On the inverse Kontorovich-Lebedev transform for distributions

Semyon B. Yakubovich *

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Abstract

We show that in a sense of distributions

$$\lim_{\varepsilon\to 0+} \frac{1}{\pi^2}\tau \sinh \pi\tau \int_{\varepsilon}^{\infty} K_{i\tau}(y)K_{ix}(y)\frac{dy}{y} = \delta(\tau - x),$$

where δ is the Dirac distribution, $\tau, x \in \mathbb{R}$ and $K_\nu(x)$ is the modified Bessel function. The convergence is in $\mathcal{E}'(\mathbb{R})$ for any even $\varphi(x) \in \mathcal{E}(\mathbb{R})$, which is a restriction to $\mathbb{R}$ of an analytic function $\varphi(z)$ in a horizontal strip $G_a = \{z \in \mathbb{C} : |\text{Im } z| \le a,\ a > 0\}$ and satisfies the condition $\varphi(z) = O\left(z^{-\alpha}\Gamma(iz + 1/2)\right),\ |\text{Re z}| \to \infty,\ \alpha > 1$ uniformly in G_a. The result is applied to prove the representation theorem for the inverse Kontorovich-Lebedev transformation on distributions.

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AMS subject classification: 46F12, 44A15, 33C10

1 Introduction

In this paper we study a natural extension to spaces of distributions for the inverse Kontorovich-Lebedev transform [4], [7]

$$F(y) = \int_{-\infty}^{\infty} K_{i\tau}(y)f(\tau)d\tau,\ y > 0, \tag{1.1}$$

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modifying our previous version of this transformation given in [5], which is based on the following expansion

$$f(x) = \frac{1}{\pi^2} \lim_{\epsilon\to 0+} \int_0^\infty K_{ix}(y) \frac{1}{y^{1-\epsilon}} \int_{-\infty}^{+\infty} \tau \sinh(\pi\tau) K_{i\tau}(y) f(\tau) d\tau dy, \tag{1.2}$$

where the limit is understood in the weak topology of distributions with compact supports $\mathcal{E}'(\mathbb{R})$. These results were initiated by the pioneer paper [8] and by evaluation of concrete distributions (cf. in [2]) useful in various applications of the Kontorovich-Lebedev transform.

As it is known [1], [4], [6], [7], the kernel $K_{i\tau}(y)$ belongs to a class of the modified Bessel functions $K_\nu(z), I_\nu(z)$, which are linear independent solutions of the Bessel differential equation

$$z^2 \frac{d^2u}{dz^2} + z\frac{du}{dz} - (z^2+\nu^2)u = 0. \tag{1.3}$$

They can be given by formulas

$$I_\nu(z) = \sum_{k=0}^\infty \frac{(z/2)^{\nu+2k}}{\Gamma(\nu+k+1)k!}, \tag{1.4}$$

where $\Gamma(w)$ is Euler's Gamma-function [1],

$$K_\nu(z) = \frac{\pi}{2\sin\pi\nu} \left[I_{-\nu}(z) - I_\nu(z)\right], \tag{1.5}$$

when $\nu \neq 0, \pm 1, \pm 2, \ldots$, and $K_n(z) = \lim_{\nu\to n} K_\nu(z),\ n = 0, \pm 1, \pm 2, \ldots$. The function $K_\nu(z)$ is called also the Macdonald function. It is even with respect to ν and has the following integral representations (cf., in [1], [3])

$$K_\nu(z) = \int_0^\infty e^{-z\cosh t} \cosh \nu t dt = \frac{1}{2}\int_0^\infty e^{-z(t+t^{-1})/2} t^{\nu-1} dt. \tag{1.6}$$

Useful relations are [1]

$$z\frac{\partial}{\partial z} K_\nu(z) = \nu K_\nu(z) - zK_{\nu+1}(z), \tag{1.7}$$

$$\int_0^\infty I_\xi(x) K_\nu(x) \frac{dx}{x} = \frac{1}{\xi^2-\nu^2},\ \mathrm{Re}\xi > |\mathrm{Re}\nu|, \tag{1.8}$$

$$I_\nu(z) = \frac{1}{\sqrt{\pi}\Gamma(\nu+1/2)} \left(\frac{z}{2}\right)^\nu \int_{-1}^1 e^{-zx}(1-x^2)^{\nu-\frac{1}{2}} dx,\ \mathrm{Re}\ \nu > -\frac{1}{2}. \tag{1.9}$$

