Voronoi- Nasim summation formulas and index transforms

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Abstract

Using L_2-theory of the Mellin and Fourier -Watson transformations we relax Nasim's conditions to prove the summation formula of Voronoi. It involves sums of the form $\sum d(n)f(n)$, where $d(n)$ is the number of divisors of n. These sums are related to the famous Dirichlet divisor problem of determining the asymptotic behaviour as $x \to \infty$ of the sum $D(x) = \sum_{n\le x} d(n)$. In particular, we generalize Koshliakov's formula, which contains the modified Bessel function of zero-index $f(x) = K_0(2\pi zx)$, (z is a parameter) on the modified Bessel function of an arbitrary complex index. Finally we apply index transforms of the Kontorovich-Lebedev type to obtain a new class of summation formulas involving Dirichlet's function $d(n)$.

Keywords: *Voronoi summation formula, Dirichlet divisor problem, Mellin transform, Fourier-Watson transformations, Riemann zeta-function, Index transforms, Kontorovich-Lebedev transform, Modified Bessel functions, Gauss hypergeometric function*

AMS subject classification: 44A15, 33C05, 33C10, 11M06, 11M36, 11N 37

1 Introduction and auxiliary results

In 1972 Nasim [4] gave a proof of the Voronoi summation formula in the form

$$\lim_{N\to\infty}\left[\sum_{n=1}^{N} d(n)f(n) - \int_{1/N}^{N}(\log x + 2\gamma)f(x)dx\right]$$

$$= \lim_{N\to\infty}\left[\sum_{n=1}^{N} d(n)g(n) - \int_{1/N}^{N}(\log x + 2\gamma)g(x)dx\right], \tag{1.1}$$

where both limits exist, $d(n)$ is the divisor function, γ is Euler's constant and f, g are Fourier cosine transformations of a certain function from a subclass of $L_2(\mathbb{R}_+)$. In this paper we will relax Nasim's conditions giving a direct proof of the formula (1.1) basing only on the L_2-theory of the Mellin transformation and properties of the Fourier-Watson kernels [8], [9] [10]. Then we will generalize the Koshliakov formula [4] involving the modified Bessel function $K_\nu(z)$ [1], Vol. II of an arbitrary index ν. Finally we will employ index transforms of the Kontorovich-Lebedev type (see in [2], [7], [11], [12]) to extend (1.1) obtaining a certain class of summation formulas involving the divisor function $d(n)$. Concerning other generalizations of classical summation formulas see, for instance, in [5].

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As usual [8], the Mellin transform in $L_2(\mathbb{R}_+)$ is defined by the integral

$$f^*(s) = \text{l.i.m.}_{N\to\infty} \int_{1/N}^{N} f(x)x^{s-1}dx, \tag{1.2}$$

where $s \in \sigma,\ \sigma = \left\{s \in \mathbb{C} : \text{Re } s = \frac{1}{2}\right\}$ and the convergence of the integral is in the mean square with respect to the norm of the space $L_2(\sigma)$. Moreover, the inversion formula takes place

$$f(x) = \text{l.i.m.}_{N\to\infty} \frac{1}{2\pi i} \int_{1/2-iN}^{1/2+iN} f^*(s)x^{-s}ds, \tag{1.3}$$

where integral (1.3) is convergent in mean with respect to the norm in $L_2(\mathbb{R}_+)$ and the generalized Parseval equality holds

$$\int_0^\infty f(xt)g(t)dt = \frac{1}{2\pi i}\int_\sigma f^*(s)g^*(1-s)x^{-s}ds. \tag{1.4}$$

In the sequel we are going to use important definitions and properties from [4], [9] of the divisor function and its relation with the Riemann zeta-function. In fact, we denote by

$$h(x) = \left\{ \sum_{n\le x} d(n) - x(\log x + 2\gamma - 1) \right\} x^{-1} = \frac{\Delta(x)}{x},\ x > 0. \tag{1.5}$$

Its Mellin transform (1.2) is calculated in [4] and we have reciprocal relations in L_2

$$h^*(s) = \frac{\zeta^2(1-s)}{1-s}, \tag{1.6}$$

$$h(x) = \text{l.i.m.}_{N\to\infty} \frac{1}{2\pi i} \int_{1/2-iN}^{1/2+iN} \frac{\zeta^2(1-s)}{1-s} x^{-s}ds, \tag{1.7}$$

where $\zeta(s)$ is the Riemann zeta-function [9], which satisfies the functional equation

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s). \tag{1.8}$$

From (1.7) by using (1.5) and a simple change of variables we easily find the representation for $\Delta(x)$, namely

$$\Delta(x) = \text{l.i.m.}_{N\to\infty} \frac{1}{2\pi i} \int_{1/2-iN}^{1/2+iN} \frac{\zeta^2(s)}{s} x^s ds. \tag{1.9}$$

2 Main results

Let $f(x)$ be defined on $\mathbb{R}_+$ and its Mellin transform (1.2) be such that $sf^*(s) \in L_2(\sigma)$. Hence it is easily seen that f is continuous on $\mathbb{R}_+$, $f(x)x^{1/2}$ tends to zero when $x \to 0$ and $x \to \infty$, $f \in L_2(\mathbb{R}_+)$, $f^*(s) \in L_2(\sigma) \cap L_1(\sigma)$ and the integral (1.3) exists as Lebesgue integral. In fact, we obtain

$$\int_\sigma |f^*(s)|^2|ds| < 2\int_\sigma |sf^*(s)|^2|ds| < \infty,$$

$$\int_\sigma |f^*(s)ds| \le \left(\int_\sigma |sf^*(s)|^2 ds\right)^{1/2} \left(\int_\sigma \frac{|ds|}{|s|^2}\right)^{1/2} < \infty.$$

Therefore formula (1.3) becomes

$$f(x) = \frac{1}{2\pi i}\int_\sigma f^*(s)x^{-s}ds, \tag{2.1}$$

and in particular we have the estimate

$$x^{1/2}\,|f(x)| \le \frac{1}{2\pi}\int_\sigma |f^*(s)ds| = O(1), \quad x>0.$$

Moreover it tends to zero when x approaches to zero and infinity via the Riemann- Lebesgue lemma. Finally, condition $sf^*(s) \in L_2(\sigma)$ implies that f is equivalent to some absolutely continuous function ρ such that for almost all $x>0$ we have

$$\left(-x\frac{d}{dx}\right)\rho(x) = \left(-x\frac{d}{dx}\right) f(x) \in L_2(\mathbb{R}_+).$$

Theorem 1. *Let $sf^*(s) \in L_2(\sigma)$. Then there exists a function $\varphi \in L_2(\mathbb{R}_+)$ and $\varphi \in L_2(\mathbb{R}_+)$ such that f, g are Fourier cosine transforms of φ in L_2, namely*

$$f(x) = (F_c\ \varphi(t))(x) = 2\ \text{l.i.m.}_{N\to\infty}\int_{1/N}^N \cos(2\pi xt)\varphi(t)dt, \tag{2.2}$$

$$g(x) = (F_c\ t^{-1}\varphi(t^{-1}))(x) = 2\ \text{l.i.m.}_{N\to\infty}\int_{1/N}^N \cos(2\pi xt)\ \frac{1}{t}\varphi\left(\frac{1}{t}\right)dt. \tag{2.3}$$

Moreover, the following identity holds

$$\int_\sigma \zeta^2(s)f^*(s)ds = \int_\sigma \zeta^2(s)g^*(s)ds, \tag{2.4}$$

where $g^*(s)$ is the Mellin transform (1.2) *of g in L_2, $sg^*(s) \in L_2(\sigma)$ and both integrals are absolutely convergent.*

Proof. In fact, the absolute convergence of integrals in (2.4) easily follows from (1.6), (1.7), conditions $sf^*(s), sg^*(s) \in L_2(\sigma)$ and the Schwarz inequality. Further, using the functional equation (1.8) for the Riemann zeta-function, the supplement and duplication formulas for Euler's gamma - function [1], Vol. I, we write the left hand-side of (2.3) in the form

$$\int_\sigma \zeta^2(s)f^*(s)ds = \int_\sigma 2^{2s}\pi^{2(s-1)}\sin^2\left(\frac{\pi s}{2}\right)\Gamma^2(1-s)\zeta^2(1-s)f^*(s)ds$$

$$= \pi\int_\sigma \pi^{-2s}\left[\frac{\Gamma(s/2)}{\Gamma((1-s)/2)}\right]^2\zeta^2(s)f^*(1-s)ds. \tag{2.5}$$

Hence denoting by $\varphi^*(s) = \pi^{-s+1/2}\frac{\Gamma(s/2)}{\Gamma((1-s)/2)}f^*(1-s)$ we easily get $\varphi^*(s) \in L_2(\sigma)$ since

$$\left|\frac{\Gamma(s/2)}{\Gamma((1-s)/2)}\right| = 1, \quad s\in\sigma.$$

Moreover there exists a function $\varphi(x) \in L_2(\mathbb{R}_+)$, which is the inverse Mellin transform (1.3) of φ^*. Appealing to the generalized Parseval equality (1.4) and an elementary integral

$$\int_0^\infty t^{s-1}\cos(2\pi xt)dt = \pi^{-s+1/2}\frac{x^{-s}}{2}\frac{\Gamma(s/2)}{\Gamma((1-s)/2)},$$

function $\varphi(x)$ can be written as the cosine Fourier transform of f, namely

$$\varphi(x) \equiv (F_c f)(x) = \text{ l.i.m.}_{N\to\infty}\ 2\int_{1/N}^{N} \cos(2\pi xt) f(t)dt. \tag{2.6}$$

Thus reciprocally we prove (2.2). Further, the right-hand side of (2.4) can be written accordingly

$$\pi\int_\sigma \pi^{-2s}\left[\frac{\Gamma(s/2)}{\Gamma((1-s)/2)}\right]^2 \zeta^2(s) f^*(1-s)ds = \int_\sigma \zeta^2(s) g^*(s)ds,$$

where

$$g^*(s) = \pi^{-s+1/2}\frac{\Gamma(s/2)}{\Gamma((1-s)/2)}\varphi^*(s) \in L_2(\sigma) \tag{2.7}$$

and $sg^*(s) \in L_2(\sigma)$. Therefore cancelling the Mellin transform in (2.7) we observe that there exists a function $g \in L_2(\mathbb{R}_+)$ satisfying (2.3). Combining with to (2.5) we get (2.4) and complete the proof of Theorem 1.

