Multidimensional Kontorovich-Lebedev transforms

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Abstract

Different approaches are used to introduce a direct product Kontorovich-Lebedev transformation, its essentially multidimensional analog and certain modifications in terms of multiple integrals over $\mathbb{R}^n_+$. Mapping properties are investigated. Inversion formulas are proved.

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1 Introduction

As it is known [6], [7], [9], [12], [13], classical Kontorovich-Lebedev transforms are of the form

$$g(\tau) = \int_0^\infty K_{i\tau}(x) f(x) dx, \quad \tau \geq 0, \tag{1.1}$$

$$g(s) = \int_0^\infty I_s(x) f(x) dx, \quad \text{Re s} > 0, \tag{1.2}$$

$$g(x) = \int_0^\infty K_{i\tau}(x) f(\tau) d\tau, \quad x > 0. \tag{1.3}$$

Here $K_s(x),\ I_s(x), K_{i\tau}(x)$ are modified Bessel functions [3, Vol. II] of the complex index and i is the imaginary unit. We note that the modified Bessel functions $K_\mu(z), I_\mu(z)$ are linear independent solutions of the Bessel differential equation

$$z^2 \frac{d^2u}{dz^2} + z\frac{du}{dz} - (z^2 + \mu^2)u = 0. \tag{1.4}$$

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They can be given by the formulas

$$I_\mu(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{\mu+2k}}{\Gamma(\mu+k+1)k!}, \tag{1.5}$$

$$K_\mu(z) = \frac{\pi}{2\sin\pi\mu}\left[I_{-\mu}(z) - I_\mu(z)\right], \tag{1.6}$$

when $\mu \neq 0, \pm1, \pm2, \ldots$, and $K_n(z) = \lim_{\mu\to n} K_\mu(z),\ n = 0, \pm1, \pm2, \ldots$. The function $K_\mu(z)$ is called also the Macdonald function and has the following integral representations (cf., in [3, Vol. II], [13])

$$K_\mu(z) = \int_0^\infty e^{-z\cosh t}\cosh\mu t dt = \frac{1}{2}\int_{-\infty}^{\infty} e^{-z\cosh t+\mu t}dt = \frac{1}{2}\int_0^\infty e^{-z(t+t^{-1})/2}t^{\mu-1}dt. \tag{1.7}$$

Useful relations are [3, Vol. II], [12]

$$2\mu K_\mu(z) = z\left[K_{\mu+1}(z) - K_{\mu-1}(z)\right], \tag{1.8}$$

$$\int_0^\infty I_\xi(x)K_\mu(x)\frac{dx}{x} = \frac{1}{\xi^2-\mu^2},\ \text{Re}\ \xi > |\text{Re}\ \mu|, \tag{1.9}$$

$$\int_0^\infty x^{\alpha-1}K_\mu(x)dx = 2^{\alpha-2}\Gamma\left(\frac{\alpha+\mu}{2}\right)\Gamma\left(\frac{\alpha-\mu}{2}\right),\ \text{Re}\ \alpha > |\text{Re}\ \mu|, \tag{1.10}$$

where $\Gamma(z)$ is Euler's gamma-function [3, Vol. I],

$$\int_0^\infty \tau\sinh\pi\tau\Gamma\left(\frac{\alpha+i\tau}{2}\right)\Gamma\left(\frac{\alpha-i\tau}{2}\right)K_{i\tau}(x)d\tau = 2^{1-\alpha}\pi^2x^\alpha,\ 0 \le \text{Re}\ \alpha < \frac{1}{2}. \tag{1.11}$$

These functions have the asymptotic behaviour [3, Vol. II]

$$K_\mu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}[1 + O(1/z)], \qquad z \to \infty, \tag{1.12}$$

$$I_\mu(z) = \frac{e^z}{\sqrt{2\pi z}}[1 + O(1/z)], \qquad z \to \infty, \tag{1.13}$$

and near the origin

$$K_\mu(z) = O\left(z^{-|Re\mu|}\right),\ z \to 0, \tag{1.14}$$

$$K_0(z) = -\log z + O(1),\ z \to 0, \tag{1.15}$$

$$I_\mu(z) = O(z^{\text{Re}\mu}), \mu \neq 0,\ z \to 0. \tag{1.16}$$

We will use in the sequel the properties of the multidimensional Mellin transform [2]. In fact, let $s^1 = \prod_{j=1}^n s_j$ for $s = (s_1, s_2, \ldots, s_n) \in \mathbb{C}^n,\ \sigma_j = \left\{s_j \in \mathbb{C},\ \text{Re}\ s_j = \frac{1}{2}\right\},\ \sigma =$

$\sigma_1 \times \sigma_2 \times \cdots \times \sigma_n$. Then the Mellin transform of a function $f : \mathbb{R}^n_+ \to \mathbb{C}$ is called the function $f^*(s)$ defined by

$$f^*(s) = \int_{\mathbb{R}^n_+} x^{s-1} f(x)\, dx, \tag{1.17}$$

where the latter integral by virtue of Fubini's theorem and under some conditions on f (see [2]) is understood as a product of simple integrals

$$f^*(s) = \int_0^\infty x_1^{s_1-1} \int_0^\infty x_2^{s_2-1} \ldots \int_0^\infty x_n^{s_n-1} f(x) dx_1 \ldots dx_n. \tag{1.18}$$

Its inversion is given accordingly [2]

$$f(x) = \frac{1}{(2\pi i)^n} \int_\sigma f^*(s) x^{-s}\, ds \tag{1.19}$$

as well as the generalized Parseval equality for two functions f, g and their Mellin transforms f^*, g^*

$$\int_{\mathbb{R}^n_+} f(x) g(x)\, dx = \frac{1}{(2\pi i)^n} \int_\sigma f^*(s) g^*(1-s) ds, \tag{1.20}$$

where $1 - s = (1 - s_1, 1 - s_2, \ldots,\ 1 - s_n)$.

Basing on formulas (1.18), (1.19) we will also treat under certain conditions from [2] the multidimensional Laplace transform

$$L[f](x) = \int_{\mathbb{R}^n_+} e^{-\langle x,t\rangle} f(t)\, dt, \tag{1.21}$$

where $\langle x, t\rangle = x_1 t_1 + \cdots + x_n t_n$ and its inverse in terms of the Mellin transform (1.19) $F = L[f]$

$$f(t) = \frac{1}{(2\pi i)^n} \int_\sigma \frac{F^*(1-s)}{\Gamma(1-s)} t^{-s}\, ds, \tag{1.22}$$

where $\Gamma(1-s) = \prod_{j=1}^n \Gamma(1-s_j)$.

We will use in the sequel the following modification of the Laplace transformation [2]

$$(\Lambda f)(x) = \int_{\mathbb{R}^n_+} \exp\left(-\max(x_1 t_1, \ldots, x_n t_n)\right) f(t) dt. \tag{1.23}$$

The Mellin transform (1.17) of the modified Laplace kernel $\exp(-\max(x_1, \ldots, x_n))$ is given by the integral (see [2], [5])

$$\int_{\mathbb{R}^n_+} x^{s-1} \exp(-\max(x_1, \ldots, x_n)) dx = \frac{\Gamma\left(1 + s_1 + \cdots + s_n\right)}{s_1 \ldots s_n}. \tag{1.24}$$

Furthermore, via (1.19) and the Mellin transform theory in $L_2(\mathbb{R}^n_+)$ equality (1.24) immediately implies the reciprocal relation for almost all $x \in \mathbb{R}^n_+$

$$\exp(-\max(x_1,\dots,x_n)) = \frac{1}{(2\pi i)^n} D^1_x \int_{(\sigma_1)} \dots \int_{(\sigma_n)} \frac{\Gamma(1+s_1+\dots+s_n)}{s^1(1-s)^1} x^{1-s} ds, \quad (1.25)$$

where we denoted by $D^1_x = \left(\frac{\partial}{x_1}, \dots, \frac{\partial}{x_n}\right)$. Moreover, by using definition (1.17) of the Mellin transform, it can be generalized on an arbitrary kernel $k(x)$ (cf. [5], [11]) for a maximum and a sum of its arguments. In particular, we have the equalities

$$\int_{\mathbb{R}^n_+} x^{s-1} k(\max(x_1,\dots,x_n)) dx = \frac{(s_1+\dots+s_n)}{s_1 \dots s_n} k^*(s_1+\dots+s_n), \quad (1.26)$$

$$\int_{\mathbb{R}^n_+} x^{s-1} k(x_1+\dots+x_n) dx = \frac{\prod_{j=1}^n \Gamma(s_j)}{\Gamma(s_1+\dots+s_n)} k^*(s_1+\dots+s_n). \quad (1.27)$$

2 A direct product Kontorovich-Lebedev transform

In this section we will expand formula (1.1) on a n-dimensional case introducing a direct product Kontorovich-Lebedev transform

$$(\mathcal{KL}f)(\tau) = \int_{\mathbb{R}^n_+} f(x) \prod_{j=1}^n K_{i\tau_j}(x_j) dx_j, \ \tau = (\tau_1, \tau_2, \dots, \tau_n) \in \mathbb{R}^n_+ \quad (2.1)$$

as the operator $\mathcal{KL} : L^\alpha_p(\mathbb{R}^n_+) \to C_0(\mathbb{R}^n_+)$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $0 < \alpha_i \le 1$, $i = 1, \dots, n$, $p > 1$. Here $L^\alpha_p(\mathbb{R}^n_+)$ denotes the weighted Lebesgue space normed by

$$||f||_{L^\alpha_p(\mathbb{R}^n_+)} = \left(\int_{\mathbb{R}^n_+} |f(x)|^p \prod_{j=1}^n K_0(\alpha_j x_j) dx \right)^{1/p}, \ p > 1$$

with respect to the measure $\prod_{j=1}^n K_0(\alpha_j x_j) dx_j$ and $C_0(\mathbb{R}^n_+)$ as usual is the space of bounded continuous functions vanishing at infinity.

