion of reduced words for the elements of $F$ and a well-defined product (associative operation) $\odot$ on the set $\mathsf{s}(F)$ of reduced words. He introduced another binary operation $\wedge$ on $\mathsf{s}(F)$ and he proved that $(s(F), \odot, \wedge)$ is a model for the bifree completely simple semigroup. Note that the class of all completely simple semigroups is an e-variety of locally inverse semigroups, and thus it has bifree objects. In that same paper, Auinger constructed a model for the bifree locally inverse semigroup. In his model for $\mathrm{BFLI}(X)$, the elements are represented as pairs $(A, u)$ where $A$ is a special subset of $s(F)$ and $u \in A$. The goal of the present section is to recall Auinger's model for the bifree locally inverse semigroup. However, we will describe it from a different perspective. This will allow us to prove more easily that our model is isomorphic to Auinger's model.

A word $u \in F$ is called reduced if it does not contain a subword of one of the following forms: $(i)$ $a(b \wedge a)$, $(ii)$ $(b \wedge a)b$, $(iii)$ $(a \wedge b)(a \wedge c)$, $(iv)$ $(c \wedge a)(b \wedge a)$ and $(v)$ $aa'$, for $a, b, c \in \overline{X}$. For a word $w \in F$ there exists a unique reduced word $\mathsf{s}(w)$ that can be obtained from $w$ by applying successively the reductions:

$(i')$ $a(b \wedge a) \to a$,
$(ii')$ $(b \wedge a)b \to b$,
$(iii')$ $(a \wedge b)(a \wedge c) \to (a \wedge c)$,
$(iv')$ $(c \wedge a)(b \wedge a) \to (c \wedge a)$,
$(v')$ $aa' \to (a \wedge a')$,

where $a, b, c \in \overline{X}$. Let $\mathsf{s}(F)$ be the set of all reduced words of $F$ and for $u, v \in \mathsf{s}(F)$, define $u \odot v = \mathsf{s}(uv)$. If we define also $u \oslash v = (\lambda u \wedge v\tau) \in \mathsf{s}(F)$ for any $u, v \in \mathsf{s}(F)$, then $(\mathsf{s}(F), \odot, \oslash)$ is a model for the bifree completely simple semigroup on the set $X$ ([2, Theorem 3.7]).

Now, we will introduce an alternative description for the reduced words used in Auinger's model. Given a word $u \in F$, let $\overline{u}$ be the string obtained form $u$ as follows:

$(a)$ delete all parentheses;
$(b)$ delete the symbols $\wedge$ whose letter following the symbol is the inverse letter of the letter preceding the symbol ($a \wedge a' \to aa'$).

Thus, $\overline{u}$ is a sequence of words from $\overline{X}^+$ separated by the symbol $\wedge$ such that any word between two consecutive $\wedge$ has length greater than 1. In fact, any string with the previous characteristics is of the form $\overline{u}$ for some $u \in F$: if, for each symbol $\wedge$, we add "(" before the letter that precedes that symbol and add ")" after the letter that follows that symbol, then we get a word $u \in F$ in the desired conditions. However, if we apply the procedure just described to some string $\overline{v}$ with $v \in F$, we may get some $u \neq v$ (obviously $\overline{v} = \overline{u}$): a substring $aa'$ of $\overline{v}$ neither preceded nor followed by the symbol $\wedge$ can come from either the subwords $aa'$ or $(a \wedge a')$. In spite of the previous observation, if $u \in F$ is reduced, it is possible recover $u$ from $\overline{u}$: we proceed as explained above for the parentheses, and then we replace each subword $aa'$ with $(a \wedge a')$; we get $u$ again. Thus, if $\overline{F}$ and $\mathsf{t}(\overline{F})$ are the sets of all strings $\overline{u}$ for $u \in F$ or $u \in \mathsf{s}(F)$, respectively, then $u \to \overline{u}$ is a bijection from $\mathsf{s}(F)$ onto $\mathsf{t}(\overline{F})$.

Both $\overline{F}$ and $\mathsf{t}(\overline{F})$ are subsets of $(\overline{X} \cup \{\wedge\})^+$. The strings of $\mathsf{t}(\overline{F})$ are the words $w \in (\overline{X} \cup \{\wedge\})^+$ with $\lambda w \neq \wedge$ and $w\tau \neq \wedge$, and not containing the following subwords:

(1) $\wedge a \wedge$ for $a \in \overline{X} \cup \{1\}$;
(2) $ab \wedge a$ and $a \wedge ba$ for $a, b \in \overline{X}$;
(3) $aa'a$ for $a \in \overline{X}$ unless this subword is preceded and followed by $\wedge$;
(4) $a \wedge a'$ for $a \in \overline{X}$.

Note that $\lambda w \neq \wedge$ and $w\tau \neq \wedge$ together with (1) mean that $w \in \overline{F}$; (2) corresponds to $(i)-(iv)$ for the case $b \neq a'$, while (3) corresponds to $(i)-(iv)$ for the case $b = a'$; and (4) corresponds to $(v)$.

If $u, v \in F$, then $\overline{u}\,\overline{v} = \overline{uv} \in \overline{F}$. Further, the reductions $(i') - (v')$ can be rewritten for $\overline{F}$:

(1′) $a \wedge ba \to a$ and $ab \wedge a \to a$ for $a, b \in \overline{X}$ with $b \neq a'$;
(2′) $aa'a \to a$ for $a \in \overline{X}$ unless $aa'a$ is preceded and followed by $\wedge$;
(3′) $a \wedge a' \to aa'$ for $a \in \overline{X}$.

In $(\mathsf{s}(F), \odot, \oslash)$, Auinger chose to use the notation $a \wedge a'$ instead of $aa'$ (note that $a \wedge a' = aa'$ in any locally inverse semigroup). In our description of Auinger's model, we choose the notation $aa'$ instead of $a \wedge a'$. Therefore, we must not consider $(v')$ but its reverse (3′) instead. Note that (1′) corresponds to $(i') - (iv')$ for the case $b \neq a'$; and (2′) together with (3′) correspond to $(i') - (iv')$ for $b = a'$.

From the previous observations, if $u \in F$, then the reduced form of $\overline{u}$ for $(1') - (3')$ exists and it is $\overline{\mathsf{s}(u)}$. We will denote by $\mathsf{t}(\overline{u})$ the reduced form of $\overline{u}$, that is, $\mathsf{t}(\overline{u}) = \overline{\mathsf{s}(u)}$. Thus $\mathsf{t}(\overline{F}) = \{\mathsf{t}(\overline{u}) \,:\, \overline{u} \in \overline{F}\}$. Define $\odot$ and $\oslash$ on $\mathsf{t}(\overline{F})$ as follows: for $u, v \in \mathsf{s}(F)$,

$$\overline{u} \odot \overline{v} = \mathsf{t}(\overline{u}\,\overline{v}) = \mathsf{t}(\overline{uv}) \quad \text{and} \quad \overline{u} \oslash \overline{v} = \begin{cases} \lambda\overline{u} \wedge \overline{v}\tau & \text{if } \overline{v}\tau \neq (\lambda\overline{u})' \\ \lambda\overline{u}\,(\lambda\overline{u})' & \text{othewise.} \end{cases}$$

Hence $\overline{u} \odot \overline{v} = \mathsf{t}(\overline{uv}) = \overline{u \odot v}$ and $\overline{u} \oslash \overline{v} = \overline{u \oslash v}$. We proved the following result:

**Proposition 2.1.** *The mapping $u \to \overline{u}$ is an isomorphism from $(\mathsf{s}(F), \odot, \oslash)$ onto $(\mathsf{t}(\overline{F}), \odot, \oslash)$, and $(\mathsf{t}(\overline{F}), \odot, \oslash)$ is a model for the bifree completely simple semigroup on $X$.*

Next, we will describe Auinger's model for the bifree locally inverse semigroup on a set $X$ using sets of words from $\mathsf{t}(\overline{F})$. The following statements are due to [2, Theorem 4.22]. Let $\hat{\mathsf{s}} : F \to \mathcal{P}(\mathsf{s}(F))$ be the mapping defined inductively as follows. Set

$$\hat{\mathsf{s}}(a) = \{a, (a \wedge a')\} \text{ and } \hat{\mathsf{s}}((a \wedge b)) = \{a, (a \wedge a'), (a \wedge b), (a \wedge b) \odot b'\}$$

for $a, b \in \overline{X}$. If $\hat{\mathsf{s}}(u)$ and $\hat{\mathsf{s}}(v)$ are already defined for $u, v \in F$, then set

$$\hat{\mathsf{s}}(uv) = \hat{\mathsf{s}}(u) \cup \mathsf{s}(u) \odot \hat{\mathsf{s}}(v)$$

where $\odot$ is used also to denote the complex product of subsets of $\mathsf{s}(F)$. The mapping $\hat{\mathsf{s}}$ is well defined, that is, it is independent of the factorization of the word $uv$.

Since $\hat{\mathsf{s}}(aa') = \{a, (a \wedge a')\} = \hat{\mathsf{s}}((a \wedge a'))$ for $a \in \overline{X}$, if $\overline{u} = \overline{v}$ for $u, v \in F$, then $\hat{\mathsf{s}}(u) = \hat{\mathsf{s}}(v)$. Thus, we can adapt $\hat{\mathsf{s}} : F \to \mathcal{P}(\mathsf{s}(F))$ to a mapping $\hat{\mathsf{t}} : \overline{F} \to \mathcal{P}(\mathsf{t}(\overline{F}))$ as follows. Set

$$\hat{\mathsf{t}}(a) = \{a, aa'\} \quad \text{and} \quad \hat{\mathsf{t}}(a \wedge b) = \{a, aa', a \wedge b, a \wedge bb'\}$$

for $a, b \in \overline{X}$ with $b \neq a'$; and if $\hat{\mathsf{t}}(\overline{u})$ and $\hat{\mathsf{t}}(\overline{v})$ are already defined for $\overline{u}, \overline{v} \in \overline{F}$, set

$$\hat{\mathsf{t}}(\overline{u}\,\overline{v}) = \hat{\mathsf{t}}(\overline{u}) \cup \mathsf{t}(\overline{u}) \odot \hat{\mathsf{t}}(\overline{v})\,.$$

Note that $\mathsf{t}(\overline{u}) \in \hat{\mathsf{t}}(\overline{u})$ and let $Y = \{\hat{\mathsf{t}}(\overline{u}) \;:\; u \in F\} \subseteq \mathcal{P}(\mathsf{t}(\overline{F}))$. Then $A \in Y$ if and only if

(*i*) $\lambda w_1 = \lambda w_2$ for any $w_1, w_2 \in A$;

(*ii*) if $w_1 w_2 \in A$ for strings $w_1, w_2 \in (\overline{X} \cup \{\wedge\})^+$ such that $w_1 \in \overline{F}$ and $w_1 \neq w_3 a a' a$ for any $w_3 \in (\overline{X} \cup \{\wedge\})^+$ and any $a \in \overline{X}$, then $w_1 \in A$;

(*iii*) if $w_1 a \in A$ for a string $w_1 \in (\overline{X} \cup \{\wedge\})^*$ and a letter $a \in \overline{X}$ such that $w_1 \tau \neq a'$, then $w_1 a a' \in A$.

Let $T$ be the set of all pairs $(A, w) \in Y \times \mathsf{t}(\overline{F})$ such that $w \in A$. Define a product in $T$ as follows: for $(A_1, w_1), (A_2, w_2) \in T$, set

$$(A_1, w_1) \cdot (A_2, w_2) = (A_1 \cup w_1 \odot A_2, w_1 \odot w_2).$$

Then $T$ becomes a locally inverse semigroup with this product. Let $\wedge$ be the pseudosemilattice operation on the locally inverse semigroup $(T, \cdot)$ induced by the product $\cdot$ . We can now restate Theorem 4.22 of [2] as follows:

**Theorem 2.2.** *The locally inverse semigroup* $(T, \cdot, \wedge)$ *is a model for the bifree locally inverse semigroup on the set* $X$.

As a consequence of this theorem and of Proposition 2.1 we obtain the following obvious corollary. Let $\rho$ be the least completely simple congruence on $T$, that is, the smallest congruence $\sigma$ on $T$ such that $T/\sigma$ is a completely simple semigroup.

**Corollary 2.3.** *Let* $(A_1, w_1)$, $(A_2, w_2) \in T$. *Then* $(A_1, w_1)\,\rho\,(A_2, w_2)$ *if and only if* $w_1 = w_2$.

We can find a concrete description for the binary operation $\wedge$ on $T$ following Auinger's results. However, giving such a description would take us some unnecessary effort since we will not need it for this paper. Note that to prove that $(S, \cdot, \wedge)$ is a model for the bifree locally inverse semigroup, it is enough to prove that $(S, \cdot)$ is isomorphic to $(T, \cdot)$ and that $\wedge$ is the pseudosemilattice operation on the locally inverse semigroup $(S, \cdot)$. Thus, we leave the description of the binary operation $\wedge$ on $T$ for the reader to find following results from [2].

We end this section with a result that follows from [2, Corollary 4.7] and Theorem 2.2.

**Proposition 2.4.** *The semigroup* $(T, \cdot)$ *is generated (as a semigroup) by the set* $A = \{(\{a, aa'\}, a),\ (\{a, aa', a \wedge b, a \wedge bb'\}, a \wedge b) \;:\; a, b \in \overline{X}$ *with* $b \neq a'\}$ .