Theorem 2. *Let $sf^*(s) \in L_2(\sigma)$. Then*

$$\frac{1}{2\pi i}\int_\sigma \zeta^2(s) f^*(s)ds = \lim_{N\to\infty}\left[\sum_{n=1}^N d(n)f(n) - \int_0^N (\log x + 2\gamma) f(x)dx\right]. \tag{2.8}$$

Proof. In fact, calling formula (1.9) we have conversely

$$\frac{\zeta^2(s)}{s} = \text{l.i.m.}_{N\to\infty}\int_{1/N}^N \Delta\left(\frac{1}{x}\right) x^{s-1}dx.$$

Hence denoting by

$$\Delta_N^*(s) = \int_{1/N}^N \Delta\left(\frac{1}{x}\right) x^{s-1}dx$$

after a simple substitution and integration by parts it becomes

$$s\Delta_N^*(s) = \int_{1/N}^N \Delta'(x)\, x^{-s}dx - \Delta(N)\ N^{-s}\ + \Delta\left(\frac{1}{N}\right)\ N^s. \tag{2.9}$$

Therefore the left-hand side of (2.8) can be represented by the ordinary limit

$$\frac{1}{2\pi i}\int_\sigma \zeta^2(s) f^*(s)ds = \lim_{N\to\infty}\frac{1}{2\pi i}\int_\sigma s\Delta_N^*(s) f^*(s)ds$$

$$= \lim_{N\to\infty}\left[\frac{1}{2\pi i}\int_\sigma\int_{1/N}^N \Delta'(x)\, x^{-s} f^*(s)dx\ ds - \Delta(N)\ f(N)\ + \Delta\left(\frac{1}{N}\right)\ f\left(\frac{1}{N}\right)\right]. \tag{2.10}$$

The latter two terms in the right-hand side of (2.10) are via formula (2.1) of the Mellin transform of $f^* \in L_1(\sigma)$ and they tend to zero when $N \to \infty$ by virtue of the discussion at the beginning of Section 2 and the Voronoi estimates (see in [9])

$$\Delta(x) = O(x^{1/3}),\ x \to \infty,\ \Delta(x) = O(x\log x),\ x \to 0.$$

So changing the order of integration in (2.10) by Fubini's theorem and applying again (2.1) it gives the equality

$$\frac{1}{2\pi i}\int_\sigma \zeta^2(s)f^*(s)ds = \lim_{N\to\infty}\frac{1}{2\pi i}\int_\sigma s\Delta_N^*(s)f^*(s)ds = \lim_{N\to\infty}\int_{1/N}^N \Delta'(x)\, f(x)dx.$$

But in the meantime from (1.5) and since $f(x)$ is continuous we have

$$\int_{1/N}^N \Delta'(x)\, f(x)dx = \int_{1/N}^N f(x)\frac{d}{dx}\left(\sum_{n\le x} d(n)\right)dx - \int_{1/N}^N (\log x + 2\gamma)f(x)dx$$

$$= \sum_{n=1}^N d(n)f(n) - \int_{1/N}^N (\log x + 2\gamma)f(x)dx.$$

Thus we come out with (2.8) and prove Theorem 2.

Corollary 1. *Let $sf^*(s)\in L_2(\sigma)$. Then the summation formula* (1.1) *takes place where f,g are related by* (2.2), (2.3) *and $\varphi\in L_2(\mathbb{R}_+)$.*

Corollary 2. *Functions f and g are Fourier- Watson L_2- transforms of each other with the kernel as a combination of Bessel functions, namely $\chi(x) = 4K_0(4\pi\sqrt{x}) - 2\pi Y_0(4\pi\sqrt{x})$.*

Proof. In fact, calling (2.5), (2.7) we write

$$g^*(s) = 4(2\pi)^{-2s}\Gamma^2(s)\cos^2\left(\frac{\pi s}{2}\right)f^*(1-s). \tag{2.11}$$

Making use the representation [8]

$$\int_0^x \left[4K_0\left(4\pi\sqrt{t}\right) - 2\pi Y_0\left(4\pi\sqrt{t}\right)\right]dt$$

$$= \text{l.i.m.}_{N\to\infty}\frac{2}{\pi i}\int_{1/2-iN}^{1/2+iN}(2\pi)^{-2s}\Gamma^2(s)\cos^2\left(\frac{\pi s}{2}\right)\frac{x^{1-s}}{1-s}ds$$

and generalized Parseval equality (1.4), relation (2.11) yields

$$g(x) = \text{l.i.m.}_{N\to\infty}\int_{1/N}^N \chi(xt)\ f(t)dt. \tag{2.12}$$

Further, solving the algebraic equation (2.11) with respect to $f^*(s)$ we easily get by straightforward calculations

$$f^*(s) = 4(2\pi)^{-2s}\Gamma^2(s)\cos^2\left(\frac{\pi s}{2}\right)g^*(1-s).$$

Therefore it corresponds the reciprocal formula

$$f(x) = \text{l.i.m.}_{N\to\infty}\int_{1/N}^N \chi(xt)\ g(t)dt$$

and completes the proof of Corollary 2.

3 A modification of Koshliakov's formula

As an example of the formula (1.1) Nasim proved in [4] the familiar Koshliakov's formula, which were established in 1928

$$\sum_{n=1}^{\infty} d(n)K_0(2\pi zn) - \frac{1}{z}\sum_{n=1}^{\infty} d(n)K_0\left(\frac{2\pi n}{z}\right)$$
$$= \frac{1}{4z}(\gamma - \log 4\pi z) - \frac{1}{4}(\gamma - \log(4\pi/z)), \ \text{Re } z > 0. \tag{3.1}$$

Here we will give another proof of (3.1) basing on the equality (2.11) and the inverse Mellin transform (1.3). In fact, appealing to the relation (2.16.2.2) in [6], Vol. 2

$$\int_0^\infty K_\nu(cx)x^{s-1}dx = 2^{s-2}c^{-s}\Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right), \ \text{Re } c > 0, \text{Re } s > |\text{Re}\nu|, \tag{3.2}$$

we have reciprocally

$$K_\nu(cx) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} 2^{s-2}\Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right)(cx)^{-s}ds, \ \text{Re } s = \alpha, \tag{3.3}$$

where both integrals (3.1), (3.2) are absolutely convergent via the asymptotic behavior of the modified Bessel functions and the Euler gamma-functions (see [1]). Consequently, taking $f(x) = K_0(2\pi zx)$, we find its Mellin transform (see (1.2), (3.2)) as $f^*(s) = \frac{1}{4}(\pi z)^{-s}\Gamma^2(s/2)$. Hence substituting in (2.11) we get $g^*(s) = \frac{1}{4}\pi^{-s}z^{s-1}\Gamma^2(s/2)$. Therefore $g(x) = z^{-1}K_0(2\pi x/z)$ and by straightforward calculations of the related absolutely convergent integrals in (1.1) (see [4]) we come out with (3.1).

However, when the index ν of the modified Bessel function is different from zero, the corresponding modification of the Koshliakov formula (3.1) needs an operational technique being used for the generalized Fourier- Watson transformations. In fact, let consider the modified Bessel function $yK_\nu(2\sqrt{xy})$, $|\text{Re } \nu| < 1, \ x, y > 0$. Then it evidently satisfies conditions of Theorem 1 and its Mellin transform by x (see (3.2)) is equal to

$$\frac{1}{2}\, y^{1-s}\, \Gamma\left(s + \frac{\nu}{2}\right)\Gamma\left(s - \frac{\nu}{2}\right). \tag{3.4}$$

Further, the left hand-side of (1.1) in this case contains a series and an integral, which converge absolutely. Calling relation (2.16.20.1) in [6], Vol.2 it can be calculated explicitly and we obtain

$$\lim_{N\to\infty}\left[y\sum_{n=1}^{N} d(n)K_\nu(2\sqrt{yn}) - y\int_{1/N}^{N}(\log x + 2\gamma)K_\nu(2\sqrt{xy})dx\right]$$
$$= y\sum_{n=1}^{\infty} d(n)K_\nu(2\sqrt{yn}) - \frac{1}{2}\left[2\gamma - \log y + \psi\left(1 + \frac{\nu}{2}\right)\right.$$
$$\left. + \psi\left(1 - \frac{\nu}{2}\right)\right]\Gamma\left(1 + \frac{\nu}{2}\right)\Gamma\left(1 - \frac{\nu}{2}\right), \tag{3.5}$$

where $\psi(z)$ is Euler's ψ-function [1], Vol. I. In order to treat the right-hand side of (1.1) we first find a representation of the reciprocal function $g_\nu(x, y)$. Appealing to (1.2), (2.5), (3.4) we derive

$$g_\nu^*(s, y) = \int_0^\infty g_\nu(x, y)x^{s-1}dx = 2^{1-2s}y^s\pi^{-2s}\cos^2\left(\frac{\pi s}{2}\right)\Gamma^2(s)$$
$$\times\Gamma\left(1 - s + \frac{\nu}{2}\right)\Gamma\left(1 - s - \frac{\nu}{2}\right)$$

$$= (2\pi)^{-2s} y^s \left(\cos \pi s + 1\right) \Gamma^2(s) \Gamma\left(1 - s + \frac{\nu}{2}\right) \Gamma\left(1 - s - \frac{\nu}{2}\right)$$

$$= (2\pi)^{-2s} y^s \Gamma^2(s) \Gamma\left(1 - s + \frac{\nu}{2}\right) \Gamma\left(1 - s - \frac{\nu}{2}\right)$$

$$+ (2\pi)^{-2s} y^s \sum_{k=0}^{\infty} \frac{(-1)^k (\pi s)^{2k}}{(2k)!} \Gamma^2(s) \Gamma\left(1 - s + \frac{\nu}{2}\right) \Gamma\left(1 - s - \frac{\nu}{2}\right). \tag{3.6}$$