Theorem 1. *The Kontorovich-Lebedev transform* (2.1) *is well-defined continuous, linear map*

$$\mathcal{KL} : L^\alpha_p(\mathbb{R}^n_+) \to C_0(\mathbb{R}^n_+),$$

with the norm at most $\prod_{j=1}^n \left(\frac{2\alpha_j}{\pi}\right)^{\frac{1-p}{p}}$, *i.e.*

$$||\mathcal{KL}|| \le \prod_{j=1}^n \left(\frac{2\alpha_j}{\pi}\right)^{\frac{1-p}{p}}.$$

When $\alpha_i = 1,\ i = 1, \dots, n$ *it has the exact value* $\left(\frac{\pi}{2}\right)^{n(p-1)/p}$.

Proof. We will appeal to the following inequality for the modified Bessel function [13]

$$|K_{i\lambda}(\beta t)| \le e^{-|\lambda| \text{ arccos } \beta} K_0(\beta t),\ x \in \mathbb{R},\ 0 < \beta \le 1. \tag{2.2}$$

Hence with the Hölder inequality for multiple integrals and relation (2.16.2.2) in [8], Vol. 2 we derive

$$||\mathcal{KL}\ f||_{C_0\left(\mathbb{R}^n_+\right)} \le \left(\int_{\mathbb{R}^n_+} \prod_{j=1}^n K_0(\alpha_j x_j) dx_j\right)^{\frac{p-1}{p}} ||f||_{L^\alpha_p(\mathbb{R}^n_+)} = \prod_{j=1}^n \left(\int_0^\infty K_0(\alpha_j\ t) dt\right)^{\frac{p-1}{p}}$$

$$\times ||f||_{L^\alpha_p(\mathbb{R}^n_+)} = \prod_{j=1}^n \left(\frac{2\alpha_j}{\pi}\right)^{\frac{1-p}{p}} ||f||_{L^\alpha_p(\mathbb{R}^n_+)}. \tag{2.3}$$

So the norm of the Kontorovich-Lebedev operator is at most $\prod_{j=1}^n \left(\frac{2\alpha_j}{\pi}\right)^{\frac{1-p}{p}}$. Moreover, using representations (1.7) for the modified Bessel function we find that the product $f(x)e^{-\langle x,\ \cosh u\rangle}$, $\cosh u = (\cosh u_1, \cosh u_2, \dots,\ \cosh u_n)$ is summable over $\mathbb{R}^n_+ \times \mathbb{R}^n$ and via Fubini's theorem we obtain the composition representation of the Kontorovich-Lebedev transform (2.1)

$$(\mathcal{KL}\ f)(\tau) = \left(\frac{\pi}{2}\right)^{n/2} F[L[f](\cosh u)](\tau) \tag{2.4}$$

in terms of the Laplace transform (1.21) in the point $\cosh u$ and the Fourier transform

$$F[f](\tau) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(t) e^{i\langle \tau, t\rangle} dt. \tag{2.5}$$

Therefore $(\mathcal{KL}\ f)(\tau)$ is continuous and it vanishes at infinity via the Riemann-Lebesgue lemma if we show that $L[f](\cosh u) \in L_1(\mathbb{R}^n)$. But the latter fact follows from (1.7) and the estimate

$$\int_{\mathbb{R}^n} |L[f](\cosh u)|\, du \le \int_{\mathbb{R}^n_+} |f(x)| \prod_{j=1}^n K_0(x_j) dx_j$$

$$\le \left(\frac{\pi}{2}\right)^{n(p-1)/p} ||f||_{L^\alpha_p(\mathbb{R}^n_+)} < \infty.$$

Finally, taking $f(x) \equiv 1$, which belongs to $L^\alpha_p(\mathbb{R}^n_+)$ and calculating its Kontorovich-Lebedev transform (2.1), which gives the value $\left(\frac{\pi}{2}\right)^n \prod_{j=1}^n [\cosh(\pi\tau_j/2)]^{-1}$ we easily find that the norm of the KL-operator attains its least value $\left(\frac{\pi}{2}\right)^{n(p-1)/p}$. Theorem 1 is proved.

Now we are going to prove the inversion theorem for the Kontorovich-Lebedev transform (2.1) in $L^\alpha_p(\mathbb{R}^n_+),\ p > 1$. To do this we will employ the inversion L_1- theorem [1] for the Fourier transform (2.5) and Sneddon's operational method, which was used formally in [9], Chapter 6 to establish an inversion formula for the ordinary Kontorovich-Lebedev transform (1.1). We have

Theorem 2. *Let* $f \in L_p^\alpha(\mathbb{R}_+^n)$, $p > 1$ *and* $(\mathcal{KL}\ f)(\tau)\ \tau^1 \exp\left(\frac{\pi}{2}\sum_{j=1}^n \tau_j\right) \in L_1(\mathbb{R}_+^n)$. *Then for almost all* $x \in \mathbb{R}_+^n$

$$f(x) = \lim_{\varepsilon\to 0+} \left(\frac{2}{\pi^2}\right)^n \int_{\mathbb{R}_+^n} e^{-\varepsilon|\tau|^2} (\mathcal{KL}\ f)(\tau) \prod_{j=1}^n \tau_j \sinh \pi\tau_j\ \frac{K_{i\tau_j}(x_j)}{x_j} d\tau_j. \tag{2.6}$$

Proof. Indeed, we take composition representation (2.4) and use the continuity of the function $L[f](\cosh u) \in L_1(\mathbb{R}^n)$ in order to apply the inversion theorem for the Fourier transform and for all $u \in \mathbb{R}^n$, $\cosh u = (\cosh u_1,\ \ldots,\ \cosh u_n)$ to get the equality

$$L[f](\cosh u) = \lim_{\varepsilon\to 0+} \frac{1}{\pi^n} \int_{\mathbb{R}^n} e^{-\varepsilon|\tau|^2 + i\langle \tau,\ u\rangle} (\mathcal{KL}\ f)(\tau) d\tau. \tag{2.7}$$

The mentioned continuity of $L[f](\cosh u)$ can be established, in turn, by the uniform convergence of the Laplace integral (1.21). In fact, by the Hölder inequality and asymptotic formulas (1.12), (1.15) for the modified Bessel functions we deduce the estimate

$$|L[f](\cosh u)| \le \int_{\mathbb{R}_+^n} e^{-\langle \cosh u, t\rangle} |f(t)|\ dt \le \left(\int_{\mathbb{R}_+^n} \frac{e^{-q\sum_{j=1}^n t_j}}{\prod_{j=1}^n K_0^{q/p}(\alpha_j t_j)} dt\right)^{1/q} ||f||_{L_p^\alpha(\mathbb{R}_+^n)}$$

$$\le \left(\int_0^\infty \frac{e^{-qt}}{K_0^{q/p}(t)} dt\right)^{n/q} ||f||_{L_p^\alpha(\mathbb{R}_+^n)} < \infty,\ q = \frac{p}{p-1}, \tag{2.8}$$

which implies the uniform convergence of the corresponding Laplace integral for all $u \in \mathbb{R}^n$ and the continuity of $L[f](\cosh u)$. Returning to (2.7) we rewrite its right-hand side by using the evenness of the integrand with respect to τ and u. Therefore we obtain

$$L[f](\cosh u) = \lim_{\varepsilon\to 0+} \left(\frac{2}{\pi}\right)^n \int_{\mathbb{R}_+^n} e^{-\varepsilon|\tau|^2} (\mathcal{KL}\ f)(\tau) \prod_{j=1}^n \cos \tau_j u_j d\tau_j,\ u \in \mathbb{R}_+^n. \tag{2.9}$$