## 3. Birooted bipartite block-trees (or rbb-trees)

In this section we introduce the graphs that will be used in the next section to describe another model for the bifree locally inverse semigroup on a set $X$. These graphs are just trees with an additional graph structure associated with its vertices that influences also the edges.

We begin by describing the vertices of our graphs. We will call them *blocks*. A block $\mathfrak{b}$ has 4 vertices which are called the *left positive*, the *left negative*, the *right positive* and the *right negative* vertices of that block. Thus, each vertex is characterized by two properties: its *side* (left or right) and its *pole* (positive or negative). For shortness, we will refer to them, respectively, as $l^+$-vertex, $l^-$-vertex, $r^+$-vertex and $r^-$-vertex. Further, let $\mathfrak{b}^{l+}$, $\mathfrak{b}^{l-}$, $\mathfrak{b}^{r+}$ and $\mathfrak{b}^{r-}$ be the notation we will use for those vertices of $\mathfrak{b}$. A block has also 4 edges connecting the vertices with opposite pole and a label, a letter $x \in X$. This label extends to the vertices of the block in the following way: positive vertices are labeled also with $x$ and negative vertices are labeled with $x'$. We denote by $\mathrm{l}(\mathfrak{b})$ the label of the block $\mathfrak{b}$ and by $\mathrm{l}(\mathfrak{b}^i)$ the label of its vertex $\mathfrak{b}^i$. Usually, we will write $\mathfrak{b}^i$ to refer to one of the vertices of the block $\mathfrak{b}$; thus $i \in \{l+, l-, r+, r-\}$.

We need to establish a way of drawing the blocks. These drawings should be as simple as possible since they will be considered the vertices of the graphs we will introduce later. Thus, we represent each block as a square with its label at the center. The corners of the square represent the four vertices of the block and the sides of the square represent the four edges of the block. We will stipulate that the upper left corner represents the $l^+$-vertex, the upper right corner represents the $r^-$-vertex, the lower right corner represents the $r^+$-vertex and the lower left corner represents the $l^-$-vertex. We illustrate the representation of a block labeled with $x \in X$ in Figure 1.

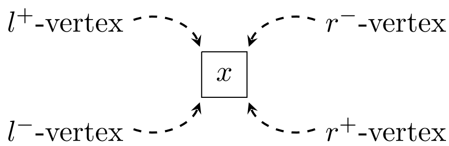



FIGURE 1. A block labeled with $x \in X$.

A *block-graph* $\gamma$ is a connected graph whose vertices are blocks and whose edges are pairs $\{\mathfrak{b}^i, \mathfrak{c}^j\}$ for $i, j \in \{l+, l-, r+, r-\}$ and blocks $\mathfrak{b}$ and $\mathfrak{c}$ of $\gamma$. The edge $\{\mathfrak{b}^i, \mathfrak{c}^j\}$ connects the blocks $\mathfrak{b}$ and $\mathfrak{c}$ of $\gamma$. In some sense we can say that we have many distinct types of edges in a block-graph. There are two other conditions we will impose to have a block-graph: $(a)$ $\gamma$ has no loops, that is, there is no edge $\{\mathfrak{b}^i, \mathfrak{b}^j\}$ in $\gamma$; and $(b)$ there is at most one edge connecting two blocks of $\gamma$ (that is, $\gamma$ is not a multigraph). A block-graph $\gamma$ will be called:

$(i)$ a *block-tree* if $\gamma$ is a tree (the cycles inside the blocks are not taken into consideration);

(*ii*) a *bipartite block-graph* if $\mathfrak{b}^i$ and $\mathfrak{c}^j$ have different sides for any edge $\{\mathfrak{b}^i, \mathfrak{c}^j\}$ of $\gamma$, that is, one of $\mathfrak{b}^i$ and $\mathfrak{c}^j$ is a left vertex while the other is a right vertex;

(*iii*) a *birooted block-graph* if it has highlighted one left vertex and one right vertex belonging to some blocks of $\gamma$ (these vertices may belong to different blocks or not).

A *birooted bipartite block-tree* is a block-graph with the previous three characteristics. For our convenience we will use the abbreviation *rbb-tree*.

We denote by $\mathfrak{l}_\gamma$ the highlighted left vertex of a block of $\gamma$ and call it the *left root* of $\gamma$. Similarly, $\mathfrak{r}_\gamma$ denotes the highlighted right vertex of a block of $\gamma$ and is called the *right root* of $\gamma$. In some places, when manipulating a rbb-tree $\gamma$, it will be useful to refer also to its roots. In those cases we will represent $\gamma$ as the triple

$$(\mathfrak{l}_\gamma, \gamma, \mathfrak{r}_\gamma)\,.$$

We denote also the block of $\gamma$ containing $\mathfrak{l}_\gamma$ by $\mathfrak{l}_\gamma^{\mathfrak{b}}$ and call it the *left block* of $\gamma$. Similarly, $\mathfrak{r}_\gamma^{\mathfrak{b}}$ denotes the block of $\gamma$ containing $\mathfrak{r}_\gamma$ and it is called the *right block* of $\gamma$. Note that, in some cases, the left block and the right block can be the same block.

We will draw a block-tree $\gamma$ as follows: each block is drawn following the conventions stated earlier for them; and each edge $\{\mathfrak{b}^i, \mathfrak{c}^j\}$, with $i, j \in \{l+, l-, r+, r-\}$, between two blocks $\mathfrak{b}$ and $\mathfrak{c}$ of $\gamma$ is drawn as a line connecting the corner representing $\mathfrak{b}^i$ in $\mathfrak{b}$ to the corner representing $\mathfrak{c}^j$ in $\mathfrak{c}$. If $\gamma$ is a rbb-tree $(\mathfrak{l}_\gamma, \gamma, \mathfrak{r}_\gamma)$, then we will draw an arrow pointing to the corner representing $\mathfrak{l}_\gamma$ in $\mathfrak{l}_\gamma^{\mathfrak{b}}$ to identify the left root of $\gamma$ and another arrow coming out from the corner representing $\mathfrak{r}_\gamma$ in $\mathfrak{r}_\gamma^{\mathfrak{b}}$ to identify the right root of $\gamma$.

To help understanding this representation, we present an example next. The rbb-tree of our example is $\eta$ composed by the four blocks $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ and $\mathfrak{d}$, and with the following set of edges:

$$\{\{\mathfrak{b}^{r-}, \mathfrak{c}^{l+}\}, \{\mathfrak{b}^{r-}, \mathfrak{a}^{l+}\}, \{\mathfrak{b}^{l-}, \mathfrak{d}^{r+}\}\}\,.$$

The left root of $\eta$ is $\mathfrak{l}_\eta = \mathfrak{b}^{l-}$ and the right root of $\eta$ is $\mathfrak{r}_\eta = \mathfrak{c}^{r+}$. Further, the labels of the blocks are $\mathrm{l}(\mathfrak{a}) = y = \mathrm{l}(\mathfrak{c})$ and $\mathrm{l}(\mathfrak{b}) = x = \mathrm{l}(\mathfrak{d})$. The graphic representation of $\eta$ appears in Figure 2.

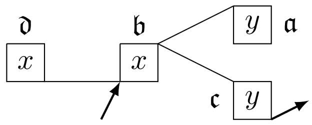



FIGURE 2. The graphic representation of the rbb-tree $\eta$ (usually we don't write the names of the blocks).

Let $\mathfrak{B}(X)$ be the set of all rbb-trees with blocks labeled with letters of $X$. We define two operations $\odot$ and $\owedge$ on $\mathfrak{B}(X)$ as follows. Let $(\mathfrak{l}_\gamma, \gamma, \mathfrak{r}_\gamma)$

and $(\mathfrak{l}_\beta, \beta, \mathfrak{r}_\beta)$ be two rbb-trees. Then

$$(\mathfrak{l}_\gamma, \gamma, \mathfrak{r}_\gamma) \odot (\mathfrak{l}_\beta, \beta, \mathfrak{r}_\beta) = (\mathfrak{l}_\gamma, \gamma \cup \{\{\mathfrak{r}_\gamma, \mathfrak{l}_\beta\}\} \cup \beta, \mathfrak{r}_\beta)$$

and

$$(\mathfrak{l}_\gamma, \gamma, \mathfrak{r}_\gamma) \owedge (\mathfrak{l}_\beta, \beta, \mathfrak{r}_\beta) = (\mathfrak{l}_\gamma, \gamma \cup \{\{\mathfrak{l}_\gamma, \mathfrak{r}_\beta\}\} \cup \beta, \mathfrak{r}_\beta) .$$

These two operations are very similar. In both cases, we take the union of the bipartite block-trees $\gamma$ and $\beta$, add one more edge (for the new graph to become connected, and thus a block-tree), and set $\mathfrak{l}_\gamma$ and $\mathfrak{r}_\beta$, respectively, as the left root and the right root of the resulting block-graph. The difference between the two operations is on the new edge added: we add $\{\mathfrak{r}_\gamma, \mathfrak{l}_\beta\}$ in the operation $\odot$ and $\{\mathfrak{l}_\gamma, \mathfrak{r}_\beta\}$ in the operation $\owedge$. Note that, in both operations, the new edge added is a pair of two vertices with different sides. Thus, both operations give us rbb-trees. There is one further observation we must do: the operation $\odot$ is clearly associative, and thus $(\mathfrak{B}(X), \odot, \owedge)$ is a binary semigroup. To illustrate the difference between these two operation, we compute both $\odot$ and $\owedge$ for the same rbb-trees in Figure 3.

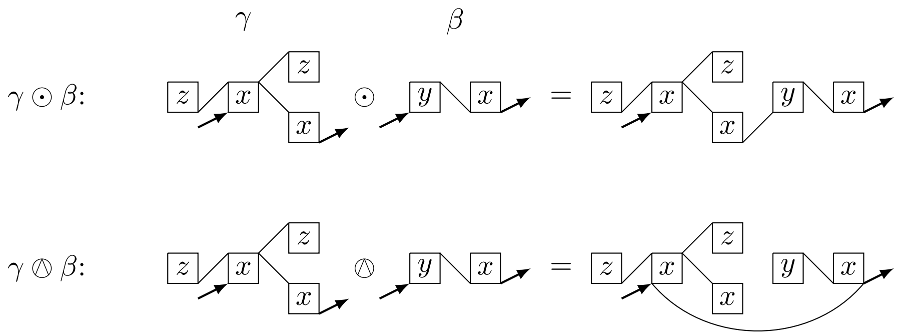
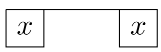


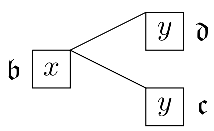

FIGURE 3. A computation of both $\odot$ and $\owedge$ operations.

A rbb-tree is called *reduced* if it does not have:

(1) an edge $\{\mathfrak{b}^i, \mathfrak{c}^j\}$ between two blocks $\mathfrak{b}$ and $\mathfrak{c}$ with the same label such that $\mathfrak{b}^i$ and $\mathfrak{c}^j$ have different poles, that is, it avoids the following subgraphs:

 and  ;

(2) two edges $\{\mathfrak{b}^i, \mathfrak{c}^j\}$ and $\{\mathfrak{b}^i, \mathfrak{d}^j\}$ with $\mathrm{l}(\mathfrak{c}) = \mathrm{l}(\mathfrak{d})$, that is, it avoids 8 'similar' subgraphs (4 possible choices for $i$ and, for each choice of $i$, 2 possible choices for $j$). We depict one of the avoided 8 subgraphs next:



.

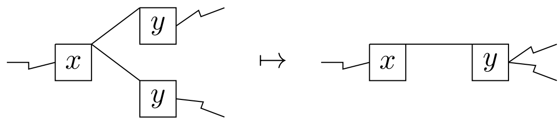
Let $\mathfrak{A}(X) \subseteq \mathfrak{B}(X)$ be the set of all reduced rbb-trees.

Let $\gamma$ be a rbb-tree. We say that two distinct blocks $\mathfrak{b}$ and $\mathfrak{c}$ of $\gamma$ are equivalent if $\mathrm{l}(\mathfrak{b}) = \mathrm{l}(\mathfrak{c})$ and there exists an edge $\{\mathfrak{b}^i, \mathfrak{c}^j\}$ in $\gamma$ with $\mathfrak{b}^i$ and $\mathfrak{c}^j$ of opposite poles. Hence, $\gamma$ avoids the subgraphs of (1) if and only if $\gamma$ has no equivalent blocks. We say that two distinct edges $\{\mathfrak{b}^i, \mathfrak{c}^j\}$ and $\{\mathfrak{b}^i, \mathfrak{d}^j\}$ (thus $\mathfrak{c} \neq \mathfrak{d}$) of $\gamma$ are equivalent if $\mathrm{l}(\mathfrak{c}) = \mathrm{l}(\mathfrak{d})$. Hence, $\gamma$ avoids the subgraphs of (2) if and only if $\gamma$ has no equivalent edges. Therefore, $\gamma$ is reduced if and only if $\gamma$ has neither equivalent blocks nor equivalent edges.

Next, we introduce two reduction rules to convert any rbb-tree $\gamma$ into a reduced rbb-tree. The rules will be called block-folding and edge-folding.