Hence cancelling the Mellin transform by using the generalized Parseval equality (1.4), relation (2.16.33.1) in [6], Vol. 2, operational properties of the Mellin transform [7], [8] and the change of the order of integration and summation via the absolute and uniform convergence with respect to $y,\ x:\ y \ge y_0 > 0,\ x \ge x_0 > 0$, we obtain correspondingly

$$g_\nu(x,y) = 2 \left[\int_0^\infty K_0\left(2\pi t \sqrt{\frac{x}{y}}\right) K_\nu(t)\ t\ dt \right.$$

$$\left. + \sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k}}{(2k)!} \left(y \frac{d}{dy}\right)^{2k} \int_0^\infty K_0\left(2\pi t \sqrt{\frac{x}{y}}\right) K_\nu(t)\ t\ dt \right]$$

$$= \left[\Gamma\left(1 + \frac{\nu}{2}\right) \Gamma\left(1 - \frac{\nu}{2}\right)\right]^2 \left[2\ {}_2F_1\left(1 + \frac{\nu}{2},\ 1 - \frac{\nu}{2};\ 2;\ 1 - \frac{4\pi^2 x}{y}\right) \right.$$

$$\left. + \sum_{k=1}^{\infty} \frac{(-1)^k \pi^{2k}}{(2k)!} \left(y \frac{d}{dy}\right)^{2k} {}_2F_1\left(1 + \frac{\nu}{2},\ 1 - \frac{\nu}{2};\ 2;\ 1 - \frac{4\pi^2 x}{y}\right) \right], \tag{3.7}$$

where ${}_2F_1(a,b;c;w)$ is the Gauss hypergeometric function [1], Vol. I. But according to relation (7.3.1.30) in [6], Vol. 3 it has a representation in terms of the series of ψ-functions

$${}_2F_1\left(1 + \frac{\nu}{2},\ 1 - \frac{\nu}{2};\ 2;\ 1 - \frac{4\pi^2 x}{y}\right) = \frac{1}{\Gamma\left(1 + \frac{\nu}{2}\right) \Gamma\left(1 - \frac{\nu}{2}\right)}$$

$$\times \sum_{m=0}^{\infty} \frac{\left(1 + \frac{\nu}{2}\right)_m \left(1 - \frac{\nu}{2}\right)_m}{(m!)^2} \left[2\ \psi(m+1) - \psi\left(1 + \frac{\nu}{2} + m\right) \right.$$

$$\left. - \psi\left(1 - \frac{\nu}{2} + m\right) - \log\left(\frac{4\pi^2 x}{y}\right) \right] \left(\frac{4\pi^2 x}{y}\right)^m,\ y > 4\pi^2 x, \tag{3.8}$$

where $(a)_m$ is the Pochhammer symbol [1], Vol. 1. Meanwhile making an induction by k it is not difficult to prove the equality

$$\left(y \frac{d}{dy}\right)^{2k} \left(y^{-\alpha} \log y\right) = y^{-\alpha} \left(\alpha^{2k} \log y - 2k\ \alpha^{2k-1}\right). \tag{3.9}$$

Hence due to the absolute and uniform convergence with respect to $y \ge y_0 > 4\pi^2 x$ the second term in the right-hand side of (3.7) becomes

$$\sum_{k=1}^{\infty} \frac{(-1)^k \pi^{2k}}{(2k)!} \left(y \frac{d}{dy}\right)^{2k} {}_2F_1\left(1 + \frac{\nu}{2},\ 1 - \frac{\nu}{2};\ 2;\ 1 - \frac{4\pi^2 x}{y}\right)$$

$$= \frac{1}{\Gamma\left(1 + \frac{\nu}{2}\right) \Gamma\left(1 - \frac{\nu}{2}\right)} \sum_{k=1}^{\infty} \frac{(-1)^k \pi^{2k}}{(2k)!}$$

$$\times \sum_{m=0}^{\infty} \frac{\left(1+\frac{\nu}{2}\right)_m \left(1-\frac{\nu}{2}\right)_m}{(m!)^2} \Big[2\ \psi(m+1) - \psi\left(1+\frac{\nu}{2}+m\right)$$

$$-\psi\left(1-\frac{\nu}{2}+m\right)\Big] m^{2k} \left(\frac{4\pi^2 x}{y}\right)^m - \frac{1}{\Gamma\left(1+\frac{\nu}{2}\right)\Gamma\left(1-\frac{\nu}{2}\right)} \sum_{k=1}^{\infty} \frac{(-1)^k \pi^{2k}}{(2k)!}$$

$$\times \sum_{m=0}^{\infty} \frac{\left(1+\frac{\nu}{2}\right)_m \left(1-\frac{\nu}{2}\right)_m}{(m!)^2} \left(m^{2k} \log\left(\frac{4\pi^2 x}{y}\right) + 2k\ m^{2k-1}\right) \left(\frac{4\pi^2 x}{y}\right)^m$$

$$= \frac{1}{\Gamma\left(1+\frac{\nu}{2}\right)\Gamma\left(1-\frac{\nu}{2}\right)} \sum_{m=0}^{\infty} \frac{\left(1+\frac{\nu}{2}\right)_m \left(1-\frac{\nu}{2}\right)_m}{(m!)^2} (\cos \pi m\ - 1)$$

$$\times \left[2\ \psi(m+1) - \psi\left(1+\frac{\nu}{2}+m\right) - \psi\left(1-\frac{\nu}{2}+m\right) - \log\left(\frac{4\pi^2 x}{y}\right)\right] \left(\frac{4\pi^2 x}{y}\right)^m$$

$$+ \frac{\pi}{\Gamma\left(1+\frac{\nu}{2}\right)\Gamma\left(1-\frac{\nu}{2}\right)} \sum_{m=0}^{\infty} \frac{\left(1+\frac{\nu}{2}\right)_m \left(1-\frac{\nu}{2}\right)_m}{(m!)^2} \sin \pi m\ \left(\frac{4\pi^2 x}{y}\right)^m$$

$$= \frac{1}{\Gamma\left(1+\frac{\nu}{2}\right)\Gamma\left(1-\frac{\nu}{2}\right)} \sum_{m=0}^{\infty} \frac{\left(1+\frac{\nu}{2}\right)_m \left(1-\frac{\nu}{2}\right)_m}{(m!)^2}$$

$$\times \left[2\ \psi(m+1) - \psi\left(1+\frac{\nu}{2}+m\right) - \psi\left(1-\frac{\nu}{2}+m\right) - \log\left(\frac{4\pi^2 x}{y}\right)\right] \left(-\ \frac{4\pi^2 x}{y}\right)^m$$

$$- {}_2F_1\left(1+\frac{\nu}{2},\ 1-\frac{\nu}{2};\ 2;\ 1-\frac{4\pi^2 x}{y}\right). \tag{3.10}$$

Substituting the latter expression into (3.7) we write $g_\nu(x, y)$ in the form

$$g_\nu(x,y) = \left[\Gamma\left(1+\frac{\nu}{2}\right)\Gamma\left(1-\frac{\nu}{2}\right)\right]^2\ {}_2F_1\left(1+\frac{\nu}{2},\ 1-\frac{\nu}{2};\ 2;\ 1-\frac{4\pi^2 x}{y}\right)$$

$$+\Gamma\left(1+\frac{\nu}{2}\right)\Gamma\left(1-\frac{\nu}{2}\right) \sum_{m=0}^{\infty} \frac{\left(1+\frac{\nu}{2}\right)_m \left(1-\frac{\nu}{2}\right)_m}{(m!)^2}$$

$$\times \Big[2\ \psi(m+1) - \psi\left(1+\frac{\nu}{2}+m\right) - \psi\left(1-\frac{\nu}{2}+m\right)$$

$$- \log\left(\frac{4\pi^2 x}{y}\right)\Big] \left(-\ \frac{4\pi^2 x}{y}\right)^m,\ y > 4\pi^2 x,\ |\mathrm{Re}\nu| < 1. \tag{3.11}$$

Calling (3.8) we slightly simplify (3.11) representing the kernel $g_\nu(x,y) \equiv S_\nu\left(\frac{4\pi^2 x}{y}\right)$ as follows

$$S_\nu(z) \equiv V_{1,\nu}(z) = 2 \sum_{m=0}^{\infty} \frac{\Gamma\left(1+2m+\frac{\nu}{2}\right)\Gamma\left(1+2m-\frac{\nu}{2}\right)}{((2m)!)^2} [2\ \psi(2m+1)$$

$$-\psi\left(1+\frac{\nu}{2}+2m\right) - \psi\left(1-\frac{\nu}{2}+2m\right) - \log z\Big] z^{2m},\ 0 < z = \frac{4\pi^2 x}{y} < 1. \tag{3.12}$$