These functions have the asymptotic behaviour [1], [7]

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}[1+O(1/z)], \qquad z\to\infty, \tag{1.10}$$

$$I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}}[1 + O(1/z)], \qquad z \to \infty, \tag{1.11}$$

and near the origin

$$K_\nu(z) = O\left(z^{-|Re\nu|}\right), \ z \to 0, \tag{1.12}$$

$$K_0(z) = -\log z + O(1), \ z \to 0, \tag{1.13}$$

$$I_\nu(z) = O(z^{\mathrm{Re}\nu}), \nu \neq 0, \ z \to 0. \tag{1.14}$$

We also mention here the value of their Wronskian [1]

$$W(K_\nu(z), I_\nu(z)) = K_\nu(z)I'_\nu(z) - I_\nu(z)K'_\nu(z) = -\frac{1}{z}, \quad z \neq 0, \quad \nu \in \mathbb{C} \tag{1.15}$$

and symbol $'$ denotes the derivative with respect to z. When the index of the Macdonald function is pure imaginary, i.e. $\nu = i\tau, \ \tau \in \mathbb{R}$ then $K_{i\tau}(y), \ y > 0$ is real-valued.

The main object of this work is to study a distributional version of the Kontorovich-Lebedev transformation (1.1) and to prove a representation theorem involving the following kernel function

$$\mathcal{K}_\varepsilon(\tau, x) = \frac{1}{\pi^2}\tau \sinh \pi\tau \int_\varepsilon^\infty K_{i\tau}(y)K_{ix}(y)\frac{dy}{y}, \ \varepsilon > 0. \tag{1.16}$$

We will prove that $\mathcal{K}_\varepsilon(\tau, x)$ converges to a shifted Dirac distribution $\delta(\tau - x)$ when $\varepsilon \to 0+$ in the sense of the convergence in $\mathcal{E}'(\mathbb{R})$. This property can be interpreted as a certain orthogonality of the modified Bessel functions with pure imaginary subscripts.

We note that $\mathcal{E}'(\mathbb{R})$ is a dual space of $\mathcal{E}(\mathbb{R})$, which in turn, is a metrizable locally convex space of infinitely differentiable functions $\varphi(x)$ with the topology generated by the collection of seminorms

$$\gamma_{p,K}(\varphi) \equiv \sup_{x \in K} | D_x^p \varphi(x) | < \infty, \tag{1.17}$$

where p is a non-negative integer number, K is a compact set on $\mathbb{R}$, and $D_x = \frac{d}{dx}$.

Along this paper by C we will denote a positive constant not necessarily the same in each occurrence.

2 Orthogonality of the Macdonald functions

The main result of this section is the following

Theorem 1. *Let $\varphi \in \mathcal{E}(\mathbb{R})$ be an even function, which is a restriction to $\mathbb{R}$ of an analytic function $\varphi(z)$ in a horizontal strip $G_a = \{z \in \mathbb{C} : |\mathrm{Im}\ z| \leq a, \ a > 0\}$ satisfying the condition $\varphi(z) = O\left(z^{-\alpha}\Gamma(1/2 + iz)\right), \ |\mathrm{Re\ z}| \to \infty, \ \alpha > 1$ uniformly in G_a. Then in $\mathcal{E}'(\mathbb{R})$*

$$\lim_{\varepsilon \to 0+} \mathcal{K}_\varepsilon(\tau, x) = \delta(\tau - x), \ \tau, x \in \mathbb{R}. \tag{2.1}$$

More precisely, for each φ under conditions of the theorem we have the equality

$$\lim_{\varepsilon\to 0+} \langle \mathcal{K}_\varepsilon(\cdot, x), \varphi\rangle = \varphi(x),\ \ x\in\mathbb{R}, \tag{2.2}$$

where the convergence is in the countably multinorm space $\mathcal{E}(\mathbb{R})$.