($i$) *Block-folding*: if $\mathfrak{b}$ and $\mathfrak{c}$ are two equivalent blocks, then delete the edge connecting them and collapse the two blocks into a single block (corresponding vertices and edges of the two blocks are identified). The new block keeps the label of both $\mathfrak{b}$ and $\mathfrak{c}$. If the left root and/or the right root of $\gamma$ belong to $\mathfrak{b}$ or $\mathfrak{c}$, then the left root and/or the right root of the new rbb-tree are the corresponding vertices in the block resulted from collapsing $\mathfrak{b}$ with $\mathfrak{c}$. We may visualize this reduction rule as follows:


($ii$) *Edge-folding*: if $\{\mathfrak{b}^i, \mathfrak{c}^j\}$ and $\{\mathfrak{b}^i, \mathfrak{d}^j\}$ are equivalent edges, then collapse the two edges into a single edge and the two blocks $\mathfrak{c}$ and $\mathfrak{d}$ into a single block (corresponding vertices and edges of the two blocks are identified). The new block keeps the label of both $\mathfrak{c}$ and $\mathfrak{d}$. If the left root and/or the right root of $\gamma$ belong to $\mathfrak{c}$ or $\mathfrak{d}$, then the left root and/or the right root of the new rbb-tree are the corresponding vertices in the block resulted from collapsing $\mathfrak{c}$ with $\mathfrak{d}$. We may visualize this reduction rule (one of the 8 'similar' cases) as follows:




Each time we apply one of these two reductions to a rbb-tree, the number of blocks diminish and so we must obtain a reduced rbb-tree after applying some consecutive reductions. Note also that any two reductions that can be applied to a rbb-tree can be commuted except if both refer to a subgraph 'similar' to the one depicted in Figure 4. However, the subgraph of Figure

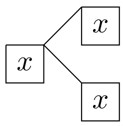



FIGURE 4. A special subgraph.

4 can be always reduced to a single block. Thus, we always get the same reduced rbb-tree from a given rbb-tree $\gamma$ (see [1, Section 4.2]). We will denote

by $\overline{\gamma}$ the reduced form of $\gamma$. The rbb-tree $\eta$ of Figure 2 is not reduced. In Figure 5 we illustrate a possible reduction for $\eta$ into its reduced rbb-tree $\overline{\eta}$.

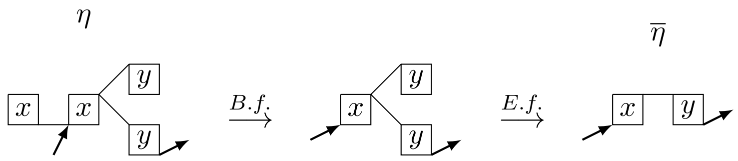



FIGURE 5. A reduction sequence for the rbb-tree $\eta$ of Figure 2.

We can introduce now two operations $\cdot$ and $\wedge$ on $\mathfrak{A}(X)$ derived from the operations $\odot$ and $\owedge$ on $\mathfrak{B}(X)$ respectively. If $\gamma, \beta \in \mathfrak{A}(X)$, then define

$$\gamma \cdot \beta = \overline{\gamma \odot \beta} \quad \text{and} \quad \gamma \wedge \beta = \overline{\gamma \owedge \beta}\,.$$

Since $\odot$ is associative and each rbb-tree has a unique reduced form, the operation $\cdot$ on $\mathfrak{A}(X)$ is associative too and the algebra $(\mathfrak{A}(X), \cdot, \wedge)$ is a binary semigroup. In the following section we will prove that $(\mathfrak{A}(X), \cdot)$ is a model for the bifree locally inverse semigroup on the set $X$.

## 4. The bifree locally inverse semigroup on a set $X$

The goal of this section is to prove that the semigroup $(\mathfrak{A}(X), \cdot)$ is a model for the bifree locally inverse semigroup on the set $X$ (as a type $\langle 2 \rangle$ algebra). We will accomplish this goal by proving that $(\mathfrak{A}(X), \cdot)$ is isomorphic to the semigroup $(T, \cdot)$ defined at the end of Section 2. We will begin by associating a set of words from $\overline{F}$ (see page 5) to each rbb-tree.

Let $\gamma$ be a rbb-tree. We will denote by $\hat{\gamma}$ the (usual) graph whose vertices are the union of the vertices of all blocks of $\gamma$ and the edges are the union of the edges of $\gamma$ with the edges of all its blocks. We will call $\hat{\gamma}$ the *hull* of $\gamma$. We continue to use for $\hat{\gamma}$ the same notations and designations used for $\gamma$. For example, a left vertex of $\hat{\gamma}$ is a left vertex of a block of $\gamma$; $\mathfrak{l}_{\hat{\gamma}}$ is the left root of $\gamma$ (viewed as a vertex in $\hat{\gamma}$); and so on.

Before we proceed, we need to introduce some graph terminology. We will consider a sequence

$$p := \quad \mathfrak{v}_1, \mathfrak{v}_2, \cdots, \mathfrak{v}_n$$

of vertices of a graph $\chi$ a *path* if there is an edge in $\chi$ connecting every two consecutive vertices of $p$. The *length* of a path is the number of vertices it has (it is $n$ for this path $p$). Then $p$ is *geodesic* if there is no path of smaller length connecting $\mathfrak{v}_1$ and $\mathfrak{v}_n$. Note that we can have several distinct geodesic paths connecting the same two vertices in a graph $\chi$. However, all these geodesic paths have the same length. Let $\mathsf{G}(\chi)$ be the set of all geodesic paths in $\chi$. Let $\sigma$ be the following equivalence relation on $\mathsf{G}(\chi)$: $p_1 \sigma p_2$ for $p_1, p_2 \in \mathsf{G}(\chi)$ if and only if both paths start at the same vertex and end also at the same vertex. We will denote by $[p]$ the $\sigma$-equivalence class of $p \in \mathsf{G}(\chi)$.

Let $\gamma$ be a rbb-tree. Then $p$ is an *admissible path* in $\hat{\gamma}$ if $p$ is a geodesic path in $\hat{\gamma}$ starting at $\mathfrak{l}_{\hat{\gamma}}$ and ending at one of the right vertices of $\hat{\gamma}$. Let $\mathsf{H}(\hat{\gamma})$ be the set of all admissible paths of $\hat{\gamma}$. Thus, all paths from $\mathsf{H}(\hat{\gamma})$ start at vertex $\mathfrak{l}_{\hat{\gamma}}$. We may have many different admissible paths in $\hat{\gamma}$ ending at a right vertex $\mathfrak{v}$ because we have always two possible choices for going from one corner to its opposite corner in each block of $\gamma$. However, this is the only possible situation that allows to construct different admissible paths in $\hat{\gamma}$ ending at the same vertex.

Next, we will associate a word $\mathsf{w}(p) \in \overline{F}$ to each $p \in \mathsf{G}(\hat{\gamma})$. Let

$$p := \ \mathfrak{v}_1,\ \mathfrak{v}_2, \cdots ,\ \mathfrak{v}_n$$

be a geodesic path in $\hat{\gamma}$. As we go through the path, we can write down the labels of the vertices and form a word from $\overline{X}^+$. We construct the word $\mathsf{w}(p)$ in a similar way (going through the path and writing down the labels of the vertices) but with the following two additional rules: let $\mathfrak{b}^i$ be a left vertex of $\hat{\gamma}$ for some block $\mathfrak{b}$ of $\gamma$; then

- $(a)$ if $\mathfrak{b}^i$, $\mathfrak{b}^j$, $\mathfrak{b}^k$ are consecutive vertices of $p$, then don't write the labels of $\mathfrak{b}^j$ and $\mathfrak{b}^k$ (or in other words, when we reach $\mathfrak{b}^k$, $\mathfrak{b}^k$ tell us to delete the label of $\mathfrak{b}^j$ instead of adding the label of $\mathfrak{b}^k$);
- $(b)$ if $\mathfrak{b}^i$, $\mathfrak{c}^j$ are two consecutive vertices of $p$ for $\mathfrak{c} \neq \mathfrak{b}$, then write the symbol $\wedge$ between the labels of $\mathfrak{b}^i$ and $\mathfrak{c}^j$.

By construction, the word $\mathsf{w}(p)$ does not begin with the symbol $\wedge$ and does not have subwords of the form $\wedge a \wedge$ for $a \in \overline{X} \cup \{1\}$; so $\mathsf{w}(p) \in \overline{F}$. Further, if $p_1$ is another geodesic path $\sigma$-equivalent to $p$, then $\mathsf{w}(p_1) = \mathsf{w}(p)$. Thus, we can define $\mathsf{w}([p])$ as the word $\mathsf{w}(p)$.

If $\gamma$ is a rbb-tree, then let $\mathsf{w}(\gamma)$ be the set of all words $\mathsf{w}([p])$ for $p \in \mathsf{H}(\hat{\gamma})$. Thus $\mathsf{w}(\gamma) \subseteq \overline{F}$. The following lemma collects some properties of the subsets $\mathsf{w}(\gamma)$ of $\overline{F}$.

**Lemma 4.1.** *If $\gamma$ is a rbb-tree, then $\mathsf{w}(\gamma)$ has the following properties:*

- $(i)$ *all words from $\mathsf{w}(\gamma)$ have the same first letter;*
- $(ii)$ *if $w_1w_2 \in \mathsf{w}(\gamma)$ for strings $w_1, w_2 \in (\overline{X} \cup \{\wedge\})^+$ such that $w_1 \in \overline{F}$ and $w_1 \neq w_3aa'a$ for any $w_3 \in (\overline{X} \cup \{\wedge\})^+$ and any $a \in \overline{X}$, then $w_1 \in \mathsf{w}(\gamma)$;*
- $(iii)$ *if $w_1a \in \mathsf{w}(\gamma)$ for a string $w_1 \in (\overline{X} \cup \{\wedge\})^*$ and a letter $a \in \overline{X}$ such that $w_1\tau \neq a'$, then $w_1aa' \in \mathsf{w}(\gamma)$.*

*If $\gamma$ is reduced too, then*

- $(iv)$ $\mathsf{w}(\gamma) \subseteq \mathsf{t}(\overline{F})$*;*
- $(v)$ *for all $u \in \mathsf{w}(\gamma)$, $u = \mathsf{w}([p])$ for a unique equivalence class $[p]$ of admissible paths in $\hat{\gamma}$.*

*Proof.* $(i)$. This property is obviously satisfied since each word $\mathsf{w}(p)$ of $\mathsf{w}(\gamma)$ begins with $\mathrm{l}(\mathfrak{l}_\gamma)$.

$(ii)$. Let $u = w_1w_2 \in \mathsf{w}(\gamma)$ such that $w_1 \in \overline{F}$ and $w_1 \neq w_3aa'a$ for any $w_3 \in (\overline{X} \cup \{\wedge\})^+$ and any $a \in \overline{X}$. Let $p := \ \mathfrak{v}_1, \cdots, \mathfrak{v}_n$ be an admissible path in $\hat{\gamma}$ such that $\mathsf{w}(p) = u$. Since $w_1$ does not end with the symbol $\wedge$, there exists some $k_1 \in \{1, \cdots, n\}$ such that $\mathsf{w}(\mathfrak{v}_1, \cdots, \mathfrak{v}_{k_1}) = w_1$ following the rules for constructing the words $\mathsf{w}$. Take $k$ the smallest $k_1 \in \{1, \cdots, n\}$ such that

$$\mathsf{w}(\mathfrak{v}_1, \cdots, \mathfrak{v}_{k_1}) = w_1 .$$

If $\mathfrak{v}_k$ is a right vertex, then $\mathfrak{v}_1, \cdots, \mathfrak{v}_k$ is an admissible path in $\hat{\gamma}$ and we are done. So, assume that $\mathfrak{v}_k$ is a left vertex and let $\mathfrak{b}$ be the block of $\gamma$ containing $\mathfrak{v}_k$. Let also $\mathfrak{v} \in \mathfrak{b}$ be one of the vertices adjacent to $\mathfrak{v}_k$ and $\mathfrak{v}' \in \mathfrak{b}$ be the vertex in the corner opposite to $\mathfrak{v}_k$. We have to split this case into three subcases.

If $k = 1$, that is, if $\mathfrak{v}_k = \mathfrak{l}_{\hat{\gamma}}$, then $w_1 = \mathrm{l}(\mathfrak{l}_{\hat{\gamma}}) = \mathsf{w}(\mathfrak{l}_{\hat{\gamma}}, \mathfrak{v}, \mathfrak{v}')$ and $\mathfrak{l}_{\hat{\gamma}}, \mathfrak{v}, \mathfrak{v}'$ is an admissible path in $\hat{\gamma}$. So, assume that $k > 1$. If $\mathfrak{v}_k$ is a left vertex and $\mathfrak{v}_{k-1} \in \mathfrak{b}$, then $\mathfrak{v}_{k-2} \neq \mathfrak{v}'$ since $w_1 \neq w_3 aa'a$ for any $w_3 \in (\overline{X} \cup \{\wedge\})^+$ and any $a \in \overline{X}$. Hence

$$p_1 \;:=\; \mathfrak{v}_1, \cdots, \mathfrak{v}_{k-1}, \mathfrak{v}' \in \mathsf{H}(\hat{\gamma}) \quad \text{ and } \quad \mathsf{w}(p_1) = w_1 .$$

If $\mathfrak{v}_k$ is a left vertex and $\mathfrak{v}_{k-1} \notin \mathfrak{b}$, then

$$p_2 \;:=\; \mathfrak{v}_1, \cdots, \mathfrak{v}_k, \mathfrak{v}, \mathfrak{v}' \in \mathsf{H}(\hat{\gamma}) \quad \text{ and } \quad \mathsf{w}(p_2) = w_1 .$$

We have proved *(ii)*.