On the other hand, the same result we obtain inverting the Mellin transform (3.6) and calculating the corresponding integral (1.3) taking into account the residues of multiple poles of gamma-functions (the so-called logarithmic case, see details in [3]). Moreover, for any $z > 0$

$$S_\nu(z) = \frac{1}{\pi i}\int_{\mu-i\infty}^{\mu+i\infty} \cos^2\left(\frac{\pi s}{2}\right)\Gamma^2(s)$$

$$\times\Gamma\left(1 - s + \frac{\nu}{2}\right)\Gamma\left(1 - s - \frac{\nu}{2}\right) z^{-s} ds, \tag{3.13}$$

where we can choose μ from the interval $\left(0, 1 - \frac{|\mathrm{Re}\ \nu|}{2}\right)$, $|\mathrm{Re}\ \nu| < 1$. Consequently, the Mellin integral (1.2) in (3.6) exists and converges absolutely when $s \in \sigma$. Further, employing the duplication formula for gamma-functions [1], Vol. 1 and making a simple change of variables we rewrite (3.13) in the form

$$S_\nu(z) = \frac{1}{2\pi i}\int_{\mu/2-i\infty}^{\mu/2+i\infty} \left[\frac{\Gamma(s)}{\Gamma\left(\frac{1}{2} - s\right)}\right]^2 \Gamma\left(\frac{1}{2} - s + \frac{\nu}{4}\right)\Gamma\left(\frac{1}{2} - s - \frac{\nu}{4}\right)$$

$$\times\Gamma\left(1 - s + \frac{\nu}{4}\right)\Gamma\left(1 - s - \frac{\nu}{4}\right) z^{-2s} ds. \tag{3.14}$$

Letting again $z = \frac{4\pi^2 x}{y}$ and considering the case $z > 1$, i.e. $0 < y < 4\pi^2 x$, $|\mathrm{Re}\ \nu| < 1$, $\nu \neq 0$ we can calculate the latter integral (3.14) via Slater's theorem [3] as a linear combination of hypergeometric functions ${}_4F_3$, [6], Vol. 3. Precisely, we obtain

$$S_\nu(z) \equiv V_{2,\nu}(z) = z^{-\frac{\nu}{2}-1} 2^{\nu+1}\pi\Gamma(-\nu) \left[\frac{\Gamma\left(\frac{1}{2} + \frac{\nu}{4}\right)}{\Gamma(-\nu/4)}\right]^2$$

$$\times {}_4F_3\left(\frac{1}{2} + \frac{\nu}{4},\ \frac{1}{2} + \frac{\nu}{4},\ 1 + \frac{\nu}{4},\ 1 + \frac{\nu}{4};\ 1 + \frac{\nu}{2},\ \frac{1}{2},\ \frac{1+\nu}{2};\ z^{-2}\right)$$

$$+ z^{\frac{\nu}{2}-1} 2^{-\nu+1}\pi\ \Gamma(\nu) \left[\frac{\Gamma\left(\frac{1}{2} - \frac{\nu}{4}\right)}{\Gamma(\nu/4)}\right]^2$$

$$\times {}_4F_3\left(\frac{1}{2} - \frac{\nu}{4},\ \frac{1}{2} - \frac{\nu}{4},\ 1 - \frac{\nu}{4},\ 1 - \frac{\nu}{4};\ 1 - \frac{\nu}{2},\ \frac{1}{2},\ \frac{1-\nu}{2};\ z^{-2}\right)$$

$$+ z^{-\frac{\nu}{2}-2} 2^{\nu+3}\pi \frac{\Gamma(-\nu)}{1+\nu} \left[\frac{\Gamma\left(1 + \frac{\nu}{4}\right)}{\Gamma\left(-\frac{1}{2} - \frac{\nu}{4}\right)}\right]^2$$

$$\times {}_4F_3\left(1 + \frac{\nu}{4},\ 1 + \frac{\nu}{4},\ \frac{3}{2} + \frac{\nu}{4},\ \frac{3}{2} + \frac{\nu}{4};\ \frac{3}{2} + \frac{\nu}{2},\ \frac{3}{2},\ 1 + \frac{\nu}{2};\ z^{-2}\right)$$

$$+ z^{\frac{\nu}{2}-2} 2^{-\nu+3}\pi \frac{\Gamma(\nu)}{1-\nu} \left[\frac{\Gamma\left(1 - \frac{\nu}{4}\right)}{\Gamma\left(-\frac{1}{2} + \frac{\nu}{4}\right)}\right]^2$$

$$\times {}_4F_3\left(1 - \frac{\nu}{4},\ 1 - \frac{\nu}{4},\ \frac{3}{2} - \frac{\nu}{4},\ \frac{3}{2} - \frac{\nu}{4};\ \frac{3}{2} - \frac{\nu}{2},\ \frac{3}{2},\ 1 - \frac{\nu}{2};\ z^{-2}\right),\ z = \frac{4\pi^2 x}{y}. \tag{3.15}$$

We note from (3.13), (3.14) that $\lim_{z\to 1-} V_{1,\nu}(z) = \lim_{z\to 1+} V_{2,\nu}(z) = S_\nu(1)$.

Meanwhile, each hypergeometric function ${}_4F_3$ can be represented as a combination of Gauss hypergeometric functions ${}_2F_1$ owing to relations (7.5.1.6), (7.5.1.7) in [6], Vol.3. So we have, for instance,

$$
{}_4F_3\left(\frac{1}{2}+\frac{\nu}{4},\ \frac{1}{2}+\frac{\nu}{4},\ 1+\frac{\nu}{4},\ 1+\frac{\nu}{4};\ 1+\frac{\nu}{2},\ \frac{1}{2},\ \frac{1+\nu}{2};\ z^{-2}\right)
$$

$$
=\frac{1}{2}\left[{}_2F_1\left(1+\frac{\nu}{2},\ 1+\frac{\nu}{2};\ 1+\nu;\ \frac{1}{z}\right)+{}_2F_1\left(1+\frac{\nu}{2},\ 1+\frac{\nu}{2};\ 1+\nu;\ -\frac{1}{z}\right)\right],
$$

$$
{}_4F_3\left(1+\frac{\nu}{4},\ 1+\frac{\nu}{4},\ \frac{3}{2}+\frac{\nu}{4},\ \frac{3}{2}+\frac{\nu}{4};\ \frac{3}{2}+\frac{\nu}{2},\ \frac{3}{2},\ 1+\frac{\nu}{2};\ z^{-2}\right)
$$

$$
=\frac{z(1+\nu)}{2\left(1+\frac{\nu}{2}\right)^2}\left[{}_2F_1\left(1+\frac{\nu}{2},\ 1+\frac{\nu}{2};\ 1+\nu;\ \frac{1}{z}\right)-{}_2F_1\left(1+\frac{\nu}{2},\ 1+\frac{\nu}{2};\ 1+\nu;\ -\frac{1}{z}\right)\right]
$$

and similar formulas can be written for two other ${}_4F_3$-functions in (3.15) changing ν on $-\nu$. Substituting this into (3.15) and making use the duplication formula for gamma-functions we come out with the final expression of the kernel $V_{2,\nu}(z),\ z=\frac{4\pi^2x}{y}>1$

$$
V_{2,\nu}(z)=z^{-\frac{\nu}{2}-1}\Gamma(-\nu)\Gamma^2\left(1+\frac{\nu}{2}\right)\left[{}_2F_1\left(1+\frac{\nu}{2},\ 1+\frac{\nu}{2};\ 1+\nu;\ \frac{1}{z}\right)\right.
$$

$$
\left.-\cos\left(\frac{\pi\nu}{2}\right){}_2F_1\left(1+\frac{\nu}{2},\ 1+\frac{\nu}{2};\ 1+\nu;\ -\frac{1}{z}\right)\right]
$$

$$
+z^{\frac{\nu}{2}-1}\Gamma(\nu)\Gamma^2\left(1-\frac{\nu}{2}\right)\left[{}_2F_1\left(1-\frac{\nu}{2},\ 1-\frac{\nu}{2};\ 1-\nu;\ \frac{1}{z}\right)\right.
$$

$$
\left.-\cos\left(\frac{\pi\nu}{2}\right){}_2F_1\left(1-\frac{\nu}{2},\ 1-\frac{\nu}{2};\ 1-\nu;\ -\frac{1}{z}\right)\right]
$$

$$
=\frac{\pi}{\sin\pi\nu}\left[\sum_{m=0}^{\infty}\frac{\Gamma\left(1+m+\frac{\nu}{2}\right)\Gamma\left(1+m+\frac{\nu}{2}\right)}{\Gamma(1+m+\nu)}\left[\cos\left(\frac{\pi\nu}{2}\right)(-1)^m-1\right]\frac{z^{-m-\frac{\nu}{2}-1}}{m!}\right.
$$

$$
\left.+\sum_{m=0}^{\infty}\frac{\Gamma\left(1+m-\frac{\nu}{2}\right)\Gamma\left(1+m-\frac{\nu}{2}\right)}{\Gamma(1+m-\nu)}\left[1-\cos\left(\frac{\pi\nu}{2}\right)(-1)^m\right]\frac{z^{-m+\frac{\nu}{2}-1}}{m!}\right]. \tag{3.16}
$$

In the meantime, another representation of the kernel $S_\nu(z)$ comes from Corollary 2. In fact, formula (2.12) and a simple substitution give

$$
S_\nu(z)=\int_0^\infty\left[2K_0\left(t\sqrt{z}\right)-\pi Y_0\left(t\sqrt{z}\right)\right]K_\nu(t)\,tdt,\ z=\frac{4\pi^2x}{y}. \tag{3.17}
$$

where the integral converges absolutely.

