NEW INVERSION, CONVOLUTION AND TITCHMARSH'S THEOREMS FOR THE HALF-HARTLEY TRANSFORM

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ABSTRACT. The generalized Parseval equality for the Mellin transform is employed to prove the inversion theorem in L_2 with the respective inverse operator related to the Hartley transform on the nonnegative half-axis (the half-Hartley transform). Moreover, involving the convolution method, which is based on the double Mellin-Barnes integrals, the corresponding convolution and Titchmarsh's theorems for the half-Hartley transform are established. Finally, necessary and sufficient conditions for a L_2-function to be self-reciprocal under the half-Hartley transform are obtained.

1. INTRODUCTION AND AUXILIARY RESULTS

The familiar reciprocal pair of the Hartley transforms

$$(\mathscr{H}f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\cos(xt) + \sin(xt)] f(t) dt, \ x \in \mathbb{R}, \tag{1.1}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\cos(xt) + \sin(xt)] (\mathscr{H}f)(t) dt \tag{1.2}$$

is well-known [1] in connection with various applications in mathematical physics. Mapping and inversion properties of these transforms in L_2 as well as their multidimensional analogs were investigated, for instance, in [2], [3], [4]. These operators were treated as the so-called bilateral Watson transform. Recently, the author found the paper [5] (see also [6], [7]), where the attempt to invert the Hartley transform with the integration over $\mathbb{R}_+$

$$(\mathscr{H}_+ f)(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} [\cos(xt) + \sin(xt)] f(t) dt, \ x \in \mathbb{R}_+, \tag{1.3}$$

was undertaken. However, the inversion formula obtained by the authors is depending on the Fourier transform of the image and, indeed, needs to be improved. Here we will achieve this main goal, proving the inversion theorem for transformation (1.3) in $L_2(\mathbb{R}_+)$. Moreover, we will construct and study properties of the convolution operator, related to the half-Hartley transform by general convolution method developed by the author in 1990, and which is based on the double Mellin-Barnes integrals [8], [9], [4]. Namely, we will prove the convolution theorem and Titchmarsh's theorem about the absence of divisors in the convolution product. Finally, we will establish necessary and sufficient conditions for a L_2-function to be self-reciprocal under the half-Hartley transform (1.3).

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We note in this section that our natural approach will involve the L_2-theory of the Mellin transform [10]

$$(\mathscr{M}f)(s) = f^*(s) = \int_0^\infty f(t)t^{s-1}dt,\ s \in \sigma = \{s \in \mathbb{C}, s = \frac{1}{2} + i\tau\}, \tag{1.4}$$

where the integral is convergent in the mean square sense with respect to the norm in $L_2(\sigma)$. Reciprocally, the inversion formula takes place

$$f(x) = \frac{1}{2\pi i}\int_\sigma f^*(s)x^{-s}ds,\ x > 0 \tag{1.5}$$

with the convergence of the integral in the mean square sense with respect to the norm in $L_2(\mathbb{R}_+)$. Furthermore, for any $f_1, f_2 \in L_2(\mathbb{R}_+)$ the generalized Parseval identity holds

$$\int_0^\infty f_1(xt) f_2(t)dt = \frac{1}{2\pi i}\int_\sigma f_1^*(s) f_2^*(1-s)x^{-s}ds,\ x > 0 \tag{1.6}$$

with Parseval's equality of squares of L_2- norms

$$\int_0^\infty |f(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^\infty \left| f^*\left(\frac{1}{2} + i\tau\right)\right|^2 d\tau. \tag{1.7}$$

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