Now we are going to differentiate by $u_j > 0$, $j = 1, \ldots,\ n$ both sides of (2.9). The left-hand side becomes

$$\frac{\partial^n}{\partial u_1 \ldots \partial u_n} L[f](\cosh u) = (-1)^n \prod_{j=1}^n \sinh u_j \int_{\mathbb{R}_+^n} e^{-\langle \cosh u, t\rangle} f(t)\ t^1 dt$$

and similarly to (2.8) we show that the latter integral is uniformly convergent by u on any compact set of $\mathbb{R}_+^n$. On the other hand, the differentiation by u in the right-hand side of (2.9) under the limit sign can be motivated considering the uniform convergence by u of the integral

$$\int_{\mathbb{R}^n_+} e^{-\varepsilon|\tau|^2}(\mathcal{KL}\ f)(\tau)\prod_{j=1}^n \sin\tau_j u_j\ \tau_j d\tau_j$$

when $\varepsilon \to 0+$. But this follows via Cauchy's criterium because (see (2.2), (2.3)) $|\mathcal{KL}\ f)(\tau)| \leq Ae^{-\sum_{j=1}^n c_j\tau_j},\ A > 0, 0 < c_j < \frac{\pi}{2},\ j = 1, \ldots, n$ and $(\varepsilon_1 < \varepsilon_2)$

$$\left|\int_{\mathbb{R}^n_+}\left[e^{-\varepsilon_1|\tau|^2} - e^{-\varepsilon_2|\tau|^2}\right](\mathcal{KL}\ f)(\tau)\prod_{j=1}^n \sin\tau_j u_j\ \tau_j d\tau_j\right|$$

$$\leq A\int_{\mathbb{R}^n_+} e^{-\varepsilon_1|\tau|^2}\left[1 - e^{-(\varepsilon_2-\varepsilon_1)|\tau|^2}\right]\prod_{j=1}^n e^{-c_j\tau_j}\ \tau_j d\tau_j$$

$$\leq A(\varepsilon_2 - \varepsilon_1)\int_{\mathbb{R}^n_+}|\tau|^2\prod_{j=1}^n e^{-c_j\tau_j}\ \tau_j d\tau_j \to 0,\ \varepsilon_2 \to 0+.$$

Consequently, (2.9) leads to the equality

$$\int_{\mathbb{R}^n_+} e^{-\langle\cosh u, t\rangle} f(t)\ t^1 dt = \lim_{\varepsilon\to 0+}\left(\frac{2}{\pi}\right)^n\int_{\mathbb{R}^n_+} e^{-\varepsilon|\tau|^2}(\mathcal{KL}\ f)(\tau)\prod_{j=1}^n \frac{\tau_j\sin\tau_j u_j}{\sinh u_j}d\tau_j. \tag{2.10}$$

Meanwhile, relation (2.16.6.1) in [8], Vol. 2 gives

$$\frac{\sin\tau_j u_j}{\sinh u_j} = \pi^{-1}\sinh\pi\tau_j\int_0^\infty e^{-t\cosh u_j}K_{i\tau_j}(t)dt.$$

Substituting this value into (2.10) and changing the order of integration in its right-hand side via Fubini's theorem, (2.10) becomes

$$\int_{\mathbb{R}^n_+} e^{-\langle\cosh u, t\rangle} f(t)\ t^1 dt = \lim_{\varepsilon\to 0+}\int_{\mathbb{R}^n_+} e^{-\langle\cosh u, t\rangle}F_\varepsilon(t)dt, \tag{2.11}$$

where

$$F_\varepsilon(t) = \left(\frac{2}{\pi^2}\right)^n\int_{\mathbb{R}^n_+} e^{-\varepsilon|\tau|^2}(\mathcal{KL}\ f)(\tau)\prod_{j=1}^n \tau_j\sinh\pi\tau_j\ K_{i\tau_j}(t_j)d\tau_j. \tag{2.12}$$

So the theorem will be proved by virtue of the uniqueness and boundedness properties of the Laplace operator $L[f] : L_1(\mathbb{R}^n_+; e^{-\alpha t}dt) \subset L^\alpha_p(\mathbb{R}^n_+) \to C(\mathbb{R}^n_+)$ (see (2.8)). In fact, appealing to the Schwarz inequality, conditions of the theorem (see also (2.2)) and relation (2.16.33.2) in [8], Vol. 2, we get

$$||F_{\varepsilon_1} - F_{\varepsilon_2}||_{L_1(\mathbb{R}^n_+; e^{-\alpha t}dt)} \leq \left(\frac{2}{\pi^2}\right)^n\int_{\mathbb{R}^n_+} e^{-\varepsilon_1|\tau|^2}\left[1 - e^{-(\varepsilon_2-\varepsilon_1)|\tau|^2}\right]|(\mathcal{KL}\ f)(\tau)|$$

$$\times \prod_{j=1}^{n} \tau_j \sinh \pi\tau_j \left(\int_0^\infty e^{-2\alpha_j t_j} dt_j \right)^{1/2} \left(\int_0^\infty |K_{i\tau_j}(t_j)|^2 dt_j \right)^{1/2} d\tau_j$$

$$\leq \frac{1}{\sqrt{2^n \alpha^1}} \int_{\mathbb{R}_+^n} \left[1 - e^{-(\varepsilon_2 - \varepsilon_1)|\tau|^2} \right] |(\mathcal{KL}\ f)(\tau)|\, \tau^1\ \exp\left(\frac{\pi}{2} \sum_{j=1}^{n} \tau_j \right) d\tau\ \to 0,$$

when $0 < \varepsilon_1 < \varepsilon_2 \to 0+$ due to the Lebesgue dominated convergence theorem. Therefore F_ε has a limit in the space $L_1(\mathbb{R}_+^n; e^{-\alpha t}dt)$, which we denote by $h(t)$. Hence we pass to the limit under the Laplace operator in the right-hand side of (2.11) and by the uniqueness property immediately obtain $h(t) = t^1 f(t)$ for almost all $t \in \mathbb{R}_+^n$. This implies the inversion formula (2.6) and completes the proof of Theorem 2.

3 Plancherel theorem

In this section we will prove an analog of the Plancherel theorem [10] for the multidimensional Kontorovich-Lebedev transform (2.1), which was established for the classical case in [9], [12], [13]. But first we will need the multidimensional analog of the Kontorovich-Lebedev convolution [12], [13] for the transformation (2.1).

Definition 1. Let f, g be functions from $\mathbb{R}_+^n$ into $\mathbb{C}$. Then the function $f * g$ defined on $\mathbb{R}_+^n$

$$(f * g)(x) = \frac{1}{2^n x^1} \int_{\mathbb{R}_+^{2n}} f(u) g(y) \exp\left(-\frac{1}{2} \sum_{j=1}^{n} \left(x_j \frac{u_j^2 + y_j^2}{u_j y_j} + \frac{y_j u_j}{x_j} \right) \right) du dy \tag{3.1}$$

is called the (Kontorovich-Lebedev) convolution of f and g (provided that it exists).

Appealing to Fubini's theorem we immediately establish a multidimensional analog of Theorem 2.1 in [14].

Theorem 3. *Let $f, g \in L_1^\alpha(\mathbb{R}_+^n)$. Then the convolution $f * g$ exists for almost all $x \in \mathbb{R}_+^n$ and it belongs to $L_1^\alpha(\mathbb{R}_+^n)$. The convolution is commutative, associative and*

$$||f * g||_{L_1^\alpha(\mathbb{R}_+^n)} \leq ||f||_{L_1^\alpha(\mathbb{R}_+^n)} ||g||_{L_1^\alpha(\mathbb{R}_+^n)}. \tag{3.2}$$

For the Kontorovich-Lebedev transform (2.1) *it has the factorization equality*

$$\mathcal{KL}[f * g] = \mathcal{KL}[f]\mathcal{KL}[g]. \tag{3.3}$$

Finally, if $0 < \alpha_j < \frac{1}{2},\ j = 1, \ldots, n$ then the following Parseval type equality holds true

$$(f * g)(x) = \left(\frac{2}{\pi^2} \right)^n \frac{1}{x^1} \int_{\mathbb{R}_+^n} \mathcal{KL}[f](\tau)\mathcal{KL}[g] \prod_{j=1}^{n} \tau_j \sinh \pi\tau_j\ K_{i\tau_j}(x_j) d\tau_j. \tag{3.4}$$

Now putting $g = \bar{f}$ in (3.4) we assume that f belongs to the space of C^∞-functions with compact support on $\mathbb{R}^n_+$. Multiplying both sides of (3.4) by $x^{\lambda-1} \exp\left(\sum_{j=1}^n x_j\right)$, $0 < \lambda < \frac{1}{2}$ and inverting the order of integration by Fubini's theorem, we calculate the corresponding Mellin transforms (1.17) using relations (2.3.16.1) in [8], Vol. 1 and (2.16.6.5) in [8], Vol. 2. Hence we arrive at the equality

$$\frac{1}{\Gamma^n\left(\frac{1}{2}-\lambda\right)} \int_{\mathbb{R}^{2n}_+} f(u)\overline{f(y)} \prod_{j=1}^n \left(\frac{u_j y_j}{|u_j - y_j|}\right)^\lambda K_\lambda(|u_j - y_j|)\, du_j dy_j$$

$$= \left(\frac{2^{-\lambda}}{\pi^2\sqrt{\pi}}\right)^n \int_{\mathbb{R}^n_+} |\mathcal{KL}[f](\tau)|^2 \prod_{j=1}^n \tau_j \sinh(2\pi\tau_j)\ |\Gamma(\lambda + i\tau)|^2 d\tau_j. \tag{3.5}$$