Proof. By using relation (1.12.3.3) in [3] we calculate the integral with respect to y in (1.16). Then invoking (1.5) with the definition of Wronskian, the kernel $\mathcal{K}_\varepsilon(\tau, x)$ can be represented as follows ($|\tau|\neq |x|$)

$$\mathcal{K}_\varepsilon(\tau, x) = \frac{\varepsilon\tau\sinh\pi\tau}{\pi^2(\tau^2-x^2)}\left[K_{ix}(\varepsilon)K'_{i\tau}(\varepsilon) - K_{i\tau}(\varepsilon)K'_{ix}(\varepsilon)\right]$$

$$= \frac{\varepsilon\tau\sinh\pi\tau}{\pi^2(\tau^2-x^2)} W\left(K_{ix}(\varepsilon), K_{i\tau}(\varepsilon)\right) = \frac{\varepsilon i\tau}{2\pi(\tau^2-x^2)}\left[W\left(K_{ix}(\varepsilon), I_{-i\tau}(\varepsilon)\right)\right.$$

$$\left. - W\left(K_{ix}(\varepsilon), I_{i\tau}(\varepsilon)\right)\right]. \tag{2.3}$$

Diagonal values $|\tau| = |x|$ of the kernel (1.16) can be easily find by its continuity on $\mathbb{R}^2$ as a function of two variables. In fact, for each $\varepsilon > 0$ the integral by y is absolutely and uniformly convergent with respect to (τ, x) on any compact subset of $\mathbb{R}^2$ by virtue of the inequality (see (1.6)) $|K_\nu(y)| \le K_{\mathrm{Re}\nu}(y)$ and asymptotic behavior (1.10). Our goal is to show that under conditions of the theorem there exists a nonnegative integer r such that

$$\max_{0\le p\le r} \sup_{x\in[-x_0, x_0]} |D_x^p(\varphi - \varphi_\varepsilon)| \to 0,\ \ \varepsilon\to 0+, \tag{2.4}$$

where $x_0 > 0$ and we denote by

$$\varphi_\varepsilon(x) = \langle \mathcal{K}_\varepsilon(\cdot, x), \varphi\rangle,\ \ \varepsilon > 0. \tag{2.5}$$

We observe that (2.5) is a regular distribution. Indeed, taking into account the evenness of φ we can write it in the form

$$\varphi_\varepsilon(x) = \frac{\varepsilon}{\pi^2}\int_{-\infty}^{\infty} \frac{\tau\sinh\pi\tau}{\tau^2-x^2} W\left(K_{ix}(\varepsilon), K_{i\tau}(\varepsilon)\right)\varphi(\tau)d\tau$$

$$= \frac{\varepsilon}{2\pi^2}\int_{-\infty}^{\infty}\left[\frac{1}{\tau-x} + \frac{1}{\tau+x}\right]\sinh\pi\tau W\left(K_{ix}(\varepsilon), K_{i\tau}(\varepsilon)\right)\varphi(\tau)d\tau$$

$$= \frac{\varepsilon}{\pi^2}\int_{-\infty}^{\infty}\frac{\sinh\pi\tau}{\tau-x} W\left(K_{ix}(\varepsilon), K_{i\tau}(\varepsilon)\right)\varphi(\tau)d\tau$$

$$= \frac{\varepsilon}{2\pi i} P.V. \int_{-\infty}^{\infty} W\left(K_{ix}(\varepsilon), I_{i\tau}(\varepsilon)\right)\frac{\varphi(\tau)}{\tau-x}d\tau$$

$$-\frac{\varepsilon}{2\pi i} P.V. \int_{-\infty}^{\infty} W\left(K_{ix}(\varepsilon), I_{-i\tau}(\varepsilon)\right) \frac{\varphi(\tau)}{\tau - x} d\tau = \varphi_{1\varepsilon}(x) - \varphi_{2\varepsilon}(x), \tag{2.6}$$

where both integrals $\varphi_{j\varepsilon}(x),\ j = 1, 2$ are understood in the principal Cauchy value. We also satisfy their absolute convergence. We take for instance, integral $\varphi_{1\varepsilon}(x)$. We have

$$\varphi_{1\varepsilon}(x) = \frac{\varepsilon}{2\pi i} \lim_{\delta \to 0+} \left(\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty} \right) W\left(K_{ix}(\varepsilon), I_{i(\tau+x)}(\varepsilon)\right) \frac{\varphi(\tau + x)}{\tau} d\tau$$