*(iii)*. Let $w_1 a \in \mathsf{w}(\gamma)$ with $a \in \overline{X}$ and $w_1 \in (\overline{X} \cup \{\wedge\})^*$ such that $w_1 \tau \neq a'$. Let

$$p \;:=\; \mathfrak{v}_1, \cdots, \mathfrak{v}_n$$

be an admissible path in $\hat{\gamma}$ such that $\mathsf{w}(p) = w_1 a$ and let $\mathfrak{v}_n = \mathfrak{b}^i$ for some block $\mathfrak{b}$ of $\gamma$. Then $\mathfrak{b}^i$ is a right vertex of $\gamma$. Let $\mathfrak{b}^j$ be the other right vertex of $\mathfrak{b}$. If $\mathfrak{v}_{n-1} \notin \mathfrak{b}$, then

$$p_1 \;:=\; \mathfrak{v}_1, \cdots, \mathfrak{v}_n, \mathfrak{b}^j \in \mathsf{H}(\hat{\gamma}) \quad \text{ and } \quad \mathsf{w}(p_1) = w_1 aa' .$$

If $\mathfrak{v}_{n-1} \in \mathfrak{b}$, then $n \geq 3$ and $\mathfrak{v}_{n-2}$ is the vertex of $\mathfrak{b}$ in the corner opposite to $\mathfrak{b}^i$ since $w_1 \tau \neq a'$. Hence, for this case,

$$p_2 \;:=\; \mathfrak{v}_1, \cdots, \mathfrak{v}_{n-2}, \mathfrak{b}^j \in \mathsf{H}(\hat{\gamma}) \quad \text{ and } \quad \mathsf{w}(p_2) = w_1 aa' .$$

We have proved *(iii)*.

*(iv)*. Assume $\gamma$ is reduced and let $p$ be an admissible path in $\hat{\gamma}$. By the block-folding reduction rule, $\mathsf{w}(p)$ does not contain subwords $a{\wedge}a'$ for $a \in \overline{X}$; and by the edge-folding reduction rule, $\mathsf{w}(p)$ does not contain subwords $ab{\wedge}a$ and $a \wedge ba$ for $a, b \in \overline{X}$ with $b \neq a'$. If $aa'a$ is a subword of $\mathsf{w}(p)$ for some $a \in \overline{X}$, then there are three consecutive vertices $\mathfrak{b}^i$, $\mathfrak{c}^j$, $\mathfrak{d}^k$ in $p$ such that $\mathrm{l}(\mathfrak{b}^i) = a = \mathrm{l}(\mathfrak{d}^k)$ and $\mathrm{l}(\mathfrak{c}^j) = a'$. Since $\gamma$ is reduced for block-folding, the blocks $\mathfrak{b}$, $\mathfrak{c}$ and $\mathfrak{d}$ of $\gamma$ must be the same block; and by *(a)*, $\mathfrak{b}^i$ must be a right vertex. Hence, $\mathfrak{b}^i$ is preceded by a left vertex $\mathfrak{a}^h$ in the path $p$ with $\mathfrak{a} \neq \mathfrak{b}$, and $aa'a$ is preceded by the symbol $\wedge$ in $\mathsf{w}(p)$ due to *(b)*. Further, $\mathfrak{d}^k$ is a left vertex and it must be followed by a right vertex $\mathfrak{a}_1^{h_1}$ in $p$ with $\mathfrak{a}_1 \neq \mathfrak{d}$. Hence, $aa'a$ must be followed by the symbol $\wedge$ in $\mathsf{w}(p)$ too. We have shown that $\mathsf{w}(p) \in \mathsf{t}(\overline{F})$, and so $\mathsf{w}(\gamma) \subseteq \mathsf{t}(\overline{F})$.

$(v)$. Assume $\gamma$ is reduced and let $[p]$ and $[q]$ be two distinct equivalence classes of admissible paths in $\hat{\gamma}$. Both paths $p$ and $q$ start with $\mathfrak{l}_\gamma$. We will assume that $p \in [p]$ and $q \in [q]$ are chosen in a way that maximizes the length of the initial sequence of vertices in common.

Note that, as we walk through the path $p$ and write down $\mathsf{w}(p)$, each vertex $\mathfrak{b}^i$ of $p$ contributes with $\mathrm{l}(\mathfrak{b}^i)$ or with $\wedge\mathrm{l}(\mathfrak{b}^i)$ to the word $\mathsf{w}(p)$ except if $\mathfrak{b}^i$ is a right vertex and $\mathfrak{b}^k$, $\mathfrak{b}^j$, $\mathfrak{b}^i$ are three consecutive vertices of $p$ from the same block. In this exceptional case, $\mathfrak{b}^i$ does not contribute to $\mathsf{w}(p)$ but tell us to delete also the contribution of $\mathfrak{b}^j$. Further, if $p$ does not end at $\mathfrak{b}^i$ and $\mathfrak{a}^h$ is the following vertex, then $\mathfrak{a} \neq \mathfrak{b}$ and $\mathrm{l}(\mathfrak{a}^h) \neq \mathrm{l}(\mathfrak{b}^j)$ since otherwise we could apply a block-folding to merge $\mathfrak{a}$ with $\mathfrak{b}$. Thus, if $q$ is contained in $p$, then either

(1) $\mathsf{w}(p) = \mathsf{w}(q)v$ with $v \in \mathsf{t}(\overline{F})$ if the first vertex of $p$ not in $q$ is not a right vertex preceded by two other vertices from the same block; or
(2) $\mathsf{w}(p) = v_1v_2$ and $\mathsf{w}(q) = v_1a'$ with $v_1, v_2 \in \mathsf{t}(\overline{F})$ ($v_2$ can be the empty word), $v_1\tau = a \in \overline{X}$ and $\lambda v_2 \neq a'$, otherwise.

Therefore, $\mathsf{w}(p) \neq \mathsf{w}(q)$.

Assume that neither of the paths $p$ and $q$ is contained in the other. Let $\mathfrak{a}^i$ be the last vertex in common, and let $\mathfrak{b}^j$ and $\mathfrak{c}^k$ be the following vertices in $p$ and $q$, respectively. Let $a = \mathrm{l}(\mathfrak{a}^i)$, $b = \mathrm{l}(\mathfrak{b}^j)$ and $c = \mathrm{l}(\mathfrak{c}^k)$, and set $v$ as the word read on the path $p$ just up to the vertex $\mathfrak{a}^i$. Thus $v \in \overline{F}$ and $v\tau = a$. If $b = c$, then $\mathfrak{a} = \mathfrak{b} = \mathfrak{c}$ and $a = b'$ since $\gamma$ is reduced. Further, the next vertex in $p$ after $\mathfrak{b}^j$ must be in another block if it exists (by the choice of $p$ and $q$ maximizing the number of vertices in common). A similar conclusion is true also for $\mathfrak{c}^k$. Then $\mathsf{w}(p) = vbv_1$ and $\mathsf{w}(q) = vb \wedge dv_2$ (or vice-versa) for $v_1, v_2 \in \mathsf{t}(\overline{F})$ ($v_1$ and $v_2$ may be empty words) and $d \in \overline{X}$ with $d \neq a = b'$. Therefore $\mathsf{w}(p) \neq \mathsf{w}(q)$.

So, assume further that $b \neq c$. If $b \neq a'$ and $c \neq a'$, then $\mathsf{w}(p) = vsv_1$ and $\mathsf{w}(q) = vsv_2$ for $v_1, v_2 \in \overline{F}$ and $s \in \{1, \wedge\}$ (1 designates the empty word) with $\lambda v_1 = b$ and $\lambda v_2 = c$. Hence $\mathsf{w}(p) \neq \mathsf{w}(q)$. Finally, assume that $c = a'$ and $b \neq a'$ (the case $b = a'$ and $c \neq a'$ is similar). Then $\mathsf{w}(p) = vsv_1$ for $s \in \{1, \wedge\}$ and $v_1 \in \overline{F}$ with $\lambda v_1 = b$. We have three distinct possibilities for $\mathsf{w}(q)$:

(1) $\mathsf{w}(q) = vv_2$ with $v_2 \in \mathsf{t}(\overline{F})$ such that $\lambda v_2 = a'$;
(2) $\mathsf{w}(q) = vv_2$ with $v_2 \in \mathsf{t}(\overline{F})$ such that $\lambda v_2 \neq a'$, but $s = \wedge$ and $\mathsf{w}(p) = v \wedge v_1$ for this case;
(3) $v = v_3a$ and $\mathsf{w}(q) = v_3v_2$ with $v_2, v_3 \in \mathsf{t}(\overline{F})$ ($v_2$ may be empty) such that $v_3\tau = a'$ and $\lambda v_2 \neq a$.

Thus $\mathsf{w}(p) \neq \mathsf{w}(q)$ and we have proved $(v)$. □

Given a rbb-tree $\gamma$, let $\mathsf{w}^c(\gamma) = \mathsf{w}(p)$ where $p$ is an admissible path in $\hat{\gamma}$ ending at $\mathfrak{r}_{\hat{\gamma}}$. Thus $\mathsf{w}^c(\gamma) \in \mathsf{w}(\gamma)$. In the next result we show that $(\mathsf{w}(\gamma), \mathsf{w}^c(\gamma)) \in T$ if $\gamma$ is reduced.

**Corollary 4.2.** *If $\gamma$ is a reduced rbb-tree, then $(\mathsf{w}(\gamma), \mathsf{w}^c(\gamma)) \in T$.*

*Proof.* By Lemma 4.1, $\mathsf{w}(\gamma) \subseteq \mathsf{t}(\overline{F})$ and it satisfies the conditions $(i)$, $(ii)$ and $(iii)$ stated on page 7. As observed above, $\mathsf{w}^c(\gamma) \in \mathsf{w}(\gamma)$. Thus $(\mathsf{w}(\gamma), \mathsf{w}^c(\gamma)) \in T$. $\square$

Let $\mathfrak{b}$ and $\mathfrak{c}$ be two equivalent blocks of a rbb-tree $\gamma$ and let $\{\mathfrak{b}^i, \mathfrak{c}^j\}$ be the edge connecting them. Let $\gamma_1$ be the rbb-tree obtained after applying a block-folding to these two blocks. Designate by $\mathfrak{d}$ the block of $\gamma_1$ that results from collapsing $\mathfrak{b}$ and $\mathfrak{c}$ into a single block. Thus $\mathfrak{b}^i$ and $\mathfrak{c}^i$ collapse into $\mathfrak{d}^i$, and so on.

Let $p$ be an admissible path in $\hat{\gamma}$. We can split $p$ into three parts

$$p := p_1, p_2, p_3$$

where $p_1$ are the vertices of $p$ that come before any vertex from $\mathfrak{b}$ and $\mathfrak{c}$ (or the entire path $p$ if $p$ does not contain vertices from $\mathfrak{b}$ and $\mathfrak{c}$), $p_2$ are the vertices of $p$ from $\mathfrak{b}$ and $\mathfrak{c}$, and $p_3$ are the vertices of $p$ that come after all the vertices from $\mathfrak{b}$ and $\mathfrak{c}$ ($p_1$, $p_2$ and $p_3$ can be empty). Let $p_2'$ be the sequence of vertices from $\mathfrak{d}$ obtained by replacing each $\mathfrak{b}^k$ and $\mathfrak{c}^k$ of $p_2$ with $\mathfrak{d}^k$, for $k \in \{l+, l-, r+, r-\}$. Designate by $p_2''$ the geodesic path inside $\mathfrak{d}$ connecting the first vertex of $p_2'$ to its last vertex (or $p_2''$ is empty if $p_2'$ is empty too). The block-folding transforms $p$ into an admissible path

$$p'' = p_1, p_2'', p_3$$

in $\hat{\gamma}_1$. Further, any admissible path in $\hat{\gamma}_1$ is obtained from an admissible path in $\hat{\gamma}$ in this way.

By an exhaustive analyses of all possible cases (there are many but all immediate) we can see that either $\mathsf{w}(p'') = \mathsf{w}(p)$ or $\mathsf{w}(p'')$ can be obtained from $\mathsf{w}(p)$ by applying one or two reductions of type $(1') - (3')$ stated on page 6. Thus, each word of $\mathsf{w}(\gamma_1)$ is either a word of $\mathsf{w}(\gamma)$ or obtained from a word of $\mathsf{w}(\gamma)$ by applying reductions of type $(1') - (3')$. We can conclude that there exists a surjective mapping $\vartheta : \mathsf{w}(\gamma) \to \mathsf{w}(\gamma_1)$ such that $u\vartheta$ is obtained from $u$ by applying a (possibly empty) sequence of reduction rules of type $(1') - (3')$.