But the left-hand side of (3.5) can be treated with the use of Theorem 65 in [10] for the Fourier convolution. Indeed, invoking with relation (2.16.6.6) in [8], Vol. 2 it becomes

$$\frac{1}{\Gamma^n\left(\frac{1}{2}-\lambda\right)} \int_{\mathbb{R}^{2n}_+} f(u)\overline{f(y)} \prod_{j=1}^n \left(\frac{u_j y_j}{|u_j - y_j|}\right)^\lambda K_\lambda(|u_j - y_j|)\, du_j dy_j$$

$$= (2^{-2\lambda}\pi)^{n/2} \int_{\mathbb{R}^n} \frac{|F[f(u)u^\lambda]\ (\xi)|^2}{\prod_{j=1}^n (1+\xi_j^2)^{1/2-\lambda}} d\xi,$$

where $F[f(u)u^\lambda]\ (\xi)$ is the Fourier transform (2.5) of $f(u)u^\lambda$. Hence combining with (3.5), we obtain

$$\int_{\mathbb{R}^n} \frac{|F[f(u)u^\lambda]\ (\xi)|^2}{\prod_{j=1}^n (1+\xi_j^2)^{1/2-\lambda}}\ d\xi = \frac{1}{\pi^{3n}} \int_{\mathbb{R}^n_+} |\mathcal{KL}[f](\tau)|^2$$

$$\times \prod_{j=1}^n \tau_j \sinh(2\pi\tau_j)\ |\Gamma(\lambda + i\tau)|^2 d\tau_j. \tag{3.6}$$

Since for any smooth f with compact support $F[f(u)u^\lambda]\ (\xi) \in L_2(\mathbb{R}^n)$ and continuous with respect to $\lambda \in \left(0, \frac{1}{2}\right]$ we can pass to the limit under the integral sign when $\lambda \to \frac{1}{2}$ and apply the Parseval identity for the Fourier transform [10]. Therefore we have

$$\lim_{\lambda \to 1/2-} \int_{\mathbb{R}^n} \frac{|F[f(u)u^\lambda]\ (\xi)|^2}{\prod_{j=1}^n (1+\xi_j^2)^{1/2-\lambda}}\ d\xi = \int_{\mathbb{R}^n_+} |f(u)|^2 u^1\ du.$$

On the other hand, similarly to [13], Chapter 2 we find that

$$\mathcal{KL}[f](\tau) = O\left((\tau^1)^{-3n/2} \exp\left(-\frac{\pi}{2}\sum_{j=1}^n \tau_j\right)\right),\ \tau \to \infty$$

Moreover, as a consequence of the Stirling formula for gamma-functions [3], Vol. 1

$$\cosh \pi v \ |\Gamma(\lambda + iv)|^2 \leq \sqrt{2\pi} v^{2\lambda - 1} e^{1/(3\lambda)}.$$

Hence one can pass to the limit when $\lambda \to \frac{1}{2}-$ under the integral sign in the right-hand side of (3.6). Consequently, by straightforward calculations we establish the Parseval identity for the Kontorovich-Lebedev transform (2.1) of smooth functions with compact support

$$\int_{\mathbb{R}^n_+} |f(u)|^2 u^1 \ du = \left(\frac{2}{\pi^2}\right)^n \int_{\mathbb{R}^n_+} |\mathcal{KL}[f](\tau)|^2 \prod_{j=1}^n \tau_j \sinh \pi\tau_j \ d\tau_j, \tag{3.7}$$

which coincides with the classical case (cf. [9], [13]) when $n = 1$. Now taking an arbitrary function $f(x) \in L_2(\mathbb{R}^n_+; x^1 dx)$ we will approximate it by a sequence $\{f_k\}$ of smooth functions with compact support in $Q_N = \left[\frac{1}{N}, N\right]^n, \ N > 0$, i.e.

$$\lim_{k\to\infty} ||f_k - f||_{L_2(Q_N;x^1 dx)} = 0. \tag{3.8}$$

Furthermore, it follows that $f \in L^\alpha_p(Q_N), \ p > 1$ and denoting by

$$g_N(\tau) = \int_{Q_N} f(x) \prod_{j=1}^n K_{i\tau_j}(x_j) dx_j, \quad g_{N,k}(\tau) = \int_{Q_N} f_k(x) \prod_{j=1}^n K_{i\tau_j}(x_j) dx_j$$

it gives by Theorem 1, estimates (2.3), (3.8) the inequality

$$|g_{N,k}(\tau) - g_N(\tau)| \leq \left(\frac{2}{\pi}\right)^{\frac{n(1-p)}{p}} ||f_k - f||_{L^\alpha_p(Q_N)} \to 0, \ k \to \infty. \tag{3.9}$$

Thus the sequence $\{g_{N,k}\}(\tau)$ converges uniformly to $g_N(x)(\tau)$ with respect to τ. Meanwhile, from (3.7) we know that

$$||f_k||^2_{L_2(Q_N;x^1 dx)} = \left(\frac{2}{\pi^2}\right)^n ||g_{N,k}||^2_{L_2(\mathbb{R}^n_+;\prod_{j=1}^n \tau_j \sinh \pi\tau_j d\tau_j)} .$$

Therefore

$$\left(\frac{2}{\pi^2}\right)^n ||g_{N,k} - g_{N,m}||^2_{L_2(\mathbb{R}^n_+;\prod_{j=1}^n \tau_j \sinh \pi\tau_j d\tau_j)} = ||f_k - f_m||^2_{L_2(Q_N;x^1 dx)}$$

and making use of (3.9) we get as $m \to \infty$

$$\left(\frac{2}{\pi^2}\right)^n ||g_{N,k} - g_N||^2_{L_2(\mathbb{R}^n_+;\prod_{j=1}^n \tau_j \sinh \pi\tau_j d\tau_j)} = ||f_k - f||^2_{L_2(Q_N;x^1 dx)} .$$

Hence with the aid of (3.8) we have

$$\left(\frac{2}{\pi^2}\right)^n ||g_N||^2_{L_2\left(\mathbb{R}^n_+;\prod_{j=1}^n \tau_j \sinh \pi\tau_j d\tau_j\right)} = ||f||^2_{L_2(Q_N;x^1 dx)}\,.$$

Analogously,

$$\left(\frac{2}{\pi^2}\right)^n ||g_N - g_{N_1}||^2_{L_2(\mathbb{R}^n_+;\prod_{j=1}^n \tau_j \sinh \pi\tau_j d\tau_j)} = ||f||^2_{L_2(Q_N\setminus Q_{N_1};x^1 dx)}\,,\ N > N_1,$$

i.e $\{g_N\}$ is a Cauchy sequence of $L_2(\mathbb{R}^n_+;\prod_{j=1}^n \tau_j \sinh \pi\tau_j d\tau_j)$ and there exists a function $G(\tau) = \mathcal{KL}[f](\tau) \in L_2(\mathbb{R}^n_+;\prod_{j=1}^n \tau_j \sinh \pi\tau_j d\tau_j)$, $G(\tau) = \text{l.i.m.}_{N\to\infty} g_N(\tau)$ and the Parseval equality

$$\left(\frac{2}{\pi^2}\right)^n ||G||^2_{L_2\left(\mathbb{R}^n_+;\prod_{j=1}^n \tau_j \sinh \pi\tau_j d\tau_j\right)} = ||f||^2_{L_2(\mathbb{R}^n_+;x^1 dx)}$$

holds. Further, from (3.7) we easily deduce

$$\int_{\mathbb{R}^n_+} f(u)\overline{h(u)}u^1\ du = \left(\frac{2}{\pi^2}\right)^n \int_{\mathbb{R}^n_+} G(\tau)\overline{H(\tau)} \prod_{j=1}^n \tau_j \sinh \pi\tau_j\ d\tau_j \tag{3.10}$$

for any two $L_2(\mathbb{R}^n_+;x^1 dx)$- functions f, h and their Kontorovich-Lebedev transforms $G, H \in L_2\left(\mathbb{R}^n_+;\prod_{j=1}^n \tau_j \sinh \pi\tau_j d\tau_j\right)$. In particular, taking $h(u) = \prod_{j=1}^n h_j(u_j)$, where

$$h_j(u_j) = \begin{cases} 1, & \text{if } u_j \in [0, x_j], \\ 0, & \text{if } u_j \in (x_j, \infty),\ j = 1, \dots, n, \end{cases}$$

we substitute it in (3.10) and we arrive after differentiation at the reciprocal inversion formula, which is valid for almost all $x \in \mathbb{R}^n_+$, namely

$$f(x) = \left(\frac{2}{\pi^2}\right)^n \frac{1}{x^1} D^1_x \int_{\mathbb{R}^n_+} G(\tau) \prod_{j=1}^n \tau_j \sinh \pi\tau_j \int_0^{x_j} K_{i\tau_j}(t)dt\ d\tau_j. \tag{3.11}$$

On the other hand, taking

$$G_N(\tau) = \begin{cases} G(\tau), & \text{if } \tau \in Q_N, \\ 0, & \text{otherwise}, \end{cases}$$

we prove similarly that the sequence $\{f_N\}$, where

$$f_N(x) = \left(\frac{2}{\pi^2}\right)^n \frac{1}{x^1} \int_{Q_N} G(\tau) \prod_{j=1}^n \tau_j \sinh \pi\tau_j K_{i\tau_j}(x_j)\ d\tau_j, \tag{3.12}$$

which is obtained from (3.11) by differentiation under the integral sign is approximating the function f, i.e. $f(x) = \text{l.i.m.}_{N\to\infty} f_N(x)$ with respect to the norm in $L_2(\mathbb{R}^n_+; x^1 dx)$. So we have proved the following Plancherel theorem for the multidimensional Kontorovich-Lebedev transform.