We note here a particular case of the kernel $S_\nu(z)$ when $\nu=0$. In fact, after straightforward calculations we obtain from the equality (3.12)

$$
S_0(z)=\frac{\log z^2}{z^2-1},\ z=\frac{4\pi^2x}{y}. \tag{3.18}
$$

On the other hand, the same result comes immediately, when we pass to the limit $\nu\to0$ through the equality (3.16) and take into account particular cases of the Gauss hypergeometric functions. Therefore combining with (3.5) and taking in mind the value $\gamma = -\psi(1)$ we have in this case an interesting modification of the Koshliakov formula (3.1), namely

$$y\sum_{n=1}^{\infty} d(n)K_0(2\sqrt{yn}) - 2y^2\sum_{n=1}^{\infty} d(n)\frac{\log(4\pi^2 n/y)}{(4\pi^2 n)^2 - y^2}$$

$$= -\log\sqrt{y} - y\int_0^{\infty}(\log x + 2\gamma)S_0\left(\frac{4\pi^2 x}{y}\right)dx.$$

But the latter integral is easily calculated with the relation (2.6.5.15) in [6], Vol. 1. So we have finally the following Koshliakov type formula

$$\sum_{n=1}^{\infty} d(n)\left[K_0(2\sqrt{yn}) - 2y\ \frac{\log(4\pi^2 n/y)}{(4\pi^2 n)^2 - y^2}\right]$$

$$= \frac{1}{8}\log\left(\frac{4\pi^2}{y}\right) - \frac{\log\sqrt{y}}{y} - \frac{\gamma}{4},\ y > 0. \tag{3.19}$$

Letting in (3.19) $y = 1$ we find the identity

$$\sum_{n=1}^{\infty} d(n)\left[K_0(2\sqrt{n}) - 2\ \frac{\log(4\pi^2 n)}{(4\pi^2 n)^2\ - 1}\right] = \frac{1}{4}(\log 2\pi - \gamma).$$

Meanwhile, singular value $y = 4\pi^2$ in (3.19) yields

$$\sum_{n=2}^{\infty} d(n)\left[4\pi^2 K_0(4\pi\sqrt{n}) - \frac{\log n^2}{n^2\ - 1}\right] = 1 - 4\pi^2 K_0(4\pi) - \log 2\pi - \gamma\pi^2.$$

Considering the general case $|\mathrm{Re}\,\nu| < 1,\ \nu \neq 0$, we appeal to Theorem 2 writing (2.7) for the function $g_\nu^*(s,y)$, namely (see (3.6), (3.11), (3.12))

$$\frac{1}{2\pi i}\int_\sigma \zeta^2(s)g_\nu^*(s,y)ds = \lim_{N\to\infty}\left[\sum_{n=1}^{N} d(n)S_\nu\left(\frac{4\pi^2 n}{y}\right) - \int_0^N(\log x + 2\gamma)S_\nu\left(\frac{4\pi^2 x}{y}\right)dx\right].$$

Moving the line of integration σ in the left hand-side of the latter equality to the right, we take the contour $\hat{\sigma} = \left\{s \in \mathbb{C},\ \mathrm{Re\ s} = \frac{3}{2}\right\}$. Doing this we encounter simple poles $s = 1 - \frac{\nu}{2},\ s = 1 + \frac{\nu}{2}$ of gamma-functions $\Gamma\left(1 - s - \frac{\nu}{2}\right),\ \Gamma\left(1 - s + \frac{\nu}{2}\right)$, respectively. Therefore we obtain

$$\frac{1}{2\pi i}\int_\sigma \zeta^2(s)g_\nu^*(s,y)ds = \frac{1}{\pi i}\int_{\hat{\sigma}}\left(\frac{4\pi^2}{y}\right)^{-s}\zeta^2(s)\cos^2\left(\frac{\pi s}{2}\right)\Gamma^2(s)$$

$$\times\Gamma\left(1 - s + \frac{\nu}{2}\right)\Gamma\left(1 - s - \frac{\nu}{2}\right)ds + \frac{y}{8\cos^2\left(\frac{\pi\nu}{4}\right)}\left[\left(\frac{4\pi^2}{y}\right)^{\nu/2}\zeta^2\left(1 - \frac{\nu}{2}\right)\frac{\Gamma(\nu)}{\Gamma^2\left(\frac{\nu}{2}\right)}\right.$$

$$\left. + \left(\frac{4\pi^2}{y}\right)^{-\nu/2}\zeta^2\left(1 + \frac{\nu}{2}\right)\frac{\Gamma(-\nu)}{\Gamma^2\left(-\frac{\nu}{2}\right)}\right]. \tag{3.20}$$

In the meantime, using the representation [9]

$$\zeta^2(s) = \sum_{n=1}^{\infty}\frac{d(n)}{n^s},\ \mathrm{Re}\ s > 1,$$

equalities (3.12), (3.13), (3.14) and the absolute convergence of the series and the integral we get

$$\frac{1}{\pi i}\int_{\hat\sigma}\left(\frac{4\pi^2}{y}\right)^{-s}\zeta^2(s)\cos^2\left(\frac{\pi s}{2}\right)\Gamma^2(s)\Gamma\left(1-s+\frac{\nu}{2}\right)\Gamma\left(1-s-\frac{\nu}{2}\right)ds$$

$$=\sum_{n=1}^{\infty}d(n)\ \frac{1}{\pi i}\int_{\hat\sigma}\left(\frac{4\pi^2 n}{y}\right)^{-s}\cos^2\left(\frac{\pi s}{2}\right)\Gamma^2(s)\Gamma\left(1-s+\frac{\nu}{2}\right)\Gamma\left(1-s-\frac{\nu}{2}\right)ds$$

$$=\sum_{n=1}^{\infty}d(n)\left[S_\nu\left(\frac{4\pi^2 n}{y}\right)+\frac{y}{8n\cos^2\left(\frac{\pi\nu}{4}\right)}\left[\left(\frac{4\pi^2 n}{y}\right)^{\nu/2}\frac{\Gamma(\nu)}{\Gamma^2\left(\frac{\nu}{2}\right)}+\left(\frac{4\pi^2 n}{y}\right)^{-\nu/2}\frac{\Gamma(-\nu)}{\Gamma^2\left(-\frac{\nu}{2}\right)}\right]\right]$$

$$=\sum_{n\le y/(4\pi^2)}d(n)\left[V_{1,\nu}\left(\frac{4\pi^2 n}{y}\right)+\frac{y}{8n\cos^2\left(\frac{\pi\nu}{4}\right)}\left[\left(\frac{4\pi^2 n}{y}\right)^{\nu/2}\frac{\Gamma(\nu)}{\Gamma^2\left(\frac{\nu}{2}\right)}+\left(\frac{4\pi^2 n}{y}\right)^{-\nu/2}\frac{\Gamma(-\nu)}{\Gamma^2\left(-\frac{\nu}{2}\right)}\right]\right]$$

$$+\sum_{n\ge y/(4\pi^2)}d(n)\left[V_{2,\nu}\left(\frac{4\pi^2 n}{y}\right)+\frac{y}{8n\cos^2\left(\frac{\pi\nu}{4}\right)}\left[\left(\frac{4\pi^2 n}{y}\right)^{\nu/2}\frac{\Gamma(\nu)}{\Gamma^2\left(\frac{\nu}{2}\right)}+\left(\frac{4\pi^2 n}{y}\right)^{-\nu/2}\frac{\Gamma(-\nu)}{\Gamma^2\left(-\frac{\nu}{2}\right)}\right]\right].$$

Hence combining with (1.1), (3.5), (3.20) we arrive at the following Koshliakov type summation formula

$$\sum_{n=1}^{\infty}d(n)K_\nu(2\sqrt{yn})-\frac{\pi\nu}{4y\sin\left(\frac{\pi\nu}{2}\right)}\left[2\gamma-\log y+\psi\left(1+\frac{\nu}{2}\right)+\psi\left(1-\frac{\nu}{2}\right)\right]$$

$$=\sum_{n\ \le\ y/(4\pi^2)}d(n)\left[\frac{1}{y}V_{1,\nu}\left(\frac{4\pi^2 n}{y}\right)+\frac{1}{8n\cos^2\left(\frac{\pi\nu}{4}\right)}\left[\left(\frac{4\pi^2 n}{y}\right)^{\nu/2}\frac{\Gamma(\nu)}{\Gamma^2\left(\frac{\nu}{2}\right)}+\left(\frac{4\pi^2 n}{y}\right)^{-\nu/2}\frac{\Gamma(-\nu)}{\Gamma^2\left(-\frac{\nu}{2}\right)}\right]\right]$$

$$+\sum_{n\ \ge\ y/(4\pi^2)}d(n)\left[\frac{1}{y}V_{2,\nu}\left(\frac{4\pi^2 n}{y}\right)+\frac{1}{8n\cos^2\left(\frac{\pi\nu}{4}\right)}\left[\left(\frac{4\pi^2 n}{y}\right)^{\nu/2}\frac{\Gamma(\nu)}{\Gamma^2\left(\frac{\nu}{2}\right)}+\left(\frac{4\pi^2 n}{y}\right)^{-\nu/2}\frac{\Gamma(-\nu)}{\Gamma^2\left(-\frac{\nu}{2}\right)}\right]\right]$$

$$+\frac{1}{8\cos^2\left(\frac{\pi\nu}{4}\right)}\left[\left(\frac{4\pi^2}{y}\right)^{\nu/2}\zeta^2\left(1-\frac{\nu}{2}\right)\frac{\Gamma(\nu)}{\Gamma^2\left(\frac{\nu}{2}\right)}+\left(\frac{4\pi^2}{y}\right)^{-\nu/2}\zeta^2\left(1+\frac{\nu}{2}\right)\frac{\Gamma(-\nu)}{\Gamma^2\left(-\frac{\nu}{2}\right)}\right], \tag{3.21}$$

where $y>0,\ |\mathrm{Re}\ \nu|<1,\ \nu\neq 0$.