**Lemma 4.3.** *If $\gamma_1$ is obtained from $\gamma$ by a sequence of block-foldings and edge-foldings, then there exists a surjective mapping $\vartheta : \mathsf{w}(\gamma) \to \mathsf{w}(\gamma_1)$ such that $u\vartheta$ is obtained from $u$ by applying a (possibly empty) sequence of reduction rules of type $(1') - (3')$.*

*Proof.* The conclusion taken above for applying one block-folding reduction to $\gamma$ can be taken also for applying one edge-folding reduction to $\gamma$. The proof follows exactly the same steps and, for that reason, we leave the details for the reader to fill in. This lemma follows now from applying these two conclusions several times. $\square$

**Corollary 4.4.** *Let $\gamma \in \mathfrak{B}(X)$. Then $\mathsf{w}(\overline{\gamma}) = \mathsf{t}(\mathsf{w}(\gamma))$ where $\mathsf{t}(\mathsf{w}(\gamma)) = \{\mathsf{t}(u) \,:\, u \in \mathsf{w}(\gamma)\}$.*

*Proof.* By Lemma 4.3 there exists a surjective mapping $\vartheta : \mathsf{w}(\gamma) \to \mathsf{w}(\overline{\gamma})$ such that $u\vartheta$ is obtained from $u \in \mathsf{w}(\gamma)$ by applying the reduction rules $(1')-(3')$. But $u\vartheta \in \mathsf{t}(\overline{F})$ by Corollary 4.1 since $\overline{\gamma}$ is reduced. Thus $u\vartheta = \mathsf{t}(u)$ and $\mathsf{w}(\overline{\gamma}) = \mathsf{t}(\mathsf{w}(\gamma))$. □

We can prove now that $(T, \cdot)$ is an isomorphic image of $(\mathfrak{A}(X), \cdot)$.

**Proposition 4.5.** *The mapping* $\varphi : \mathfrak{A}(X) \to T$, $\gamma \mapsto (\mathsf{w}(\gamma), \mathsf{w}^c(\gamma))$ *is a semigroup isomorphism.*

*Proof.* The mapping $\varphi$ is well defined by Corollary 4.2. Let $\gamma, \beta \in \mathfrak{A}(X)$ and note that

$$\mathsf{w}(\gamma \odot \beta) = \mathsf{w}(\gamma) \cup \mathsf{w}^c(\gamma)\mathsf{w}(\beta) \quad \text{and} \quad \mathsf{t}(\mathsf{w}^c(\gamma)\mathsf{w}(\beta)) = \mathsf{w}^c(\gamma) \odot \mathsf{w}(\beta)$$

where $\mathsf{w}^c(\gamma)\mathsf{w}(\beta) = \{\mathsf{w}^c(\gamma)\mathsf{w}(p) \,:\, p \in \mathsf{H}(\hat{\beta})\}$. Then, by Corollary 4.4,

$$\begin{aligned}\mathsf{w}(\gamma \cdot \beta) &= \mathsf{w}(\overline{\gamma \odot \beta}) = \mathsf{t}(\mathsf{w}(\gamma \odot \beta)) = \mathsf{t}(\mathsf{w}(\gamma) \cup \mathsf{w}^c(\gamma)\mathsf{w}(\beta)) = \\ &= \mathsf{w}(\gamma) \cup \mathsf{w}^c(\gamma) \odot \mathsf{w}(\beta)\,.\end{aligned}$$

Further, $\mathsf{w}^c(\gamma \cdot \beta) = \mathsf{t}(\mathsf{w}^c(\gamma)\mathsf{w}^c(\beta)) = \mathsf{w}^c(\gamma) \odot \mathsf{w}^c(\beta)$. We have shown that

$$(\gamma \cdot \beta)\varphi = (\mathsf{w}(\gamma \cdot \beta), \mathsf{w}^c(\gamma \cdot \beta)) = (\mathsf{w}(\gamma), \mathsf{w}^c(\gamma)) \cdot (\mathsf{w}(\beta), \mathsf{w}^c(\beta)) = \gamma\varphi \cdot \beta\varphi\,.$$

Hence, $\varphi$ is a semigroup homomorphism.

Let $\mathfrak{a}$ and $\mathfrak{b}$ be blocks labeled with $x$ and $y$, respectively. Let $a \in \{x, x'\}$ and $b \in \{y, y'\}$ such that $b \neq a'$. Define $\gamma_a \in \mathfrak{A}(X)$ as the rbb-tree with the block $\mathfrak{a}$ only and roots (left and right) set as the vertices labeled with $a$. Define also $\gamma_{a\wedge b}$ as the rbb-tree with blocks $\mathfrak{a}$ and $\mathfrak{b}$, left root the left vertex $\mathfrak{a}^i$ of $\mathfrak{a}$ with label $a$, right root the right vertex $\mathfrak{b}^j$ of $\mathfrak{b}$ with label $b$, and the edge $\{\mathfrak{a}^i, \mathfrak{b}^j\}$. Then $\gamma_a$ and $\gamma_{a\wedge b}$ are reduced, $(\gamma_a)\varphi = (\{a, aa'\}, a)$ and $(\gamma_{a\wedge b})\varphi = (\{a, aa', a \wedge b, a \wedge bb'\}, a \wedge b)$. By Proposition 2.4, the homomorphism $\varphi$ is surjective.

Note that $\gamma_a$ is the only reduced rbb-tree $\gamma$ such that $\gamma\varphi = (\{a, aa'\}, a)$. Let $\gamma_{aa'}$ be the rbb-tree obtained from $\gamma_a$ by setting $\mathfrak{r}_{\gamma_{aa'}}$ as the right vertex labeled with $a'$ (note that $\mathfrak{r}_{\gamma_a}$ is the right vertex labeled with $a$). It is also clear that $\gamma_{aa'}$ is the only reduced rbb-tree such that $(\gamma_{aa'})\varphi = (\{a, aa'\}, aa')$. Now, let $(A, w) \in T$ such that $A$ has more than 2 words. We will prove by induction on the number of words of $A$ that there exists only one $\gamma \in \mathfrak{A}(X)$ such that $\gamma\varphi = (A, w)$.

Let $u$ be a word of maximal length among all words from $A$. Then either $u = vaa'$, $u = v \wedge aa'$, or $u = vb'b \wedge aa'$ for some $a \in \overline{X}$ and $v \in A$ such that $v\tau = b \neq a'$. Let $u' \in \mathsf{t}(\overline{F})$ such that $u = u'a'$ and set $A_1 = A \setminus \{u, u'\}$. Set $w_1 = v$ if $w \in \{u, u'\}$, and set $w_1 = w$ otherwise. Then $w_1 \in A_1$ and $(A_1, w_1) \in T$.

Let $\beta \in \mathfrak{A}(X)$ such that $\beta\varphi = (A, w)$ and let $p \in \mathsf{H}(\hat{\beta})$ such that $\mathsf{w}(p) = u$. Denote by $\mathfrak{a}$ the block of $\beta$ containing the last vertex of the path $p$. Then $\mathfrak{a}$ is a degree one vertex of $\beta$ by the maximality of the length of $u$. Let $\beta_1$ be the reduced rbb-tree obtained from $\beta$ by deleting the block $\mathfrak{a}$ (and the only edge with $\mathfrak{a}$ at one of its endpoints); and keeping the right root of $\beta$ if $w_1 = w$, or setting the right root of $\beta_1$ to be the right vertex of the block $\mathfrak{b}$ adjacent to $\mathfrak{a}$ labeled with $b$ otherwise. By construction of $\beta_1$, $\beta_1\varphi = (A_1, w_1)$. Further, we recover $\beta$ from $\beta_1$ in a unique manner: add the block $\mathfrak{a}$ with the same label, reset the right root, and add the edge eliminated. We just need to take some caution with the edge we need to add as it differs according to whether $u = vaa'$, or $u = v \wedge aa'$ or $u = vb'b \wedge aa'$. Nevertheless, once we know if $u = vaa'$ or if $u = v \wedge aa'$ or $u = vb'b \wedge aa'$, then this edge is completely determined.

Let $\delta$ be another reduced rbb-tree such that $\delta\varphi = (A, w)$. Apply the same procedure described above and construct another $\delta_1 \in \mathfrak{A}(X)$ such that $(\delta_1)\varphi = (A_1, w_1)$. By induction, $\beta_1 = \delta_1$. Call $\beta_1$ and $\delta_1$ by $\gamma_1$. By Proposition 4.1, there is a unique $q \in \mathsf{H}(\hat{\gamma}_1)$ (up to equivalence) such that $\mathsf{w}(q) = v$. Then the block $\mathfrak{b}$ of $\gamma_1$ must be the block containing the last vertex of $q$. Finally, we can conclude that $\beta = \delta$ because both of them are constructed in the same manner from the block $\mathfrak{b}$ of $\gamma_1$. We have proved that $\varphi$ is injective, whence $\varphi$ is a semigroup isomorphism. $\square$

A particular consequence of this last result is that $\mathfrak{A}(X)$ is a locally inverse semigroup. In the next section we characterize the idempotents, the Green's relations and the natural partial order of $\mathfrak{A}(X)$, and the inverses of each $\gamma \in \mathfrak{A}(X)$. It will then follow that the operation $\wedge$ on $\mathfrak{A}(X)$ introduced in Section 3 is the desired pseudosemilattice operation on the locally inverse semigroup $\mathfrak{A}(X)$. Thus, $(\mathfrak{A}(X), \cdot, \wedge)$ is a model for the bifree locally inverse semigroup on the set $X$ (seen as a type $\langle 2, 2\rangle$ algebra).

## 5. The structure of $\mathfrak{A}(X)$

We begin this section by defining the concepts of morphism and substructure for rbb-trees. A *morphism* between two rbb-trees $\gamma$ and $\beta$ is a usual graph-morphism $\varphi : \gamma \to \beta$ such that

$$\mathrm{l}(\mathfrak{b}\varphi) = \mathrm{l}(\mathfrak{b}) \qquad \text{and} \qquad (\{\mathfrak{b}^i, \mathfrak{c}^j\})\varphi = \{(\mathfrak{b}\varphi)^i, (\mathfrak{c}\varphi)^j\},$$

that is, it preserves the labels and sends each edge into another edge of the same "type". Thus, $\varphi$ induces the following graph-morphism $\hat{\varphi} : \hat{\gamma} \to \hat{\beta}$ defined by $(\mathfrak{b}^i)\hat{\varphi} = (\mathfrak{b}\varphi)^i$ which preserves labels also. If $\varphi$ preserves also the roots (the images of the left and right roots of $\hat{\gamma}$ under $\hat{\varphi}$ are the left and right roots of $\hat{\beta}$, respectively), then we call $\varphi$ a *bi-morphism*. The left and right variants are called *left-morphism* and *right-morphism*, respectively.

Let $\varphi : \gamma \to \beta$ be a morphism between two rbb-trees $\gamma$ and $\beta$. Note that no two adjacent blocks of $\gamma$ can be sent into the same block of $\beta$ by $\varphi$ since otherwise we would have a loop in $\beta$ (the image of the edge connecting the two blocks of $\gamma$ becomes a loop in $\beta$). Thus, if $\gamma$ is not reduced for block-folding, then neither is $\beta$.

Assume that there are two distinct blocks $\mathfrak{b}$ and $\mathfrak{c}$ of $\gamma$ adjacent to another block $\mathfrak{a}$ such that $\mathfrak{b}\varphi = \mathfrak{c}\varphi$. Then $\mathrm{l}(\mathfrak{b}) = \mathrm{l}(\mathfrak{c})$. Let $\{\mathfrak{a}^i, \mathfrak{b}^j\}$ and $\{\mathfrak{a}^h, \mathfrak{c}^k\}$ be the edges connecting $\mathfrak{a}$ to $\mathfrak{b}$ and $\mathfrak{c}$, respectively. Then, the image of both edges is the same ($\beta$ has at most one edge connecting each two blocks), and so $i = h$ and $j = k$. Hence, we can apply an edge-folding in $\gamma$ to these two edges. Therefore, if $\gamma$ is reduced, then $\varphi$ is a one-to-one mapping by the observations just made and since $\gamma$ and $\beta$ are trees.

If $\beta$ is reduced, then $\gamma$ must be reduced for block-folding. Further, the morphism $\varphi$ becomes completely determined once we know the image of one block of $\gamma$ (or the image of one of the vertices of $\hat{\gamma}$ under $\hat{\varphi}$) since $\beta$ is reduced for edge-folding. Therefore, a morphism between two reduced rbb-trees is always one-to-one and becomes completely determined once we know the image of a block. In particular, there is at most one bi-morphism [left-morphism, right-morphism] between two reduced rbb-trees. We leave this result registered in the following lemma.

**Lemma 5.1.** *Let $\gamma, \beta \in \mathfrak{A}(X)$. Then there exists at most one bi-morphism [left-morphism, right-morphism] from $\gamma$ into $\beta$.*

Two rbb-trees $\gamma$ and $\beta$ are called *isomorphic* [bi-isomorphic, left-isomorphic, right-isomorphic] if there exists a bijective morphism [bi-morphism, left-morphism, right-morphism] between them (it is enough to be bijective on the vertices). We will write $\gamma \approx \beta$, $\gamma \approx_l \beta$ and $\gamma \approx_r \beta$ to indicate that $\gamma$ and $\beta$ are isomorphic, left-isomorphic and right-isomorphic, respectively. For practical reasons we will say that they are the same rbb-tree (and write $\gamma = \beta$) if they are bi-isomorphic.