Theorem 4. *The Kontorovich-Lebedev transform* (2.1) *written in the form*

$$(\mathcal{KL}f)(\tau) \equiv G(\tau) = l.i.m._{N\to\infty} \int_{Q_N} f(x) \prod_{j=1}^{n} K_{i\tau_j}(x_j) dx_j$$

forms an isometric isomorphism

$$\mathcal{KL} : L_2(\mathbb{R}^n_+; x^1 dx) \to L_2(\mathbb{R}^n_+; \prod_{j=1}^{n} \tau_j \sinh \pi\tau_j d\tau_j)$$

and the Plancherel identity (3.7) *and the Parseval equality* (3.10) *for two functions $f, h \in L_2(\mathbb{R}^n_+; x^1 dx)$ hold. Furthermore, the inverse operator has the form*

$$(\mathcal{KL}^{-1}\ G)(x) \equiv f(x) = l.i.m._{N\to\infty} \left(\frac{2}{\pi^2}\right)^n \frac{1}{x^1} \int_{Q_N} G(\tau) \prod_{j=1}^{n} \tau_j \sinh \pi\tau_j K_{i\tau_j}(x_j)\ d\tau_j,$$

which is equivalent to formula (3.11) *for almost all $x \in \mathbb{R}^n_+$.*

4 Essentially multidimensional transforms

In this section we will consider an analog of the essentially multidimensional Kontorovich-Lebedev transform, which was announced formally in [13], basing on the modified Laplace transform (1.23). We note, that an analog of the multidimensional Watson transforms was investigated in [11].

Let us consider the following kernel function

$$K_{\max}(x, \tau) = \frac{1}{2^n} \int_{\mathbb{R}^n} \exp\left(-\max(x_1 \cosh u_1, \ldots, x_n \cosh u_n) + i\langle \tau, u\rangle\right) du, \tag{4.1}$$

where $x \in \mathbb{R}^n_+,\ \tau \in \mathbb{R}^n_+$. Letting $n = 1$ we easily arrive at the integral representation (1.7) of the modified Bessel function, which is, in turn, the kernel of the Kontorovich-Lebedev transforms (1.1), (1.3). From (4.1) and the estimate

$$|K_{\max}(x, \tau)| \le \int_{\mathbb{R}^n_+} \exp\left(-\max(x_1 \cosh u_1, \ldots, x_n \cosh u_n)\right) du \tag{4.2}$$

it follows that this kernel is continuous by τ for a fixed $x \in \mathbb{R}^n_+$ if the integral (4.2) is finite. However, the latter fact will be verified via Fubini's theorem and from the formula for the Mellin transform (1.17) of $K_{\max}(x,\tau)$, which can be calculated making use (1.24) and the relation (2.5.46.6) in [8], Vol. 1. Precisely we obtain

$$K^*_{\max}(s,\tau) = \int_{\mathbb{R}^n_+} K_{\max}(x,\tau)x^{s-1}dx = \Gamma\left(1+s_1+\cdots+s_n\right)$$

$$\times \prod_{j=1}^{n} \frac{2^{s_j-2}}{\Gamma(s_j+1)}\Gamma\left(\frac{s_j+i\tau_j}{2}\right)\Gamma\left(\frac{s_j-i\tau_j}{2}\right), \tag{4.3}$$

where $s_j \in \mathbb{C},\ \mathrm{Re} s_j = \nu_j > 0,\ j = 1,\ldots,n$ and the corresponding change of the order of integration is possible due to the estimate (see [2], Chapter 4)

$$\int_{\mathbb{R}^n_+} \left|K_{\max}(x,\tau)x^{s-1}dx\right| \le \int_{\mathbb{R}^n_+} \exp\left(-\max(t_1,\ldots,t_n)\right)t^{\nu-1}dt$$

$$\times \prod_{j=1}^{n}\int_0^\infty \frac{du}{\cosh^{\nu_j} u} < \infty.$$

Hence

$$\int_{\mathbb{R}^n_+} dx \int_{\mathbb{R}^n_+} \exp\left(-\max(x_1\cosh u_1,\ldots,x_n\cosh u_n)\right)du$$

$$= \int_{\mathbb{R}^n_+} \exp\left(-\max(t_1,\ldots,t_n)\right)dt\ \left(\int_0^\infty \frac{du}{\cosh u}\right)^n = n!\left(\frac{\pi}{2}\right)^n$$

and therefore the integral in the right-hand side of (4.2) is finite for almost all $x \in \mathbb{R}^n_+$. Moreover, if it is convergent for some $a = (a_1,\ldots,a_n),\ a_j > 0,\ j = 1,\ldots,n$ then it is evidently convergent for all $x = (x_1,\ldots,x_n),\ x_j \ge a_j,\ j = 1,\ldots,n$. Thus the kernel (4.1) is continuous by $x = (x_1,\ldots,x_n),\ x_j \ge a_j > 0,\ j = 1,\ldots,n$ and $\tau \in \mathbb{R}^n_+$ and tends to zero when $|\tau| \to \infty$. Returning to (4.3) we observe with the use of the Stirling asymptotic formula for gamma-functions [3], Vol. I that its right-hand side is summable by s for a fixed τ. Consequently, owing to (1.19) it leads to the representation

$$K_{\max}(x,\tau) = \frac{4^{-n}}{(2\pi i)^n}\int_{(\nu_1)}\cdots\int_{(\nu_n)} \frac{\Gamma\left(1+s_1+\cdots+s_n\right)}{\Gamma(s_1+1)\ldots\Gamma(s_n+1)}$$

$$\times \prod_{j=1}^{n}\Gamma\left(\frac{s_j+i\tau_j}{2}\right)\Gamma\left(\frac{s_j-i\tau_j}{2}\right)\left(\frac{x_j}{2}\right)^{-s_j}ds_j, \tag{4.4}$$

which can be written in terms of the multiple hypergeometric functions [4]. However, in the meantime we can express the kernel (4.1) as a one-dimensional integral containing a product of the associated Legendre functions [3], Vol. I of different arguments and indices.

In fact, appealing to relation (8.4.41.4) in [8], Vol. 3 and making use Euler's integral and duplication formula for gamma-function we derive the equality

$$K_{\max}(x,\tau) = \left(\frac{\sqrt{\pi}}{2}\right)^n \frac{1}{x^1} \int_{\max(x_1,\dots,x_n)}^{\infty} e^{-y}\, y^{n/2}$$

$$\times \prod_{j=1}^{n} \left(y^2 - x_j^2\right)^{1/4} P^{-1/2}_{-(1-i\tau_j)/2}\left(\frac{2y^2}{x_j^2} - 1\right) dy. \tag{4.5}$$

Again, letting $n = 1$ in (4.5) by straightforward calculations it gives the value $K_{i\tau}(x)$, which coincides with the kernel of the Kontorovich-Lebedev transforms (1.1), (1.3).