Finally in this section we will show how to come to formula (3.19) from (3.21) considering the limit case $\nu\to 0$. In fact, we can easily pass to the limit in the left-hand side of (3.21) via the absolute and uniform convergence by ν of the series (see asymptotic properties and elementary inequalities for the modified Bessel functions in [1], Vol. II, [3], [11], [12]). On the other hand, involving the series expansions of the Riemann zeta-function [1], Vol. I and the reduction formula for gamma-functions it is not difficult to prove the limit equality

$$\lim_{\nu\to 0}\ \frac{1}{8\cos^2\left(\frac{\pi\nu}{4}\right)}\left[\left(\frac{4\pi^2}{y}\right)^{\nu/2}\zeta^2\left(1-\frac{\nu}{2}\right)\frac{\Gamma(\nu)}{\Gamma^2\left(\frac{\nu}{2}\right)}+\left(\frac{4\pi^2}{y}\right)^{-\nu/2}\zeta^2\left(1+\frac{\nu}{2}\right)\frac{\Gamma(-\nu)}{\Gamma^2\left(-\frac{\nu}{2}\right)}\right]$$

$$=\frac{1}{8}\log\left(\frac{4\pi^2}{y}\right)-\frac{\gamma}{4}.$$

Therefore in order to obtain (3.19) from (3.21) we only must calculate the limit of two series in the right hand -side of (3.21) when $\nu\to 0$. Indeed, assuming $|\nu|\le\delta$ for a small positive δ and substituting these two terms by series

$$J_\nu(y)=\sum_{n=1}^{\infty}d(n)\ \frac{1}{\pi i}\int_{\hat\sigma}\left(\frac{4\pi^2 n}{y}\right)^{-s}\cos^2\left(\frac{\pi s}{2}\right)\Gamma^2(s)$$

$$\times\Gamma\left(1-s+\frac{\nu}{2}\right)\Gamma\left(1-s-\frac{\nu}{2}\right)ds, \tag{3.22}$$

we will first establish its absolute and uniform convergence with respect to ν. With the Schwarz inequality, the Parseval equality for the Mellin transform [8] and Stirling's asymptotic formula for gamma-functions [1], Vol. I we have the estimates ($|\nu|<\delta$)

$$\sum_{n=1}^{\infty} d(n)\ \frac{1}{\pi}\int_{\hat{\sigma}}\left|\left(\frac{4\pi^2 n}{y}\right)^{-s}\cos^2\left(\frac{\pi s}{2}\right)\Gamma^2(s)\Gamma\left(1-s+\frac{\nu}{2}\right)\Gamma\left(1-s-\frac{\nu}{2}\right)ds\right|$$

$$\leq \frac{4}{\pi}\left(\frac{4\pi^2}{y}\right)^{-3/2}\sum_{n=1}^{\infty}\frac{d(n)}{n^{3/2}}\int_{-\infty}^{\infty}\left|\cos^2\left(\frac{\pi}{2}\left(\frac{3}{2}+it\right)\right)\Gamma^2\left(\frac{3}{2}+it\right)\right|$$

$$\times\left|\frac{\Gamma\left((1+\nu-2it)/2\right)\Gamma\left((1-\nu-2it)/2\right)}{(1-\nu+2it)(1+\nu+2it)}\right|\ dt$$

$$\leq C_y\int_{-\infty}^{\infty}\frac{|3/2+it|^2}{|1+2it|^2-\delta^2}\left|\Gamma\left((1+\nu-2it)/2\right)\Gamma\left((1-\nu-2it)/2\right)\right|\ dt$$

$$\leq C_{y,\delta}\left(\int_{-\infty}^{\infty}\left|\Gamma\left((1+\nu-2it)/2\right)\right|^2 dt\right)^{1/2}\left(\int_{-\infty}^{\infty}\left|\Gamma\left((1-\nu-2it)/2\right)\right|^2\right)^{1/2}\ dt$$

$$=2\pi\ C_{y,\delta}\left(\int_0^{\infty}e^{-2x}x^{\mathrm{Re}\ \nu}dx\right)^{1/2}\left(\int_0^{\infty}e^{-2x}x^{-\mathrm{Re}\ \nu}dx\right)^{1/2}$$

$$=\pi\ C_{y,\delta}\Gamma(1+\mathrm{Re}\ \nu)\Gamma(1-\mathrm{Re}\ \nu)<D_{y,\delta}<\infty,$$

where $C_y, C_{y,\delta}, D_{y,\delta}$ are absolute positive constants, which do not depend on ν. Thus we have proved the absolute and uniform convergence of the series and the integral in (3.22). Hence passing to the limit when $\nu\to 0$ under the series and the integral signs in (3.22), we use relation (8.4.6.11) in [6], Vol. 3 to calculate the corresponding Mellin transform and come out with the equality

$$\lim_{\nu\to 0} J_\nu(y)=2y\sum_{n=1}^{\infty} d(n)\frac{\log(4\pi^2 n/y)}{(4\pi^2 n)^2-y^2},\ y>0.$$

Hence taking into account our motivations above we arrive again at the Koshliakov type formula (3.19).

4 An application of index transforms

In this last section we will extend Koshliakov's type formula (3.21) on a family of Voronoi-Nasim summation formulas applying the modified Kontorovich-Lebedev transformation (see [2], [7], [11], [12], [13])

$$(KLf)(x)=\mathrm{l.i.m.}_{N\to\infty}\int_0^N K_{i\tau}(2\sqrt{x})f(\tau)\ d\tau,\ x>0, \tag{4.1}$$

which involves an integration with respect to the pure imaginary index $i\tau$ of the real-valued modified Bessel function $K_{i\tau}(2\sqrt{x})$ given by the following Fourier integral [7]

$$K_{i\tau}(2\sqrt{x})=\int_0^{\infty}e^{-2\sqrt{x}\cosh u}\cos\tau u\ du. \tag{4.2}$$

The transform (4.1) forms an isometric isomorphism

$$KL : L_2\left(\mathbb{R}+; \frac{1}{\tau\sinh\pi\tau}d\tau\right) \leftrightarrow L_2\left(\mathbb{R}+; \frac{dx}{x}\right),$$

where integral (4.1) converges with respect to the norm in $L_2\left(\mathbb{R}+; \frac{dx}{x}\right)$. Furthermore, the reciprocal inversion is given by the formula

$$f(\tau) = \text{l.i.m.}_{N\to\infty}\frac{1}{\pi^2}\tau\sinh\pi\tau\int_{1/N}^{\infty} K_{i\tau}(2\sqrt{x})(KLf)(x)\frac{dx}{x}, \tag{4.3}$$

where integral (4.2) converges with respect to the norm in $L_2\left(\mathbb{R}+; \frac{1}{\tau\sinh\pi\tau}d\tau\right)$ and the Parseval equality holds

$$\int_0^{\infty} |(KLf)(x)|^2\frac{dx}{x} = \pi^2\int_0^{\infty}\frac{|f(\tau)|^2}{\tau\sinh\pi\tau}\,d\tau. \tag{4.4}$$

However, integral (4.1) exists in the Lebesgue sense for a wide class of functions. Indeed, this fact can be easily verified using the Schwarz inequality and the uniform inequality for the modified Bessel functions [11], [12]

$$\left|K_{i\tau}(2\sqrt{x})\right| \le e^{-\pi\tau/3}K_0(\sqrt{x}), \quad x > 0,\ \tau \ge 0. \tag{4.5}$$

For instance, taking a simple function $f(\tau) \equiv 1$ and using (4.2) with the reciprocal formula for cosine Fourier transform we get $(KLf)(x) = \frac{\pi}{2}e^{-2\sqrt{x}}$. Hence letting $y = 1,\ \nu = i\tau$ and writing the Koshliakov type formula (3.21) in the form (see (3.20))

$$\sum_{n=1}^{\infty} d(n)K_{i\tau}(2\sqrt{n}) - \frac{\pi\tau}{4\sinh\left(\frac{\pi\tau}{2}\right)}\left[2\gamma + \psi\left(1+\frac{i\tau}{2}\right) + \psi\left(1-\frac{i\tau}{2}\right)\right]$$
$$= \frac{1}{\pi i}\int_{\sigma}\left(4\pi^2\right)^{-s}\zeta^2(s)\cos^2\left(\frac{\pi s}{2}\right)\Gamma^2(s)\Gamma\left(1-s+\frac{i\tau}{2}\right)\Gamma\left(1-s-\frac{i\tau}{2}\right)ds, \tag{4.6}$$

we will integrate it through with respect to τ. We note that the change of the order of integration and summation can be easily motivated via the absolute convergence and Fubini's theorem (for the latter integral one can use the Schwarz inequality, inequality (4.5) and the asymptotic behavior of the Riemann zeta-function on the critical line $\zeta(1/2+it) = O(t^{1/6}\log t),\ t\to\infty$ (cf. [9])). Therefore calculating integrals by τ invoking with relations (2.5.46.13), (2.5.46.15) in [6], Vol. 1 we obtain from the latter equality

$$\sum_{n=1}^{\infty} d(n)e^{-2\sqrt{n}} - \gamma - \frac{1}{\pi}\int_0^{\infty}\left|\Gamma\left(1+\frac{i\tau}{2}\right)\right|^2\left[\psi\left(1+\frac{i\tau}{2}\right)+\psi\left(1-\frac{i\tau}{2}\right)\right]d\tau$$
$$= \frac{1}{\pi i}\int_{\sigma}\pi^{-2s}\zeta^2(s)\cos^2\left(\frac{\pi s}{2}\right)\Gamma^2(s)\Gamma(2(1-s))ds. \tag{4.7}$$

Meanwhile, employing the definition of the psi-function $\psi(z) = \frac{d}{dz}\log\left[\Gamma(z)\right]$ and integrating by parts we easily deduce the value of the following index integral

$$\int_0^{\infty}\left|\Gamma\left(1+\frac{i\tau}{2}\right)\right|^2\left[\psi\left(1+\frac{i\tau}{2}\right)+\psi\left(1-\frac{i\tau}{2}\right)\right]d\tau = -1.$$