Hence it is sufficient to guarantee the estimate

$$\int_{|\tau| \geq M} \left| W\left(K_{ix}(\varepsilon), I_{i(\tau+x)}(\varepsilon)\right) \frac{\varphi(\tau + x)}{\tau} \right| d\tau < \infty,\ \varepsilon > 0,\ x \in [-x_0, x_0],$$

where $M > x_0 \geq |x|$ is large enough. In fact, from (1.9) we immediately obtain the following estimates of the modified Bessel function and its derivative with respect to an argument

$$|I_\nu(y)| \leq \frac{\Gamma(\mathrm{Re}\nu + 1/2)}{\Gamma(\mathrm{Re}\nu + 1)|\Gamma(\nu + 1/2)|} e^y \left(\frac{y}{2}\right)^{\mathrm{Re}\nu}, \quad y > 0, \quad \mathrm{Re}\nu > -\frac{1}{2}, \tag{2.7}$$

$$|I'_\nu(y)| \leq \frac{\Gamma(\mathrm{Re}\nu + 1/2)}{\Gamma(\mathrm{Re}\nu + 1)|\Gamma(\nu + 1/2)|} \left(\frac{|\nu|}{y} + 1\right) e^y \left(\frac{y}{2}\right)^{\mathrm{Re}\nu}. \tag{2.8}$$

Consequently, taking into account conditions of the theorem we derive

$$\int_{|\tau| \geq M} \left| W\left(K_{ix}(\varepsilon), I_{i(\tau+x)}(\varepsilon)\right) \frac{\varphi(\tau + x)}{\tau} \right| d\tau \leq C e^{\varepsilon} \left[\left(\frac{2}{y} + \frac{1}{M} \right) |K_{ix}(\varepsilon)| + \frac{|K'_{ix}(\varepsilon)|}{M} \right]$$

$$\times \int_{|\tau| \geq M} \frac{d\tau}{(|\tau| - x_0)^\alpha} < \infty,\ \alpha > 1.$$

Analogously we treat $\varphi_{2\varepsilon}(x)$. Thus (2.5) is a regular distribution and we have the representation (2.6). It is easily seen by an elementary substitution in the integral that $\varphi_{1\varepsilon}(x) = -\varphi_{2\varepsilon}(-x)$. Hence we have $\varphi_\varepsilon(x) = -\varphi_{2\varepsilon}(x) - \varphi_{2\varepsilon}(-x)$ and we will prove that

$$\max_{0 \leq p \leq r} \sup_{x \in [-x_0, x_0]} \left| D_x^p \left(\frac{\varphi}{2} + \varphi_{2\varepsilon} \right) \right| \to 0,\ \varepsilon \to 0+. \tag{2.9}$$

Taking then into account the evenness of φ we will conclude (2.4) and therefore will achieve our goal.

In order to establish (2.9) we will appeal to analytic properties of $\varphi(z)$ in the strip G_a. Precisely, via Cauchy's theorem we take a big positive R and a small $\delta > 0$ to write the equality

$$\frac{\varepsilon}{2\pi i} \left(\int_{-R}^{-\delta} + \int_{\delta}^{R} + \int_{R}^{R+ia} + \int_{R+ia}^{-R+ia} + \int_{-R+ia}^{-R} \right) W\left(K_{ix}(\varepsilon), I_{-i(z+x)}(\varepsilon)\right) \frac{\varphi(z + x)}{z} dz$$

$$+\frac{\varepsilon}{2\pi}\int_{\pi}^{0} W\left(K_{ix}(\varepsilon), I_{-i(\delta e^{i\theta}+x)}(\varepsilon)\right)\varphi\left(\delta e^{i\theta}+x\right)d\theta = 0. \tag{2.10}$$