Our goal is to study the mapping properties and prove the inversion formula of the following transformation

$$K_{\max}[f](x) = \int_{\mathbb{R}^n_+} K_{\max}(x,\tau) f(\tau) d\tau. \tag{4.6}$$

Let $f \in L_p\left(\mathbb{R}^n_+\right),\ 1 < p \leq 2$. Then making use the Hölder inequality and the Hausdorff-Young inequality for the cosine Fourier transform [10] we will show that integral (4.6) exists in the Lebesgue sense. In fact, we have

$$|K_{\max}[f](x)| \leq \left(\int_{\mathbb{R}^n_+} |K_{\max}(x,\tau)|^q\, d\tau\right)^{1/q} \left(\int_{\mathbb{R}^n_+} |f(\tau)|^p\, d\tau\right)^{1/p}$$

$$\leq \left(\frac{2}{\pi}\right)^{-n/q} ||f||_{L_p(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n_+} \exp\left(-p\max(x_1\cosh u_1,\dots,x_n\cosh u_n)\right) du\right)^{1/p} < \infty. \tag{4.7}$$

In order to estimate the latter integral in (4.7) we will employ representation (4.4). In particular, with the duplication formula for gamma-functions [3], Vol. 1 we obtain

$$\left(\int_{\mathbb{R}^n_+} \exp\left(-p\max(x_1\cosh u_1,\dots,x_n\cosh u_n)\right) du\right)^{1/p} = K^{1/p}_{\max}(px,0)$$

$$\leq \frac{\Gamma\left(1+\sum\nu_j\right)}{(8\sqrt{\pi})^{n/p}} \prod_{j=1}^{n} \left(\int_{(\nu_j)} \left|\frac{\Gamma\left(s_j/2\right)}{s_j\Gamma\left((s_j+1)/2\right)}\right| x_j^{-\nu_j} |ds_j|\right)^{1/p}.$$

Let f belong to the space of C^∞-functions with compact support on $\mathbb{R}^n_+$. Substituting (4.4) in (4.6) and changing the order of integration by Fubini's theorem we derive (see (1.18), (1.19))

$$K_{\max}[f](x) = \frac{1}{(2\pi i)^n} \int_{(\nu_1)} \dots \int_{(\nu_n)} \Gamma\left(1+s_1+\dots+s_n\right) \Theta_f(s) x^{-s} ds, \tag{4.8}$$

where

$$\Theta_f(s) = 4^{-n} \int_{\mathbb{R}_+^n} \prod_{j=1}^{n} 2^{s_j} \frac{\Gamma((s_j + i\tau_j)/2)\,\Gamma((s_j - i\tau_j)/2)}{\Gamma(s_j + 1)} f(\tau) d\tau.$$

However, calling again the relation (2.5.46.6) in [8], Vol. 1 we integrate by parts in the corresponding integral (see [15]). Hence the function $\Theta_f(s)$ can be easily rewritten in the form

$$\Theta_f(s) = \left(\frac{2}{\pi}\right)^{-n/2} \int_{\mathbb{R}_+^n} F_s\left[\frac{f}{\tau^1}\right](u) \prod_{j=1}^{n} \frac{\tanh u_j \; du_j}{\cosh^{s_j} u_j}, \tag{4.9}$$

where

$$F_s[h](x) = \left(\frac{2}{\pi}\right)^{n/2} \int_{\mathbb{R}_+^n} h(y) \prod_{j=1}^{n} \sin x_j y_j dy_j$$

is the sine Fourier transform. Therefore (4.6) becomes

$$K_{\max}[f](x) = \frac{1}{(2\pi i)^n} \left(\frac{2}{\pi}\right)^{-n/2} \int_{(\nu_1)} \cdots \int_{(\nu_n)} \Gamma(1 + s_1 + \cdots + s_n)\, x^{-s}$$
$$\times \int_{\mathbb{R}_+^n} F_s\left[\frac{f}{\tau^1}\right](u) \prod_{j=1}^{n} \frac{\tanh u_j}{\cosh^{s_j} u_j} du_j ds. \tag{4.10}$$

Then making the substitution $e^{\xi_j} = \cosh u_j,\ j = 1, \ldots, n$ in the latter integral with respect to u we appeal to the Parseval equalities for the Mellin and the Fourier transforms [10] to represent the L_2-norm of the operator $K_{\max}$ as

$$||K_{\max}[f]||^2_{L_2\left(\mathbb{R}_+^n; x^{2\nu-1}dx\right)} = \int_{\mathbb{R}_+^n} |K_{\max}[f](x)|^2 x^{2\nu-1} dx$$

$$= 2^{-2n} \int_{\mathbb{R}^n} \left|\Gamma\left(1 + \sum_{j=1}^{n} (\nu_j + it_j)\right)\right|^2 \left|\int_{\mathbb{R}_+^n} e^{-\langle \nu + it, \xi\rangle} \left(F_s\left[\frac{f}{\tau^1}\right]\right)(\operatorname{arccosh} e^{\xi}) d\xi\right|^2 dt$$

$$\leq 2^{-2n} \left[\Gamma\left(1 + \sum_{j=1}^{n} \nu_j\right)\right]^2 \int_{\mathbb{R}^n} \left|\int_{\mathbb{R}_+^n} e^{-\langle \nu + it, \xi\rangle} \left(F_s\left[\frac{f}{\tau^1}\right]\right)(\operatorname{arccosh} e^{\xi}) d\xi\right|^2 dt$$

$$= \left(\frac{\pi}{2}\right)^n \left[\Gamma\left(1 + \sum_{j=1}^{n} \nu_j\right)\right]^2 \int_{\mathbb{R}_+^n} e^{-2\langle \nu, \xi\rangle} \left|\left(F_s\left[\frac{f}{\tau^1}\right]\right)(\operatorname{arccosh} e^{\xi})\right|^2 d\xi$$

$$\leq \left(\frac{\pi}{2}\right)^n \left|\Gamma\left(1 + \sum_{j=1}^{n} \nu_j\right)\right|^2 \int_{\mathbb{R}_+^n} \left|\left(F_s\left[\frac{f}{\tau^1}\right]\right)(\operatorname{arccosh} e^{\xi})\right|^2 d\xi$$

$$= \left(\frac{\pi}{2}\right)^n \left[\Gamma\left(1+\sum_{j=1}^n \nu_j\right)\right]^2 \int_{\mathbb{R}_+^n} \left| \left(F_s\left[\frac{f}{\tau^1}\right]\right)(u)\right|^2 \prod_{j=1}^n \tanh u_j du_j$$

$$\leq \left(\frac{\pi}{2}\right)^n \left|\Gamma\left(1+\sum_{j=1}^n \nu_j\right)\right|^2 \int_{\mathbb{R}_+^n} \left| \left(F_s\left[\frac{f}{\tau^1}\right]\right)(u)\right|^2 du$$

$$= \left(\frac{\pi}{2}\right)^n \left[\Gamma\left(1+\sum_{j=1}^n \nu_j\right)\right]^2 \int_{\mathbb{R}_+^n} |f(\tau)|^2 \frac{d\tau}{(\tau^1)^2}$$

$$= \left(\frac{\pi}{2}\right)^n \left[\Gamma\left(1+\sum_{j=1}^n \nu_j\right)\right]^2 ||f||^2_{L_2\left(\mathbb{R}_+^n;\ (\tau^1)^{-2}d\tau\right)}. \tag{4.11}$$

Now via Fatou's lemma we observe that when $\nu \to 0+$ $(\nu_j \to 0+, j = 1, \dots, n)$ we arrive at the following norm inequality

$$||K_{\max}[f]||_{L_2\left(\mathbb{R}_+^n; x^{-1}dx\right)} \leq \left(\frac{\pi}{2}\right)^{n/2} ||f||_{L_2\left(\mathbb{R}_+^n;\ (\tau^1)^{-2}d\tau\right)}, \tag{4.12}$$

which is valid for any C^∞-function with compact support on $\mathbb{R}_+^n$. Taking an arbitrary function $f \in L_2\left(\mathbb{R}_+^n;\ (\tau^1)^{-2}d\tau\right)$ we approximate it by a sequence $\{f_N\}$ of C^∞-function with compact support. From (4.12) it follows that $\{K_{\max}[f_N]\}$ is a Cauchy sequence in $L_2\left(\mathbb{R}_+^n; x^{-1}dx\right)$, which converges to a limit, say, $K_{\max}[f]$. But from (4.6)

$$K_{\max}[f_N](x) = \int_{\mathbb{R}_+^n} K_{\max}(x,\tau) f_N(\tau) d\tau$$

and

$$\int_{\prod_{j=1}^n [0,x_j]} (x-t)^1 K_{\max}[f_N](t) dt = \int_{\mathbb{R}_+^n} \int_{\prod_{j=1}^n [0,x_j]} (x-t)^1 K_{\max}(t,\tau) f_N(\tau)\ dt d\tau. \tag{4.13}$$

Appealing to the Schwarz inequality by straightforward estimation we can pass to the limit under the integral sign in the left-hand side of (4.13). The same can be done in the right-hand side since (see (4.4))

$$\left(\int_{\mathbb{R}_+^n} (\tau^1)^2 \left| \int_{\prod_{j=1}^n [0,x_j]} (x-t)^1 K_{\max}(t,\tau) dt\right|^2 d\tau\right)^{1/2}$$

$$= \left(\int_{\mathbb{R}_+^n} (\tau^1)^2 \left| \frac{1}{(2\pi i)^n} \int_{(\nu_1)} \cdots \int_{(\nu_n)} \frac{\Gamma(1+s_1+\cdots+s_n)}{(1-s)^1(2-s)^1}\right.\right.$$

$$\times \prod_{j=1}^{n} \frac{2^{s_j-2}}{\Gamma(s_j+1)} \Gamma\left(\frac{s_j+i\tau_j}{2}\right) \Gamma\left(\frac{s_j-i\tau_j}{2}\right) x_j^{2-s_j} ds_j \Bigg|^2 d\tau \Bigg)^{1/2}$$

$$\leq \frac{(x/2)^{2-\nu}}{(2\pi)^n} \int_{(\nu_1)} \dots \int_{(\nu_n)} \frac{|\Gamma(1+s_1+\dots+s_n)|}{|(1-s)^1(2-s)^1|}$$

$$\times \left(\int_{\mathbb{R}_+^n} (\tau^1)^2 \left| \prod_{j=1}^{n} \frac{\Gamma((s_j+i\tau_j)/2)\,\Gamma((s_j-i\tau_j)/2)}{\Gamma(s_j+1)} \right|^2 d\tau \right)^{1/2} |ds|,\ 0 < \nu_j < 1,\ j = 1, \dots, n.$$