Thus (4.7) yields the equality

$$\sum_{n=1}^{\infty} d(n)e^{-2\sqrt{n}} + \frac{1}{\pi} - \gamma = \frac{1}{\pi i}\int_{\sigma}\pi^{-2s}\zeta^2(s)\cos^2\left(\frac{\pi s}{2}\right)\Gamma^2(s)\Gamma(2(1-s))ds$$

$$= \frac{2}{\sqrt{\pi}\ \pi i}\int_{\sigma}(2\pi)^{-2s}\zeta^2(s)\cos^2\left(\frac{\pi s}{2}\right)\Gamma^2(s)\Gamma(1-s)\Gamma\left(\frac{3}{2}-s\right)ds. \tag{4.8}$$

Further, the right-hand side of (4.8) we treat in a similar manner as above moving the contour of integration to the right taking the line $(\mu - i\infty, \mu + i\infty)$, $1 < \text{Re}\ \mu < \frac{3}{2}$ and encountering a simple pole $s = 1$ of the gamma-function $\Gamma(1-s)$ (for the product $\zeta^2(s)\cos^2\left(\frac{\pi s}{2}\right)$ the point $s = 1$ is a removable singularity). Hence with the residue theorem

$$\frac{1}{\pi i}\int_{\sigma}\pi^{-2s}\zeta^2(s)\cos^2\left(\frac{\pi s}{2}\right)\Gamma^2(s)\Gamma(2(1-s))ds$$

$$= \sum_{n=1}^{\infty} d(n)\ \frac{\sqrt{\pi}}{\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\left(4\pi^2 n\right)^{-s}\cot\left(\frac{\pi s}{2}\right)\Gamma(s)\Gamma\left(\frac{3}{2}-s\right)ds + \frac{1}{4}. \tag{4.9}$$

The latter integral with respect to s we calculate (appealing again to Slater's theorem [3]) as a sum of residues in the right-hand simple poles $s = 1 + k,\ s = \frac{3}{4} + k,\ \frac{5}{4} + k,\ k = 0, 1, 2, \ldots$ of gamma-functions $\Gamma(1-s), \Gamma\left(\frac{3}{4}-s\right),\ \Gamma\left(\frac{5}{4}-s\right)$, respectively. In fact, expressing straightforward the obtained hypergeometric series and using relations (7.3.1.106), (7.3.1.107) in [6], Vol. 3, we derive

$$\frac{\sqrt{\pi}}{\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\left(4\pi^2 n\right)^{-s}\cot\left(\frac{\pi s}{2}\right)\Gamma(s)\Gamma\left(\frac{3}{2}-s\right)ds$$

$$= \frac{\sqrt{2}}{\pi i\sqrt{\pi}}\int_{\mu/2-i\infty}^{\mu/2+i\infty}\left(4\pi^2 n\right)^{-2s}\frac{\Gamma^2(s)}{\Gamma\left(\frac{1}{2}-s\right)}\Gamma(1-s)\Gamma\left(\frac{3}{4}-s\right)\Gamma\left(\frac{5}{4}-s\right)ds$$

$$= \frac{1}{2\pi^4 n^2}\ {}_3F_2\left(1,\ 1,\ \frac{3}{2};\ \frac{3}{4},\ \frac{5}{4};\ \frac{1}{16\pi^4 n^2}\right)$$

$$-\ \frac{\pi}{\sqrt{2}(16\pi^4 n^2 -\ 1)^{3/2}}\left[\left(1+\frac{1}{\sqrt{2}}\right)(4\pi^2 n+1)^{3/2}\ +\ \left(1-\frac{1}{\sqrt{2}}\right)(4\pi^2 n-1)^{3/2}\right]. \tag{4.10}$$

Hence substituting the latter expression into the right-hand side of (4.9), changing the order of summation in the double series and using a relation for $\zeta^2(2(1+k)),\ k \in \mathbb{N}_0$ [1], Vol.1 in terms of squares of Bernoulli numbers $B^2_{2(k+1)}$, we obtain finally

$$\frac{1}{\pi i}\int_{\sigma}\pi^{-2s}\zeta^2(s)\cos^2\left(\frac{\pi s}{2}\right)\Gamma^2(s)\Gamma(2(1-s))ds = \frac{1}{4} + \sum_{k=0}^{\infty}\frac{2^{4k-1}\ B^2_{2(k+1)}}{(4k+1)!(k+1)^2(2k+1)}$$

$$-\frac{\pi}{\sqrt{2}}\sum_{n=1}^{\infty}\frac{d(n)}{(16\pi^4 n^2 -\ 1)^{3/2}}\left[\left(1+\frac{1}{\sqrt{2}}\right)(4\pi^2 n+1)^{3/2}\ +\ \left(1-\frac{1}{\sqrt{2}}\right)(4\pi^2 n-1)^{3/2}\right].$$

Combining with (4.8) we get the following summation formula of Voronoi's type

$$\sum_{n=1}^{\infty} d(n)\left[e^{-2\sqrt{n}} + \frac{\pi}{\sqrt{2}\ (16\pi^4 n^2 -\ 1)^{3/2}}\left[\left(1+\frac{1}{\sqrt{2}}\right)(4\pi^2 n+1)^{3/2}\ +\ \left(1-\frac{1}{\sqrt{2}}\right)(4\pi^2 n-1)^{3/2}\right]\right]$$

$$= \gamma - \frac{1}{\pi} + \frac{1}{4} + \sum_{n=0}^{\infty}\frac{2^{4n-1}\ B^2_{2(n+1)}}{(4n+1)!(n+1)^2(2n+1)}. \tag{4.11}$$

Generally, let f be absolutely continuous and belong to $L_2(\mathbb{R}_+)$. Returning to (4.6) we multiply both sides by $f(\tau)$ and integrate with respect to τ. Changing the order of integration and summation via

(4.5) and the absolute convergence and observing that integral (4.1) converges absolutely under these conditions, we find

$$\sum_{n=1}^{\infty} d(n)(KLf)(n) - \frac{\pi}{4}\int_0^{\infty} \frac{\tau f(\tau)}{\sinh(\pi\tau/2)}\left[2\gamma + \psi\left(1+\frac{i\tau}{2}\right) + \psi\left(1-\frac{i\tau}{2}\right)\right] d\tau$$

$$= \frac{1}{\pi i}\int_{\sigma}\int_0^{\infty} \left(4\pi^2\right)^{-s} \zeta^2(s)\cos^2\left(\frac{\pi s}{2}\right)\Gamma^2(s)\Gamma\left(1-s+\frac{i\tau}{2}\right)\Gamma\left(1-s-\frac{i\tau}{2}\right) f(\tau)d\tau\, ds. \tag{4.12}$$

However, by virtue of the Parseval equality for the cosine Fourier transform (2.6) (see [8]) and relation (2.5.46.13), (2.5.46.15) in [6], Vol. 1 we derive

$$\int_0^{\infty} \frac{\tau f(\tau)}{\sinh(\pi\tau/2)} d\tau = 2\int_0^{\infty} \frac{(F_c f)(x)}{\cosh^2(2\pi x)} dx,$$

$$\frac{\pi}{2}\int_0^{\infty} \frac{\tau f(\tau)}{\sinh(\pi\tau/2)}\left[\psi\left(1+\frac{i\tau}{2}\right) + \psi\left(1-\frac{i\tau}{2}\right)\right] d\tau = -f(0) - \frac{\pi}{2}\int_0^{\infty} \frac{\tau f'(\tau)}{\sinh(\pi\tau/2)} d\tau$$

$$= -f(0) - \pi\int_0^{\infty} \frac{(F_c f')(x)}{\cosh^2(2\pi x)} dx,$$

$$\int_0^{\infty} \Gamma\left(1-s+\frac{i\tau}{2}\right)\Gamma\left(1-s-\frac{i\tau}{2}\right) f(\tau)d\tau = 4^s\, \pi\, \Gamma(2(1-s))\int_0^{\infty} \frac{(F_c f)(x)}{[\cosh(2\pi x)]^{2(1-s)}} dx,$$

where $'$ denotes the derivative of f, which belongs to $L_1(\mathbb{R})$ via the absolute continuity. Consequently, equality (4.12) takes the form

$$\sum_{n=1}^{\infty} d(n)(KLf)(n) - \frac{f(0)}{2} - \frac{\pi}{2}\int_0^{\infty} \frac{(F_c(f'+2\gamma\, f))(x)}{\cosh^2(2\pi x)} dx$$

$$= \frac{1}{\pi i}\int_{\sigma} \pi^{-2s+1}\zeta^2(s)\cos^2\left(\frac{\pi s}{2}\right)\Gamma^2(s)\Gamma(2(1-s))\int_0^{\infty} \frac{(F_c f)(x)}{[\cosh(2\pi x)]^{2(1-s)}} dx\, ds$$

$$= \frac{1}{\pi i}\int_0^{\infty} \frac{(F_c f)(x)}{\cosh(2\pi x)}\int_{\sigma}\left(\frac{\pi}{\cosh(2\pi x)}\right)^{-2s+1} \zeta^2(s)\cos^2\left(\frac{\pi s}{2}\right)\Gamma^2(s)\Gamma(2(1-s))ds\, dx. \tag{4.13}$$