Hence letting $R\to\infty$ we observe that integrals over $(R, R+ia)$ and $(-R+ia, -R)$ tend to zero due to asymptotic behavior of the function φ in the strip G_a. Then we let $\delta\to 0$ to obtain (see (2.6))

$$\varphi_{2\varepsilon}(x) = -\frac{\varepsilon}{2\pi}\lim_{\delta\to 0+}\int_{\pi}^{0} W\left(K_{ix}(\varepsilon), I_{-i(\delta e^{i\theta}+x)}(\varepsilon)\right)\varphi\left(\delta e^{i\theta}+x\right)d\theta$$

$$+\frac{\varepsilon}{2\pi i}\int_{-\infty}^{\infty} W\left(K_{ix}(\varepsilon), I_{a-i\tau}(\varepsilon)\right)\frac{\varphi(\tau+ia)}{\tau-x+ia}d\tau, \quad a>0. \tag{2.11}$$

Meanwhile, we can pass to the limit when $\delta\to 0+$ under the integral sign in (2.11) via the dominated convergence theorem. Hence invoking the evenness of the function $K_{ix}(\varepsilon)$ with respect to x and combining with the value of the Wronskian (1.15) we derive the equality

$$\varphi_{2\varepsilon}(x)+\frac{\varphi(x)}{2} = \frac{\varepsilon}{2\pi i}\int_{-\infty}^{\infty} W\left(K_{ix}(\varepsilon), I_{a-i\tau}(\varepsilon)\right)\frac{\varphi(\tau+ia)}{\tau-x+ia}d\tau. \tag{2.12}$$

Hence differentiating through (2.12) with respect to x we put derivatives inside the integral via the uniform convergence on the compact $[-x_0, x_0]$ to find

$$D_x^p\left(\frac{\varphi}{2}+\varphi_{2\varepsilon}\right) = \frac{\varepsilon}{2\pi i}\sum_{l=0}^{p}\frac{p!}{(p-l)!}\int_{-\infty}^{\infty} W\left(D_x^{p-l}K_{ix}(\varepsilon), I_{a-i\tau}(\varepsilon)\right)$$

$$\times\frac{\varphi(\tau+ia)}{(\tau-x+ia)^{l+1}}d\tau. \tag{2.13}$$

In the meantime, appealing to representations (1.6) and assuming $0<\varepsilon<1$ we have the uniform estimates

$$\left|D_x^{p-l}K_{ix}(\varepsilon)\right| \le \int_0^{\infty} e^{-\frac{\varepsilon}{2}e^t}t^{p-l}dt = \int_{\varepsilon/2}^{\infty} e^{-u}\log^{p-l}\left(\frac{2u}{\varepsilon}\right)\frac{du}{u}$$

$$\le \int_{\varepsilon/2}^{1/2}\log^{p-l}\left(\frac{2u}{\varepsilon}\right)\frac{du}{u} + \int_{1/2}^{\infty} e^{-u}\left(\log 2u+\log\varepsilon^{-1}\right)^{p-l}\frac{du}{u} = O\left(\log^{p-l+1}\varepsilon^{-1}\right).$$

Analogously, for $p>l$ we obtain

$$\varepsilon\left|D_x^{p-l}K'_{ix}(\varepsilon)\right| \le \varepsilon\int_0^{\infty} e^{-\frac{\varepsilon}{2}e^t}e^t t^{p-l}dt = 2(p-l)\int_0^{\infty} e^{-\frac{\varepsilon}{2}e^t}t^{p-l-1}dt$$

$$= O\left(\log^{p-l}\varepsilon^{-1}\right).$$

When $p = l$ evidently (see (1.6), (1.12))

$$\varepsilon \left| D_x^{p-l} K'_{ix}(\varepsilon) \right| = \varepsilon \left| K'_{ix}(\varepsilon) \right| < \varepsilon \int_0^\infty e^{-\varepsilon \sinh t} \cosh t dt = 1.$$

Combining with estimates (2.7), (2.8) we return to (2.13) to derive

$$\max_{0\le p\le r} \sup_{x\in[-x_0,x_0]} \left| D_x^p \left(\frac{\varphi}{2} + \varphi_{2\varepsilon} \right) \right| \le \max_{0\le p\le r} \sup_{x\in[-x_0,x_0]} \frac{\varepsilon}{2\pi} \sum_{l=0}^{p} \frac{p!}{(p-l)!}$$