The inner integral with respect to τ can be calculated employing again relation (2.5.46.6) in [8], Vol. 1 (see also [15]). Therefore it gives

$$\left(\int_{\mathbb{R}_+^n} (\tau^1)^2 \left| \int_{\prod_{j=1}^n [0,x_j]} (x-t)^1 K_{\max}(t,\tau) dt \right|^2 d\tau \right)^{1/2}$$

$$\leq \frac{x^{2-\nu}}{(2\sqrt{2\pi})^n} \int_{(\nu_1)} \dots \int_{(\nu_n)} \frac{|\Gamma(1+s_1+\dots+s_n)|}{|(1-s)^1(2-s)^1|} |ds| \prod_{j=1}^{n} \left(\int_0^\infty \frac{du}{\cosh^{2\nu_j} u} \right)^{1/2} < \infty.$$

Thus passing to the limit through equality (4.13) and differentiating then twice with respect to x we obtain that for any $f \in L_2\left(\mathbb{R}_+^n;\ (\tau^1)^{-2} d\tau\right)$ and for almost all $x \in \mathbb{R}_+^n$ transformation (4.6) takes the form

$$K_{\max}[f](x) = \left(D_x^1\right)^2 \int_{\mathbb{R}_+^n} \int_{\prod_{j=1}^n [0,x_j]} (x-t)^1 K_{\max}(t,\tau) f(\tau)\ dt d\tau. \tag{4.14}$$

So we arrive at

Theorem 5. *Integral operator* $K_{\max} : L_2\left(\mathbb{R}_+^n;\ (\tau^1)^{-2} d\tau\right) \to L_2\left(\mathbb{R}_+^n; x^{-1} dx\right)$ *is bounded and its norm* $||K_{\max}|| \leq \left(\frac{\pi}{2}\right)^{n/2},\ n \in \mathbb{N}$ *by inequality* (4.12). *Moreover, for almost all* $x \in \mathbb{R}_+^n$ *representation* (4.14) *holds true.*

Remark 1. It is not difficult to verify that for the one-dimensional case ($n = 1$) one can differentiate twice in (4.14) under the integral sign and derive the classical Kontorovich-Lebedev transform (1.3).

In order to derive an inversion formula for the transformation (4.6) let us consider the following one-parametric family of auxiliary operators

$$K_\delta[f](x) = \int_{\mathbb{R}_+^n} K_\delta(x,\tau) f(\tau) d\tau, \quad \delta > 0,\ x \in \mathbb{R}_+^n, \tag{4.15}$$

where

$$K_\delta(x,\tau) = \frac{1}{(2\pi i)^n} \int_{(\nu_1)} \dots \int_{(\nu_n)} \frac{\Gamma(1+s_1)\dots\Gamma(1+s_n)}{\Gamma(1+s_1+\dots+s_n)}$$

$$\times \prod_{j=1}^{n} 2^{\delta-s_j-2}\Gamma\left(\frac{\delta-s_j+i\tau_j}{2}\right)\Gamma\left(\frac{\delta-s_j-i\tau_j}{2}\right) x_j^{s_j} ds_j,\ 0<\nu_j<\delta,\ j=1,\dots,n. \quad (4.16)$$

Hence for smooth functions with compact support f, g taking into account the Mellin - Parseval equality (1.20), representation (1.10) and a direct product analog of the Kontorovich-Lebedev transformation (1.3) we come out with the equalities

$$\int_{\mathbb{R}_+^n} K_{\max}[f](x)\ K_\delta[g](x)\ \frac{dx}{x^1} = \int_{\mathbb{R}_+^{2n}} f(\tau)g(u)$$

$$\times \frac{2^{n(\delta-4)}}{(2\pi i)^n}\int_{(\nu_1)}\dots\int_{(\nu_n)}\prod_{j=1}^{n}\Gamma\left(\frac{s_j+i\tau_j}{2}\right)\Gamma\left(\frac{s_j-i\tau_j}{2}\right)$$

$$\times\Gamma\left(\frac{\delta-s_j+iu_j}{2}\right)\Gamma\left(\frac{\delta-s_j-iu_j}{2}\right) ds_j\ d\tau\ du$$

$$= \int_{\mathbb{R}_+^n} (KLf)(x)\ (KLg)(x)\ x^{\delta-1}dx, \quad (4.17)$$

where

$$(KLf)(x) = \int_{\mathbb{R}_+^n}\prod_{j=1}^{n} K_{i\tau_j}(x_j)\ f(\tau)\ d\tau. \quad (4.18)$$

Meanwhile the kernel (4.16) can be expressed in terms of the modified Bessel functions by using (1.20), (1.27). Therefore we find

$$K_\delta(x,\tau) = x^1 D_x^1\ (x^1)^\delta \int_{\substack{y_1\ge 0,\ \dots,\ y_n\ge 0,\\ y_1+\dots+y_n\le 1}} \prod_{j=1}^{n} K_{i\tau_j}(x_jy_j)\ y^{\delta-1}dy. \quad (4.19)$$

Further, similarly, as in (4.10), (4.11) we deduce

$$||K_\delta[f]||^2_{L_2\left(\mathbb{R}_+^n;x^{2\nu-1}dx\right)} = 2^{-2n}\int_{\mathbb{R}^n}\left|\frac{\prod_{j=1}^n \Gamma(1-\nu_j-it_j)\Gamma(1+\delta+\nu_j+it_j)}{\Gamma\left(1-\sum_{j=1}^n(\nu_j+it_j)\right)}\right|^2$$

$$\times\left|\int_{\mathbb{R}_+^n} e^{-\langle\delta+\nu+it,\xi\rangle}\left(F_s\left[\frac{f}{\tau^1}\right]\right)(\operatorname{arccosh}\, e^{\xi})d\xi\right|^2 dt$$

$$\le 2^{-2n}\left[\frac{\prod_{j=1}^n\Gamma(1-\nu_j)\Gamma(1+\delta+\nu_j)}{\Gamma\left(n-\sum_{j=1}^n\nu_j\right)\Gamma\left(n(1+\delta)+\sum_{j=1}^n\nu_j\right)}\right]^2$$

$$\times \int_{\mathbb{R}^n} \left| \frac{\Gamma\left(n - \sum_{j=1}^n (\nu_j + it_j)\right) \Gamma\left(n(1+\delta) + \sum_{j=1}^n (\nu_j + it_j)\right)}{\Gamma\left(1 - \sum_{j=1}^n (\nu_j + it_j)\right)} \right|^2$$

$$\times \left| \int_{\mathbb{R}^n_+} e^{-\langle \delta+\nu+it, \xi \rangle} \left(F_s \left[\frac{f}{\tau^1} \right] \right) (\text{arccosh } e^{\xi}) d\xi \right|^2 dt. \tag{4.20}$$

But the gamma-ratio under the integral sign in (4.20) is uniformly bounded with respect to $t_j > 0,\ j = 1, \ldots, n$ owing to the Mellin transform of the Kummer function ${}_1F_1(a; c; x)$ and its asymptotic behavior (see in [3], Vol. I). Precisely, we have

$$\left| \frac{\Gamma\left(n - \sum_{j=1}^n (\nu_j + it_j)\right) \Gamma\left(n(1+\delta) + \sum_{j=1}^n (\nu_j + it_j)\right)}{\Gamma\left(1 - \sum_{j=1}^n (\nu_j + it_j)\right)} \right| = \frac{\Gamma(n(2+\delta))}{\Gamma\left(n + 1 - 2\sum_{j=1}^n \nu_j\right)}$$

$$\times \left| \int_0^\infty e^{-x} \ {}_1F_1 \left(1 - n(1+\delta) - 2\sum_{j=1}^n \nu_j;\ n + 1 - 2\sum_{j=1}^n \nu_j;\ x \right) x^{n-1-\sum_{j=1}^n (\nu_j + it_j)} dx \right|$$

$$\leq \frac{\Gamma(n(2+\delta))}{\Gamma\left(n + 1 - 2\sum_{j=1}^n \nu_j\right)} \int_0^\infty e^{-x} \left| {}_1F_1 \left(1 - n(1+\delta) - 2\sum_{j=1}^n \nu_j;\ n + 1 - 2\sum_{j=1}^n \nu_j;\ x \right) \right|$$

$$\times x^{n-1-\sum_{j=1}^n \nu_j} dx = C_{n,\nu,\delta} < \infty.$$

Hence combining with (4.20) and passing $\nu_j,\ j = 1, \ldots, n$ to zero, we obtain as in (4.12) the final estimate for each $\delta > 0$

$$||K_\delta[f]||_{L_2\left(\mathbb{R}^n_+; x^{-1}dx\right)} \leq A_{n,\delta} ||f||_{L_2\left(\mathbb{R}^n_+;\ (\tau^1)^{-2} d\tau\right)}, \tag{4.21}$$

where

$$A_{n,\delta} = \left(\frac{\pi}{2}\right)^n \frac{\Gamma^n(1+\delta)(n(1+\delta))_n}{n!(n-1)!} \int_0^\infty e^{-x} \ \left| {}_1F_1 \left(1 - n(1+\delta);\ n+1;\ x\right) \right| x^{n-1} dx,$$