A subgraph $\beta_1$ of a rbb-tree $\beta$ will be called a *rbb-subtree* if $\beta_1$ is a rbb-tree, that is, if it is connected and has defined a left root and a right root. Note that the left and right roots of $\beta_1$ may be different from the left and right roots of $\beta$. If the left and right roots of $\beta_1$ are the same as the ones of $\beta$ (and so $\beta_1$ contains the roots of $\beta$), then we say that $\beta_1$ is a *rbb-bi-subtree*. Let *rbb-left-subtree* and *rbb-right-subtree* be the expected left and right variants of the previous definition. Thus, there is a morphism [bi-morphism, left-morphism, right-morphism] $\varphi : \gamma \to \beta$ between two reduced rbb-trees if and only if $\gamma$ is bi-isomorphic to a rbb-subtree [rbb-bi-subtree, rbb-left-subtree, rbb-right-subtree] of $\beta$.

We will write $\gamma \preccurlyeq \beta$, $\gamma \preccurlyeq_l \beta$, $\gamma \preccurlyeq_r \beta$ and $\gamma \preccurlyeq_b \beta$ to indicate that $\gamma$ is (bi-isomorphic to) a rbb-subtree, a rbb-left-subtree, a rbb-right-subtree and a rbb-bi-subtree of $\beta$, respectively. Thus $\preccurlyeq$, $\preccurlyeq_l$ and $\preccurlyeq_r$ are quasi-orders, while $\preccurlyeq_b$ is a partial order (the former three relations are not partial orders because we can have different reduced rbb-trees with the same underlying graph but different roots).

We have now the tools needed for studying more deeply the structure of $\mathfrak{A}(X)$. We begin with an obvious lemma.

**Lemma 5.2.** *Let $\gamma, \beta \in \mathfrak{A}(X)$. Then $\gamma \preccurlyeq_l \gamma \cdot \beta$ and $\beta \preccurlyeq_r \gamma \cdot \beta$. Further, any block of $\gamma \cdot \beta$ comes from a block of $\gamma$ or $\beta$.*

*Proof.* By construction, $\gamma \preccurlyeq_l \gamma \odot \beta$ and $\beta \preccurlyeq_r \gamma \odot \beta$. Since $\gamma$ and $\beta$ are reduced, the reduction of $\gamma \odot \beta$ into $\gamma \cdot \beta$ does not modify the rbb-subtrees $\gamma$ and $\beta$. Thus $\gamma \preccurlyeq_l \gamma \cdot \beta$ and $\beta \preccurlyeq_r \gamma \cdot \beta$. The second statement is obvious since the blocks of $\gamma \odot \beta$ are the (disjoint) union of the blocks of $\gamma$ and $\beta$. □

In the next result we will prove that $\gamma \in \mathfrak{A}(X)$ is an idempotent if and only if $\{\mathfrak{l}_\gamma, \mathfrak{r}_\gamma\}$ is an edge of $\gamma$ or $\mathfrak{l}_\gamma$ and $\mathfrak{r}_\gamma$ are adjacent vertices from the same block. We present a clearer description using $\hat{\gamma}$.

**Proposition 5.3.** *Let $\gamma \in \mathfrak{A}(X)$. Then $\gamma$ is an idempotent if and only if $\{\mathfrak{l}_{\hat{\gamma}}, \mathfrak{r}_{\hat{\gamma}}\}$ is an edge of $\hat{\gamma}$. Further, if $\gamma, \beta \in \mathfrak{A}(X)$ with $\gamma$ idempotent, then*

(*i*) *$\gamma \cdot \beta = \beta$ if and only if $\gamma \preccurlyeq_l \beta$;*
(*ii*) *$\beta \cdot \gamma = \beta$ if and only if $\gamma \preccurlyeq_r \beta$.*

*Proof.* By Proposition 4.5 and [2, Corollary 4.24], $\gamma$ is an idempotent if and only if

$$\mathsf{w}^c(\gamma) \in \{aa', a \wedge b \;:\; a, b \in \overline{X} \text{ with } b \neq a'\},$$

that is, if and only if $\{\mathfrak{l}_{\hat{\gamma}}, \mathfrak{r}_{\hat{\gamma}}\}$ is an edge of $\hat{\gamma}$ since $\gamma$ is reduced. Since (*ii*) is the left-right dual of (*i*), we will prove only (*i*). The direct implication follows from Lemma 5.2. Thus, we just need to prove the other implication.

So, assume that $\gamma \preccurlyeq_l \beta$ and consider the graph $\gamma \odot \beta$. For our convenience, for each block $\mathfrak{b}$ of $\gamma$ in $\gamma \odot \beta$, denote by $\mathfrak{b}'$ the corresponding block of $\beta$ in $\gamma \odot \beta$. Since $\{\mathfrak{l}_{\hat{\gamma}}, \mathfrak{r}_{\hat{\gamma}}\}$ is an edge of $\hat{\gamma}$, we can apply a first block-folding (if $\mathfrak{l}_\gamma$ and $\mathfrak{r}_\gamma$ belong to the same block) or a first edge-folding (if $\{\mathfrak{l}_\gamma, \mathfrak{r}_\gamma\}$ is an edge of $\gamma$) to collapse $\mathfrak{l}_\gamma^{\mathfrak{b}}$ with $\mathfrak{l}_\beta^{\mathfrak{b}}$ in $\gamma \odot \beta$. Let $\beta_1$ be the rbb-tree obtained after this first reduction and note that the roots of $\beta_1$ are the roots of $\beta$. We can now prove easily by induction on the distance to the left block of $\beta_1$ that each block $\mathfrak{b}$ of $\gamma$ in $\beta_1$ can be collapsed with the block $\mathfrak{b}'$ of $\beta$ in $\beta_1$ by a sequence of reductions. Thus, we can collapse the entire $\gamma$ into $\beta$ inside $\beta_1$ by a sequence of reductions, and $\beta_1$ reduces into $\beta$. Hence $\gamma \odot \beta = \beta$ as wanted. □

We can now easily describe the sets $(\gamma]_{\mathscr{R}}$, $(\gamma]_{\mathscr{L}}$ and $(\gamma]_{\leq}$ for an idempotent $\gamma$ of $\mathfrak{A}(X)$.

**Corollary 5.4.** *Let $\gamma \in E(\mathfrak{A}(X))$.*

(*i*) $(\gamma]_{\mathscr{R}} = \{\beta \in E(\mathfrak{A}(X)) \;:\; \gamma \preccurlyeq_l \beta\}$.
(*ii*) $(\gamma]_{\mathscr{L}} = \{\beta \in E(\mathfrak{A}(X)) \;:\; \gamma \preccurlyeq_r \beta\}$.
(*iii*) $(\gamma]_{\leq} = \{\beta \in E(\mathfrak{A}(X)) \;:\; \gamma \preccurlyeq_b \beta\}$.

*Proof.* (*i*) and (*ii*) follow, respectively, from (*i*) and (*ii*) of Proposition 5.3. Further, $\beta \in (\gamma]_{\leq}$ if and only if $\gamma \preccurlyeq_l \beta$ and $\gamma \preccurlyeq_r \beta$. So, we just need to prove that $\gamma \preccurlyeq_b \beta$ if $\gamma \preccurlyeq_l \beta$ and $\gamma \preccurlyeq_r \beta$. Assume that $\gamma \preccurlyeq_l \beta$ and $\gamma \preccurlyeq_r \beta$. Since $\gamma \preccurlyeq_l \beta$, there is a left-morphism $\varphi$ from $\gamma$ into $\beta$. Thus $\mathfrak{l}_{\hat{\gamma}}\hat{\varphi} = \mathfrak{l}_{\hat{\beta}}$ and $\{\mathfrak{l}_{\hat{\beta}}, \mathfrak{r}_{\hat{\gamma}}\hat{\varphi}\}$ is an edge of $\hat{\beta}$. But $\mathrm{l}(\mathfrak{r}_{\hat{\gamma}}) = \mathrm{l}(\mathfrak{r}_{\hat{\beta}})$ because $\gamma \preccurlyeq_r \beta$. Since $\beta$ is reduced, we conclude that $\mathfrak{r}_{\hat{\gamma}}\hat{\varphi} = \mathfrak{r}_{\hat{\beta}}$. Hence $\varphi$ is a bi-morphism and $\gamma \preccurlyeq_b \beta$. □

Next, we will describe the Green's relations $\mathscr{R}$, $\mathscr{L}$ and $\mathscr{D}$ on $\mathfrak{A}(X)$.

**Proposition 5.5.** *Let $\gamma, \beta \in \mathfrak{A}(X)$.*

*(i) $\gamma \mathrel{\mathscr{R}} \beta$ if and only if $\gamma \approx_l \beta$.*
*(ii) $\gamma \mathrel{\mathscr{L}} \beta$ if and only if $\gamma \approx_r \beta$.*
*(iii) $\gamma \mathrel{\mathscr{D}} \beta$ if and only if $\gamma \approx \beta$.*

*Proof.* We just have to prove $(i)$ since $(ii)$ is proven dually and $(iii)$ follows from both $(i)$ and $(ii)$. Further, since each $\mathscr{R}$-class contains an idempotent, we can assume that $\gamma$ is an idempotent. So, let $\gamma$ be an idempotent of $\mathfrak{A}(X)$. If $\gamma \mathrel{\mathscr{R}} \beta$, then $\gamma \preccurlyeq_l \beta$ by Proposition 5.3 ($\gamma \cdot \beta = \beta$). But there exists some $\beta' \in \mathfrak{A}(X)$ such that $\beta \cdot \beta' = \gamma$. Since $\beta \preccurlyeq_l \beta \odot \beta'$, $\gamma = \overline{\beta \odot \beta'}$ and $\beta$ is reduced, we conclude that $\beta \preccurlyeq_l \gamma$. We have shown that $\gamma \approx_l \beta$ because $\gamma$ and $\beta$ are finite.

If $\gamma \approx_l \beta$, then $\gamma \cdot \beta = \beta$ again by Proposition 5.3. Let $\mathfrak{b}^i$ be the left vertex of $\mathfrak{b} = \mathfrak{r}_\beta^{\mathfrak{b}}$ adjacent to $\mathfrak{r}_\beta$ and set $\beta'$ as the rbb-tree right-isomorphic to $\gamma$ but with $\mathfrak{l}_{\beta'} = \mathfrak{b}^i$. Let $\varphi$ be the isomorphism from $\beta$ onto $\beta'$ and set $\delta = \beta \odot \beta'$. Note that the isomorphism $\varphi$ induces a partial mapping from the blocks of $\beta$ in $\delta$ into the blocks of $\beta'$ in $\delta$. We continue to denote by $\varphi$ this partial mapping inside $\delta$. By the choice of the left vertex of $\beta'$, we can collapse $\mathfrak{r}_\beta^{\mathfrak{b}}$ with $\mathfrak{l}_{\beta'}^{\mathfrak{b}}$ inside $\delta$ by a block-folding. Then, by induction on the distance to $\mathfrak{r}_\beta^{\mathfrak{b}}$, we can prove that each block $\mathfrak{b}$ of $\beta$ is collapsed with the block $\mathfrak{b}\varphi$ of $\beta'$ inside $\delta$ by a sequence of edge-foldings. At the end, the entire $\beta$ is collapsed with $\beta'$ inside $\delta$ by a sequence of reductions. Hence $\overline{\delta} = \beta \cdot \beta'$ is isomorphic $\beta$ with $\mathfrak{l}_{\overline{\delta}} = \mathfrak{l}_\beta = \mathfrak{l}_\gamma$ and $\mathfrak{r}_{\overline{\delta}} = \mathfrak{r}_{\beta'} = \mathfrak{r}_\gamma$, that is, $\beta \cdot \beta' = \gamma$. We have shown that $\gamma \mathrel{\mathscr{R}} \beta$ as desired. $\square$

The well known result which states that the Green's $\mathscr{H}$-relation on the bifree locally inverse semigroup is trivial is now an obvious consequence of $(i)$ and $(ii)$. Note also that the relations $\mathscr{R}$ and $\mathscr{L}$ have a right-left dual description in Proposition 5.5 reflecting the right-left dual nature of these two relations on $\mathfrak{A}(X)$. However, that is not the case for the description of these relations in the model for the bifree locally inverse semigroup presented in [2]. In fact, the description of the $\mathscr{L}$-relation in [2] and its proof is more complex and harder than the description of the $\mathscr{R}$-relation and its proof. Nevertheless, the case $(i)$ above could be proved easily using results from [2]. However, we could not argue that $(ii)$ would be proved by duality.

Given a graph $\chi$, we will denote by $\mathrm{V}(\chi)$ and $\mathrm{E}(\chi)$ the set of vertices and edges of $\chi$, respectively, and by $\mathrm{v}(\chi)$ and $\mathrm{e}(\chi)$ the number of vertices and edges of $\chi$, respectively. If $a$ is a vertex of $\chi$, let $\deg(a)$ denotes the degree of $a$ in $\chi$, that is, the number of edges with $a$ at one of its endpoints.