$$\times \int_{-\infty}^{\infty} |D_x^{p-l} K_{ix}(\varepsilon) I'_{a-i\tau}(\varepsilon)| \frac{|\varphi(\tau + ia)|}{|\tau - x + ia|^{l+1}} d\tau + \max_{0\le p\le r} \sup_{x\in[-x_0,x_0]} \frac{\varepsilon}{2\pi} \sum_{l=0}^{p} \frac{p!}{(p-l)!}$$

$$\times \int_{-\infty}^{\infty} |D_x^{p-l} K'_{ix}(\varepsilon) I_{a-i\tau}(\varepsilon)| \frac{|\varphi(\tau + ia)|}{|\tau - x + ia|^{l+1}} d\tau \le C \left(\frac{\varepsilon}{2} \right)^a e^\varepsilon \log \varepsilon^{-1}$$

$$\max_{0\le p\le r} \sum_{l=0}^{p} \frac{p!}{(p-l)!} a^{-l} \log^{p-l} \varepsilon^{-1} \int_{-\infty}^{\infty} \frac{(|a - i\tau| + \varepsilon) |\Gamma(i\tau - a + 1/2)|}{|\Gamma(a - i\tau + 1/2)||\tau + ia|^\alpha |\tau - x + ia|} d\tau$$

$$+C \left(\frac{\varepsilon}{2} \right)^a e^\varepsilon \max_{0\le p\le r} \sum_{l=0}^{p} \frac{p!}{(p-l)!} a^{-l} \log^{p-l} \varepsilon^{-1} \int_{-\infty}^{\infty} \frac{|\Gamma(i\tau - a + 1/2)|}{|\Gamma(a - i\tau + 1/2)||\tau + ia|^\alpha} d\tau$$

$$< C\varepsilon^a \left(\log \varepsilon^{-1} + \frac{1}{a} \right)^r \left(\log \varepsilon^{-1} + 1 \right) \to 0, \ \varepsilon \to 0+.$$

The latter integrals are indeed bounded since due to Stirling's formula for Gamma-functions [1]

$$\left| \frac{\Gamma(i\tau - a + 1/2)}{\Gamma(a - i\tau + 1/2)} \right| = O(|\tau|^{-2a}), \quad |\tau| \to \infty,$$

and therefore

$$\int_{-\infty}^{\infty} \frac{|\Gamma(i\tau - a + 1/2)|}{|\Gamma(a - i\tau + 1/2)||\tau + ia|^\alpha} d\tau = O(1) + O\left(\int_{|\tau|\ge M} \frac{d\tau}{|\tau|^{2a+\alpha}} \right) = O(1).$$

Meanwhile,

$$\int_{-\infty}^{\infty} \frac{(|a - i\tau| + \varepsilon) |\Gamma(i\tau - a + 1/2)|}{|\Gamma(a - i\tau + 1/2)||\tau + ia|^\alpha |\tau - x + ia|} d\tau = O(1)$$

$$+O\left(\int_{|\tau|\ge M > x_0} \frac{d\tau}{|\tau|^{2a+\alpha-1}(|\tau| - x_0)} \right) = O(1).$$

Thus we establish (2.9), which implies (2.4). Theorem 1 is proved.

3 Representation theorem

We define a complex analog of the Kontorovich-Lebedev transform (1.1) on distributions $f \in \mathcal{E}'(\mathbb{R})$ by

$$F(z) = \langle f, K_{i\cdot}(z) \rangle, \ z \in \mathbb{C}. \tag{3.1}$$

From representations (1.6) it follows that $K_{i\tau}(z)$ is infinitely differentiable with respect to $\tau \in \mathbb{R}$ and analytic with respect to z in the right half-plane Re$z > 0$. Thus $\mathcal{E}(\mathbb{R})$ contains $K_{i\tau}(z)$ for various values of the complex parameter z. We will prove that $F(z)$ is an analytic function in the right-half plane and satisfies there an appropriate estimate. Precisely, we have

Theorem 2. *For each $f \in \mathcal{E}'(\mathbb{R})$ $F(z)$ is analytic on the right half-plane* Re$z > 0$ *and its derivatives*

$$D_z^p F := \frac{(-1)^p}{2^p} \sum_{l=0}^{p} \binom{p}{l} \langle f, K_{i\cdot\ -p+2l}(z) \rangle, \ p \in \mathbb{N}_0. \tag{3.2}$$