$(a)_n$ is the Pochhammer symbol [3], Vol. I and the latter constant is uniformly bounded by a small positive δ. Similarly as above we extend (4.21) for an arbitrary $f \in L_2\left(\mathbb{R}^n_+;\ (\tau^1)^{-2} d\tau\right)$. Further, returning to (4.17) and using the Plancherel theorem for the one-dimensional Kontorovich-Lebedev transform (4.18) (cf. [15]), we can pass to the limit when $\delta \to 0$ to get the equality

$$\lim_{\delta \to 0+} \int_{\mathbb{R}^n_+} K_{\max}[f](x)\ K_\delta[g](x)\ \frac{dx}{x^1} = \left(\frac{\pi^2}{2}\right)^n \int_{\mathbb{R}^n_+} f(\tau) g(\tau) \prod_{j=1}^n \frac{d\tau_j}{\tau_j \sinh \pi \tau_j} \tag{4.22}$$

and the latter integral in (4.22) with respect to τ is absolutely convergent since $f, g \in L_2\left(\mathbb{R}^n_+;\ (\tau^1)^{-2} d\tau\right)$. On the other hand the Mellin-Parseval equality [10] and estimates (4.20), (4.21) say

$$||K_\delta[f]||_{L_2\left(\mathbb{R}^n_+; x^{2\nu-1}dx\right)} = \frac{1}{(2\pi)^{n/2}} ||K^*_\delta[f]||_{L_2\left(\prod_{j=1}^n (\nu_j);\ dt\right)} < M \tag{4.23}$$

when $M > 0$ is an absolute constant. Furthermore, using (4.15), (4.16) we show that the Mellin transform

$$K^*_\delta[f](s) = \frac{\Gamma(1-s_1)\dots\Gamma(1-s_n)}{\Gamma(1-s_1-\dots-s_n)}$$

$$\times \int_{\mathbb{R}^n_+} \prod_{j=1}^n 2^{\delta+s_j-2} \Gamma\left(\frac{\delta+s_j+i\tau_j}{2}\right) \Gamma\left(\frac{\delta+s_j-i\tau_j}{2}\right) f(\tau) d\tau,\ \delta > 0, \tag{4.24}$$

is analytic in the domain $s \in \mathbb{C}^n,\ -\delta < \text{Re } s_j < 1,\ j = 1, \dots, n$ and via (4.23) attains its limit when $\delta \to 0,\ \text{Re } s_j = \nu_j = 0,\ j = 1, \dots, n$ in the mean square sense and almost everywhere (see in [10]). Hence equality (4.23) guarantees the existence of the limit in mean of $K_\delta[f](x)$ when $\delta \to 0$, which we call $\hat{K}[f](x)$. Therefore (4.22) implies

$$\int_{\mathbb{R}^n_+} K_{\max}[f](x)\ \hat{K}[g](x)\ \frac{dx}{x^1} = \left(\frac{\pi^2}{2}\right)^n \int_{\mathbb{R}^n_+} f(\tau) g(\tau) \prod_{j=1}^n \frac{d\tau_j}{\tau_j \sinh \pi\tau_j} \tag{4.25}$$

for any $f, g \in L_2\left(\mathbb{R}^n_+;\ (\tau^1)^{-2} d\tau\right)$. Let, in particular,

$$g(u) = \begin{cases} u^1, & \text{if } u_j \in [0, \tau_j], \\ 0, & \text{if } u_j \in (\tau_j, \infty),\ j = 1, \dots, n. \end{cases}$$

Then it belongs to $L_2\left(\mathbb{R}^n_+;\ (\tau^1)^{-2} d\tau\right)$ and from (4.25) after differentiation of its both sides with respect to τ we get an inversion formula of the transformation (4.6) ((4.14)) for almost all $\tau \in \mathbb{R}^n_+$

$$f(\tau) = \left(\frac{2}{\pi^2}\right)^n \prod_{j=1}^n \sinh \pi\tau_j\ D^1_\tau \int_{\mathbb{R}^n_+} K_{\max}[f](x)\ \hat{K}_\tau(x)\ \frac{dx}{x^1}. \tag{4.26}$$

Here $\hat{K}_\tau(x)$ is the corresponding kernel, which can be represented due to (4.24) and Fubini's theorem (see also (4.9)) in the form

$$\hat{K}_\tau(x) = \int_{\mathbb{R}^n_+} \prod_{j=1}^n \tanh v_j\ \frac{1 - \cos \tau_j v_j}{v_j}$$

$$\times \frac{1}{(2\pi i)^n} \int_{(\nu_1)} \cdots \int_{(\nu_n)} \frac{\prod_{j=1}^n \Gamma(1-s_j)\Gamma(1+s_j)\ (x_j \cosh v_j)^{-s_j}}{\Gamma\left(1-s_1-\cdots-s_n\right)} ds\ dv, \tag{4.27}$$

where $0 < \nu_j < 1,\ j = 1, \ldots, n$. Hence choosing $0 < \nu_j < \frac{1}{n},\ j = 1, \ldots, n$ the inner integral with respect to s will be treated by using again the Mellin transform of the Kummer function. We have

$$\frac{1}{(2\pi i)^n} \int_{(\nu_1)} \cdots \int_{(\nu_n)} \frac{\prod_{j=1}^n \Gamma(1-s_j)\Gamma(1+s_j)\ (x_j \cosh v_j)^{-s_j}}{\Gamma\left(1-s_1-\cdots-s_n\right)} ds$$

$$= \frac{1}{(2\pi i)^n} \int_{(\nu_1)} \cdots \int_{(\nu_n)} \frac{\prod_{j=1}^n \Gamma\left(\frac{1}{n}-s_j\right)}{\Gamma\left(2-s_1-\cdots-s_n\right)}$$

$$\times \prod_{j=1}^n \frac{\Gamma(1-s_j)\Gamma(1+s_j)}{\Gamma\left(\frac{1}{n}-s_j\right)} (x_j \cosh v_j)^{-s_j}\ \ (1-s_1-\cdots-s_n)\, ds$$

$$= \left[\Gamma\left(\frac{1}{n}+1\right)\right]^{-1} \left(I + \sum_{m=1}^n x_m \frac{\partial}{\partial x_m}\right)\ x^1 \int_{\substack{y_1 \geq 0,\ \ldots,\ y_n \geq 0,\\ y_1+\cdots+y_n \leq 1}} y^{\frac{1}{n}} \exp\left(-\sum_{j=1}^n x_j\ y_j \cosh v_j\right)$$

$$\times \prod_{j=1}^n {}_1F_1\left(-1+\frac{1}{n};\ 1+\frac{1}{n};\ x_j\ y_j \cosh v_j\right)\ \cosh v_j\ dy,$$

where I is the identity operator. Substituting this expression into (4.27) and changing the order of integration and differentiation owing to the uniform convergence by x from any compact set of $\mathbb{R}_+^n$, we get the formula for the kernel $\hat{K}_\tau(x)$ of the inverse transform (4.26), namely

$$\hat{K}_\tau(x) = \left[\Gamma\left(\frac{1}{n}+1\right)\right]^{-1} \left(I + \sum_{m=1}^n x_m \frac{\partial}{\partial x_m}\right)\ x^1 \int_{\mathbb{R}_+^n} \prod_{j=1}^n \sinh v_j\ \frac{1-\cos \tau_j v_j}{v_j}$$

$$\times \int_{\substack{y_1 \geq 0,\ \ldots,\ y_n \geq 0,\\ y_1+\cdots+y_n \leq 1}} \exp\left(-\sum_{j=1}^n x_j\ y_j \cosh v_j\right)$$

$$\times \prod_{j=1}^n {}_1F_1\left(-1+\frac{1}{n};\ 1+\frac{1}{n};\ x_j\ y_j \cosh v_j\right)\ y_j^{\frac{1}{n}}\ dy_j\ dv_j. \tag{4.28}$$

It is not difficult to derive from (4.28) that the case $n = 1$ corresponds to the kernel $\hat{K}_\tau(x) = \int_0^\tau v K_{iv}(x) dv$, which is related to the classical Kontorovich-Lebedev transform.

We summarize these results by

Theorem 6. *Let* $L_2\left(\mathbb{R}_+^n;\ (\tau^1)^{-2} d\tau\right)$. *Then for almost all* $\tau \in \mathbb{R}_+^n$ *the essentially multidimensional Kontorovich-Lebedev transformation* (4.6) *can be inverted by formula* (4.26) *with the kernel* (4.28).

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