**Corollary 5.6.** *Let $\gamma \in \mathfrak{A}(X)$.*

*(i) The $\mathscr{R}$-class and the $\mathscr{L}$-class of $\gamma$ have both $2\mathrm{v}(\gamma)$ elements.*
*(ii) The $\mathscr{D}$-class of $\gamma$ has $4(\mathrm{v}(\gamma))^2$ elements.*
*(iii) The $\mathscr{R}$-class of $\gamma$ has $\deg(\mathfrak{l}_{\hat{\gamma}}) - 1$ idempotents.*

(*iv*) *The* $\mathscr{L}$*-class of* $\gamma$ *has* $\deg(\mathfrak{r}_{\hat{\gamma}})-1$ *idempotents.*
(*v*) *The* $\mathscr{D}$*-class of* $\gamma$ *has* $\mathrm{e}(\gamma)+2\mathrm{v}(\gamma)=\mathrm{e}(\hat{\gamma})-2\mathrm{v}(\gamma)$ *idempotents.*

*Proof.* (*i*) and (*ii*) are obvious from Proposition 5.5: the number of elements of the $\mathscr{R}$-class of $\gamma$ is the number of left vertices from all the blocks of $\gamma$, the number of elements of the $\mathscr{L}$-class of $\gamma$ is the number of right vertices from all the blocks of $\gamma$, and the number of elements of the $\mathscr{D}$-class of $\gamma$ is the number of possible pairs of block-vertices from $\gamma$ with different sides. (*iii*) follows from Propositions 5.5 and 5.3: note that we must subtract 1 to $\deg(\mathfrak{l}_{\hat{\gamma}})$ because we cannot consider the edge of $\hat{\gamma}$ connecting $\mathfrak{l}_{\hat{\gamma}}$ to the other left vertex of $\mathfrak{l}_{\gamma}^{\mathfrak{b}}$. (*iv*) is the dual of (*iii*). Finally, by Propositions 5.5 and 5.3 again, the idempotents of the $\mathscr{D}$-class of $\gamma$ are in a one-to-one correspondence with the edges of $\hat{\gamma}$ connecting vertices with different sides. That number is $\mathrm{e}(\gamma)+2\mathrm{v}(\gamma)$: $\mathrm{e}(\gamma)$ is the number of edges connecting different blocks of $\gamma$ and $2\mathrm{v}(\gamma)$ is the number of edges inside the blocks connecting vertices with different sides. Clearly $\mathrm{e}(\gamma)+2\mathrm{v}(\gamma)=\mathrm{e}(\hat{\gamma})-2\mathrm{v}(\gamma)$. □

We can prove now that the operation $\wedge$ on $\mathfrak{A}(X)$ introduced in Section 3 is its pseudosemilattice operation.

**Theorem 5.7.** *The algebra* $(\mathfrak{A}(X),\cdot,\wedge)$ *introduced at the end of Section 3 is a model for the bifree locally inverse semigroup on the set* $X$ *(as a type* $\langle 2,2\rangle$ *algebra).*

*Proof.* By Proposition 4.5 we just have to show that $\wedge$ is the pseudosemilattice operation on $\mathfrak{A}(X)$. Let $\gamma_1,\beta_1\in\mathfrak{A}(X)$ and $\gamma,\beta\in E(\mathfrak{A}(X))$ such that $\gamma_1\ \mathscr{R}\ \gamma$ and $\beta_1\ \mathscr{L}\ \beta$, and set $\delta=\gamma\wedge\beta=\overline{\gamma\oslash\beta}$. Thus,

$$\gamma_1\wedge\beta_1=\gamma\wedge\beta=\delta$$

by definition of $\wedge$ and the Proposition 5.5. The result becomes proved now once we show that $\delta\in E(\mathfrak{A}(X))$ and $(\gamma]_{\mathscr{R}}\cap(\beta]_{\mathscr{L}}=(\delta]_{\leq}$.

By construction, $\mathfrak{l}_{\gamma\oslash\beta}=\mathfrak{l}_\gamma$, $\mathfrak{r}_{\gamma\oslash\beta}=\mathfrak{r}_\beta$ and

$$\gamma\oslash\beta=\gamma\cup\beta\cup\{\{\mathfrak{l}_\gamma,\mathfrak{r}_\beta\}\}\,.$$

If $\mathrm{l}(\mathfrak{r}_\beta)=(\mathrm{l}(\mathfrak{l}_\gamma))'$, then we can collapse $\mathfrak{l}_\gamma^{\mathfrak{b}}$ with $\mathfrak{r}_\beta^{\mathfrak{b}}$ inside $\gamma\oslash\beta$ using a block-folding. Let $\delta_1$ be the rbb-tree obtained after applying this block-folding to $\gamma\oslash\beta$ if $\mathrm{l}(\mathfrak{r}_\beta)=(\mathrm{l}(\mathfrak{l}_\gamma))'$, or let $\delta_1=\gamma\oslash\beta$ otherwise. Note that $\{\mathfrak{l}_{\delta_1},\mathfrak{r}_{\delta_1}\}$ is always an edge of $\widehat{\delta_1}$. Further, $\delta=\overline{\delta_1}$ and no reduction applied to $\delta_1$ changes the fact stated in the previous sentence. Hence, $\{\mathfrak{l}_\delta,\mathfrak{r}_\delta\}$ is an edge of $\hat{\delta}$ and $\delta\in E(\mathfrak{A}(X))$.

Let $\alpha\in(\gamma]_{\mathscr{R}}\cap(\beta]_{\mathscr{L}}$. Then $\gamma\preccurlyeq_l\alpha$ and $\beta\preccurlyeq_r\alpha$. Let $\varphi_1$ be the left-morphism from $\gamma$ into $\alpha$ and let $\varphi_2$ be the right-morphism from $\beta$ into $\alpha$. Then $\{\mathfrak{l}_\gamma\varphi_1,\mathfrak{r}_\beta\varphi_2\}=\{\mathfrak{l}_\alpha,\mathfrak{r}_\alpha\}$ is an edge of $\hat{\alpha}$ since $\alpha$ is an idempotent. If the roots of $\alpha$ belong to different blocks, then $\delta_1=\gamma\oslash\beta$ and define $\varphi:\delta_1\to\alpha$

as follows: for a block or an edge $a$ of $\delta_1$, set

$$a\varphi = \begin{cases} a\varphi_1 & \text{if } a \in \gamma\,; \\ a\varphi_2 & \text{if } a \in \beta\,; \\ \{\mathfrak{l}_\alpha, \mathfrak{r}_\alpha\} & \text{if } a \text{ is the edge } \{\mathfrak{l}_\gamma, \mathfrak{r}_\beta\} \text{ of } \delta_1\,. \end{cases}$$

Clearly $\varphi$ is a well-defined bi-morphism from $\delta_1$ into $\alpha$. If the roots of $\alpha$ belong to the same block, then $\mathfrak{l}^{\mathfrak{b}}_\gamma\varphi_1 = \mathfrak{r}^{\mathfrak{b}}_\beta\varphi_2$ and define $\varphi : \delta_1 \to \alpha$ as follows instead:

$$a\varphi = \begin{cases} a\varphi_1 & \text{if } a \in \gamma\,; \\ a\varphi_2 & \text{if } a \in \beta\,. \end{cases}$$

The mapping $\varphi$ is well-defined because $\mathfrak{l}^{\mathfrak{b}}_\gamma\varphi_1 = \mathfrak{r}^{\mathfrak{b}}_\beta\varphi_2$, and clearly it is a bi-morphism.

We have constructed in the previous paragraph a bi-morphism from $\delta_1$ into $\alpha$. Since $\alpha$ is reduced and $\delta = \overline{\delta_1}$, there exists also another bi-morphism from $\delta$ into $\alpha$. Hence $\delta \preccurlyeq_b \alpha$ and $\alpha \in (\delta]_\leq$. Finally, note that $\gamma \preccurlyeq_l \gamma \circledS \beta$ and $\beta \preccurlyeq_r \gamma \circledS \beta$, whence $\gamma \preccurlyeq_l \delta$ and $\beta \preccurlyeq_r \delta$ because $\gamma$ and $\beta$ are reduced. Therefore $\delta \in (\gamma]_{\mathscr{R}} \cap (\beta]_{\mathscr{L}}$, and we have proved that $(\gamma]_{\mathscr{R}} \cap (\beta]_{\mathscr{L}} = (\delta]_\leq$. □

Next, we will use Propositions 5.3 and 5.5 to describe the inverses of each $\gamma \in \mathfrak{A}(X)$. As before, the description is easier to make if we use $\hat{\gamma}$ instead of $\gamma$.

**Proposition 5.8.** *Let $\beta, \gamma \in \mathfrak{A}(X)$. Then $\beta$ is an inverse of $\gamma$ if and only if*

(*i*) *$\beta \approx \gamma$;*

(*ii*) *$\{\mathfrak{l}_{\hat{\gamma}}, \mathfrak{r}_{\hat{\beta}}\}$ and $\{\mathfrak{r}_{\hat{\gamma}}, \mathfrak{l}_{\hat{\beta}}\}$ are edges of $\hat{\gamma}$ (here, $\mathfrak{r}_{\hat{\beta}}$ and $\mathfrak{l}_{\hat{\beta}}$ denote in fact the images of $\mathfrak{r}_{\hat{\beta}}$ and $\mathfrak{l}_{\hat{\beta}}$ under the isomorphism from $\hat{\beta}$ onto $\hat{\gamma}$ induced by the isomorphism from $\beta$ onto $\gamma$).*

*Proof.* Two reduced rbb-trees $\gamma$ and $\beta$ are inverses of each other if and only if there are idempotents $\delta$ and $\alpha$ such that

$$\gamma \mathrel{\mathscr{R}} \delta \mathrel{\mathscr{L}} \beta \mathrel{\mathscr{R}} \alpha \mathrel{\mathscr{L}} \gamma$$

since $\mathfrak{A}(X)$ is combinatorial. By Proposition 5.3 and 5.5, this occurs if and only if the four rbb-trees are isomorphic and

(*a*) $\mathfrak{l}_\gamma = \mathfrak{l}_\delta$, $\mathfrak{r}_\delta = \mathfrak{r}_\beta$, $\mathfrak{l}_\beta = \mathfrak{l}_\alpha$ and $\mathfrak{r}_\alpha = \mathfrak{r}_\gamma$; and

(*b*) $\{\mathfrak{l}_{\hat{\delta}}, \mathfrak{r}_{\hat{\delta}}\}$ is an edge of $\hat{\delta}$ and $\{\mathfrak{l}_{\hat{\alpha}}, \mathfrak{r}_{\hat{\alpha}}\}$ is an edge of $\hat{\alpha}$.

Thus, if and only if $\gamma$ and $\beta$ are isomorphic and $\{\mathfrak{l}_{\hat{\gamma}}, \mathfrak{r}_{\hat{\beta}}\}$ and $\{\mathfrak{r}_{\hat{\gamma}}, \mathfrak{l}_{\hat{\beta}}\}$ are edges of $\hat{\gamma}$. □

The following corollary is obvious from the previous proposition, but also from the fact that the number of inverses of a given $\gamma \in \mathfrak{A}(X)$ is equal to $ab$ where $a$ and $b$ are the number of idempotents in the $\mathscr{R}$-class and in the $\mathscr{L}$-class of $\gamma$, respectively.

**Corollary 5.9.** *Let $\gamma \in \mathfrak{A}(X)$. The number of inverses of $\gamma$ in $\mathfrak{A}(X)$ is*

$$(\deg(\mathfrak{l}_{\hat{\gamma}})-1)(\deg(\mathfrak{r}_{\hat{\gamma}})-1)\,.$$

Let $S$ be a regular semigroup. The *natural partial order* on $S$ [12] is the relation $\leq$ defined as follows: for $a, b \in S$,

$$a \leq b \quad \text{if and only if} \quad a = be = fb \text{ for some } e, f \in E(S)\,.$$

Then, for $f \in E(S)$, $(f]_{\leq} = \{e \in E(S) \; : \; e \leq f\}$. In the following proposition we describe the natural partial order on $\mathfrak{A}(X)$.

**Proposition 5.10.** *Let $\gamma, \beta \in \mathfrak{A}(X)$. Then $\beta \leq \gamma$ if and only if $\gamma \preccurlyeq_b \beta$.*

*Proof.* Assume first that $\beta \leq \gamma$. Then, there are $\delta, \alpha \in E(\mathfrak{A}(X))$ such that $\beta = \gamma \cdot \delta = \alpha \cdot \gamma$. In particular, $\mathrm{l}(\mathfrak{l}_\gamma) = \mathrm{l}(\mathfrak{l}_\beta)$ and $\mathrm{l}(\mathfrak{r}_\gamma) = \mathrm{l}(\mathfrak{r}_\beta)$. Take $\beta_1 = \gamma \odot \delta$ and observe that $\{\mathfrak{l}_{\hat{\delta}}, \mathfrak{r}_{\hat{\gamma}}\}$ and $\{\mathfrak{l}_{\hat{\delta}}, \mathfrak{r}_{\hat{\delta}}\}$ are edges of $\hat{\beta}_1$ due to the definition of $\odot$ and the Proposition 5.3, respectively. Further, since $\beta = \overline{\beta_1}$ and $\mathrm{l}(\mathfrak{r}_\gamma) = \mathrm{l}(\mathfrak{r}_\beta) = \mathrm{l}(\mathfrak{r}_\delta)$, we can collapse $\mathfrak{r}^{\mathfrak{b}}_\gamma$ with $\mathfrak{r}^{\mathfrak{b}}_\delta$ inside $\beta_1$ by an edge-folding or a block-folding. Let $\beta_2$ be the resulting rbb-tree. Clearly, $\gamma \preccurlyeq_b \beta_2$ and so $\gamma \preccurlyeq_b \beta$ because $\gamma$ is reduced.