Furthermore, the following estimates are true

$$|F(z)| = O\left(\log^{r+1}\left(\frac{1}{\text{Re}z}\right)\right), \ \text{Re}z \to 0+, \ r \in \mathbb{N}_0, \tag{3.3}$$

$$|F(z)| = O\left(\frac{e^{-\text{Re}z}}{\sqrt{\text{Re}z}}\right), \ \text{Re}z \to +\infty. \tag{3.4}$$

Proof. Let z be an arbitrary fixed point in the right half-plane with Re$z \geq y_0 > 0$. Taking a complex increment $\Delta z \neq 0$ such that $z, z+\Delta z$ belong to the right half-plane, we show that $F(z)$ admits a derivative in each inner half-plane. In view of our freedom to choose y_0 arbitrarily close to zero we will establish the analyticity of $F(z)$ on the right half-plane.

Indeed, invoking definition (3.1) of $F(z)$ we write

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} - \langle f, D_z K_{i\cdot}(z) \rangle = \langle f, \Psi_{\Delta z}(\cdot) \rangle, \tag{3.5}$$

where

$$\Psi_{\Delta z}(\tau) = \frac{1}{\Delta z}\left[K_{i\tau}(z+\Delta z) - K_{i\tau}(z)\right] - D_z K_{i\tau}(z).$$

Thus our aim is to verify that there exists an integer $r \in \mathbb{N}_0$ such that for any compact $T \in \mathbb{R}$

$$\max_{0 \leq p \leq r} \sup_{\tau \in T} |D_\tau^p \Psi_{\Delta z}(\tau)| \to 0, \ |\Delta z| \to 0. \tag{3.6}$$

To do this we employ again representations (1.6). Hence we put derivatives inside of the integral via its uniform convergence and after simple manipulations we arrive at the estimate

$$|D_\tau^p \Psi_{\Delta z}(\tau)| \leq \int_0^\infty t^p e^{-y_0 \cosh t} \frac{\left|e^{-\Delta z \cosh t} - 1 + \Delta z \cosh t\right|}{|\Delta z|} dt$$

$$= \int_0^\infty t^p e^{-y_0 \cosh t} \left| \sum_{n=2}^\infty \frac{(\Delta z)^{n-1} \cosh^n t}{n!} \right| dt \leq \int_0^\infty t^p e^{-y_0 \cosh t} \sum_{n=2}^\infty \frac{|\Delta z|^{n-1} \cosh^n t}{n!} dt.$$

The latter series can be taken out of the integral by virtue of the Levi theorem and we find

$$|D_\tau^p \Psi_{\Delta z}(\tau)| \leq \sum_{n=2}^\infty \frac{|\Delta z|^{n-1}}{n!} \left(\int_0^1 + \int_1^\infty \right) t^p e^{-y_0 e^t/2} e^{nt} dt$$

$$\leq \sum_{n=2}^\infty \frac{|\Delta z|^{n-1} e^n}{n!} + \sum_{n=2}^\infty \frac{|\Delta z|^{n-1}(n+1)}{(y_0/2)^{n+1}} \int_1^\infty e^{-t} t^p dt < C|\Delta z| \to 0, \ |\Delta z| \to 0.$$

Thus we establish (3.6). Hence by using an inductive argument we get the existence of p-th derivative with respect to z. Finally we invoke the relation (cf. [1, 3])

$$D_z^p K_\mu(z) = \frac{(-1)^p}{2^p} \sum_{l=0}^p \binom{p}{l} K_{\mu-p+2l}(z),$$

and we come out with (3.2).

In order to prove (3.3) we appeal to the fact that $F(z)$ is a continuous linear functional on countably multinormed space $\mathcal{E}(\mathbb{R})$. Hence there exists a positive constant C and a nonnegative integer r, which depend on f such that for $0 < \mathrm{Re} z < 1$ we derive

$$|F(z)| \leq C \ \max_{0\leq p\leq r} \sup_{\tau \in T} |D_\tau^p K_{i\tau}(z)| \leq C \max_{0\leq p\leq r} \int_0^\infty e^{-\mathrm{Re} z \cosh t} t^p dt$$

$$\leq C \max_{0\leq p\leq r} \left[\int_{\mathrm{Re} z/2}^{1/2} \log^p \left(\frac{2u}{\mathrm{Re} z} \right) \frac{du}{u} + \int_{1/2}^\infty e^{-u} \log^p \left(\frac{2u}{\mathrm{Re} z} \right) \frac{du}{u} \right]$$

$$= \left(\log^{r+1} \left(\frac{1}{\mathrm{Re} z} \right) \right), \ \mathrm{Re} z \to 0+.$$