But (see (4.9))

$$\frac{1}{\pi i}\int_{\sigma}\left(\frac{\pi}{\cosh(2\pi x)}\right)^{-2s+1} \zeta^2(s)\cos^2\left(\frac{\pi s}{2}\right)\Gamma^2(s)\Gamma(2(1-s))ds$$

$$= \frac{1}{\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\left(\frac{\pi}{\cosh(2\pi x)}\right)^{-2s+1} \zeta^2(s)\cos^2\left(\frac{\pi s}{2}\right)\Gamma^2(s)\Gamma(2(1-s))ds + \frac{\pi}{4}\cosh(2\pi x),\ 1 < \mu < \frac{3}{2}.$$

Denoting by

$$\Phi(z) = \frac{1}{\pi i}\int_{\mu-i\infty}^{\mu+i\infty} z^{-2s+1}\zeta^2(s)\cos^2\left(\frac{\pi s}{2}\right)\Gamma^2(s)\Gamma(2(1-s))ds$$

we have

$$\Phi(z) = \frac{z\sqrt{2}}{\pi i}\sum_{n=1} d(n)\int_{\mu/2-i\infty}^{\mu/2+i\infty} \left(16 z^4 n^2\right)^{-s} \frac{\Gamma^2(s)}{\Gamma\left(\frac{1}{2}-s\right)}\Gamma(1-s)\Gamma\left(\frac{3}{4}-s\right)\Gamma\left(\frac{5}{4}-s\right) ds$$

$$= 2\sqrt{2}\ z \sum_{n=1}^{\infty} d(n)\ \Psi(16z^4n^2), \tag{4.14}$$

where

$$\Psi(u) = \frac{1}{2\pi i}\int_{\mu/2-i\infty}^{\mu/2+i\infty} u^{-s}\frac{\Gamma^2(s)}{\Gamma\left(\frac{1}{2}-s\right)}\Gamma(1-s)\Gamma\left(\frac{3}{4}-s\right)\Gamma\left(\frac{5}{4}-s\right)ds = \begin{cases}\Psi_1(u), & \text{if } u\in(0,1),\\ \Psi_2(u), & \text{if } u\in(1,\infty)\end{cases}$$

and $\lim_{u\to 1-}\Psi_1(u) = \lim_{u\to 1+}\Psi_2(u) = \Psi(1)$. According to Slater's theorem, $\Psi_1(u)$ will be a sum of residues in the left- hand double poles $s=-k,\ k=0,1,2\ldots,$ of $\Gamma^2(s)$. Therefore, after straightforward calculations we obtain

$$\Psi_1(u) = \frac{1}{2}\sqrt{\frac{\pi}{2}}\sum_{k=0}^{\infty}\frac{(4k+1)!}{[(2k)!]^2}\left(\frac{u}{16}\right)^k\left[\psi(1+k)-\psi\left(\frac{3}{4}+k\right)-\psi\left(\frac{5}{4}+k\right)\right.$$

$$\left.+\psi\left(\frac{1}{2}+k\right)-\ \log u\right],\quad 0<u<1. \tag{4.15}$$

The value of $\Psi_2(u)$, in turn, will be derived easily from (4.10). Precisely, we find

$$\Psi_2(u) = \frac{2\sqrt{2\pi}}{u}{}_3F_2\left(1,\ 1,\ \frac{3}{2};\ \frac{3}{4},\ \frac{5}{4};\frac{1}{u}\right)$$

$$-\ \frac{\pi\sqrt{\pi}}{4(u-1)^{3/2}}\left[\left(1+\frac{1}{\sqrt{2}}\right)(\sqrt{u}+1)^{3/2}\ +\ \left(1-\frac{1}{\sqrt{2}}\right)(\sqrt{u}-1)^{3/2}\right],\ u>1. \tag{4.16}$$

Hence combining with (4.14) and using the value $z=\pi[\cosh(2\pi x)]^{-1}$ we represent the right-hand side of (4.13) as follows

$$\frac{1}{\pi i}\int_0^{\infty}\frac{(F_cf)(x)}{\cosh(2\pi x)}\int_{\sigma}\left(\frac{\pi}{\cosh(2\pi x)}\right)^{-2s+1}\zeta^2(s)\cos^2\left(\frac{\pi s}{2}\right)\Gamma^2(s)\Gamma(2(1-s))ds\ dx$$

$$= \lim_{N\to\infty}\int_0^N\frac{(F_cf)(x)}{\cosh(2\pi x)}\left[\frac{2\pi\sqrt{2}}{\cosh(2\pi x)}\left[\sum_{n\le\ \cosh^2(2\pi x)/4\pi^2} d(n)\ \Psi_1\left(\frac{16\pi^4n^2}{\cosh^4(2\pi x)}\right)\right.\right.$$

$$\left.\left.+\sum_{n\ge\ \cosh^2(2\pi x)/4\pi^2} d(n)\ \Psi_2\left(\frac{16\pi^4n^2}{\cosh^4(2\pi x)}\right)\right]+\frac{\pi}{4}\cosh(2\pi x)\right]$$

$$= 2\pi\sqrt{2}\int_0^{\infty}\frac{(F_cf)(x)}{\cosh^2(2\pi x)}\left[\sum_{4\pi^2n\le\ \cosh^2(2\pi x)} d(n)\ \Psi_1\left(\frac{16\pi^4n^2}{\cosh^4(2\pi x)}\right)\right.$$

$$\left.+\sum_{4\pi^2n\ge\ \cosh^2(2\pi x)} d(n)\ \Psi_2\left(\frac{16\pi^4n^2}{\cosh^4(2\pi x)}\right)\right]dx\ +\ \frac{\pi}{4}\lim_{N\to\infty}\int_0^N(F_cf)(x)dx.$$

Finally we will prove that under our conditions the latter limit exists and is equal to $\frac{1}{2}\ f(0)$. In fact since f is absolutely continuous and belongs to $L_2(\mathbb{R}_+)$ we write for each $N>0$ (see (2.6))

$$\int_0^N(F_cf)(x)dx = \lim_{m\to\infty}2\int_0^N\int_0^m f(t)\cos(2\pi xt)dt\ dx$$

$$= \lim_{m\to\infty} \frac{1}{\pi}\int_0^m f(t)\frac{\sin(2\pi Nt)}{t}dt = \frac{1}{\pi}\int_0^\infty f(t)\frac{\sin(2\pi Nt)}{t}dt.$$

Hence

$$\lim_{N\to\infty} \frac{1}{\pi}\int_0^\infty f(t)\frac{\sin(2\pi Nt)}{t}dt = \lim_{N\to\infty} \frac{1}{\pi}\int_0^\infty \frac{f(t)-f(0)}{t}\ \sin(2\pi Nt)dt\ + \frac{1}{2}f(0)$$

$$\lim_{N\to\infty} \frac{1}{\pi}\int_0^\delta \frac{f(t)-f(0)}{t}\ \sin(2\pi Nt)dt\ + \lim_{N\to\infty} \frac{1}{\pi}\int_\delta^\infty \frac{f(t)-f(0)}{t}\ \sin(2\pi Nt)dt + \frac{1}{2}f(0)$$

$$= \lim_{N\to\infty} \frac{1}{\pi}\int_0^\delta \frac{f(t)-f(0)}{t}\ \sin(2\pi Nt)dt + \frac{1}{2}f(0),$$

for a small positive δ since the second limit is zero via the Riemann-Lebesgue lemma. But $f(t)-f(0)$ is of bounded variation in the interval $(0,\delta)$ and can be represented by $f(t)-f(0) = \psi_1(t)-\psi_2(t)$, where $\psi_i(t),\ i=1,2$ are positive nondecreasing bounded functions, which tend to zero when $t\to 0$. Therefore for each positive ε there exists a number η, such that $\psi_1(t)\le\varepsilon$, when $t\le\eta$ and by the second mean value theorem we have, for instance,

$$\int_0^\eta \frac{\psi_1(t)}{t}\ \sin(2\pi Nt)dt = \psi_1(\eta)\int_{2\pi N\xi}^{2\pi N\eta}\frac{\sin v}{v}dv.$$

Thus

$$\left|\int_0^\eta \frac{\psi_1(t)}{t}\ \sin(2\pi Nt)dt\right| \le A\varepsilon,\ A>0$$

for any $N>0$. Further, $\psi_1(t)/t\in L_1(\eta,\delta)$. Since ε is arbitrary small we find

$$\lim_{N\to\infty} \frac{1}{\pi}\int_0^\delta \frac{\psi_1(t)}{t}\ \sin(2\pi Nt)dt = 0.$$

Analogously we treat the integral with ψ_2. Consequently, we prove the equality $\lim_{N\to\infty}\int_0^N (F_c f)(x)dx = \frac{1}{2}f(0)$ and establish our final result.

Theorem 3. *Let f be absolutely continuous on $\mathbb{R}_+$ and belong to $L_2(\mathbb{R}_+)$. Then the following Voronoi-Nasim type summation formula holds*

$$\sum_{n=1}^\infty d(n)(KLf)(n) - \left(1+\frac{\pi}{4}\right)\ \frac{f(0)}{2} - \frac{\pi}{2}\int_0^\infty \frac{(F_c(f'+2\gamma\ f))(x)}{\cosh^2(2\pi x)}dx$$

$$= 2\pi\sqrt{2}\int_0^\infty \frac{(F_c f)(x)}{\cosh^2(2\pi x)}\left[\sum_{4\pi^2 n\le\ \cosh^2(2\pi x)} d(n)\ \Psi_1\left(\frac{16\pi^4 n^2}{\cosh^4(2\pi x)}\right) + \sum_{4\pi^2 n\ge\ \cosh^2(2\pi x)} d(n)\right.$$

$$\left.\times\ \Psi_2\left(\frac{16\pi^4 n^2}{\cosh^4(2\pi x)}\right)\right] dx,$$

where F_c is the cosine Fourier transform (2.6), *KL is the modified Kontorovich-Lebedev transform* (4.1) *and functions $\Psi_i(z),\ i=1,2$ are defined by* (4.15), (4.16).

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