NEW INVERSION, CONVOLUTION AND TITCHMARSH'S THEOREMS FOR THE HALF-HILBERT TRANSFORM

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ABSTRACT. While exploiting the generalized Parseval equality for the Mellin transform, we derive the reciprocal inverse operator in the weighted L_2-space related to the Hilbert transform on the nonnegative half-axis. Moreover, employing the convolution method, which is based on the Mellin-Barnes integrals, we prove the corresponding convolution and Titchmarsh's theorems for the half-Hilbert transform. Some applications to the solvability of a new class of singular integral equations are demonstrated. Our technique does not require the use of methods of the Riemann-Hilbert boundary value problems for analytic functions. The same approach will be applied in the forthcoming research to invert the half-Hartley transform and to establish its convolution theorem.

1. INTRODUCTION AND AUXILIARY RESULTS

The main object of the present paper is the Hilbert transform on the half-axis (the half-Hilbert transform)

$$(H_+f)(x) \equiv h(x) = \frac{1}{\pi} PV \int_0^\infty \frac{f(t)}{t-x}\, dt, \quad x \in \mathbb{R}_+, \tag{1.1}$$

where $f(t)$ is a complex-valued function of the space $L_2(\mathbb{R}_+; t^{2\alpha-1}dt),\ 0 < \alpha < 1/2$ with the norm

$$||f||_{L_2(\mathbb{R}_+;\, x^{2\alpha-1}dx)} = \left(\int_0^\infty |f(t)|^2 t^{2\alpha-1} dt \right)^{1/2}. \tag{1.2}$$

Integral (1.1) is understood in the principal value sense. Mapping and inversion properties of the Hilbert transform with the integration over $\mathbb{R}$ are well-known, for instance, in L_p-spaces [1] and in connection with the solvability of singular integral equations related to the Riemann-Hilbert boundary value problems for analytic functions (see in [2]). Nevertheless the use of this theory to investigate operator (1.1) meets certain obstacles, in particular, to find its reciprocal inverse. The problem is indeed important in mathematical physics applications. This is why some attempts to solve it were undertaken, for instance, in [3], [4], [5], reducing to the corresponding Riemann-Hilbert boundary value problem.

Our natural approach will involve the L_2-theory of the Mellin transform [1]

$$(\mathscr{M}f)(s) = f^*(s) = \int_0^\infty f(t) t^{s-1} dt,\ s \in \sigma_\alpha = \{s \in \mathbb{C}, s = \alpha + i\tau\}, \tag{1.3}$$

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where the integral is convergent in the mean square sense with respect to the norm in $L_2(\sigma_\alpha)$. Reciprocally, the inversion formula takes place

$$f(x) = \frac{1}{2\pi i}\int_{\sigma_\alpha} f^*(s)x^{-s}ds,\ x>0 \tag{1.4}$$

with the convergence of the integral in the mean square sense with respect to the norm in $L_2(\mathbb{R}_+;\ x^{2\alpha-1}dx)$. Furthermore, for any $f_1, f_2 \in L_2(\mathbb{R}_+; x^{2\alpha-1}dx)$ the generalized Parseval identity holds

$$\int_0^\infty f_1\left(\frac{x}{t}\right) f_2(t)\frac{dt}{t} = \frac{1}{2\pi i}\int_{\sigma_\alpha} f_1^*(s)f_2^*(s)x^{-s}ds,\ x>0 \tag{1.5}$$

and Parseval's equality of squares of L_2- norms (see (1.2))

$$\int_0^\infty |f(x)|^2 x^{2\alpha-1}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |f^*(\alpha+i\tau)|^2 d\tau. \tag{1.6}$$

The paper is organized as follows. In Section 2 we prove an analog of the Titchmarsh theorem (see [1], Th. 90) that the half-Hilbert transform (1.1) extends to a bounded invertible map $H_+ : L_2(\mathbb{R}_+;\ x^{2\alpha-1}dx) \to L_2(\mathbb{R}_+;\ x^{2\alpha-1}dx),\quad 0<\alpha<1/2$. In Section 3 we construct a convolution operator related to the half-Hilbert transform by a method, based on the double Mellin-Barnes integrals and developed by the author in 1990 [6], [7], [8]. As a conclusion the convolution theorem and Titchmarsh's theorem about the absence of divisors of zero in the convolution product will be proved. Finally in Section 4 we consider the solvability of a new class of singular integral equations, which can be solved via the half-Hilbert transform (1.1).

As the author observed, the same technique can be applied and will be done in the forthcoming paper to study mapping, convolution properties and find the reciprocal inverse for the half-Hartley transform

$$g(x) = \int_{\mathbb{R}_+} f(t)\left[\cos(xt)+\sin(xt)\right]dt, \quad x>0,$$

giving a rigorous motivation of these results and simplifying the inversion formula in [9].

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