Assume now that $\gamma \preccurlyeq_b \beta$. Let $\delta \approx \beta$ such that $\mathfrak{r}_\delta = \mathfrak{r}_\beta$ and $\mathfrak{l}_\delta$ is the left vertex of the block $\mathfrak{r}^{\mathfrak{b}}_\delta$ adjacent to $\mathfrak{r}_\delta$. Then $\delta$ is an idempotent by Proposition 5.3. Further, we can collapse $\mathfrak{r}^{\mathfrak{b}}_\gamma$ with $\mathfrak{l}^{\mathfrak{b}}_\delta = \mathfrak{r}^{\mathfrak{b}}_\delta$ inside $\gamma \odot \delta$ by a block-folding. Let $\beta_1$ be the rbb-tree obtained from $\gamma \odot \delta$ after applying this reduction. Once again by induction on the distance to the block $\mathfrak{r}^{\mathfrak{b}}_{\beta_1}$, we can prove that each block of $\gamma$ can be collapsed with its counterpart from $\delta$ inside $\beta_1$. Thus $\gamma \cdot \delta = \overline{\beta_1} = \beta$. In a dual manner we can define an idempotent $\alpha \in \mathfrak{A}(X)$ such that $\alpha \cdot \gamma = \beta$. Hence $\beta \leq \gamma$. ☐

The usual definition of the $\mathscr{R}$-relation in a semigroup $S$ is the following: for $a, b \in S$,

$$a \,\mathscr{R}\, b \text{ if and only if } aS^1 = bS^1\,,$$

that is, if and only if $a$ and $b$ generate the same principal right ideal of $S$. The definition of the $\mathscr{L}$-relation is the left-right dual, and $a \;\mathscr{J}\; b$ if and only if $a$ and $b$ generate the same principal ideal of $S$. We will denote by $\mathrm{R}_a$, $\mathrm{L}_a$ and $\mathrm{J}_a$ the $\mathscr{R}$-class, the $\mathscr{L}$-class and the $\mathscr{J}$-class of $a$ in $S$, respectively. There is a natural partial order on the set of $\mathscr{R}$-classes of $S$:

$$\mathrm{R}_a \leq_{\mathscr{R}} \mathrm{R}_b \text{ if and only if } aS^1 \subseteq bS^1\,.$$

It is well know that if $S$ is a regular semigroup, then $\mathrm{R}_a \leq_{\mathscr{R}} \mathrm{R}_b$ if and only if there are idempotents $e \in \mathrm{R}_a$ and $f \in \mathrm{R}_b$ such that $e \leq f$ (for the natural partial order $\leq$ on $S$). Similarly definitions and conclusions can be taken for the set of $\mathscr{L}$-classes and the set of $\mathscr{J}$-classes of $S$. Let $\leq_{\mathscr{L}}$ and $\leq_{\mathscr{J}}$ denote the natural partial orders on the sets of all $\mathscr{L}$-classes and of all $\mathscr{J}$-classes of $S$, respectively.

Obviously, $\mathscr{D}$=$\mathscr{R} \vee \mathscr{L}$ is always contained in $\mathscr{J}$. Although there are cases where this inclusion is strict, we know that $\mathscr{D}$=$\mathscr{J}$ in the bifree locally inverse semigroups. The following result characterizes the relations $\leq_{\mathscr{R}}$, $\leq_{\mathscr{L}}$ and $\leq_{\mathscr{J}}$ on $\mathfrak{A}(X)$.

**Proposition 5.11.** *Let $\gamma, \beta \in \mathfrak{A}(X)$.*

*(i)* $\mathrm{R}_\gamma \leq_{\mathscr{R}} \mathrm{R}_\beta$ *if and only if* $\beta \preccurlyeq_l \gamma$.
*(ii)* $\mathrm{L}_\gamma \leq_{\mathscr{L}} \mathrm{L}_\beta$ *if and only if* $\beta \preccurlyeq_r \gamma$.
*(iii)* $\mathrm{J}_\gamma \leq_{\mathscr{J}} \mathrm{J}_\beta$ *if and only if* $\beta \preccurlyeq \gamma$.

*Proof.* Once again we just have to prove $(i)$ since $(ii)$ is its dual and $(iii)$ follows from both $(i)$ and $(ii)$. As observed above, $\mathrm{R}_\gamma \leq_{\mathscr{R}} \mathrm{R}_\beta$ if and only if there are idempotents $\delta \in \mathrm{R}_\gamma$ and $\alpha \in \mathrm{R}_\beta$ such that $\delta \leq \alpha$; and by Propositions 5.5 and 5.10, if and only if

$$\beta \approx_l \alpha \preccurlyeq_b \delta \approx_l \gamma\,,$$

that is, $\beta \preccurlyeq_l \gamma$. □

## 6. The bifree completely simple semigroup on a set $X$

Let $\gamma \in \mathfrak{A}(X)$. Denote by $\gamma_{cs}$ the smallest rbb-bi-subtree of $\gamma$. Thus $\gamma_{cs}$ is constituted by the simple path in $\gamma$ from $\mathfrak{l}^{\mathfrak{b}}_\gamma$ to $\mathfrak{r}^{\mathfrak{b}}_\gamma$. Thus $\gamma_{cs} \in \mathfrak{A}(X)$ and $\gamma_{cs}$ is a chain (a straight path from one endpoint to the other endpoint) with the left root in one of its endpoints and the right root in the other endpoint. Let $\rho$ be the smallest completely simple congruence on $\mathfrak{A}(X)$ and let $\theta$ be the equivalence relation on $\mathfrak{A}(X)$ defined by

$$\gamma\theta\beta \quad \Leftrightarrow \quad \gamma_{cs} = \beta_{cs}\,,$$

for $\gamma, \beta \in \mathfrak{A}(X)$.

**Proposition 6.1.** $\rho = \theta$.

*Proof.* Let $\gamma, \beta \in \mathfrak{A}(X)$. By Proposition 4.5 and Corollary 2.3, $\gamma\rho\beta$ if and only if $\mathsf{w}^c(\gamma) = \mathsf{w}^c(\beta)$, and so if and only if $\mathsf{w}^c(\gamma_{cs}) = \mathsf{w}^c(\beta_{cs})$ because $\mathsf{w}^c(\gamma) = \mathsf{w}^c(\gamma_{cs})$ and $\mathsf{w}^c(\beta) = \mathsf{w}^c(\beta_{cs})$. Thus $\theta \subseteq \rho$.

Assume now that $\gamma\rho\beta$ and let $\gamma_1$ be the rbb-tree left-isomorphic to $\gamma_{cs}$ with $\mathfrak{r}_{\gamma_1}$ the right vertex of $\mathfrak{l}^{\mathfrak{b}}_\gamma$ adjacent to $\mathfrak{l}_{\gamma_{cs}} = \mathfrak{l}_{\gamma_1}$. Set $\beta_1$ as the rbb-tree obtained from $\gamma_1 \odot \beta_{cs}$ after collapsing $\mathfrak{l}^{\mathfrak{b}}_{\gamma_1}$ with $\mathfrak{l}^{\mathfrak{b}}_{\beta_{cs}}$ by a block-folding. Observe that there are two admissible paths $p$ and $p_1$ in $\hat{\beta}_1$ with $\mathsf{w}(p) = \mathsf{w}(p_1) = \mathsf{w}^c(\gamma)$, namely the admissible paths that end at the vertices $\mathfrak{r}_{\gamma_{cs}}$ and $\mathfrak{r}_{\beta_{cs}}$ of $\hat{\beta}_1$. Hence, the two blocks $\mathfrak{r}^{\mathfrak{b}}_{\gamma_{cs}}$ and $\mathfrak{r}^{\mathfrak{b}}_{\beta_{cs}}$ of $\beta_1$ must be collapsed into a single block inside $\overline{\beta_1}$ by Lemma 4.1. Since they are the other endpoints of the chains $\gamma_{cs}$ and $\beta_{cs}$, these two chains must collapse into a single chain, and $\gamma_{cs} = \beta_{cs} = \overline{\beta_1}$ as wanted. □

Set $\mathfrak{A}_{cs}(X) = \{\gamma_{cs} \,:\, \gamma \in \mathfrak{A}(X)\}$. Clearly $\mathfrak{A}_{cs}(X)$ is the set of all reduced rbb-trees that are chains with the left block and the right block on its opposite endpoints, and each $\theta$-class of $\mathfrak{A}(X)$ has a unique representative in

$\mathfrak{A}_{cs}(X)$. The previous result tell us also that $\theta$ is a congruence on $\mathfrak{A}(X)$. Thus

$$(\gamma_{cs} \cdot \beta_{cs})_{cs} = (\gamma \cdot \beta)_{cs} \quad \text{and} \quad (\gamma_{cs} \wedge \beta_{cs})_{cs} = (\gamma \wedge \beta)_{cs}$$

for any $\gamma, \beta \in \mathfrak{A}(X)$. Consider the following two operation $\cdot_{cs}$ and $\wedge_{cs}$ on $\mathfrak{A}_{cs}(X)$: for $\gamma, \beta \in \mathfrak{A}_{cs}(X)$, set

$$\gamma \cdot_{cs} \beta = (\gamma \cdot \beta)_{cs} \quad \text{and} \quad \gamma \wedge_{cs} \beta = (\gamma \wedge \beta)_{cs} .$$

Since $\theta$ is a congruence on $\mathfrak{A}(X)$, the mapping

$$\mathfrak{A}(X) \to \mathfrak{A}_{cs}(X) \, , \; \gamma \to \gamma_{cs}$$

is a homomorphism from $(\mathfrak{A}(X), \wedge, \cdot)$ onto $(\mathfrak{A}_{cs}(X), \cdot_{cs}, \wedge_{cs})$ whose kernel is $\theta$. The following result is now an obvious consequence of Proposition 6.1.

**Proposition 6.2.** *The algebra $(\mathfrak{A}_{cs}(X), \cdot_{cs}, \wedge_{cs})$ is a model for the bifree completely simple semigroup on the set $X$.*

These two last results show us how similar is the theory developed here for the bifree locally inverse semigroup with the Munn's theory for the free inverse monoid. In the Munn trees, the 'chain' (straight path) connecting the two distinguished vertices gives us a 'group word' and this 'group word' determines the least group congruence on the free inverse monoid. Here, the 'chain' connecting the two distinguished blocks gives us a 'completely simple word' (the word from the left root to the right root) and this 'completely simple word' determines the least completely simple congruence on the bifree locally inverse semigroup. Further, in both cases, we can naturally modify the usual product to a product of 'chains' in a way that, in the Munn tree case, we obtain a model for the free group, and in our case we obtain a model for the bifree completely simple semigroup.

The next result describes the idempotents and the Green's relations $\mathscr{R}$, $\mathscr{L}$ and $\mathscr{H}$ on $\mathfrak{A}_{cs}(X)$. We omit its proof since the statements follow easily from the results on $\mathfrak{A}(X)$ and from the fact that $\mathfrak{A}_{cs}(X)$ is a completely simple semigroup.

**Proposition 6.3.** *Let $\gamma$ and $\beta$ be two elements of the completely simple semigroup $\mathfrak{A}_{cs}(X)$.*

- *(i)* $\gamma \mathscr{R} \beta$ *if and only if* $\mathrm{l}(\mathfrak{l}_\gamma) = \mathrm{l}(\mathfrak{l}_\beta)$ .
- *(ii)* $\gamma \mathscr{L} \beta$ *if and only if* $\mathrm{l}(\mathfrak{r}_\gamma) = \mathrm{l}(\mathfrak{r}_\beta)$ .
- *(iii)* $\gamma \mathscr{H} \beta$ *if and only if* $\mathrm{l}(\mathfrak{l}_\gamma) = \mathrm{l}(\mathfrak{l}_\beta)$ *and* $\mathrm{l}(\mathfrak{r}_\gamma) = \mathrm{l}(\mathfrak{r}_\beta)$ .
- *(iv)* $\gamma$ *is an idempotent if and only if either* $\gamma$ *has only one block with roots of opposite pole, or* $\gamma$ *has only two blocks and one edge, the edge* $\{\mathfrak{l}_\gamma, \mathfrak{r}_\gamma\}$ .

We end this paper with a remark and a question. In [14] and [3] we have described models for the free idempotent generated locally inverse semigroups and for the free pseudosemilattices, respectively, using graphs. Looking back to those models, one can see that they resemble block-graphs.

In fact, if we consider the blocks with the structure $\underset{x}{\bullet}\!\!\longrightarrow\!\!\underset{x}{\bullet}$, instead of being squares, then the theory developed in the present paper and the theory developed in those two papers are very much alike.

There are two important classes of e-varieties of regular semigroups that have bifree objects on any set [19]: the e-varieties of locally inverse semigroups and the e-varieties of regular $E$-solid semigroups. Both classes contain the e-variety of all inverse semigroups. In the present paper we have generalized, for the bifree locally inverse semigroups, the Munn's approach to the free inverse monoids. The natural question to ask now is the following: Is there a similar generalization for the bifree regular $E$-solid semigroups?

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