Analogously, since

$$\int_0^\infty e^{-\mathrm{Re} z \cosh t} t^p dt = e^{-\mathrm{Re} z} \int_0^\infty e^{-2\mathrm{Re} z \sinh^2(t/2)} t^p dt$$

$$\leq e^{-\mathrm{Re} z} \int_0^\infty e^{-\mathrm{Re} z t^2/2} t^p dt = 2^{(p-1)/2} \Gamma \left(\frac{p+1}{2} \right) e^{-\mathrm{Re} z} (\mathrm{Re} z)^{-(p+1)/2},$$

we easily get (3.4). Theorem 2 is proved.

Now we are ready to prove the representation theorem for the Kontorovich-Lebedev transform (3.1) of real positive variable.

Theorem 3. *Let $f \in \mathcal{E}'(\mathbb{R})$ and $\varphi \in \mathcal{E}(\mathbb{R})$ satisfy conditions of Theorem 1 with $\alpha > 2$. Then*

$$\lim_{\varepsilon\to 0+} \left\langle \frac{1}{\pi^2}\tau \sinh \pi\tau \int_\varepsilon^\infty F(y)K_{i\tau}(y)\frac{dy}{y}, \varphi(\tau)\right\rangle = \langle f, \varphi\rangle. \tag{3.7}$$

Proof. Taking into account asymptotic behaviour (1.10), (1.13) of the Macdonald function, estimates (3.3), (3.4) and elementary inequality (see (1.6)) $K_{i\tau}(y) \le K_0(y)$ we conclude that the latter integral in (3.7) is absolutely and uniformly convergent with respect to $\tau \in \mathbb{R}$ for each $\varepsilon > 0$. Moreover, it can be treated as a Riemann improper integral. Furthermore, we show that (3.7) is a regular distribution if φ satisfies conditions of Theorem 1 with $\alpha > 2$. In fact, by using (1.5) and the evenness of φ we write

$$\frac{1}{\pi^2}\left\langle \tau \sinh \pi\tau \int_\varepsilon^\infty F(y)K_{i\tau}(y)\frac{dy}{y}, \varphi(\tau)\right\rangle = \frac{1}{2\pi}\int_{-\infty}^\infty \tau\varphi(\tau)\int_\varepsilon^\infty I_{i\tau}(y)F(y)\frac{dy}{y}d\tau. \tag{3.8}$$

Hence appealing to estimates (2.7), (3.3), (3.4) we easily verify the absolute convergence of the iterated integral in the right-hand side of (3.8). Precisely, we obtain

$$\int_{-\infty}^\infty |\tau\varphi(\tau)| \int_\varepsilon^\infty |I_{i\tau}(y)F(y)|\frac{dy}{y}d\tau < C\int_{-\infty}^\infty \frac{|\tau\varphi(\tau)|}{|\Gamma(i\tau+1/2)|}d\tau \int_\varepsilon^\infty e^y|F(y)|\frac{dy}{y}$$

$$< C\int_\varepsilon^\infty \frac{dy}{y^{3/2}}\left[\int_{|\tau|<M}\frac{|\tau\varphi(\tau)|}{|\Gamma(i\tau+1/2)|}d\tau + \int_{|\tau|>M}\frac{d\tau}{|\tau|^{\alpha-1}}\right] < \infty, \ \alpha > 2.$$

Thus the left-hand side of (3.8) is a regular distribution and the corresponding integral in its right-hand side can be approximated by Riemann's sums. Therefore invoking (3.1) we have

$$\frac{1}{2\pi}\int_{-\infty}^\infty \tau\varphi(\tau)\int_\varepsilon^\infty I_{i\tau}(y)F(y)\frac{dy}{y}d\tau = \lim_{N\to\infty}\frac{1}{2\pi}\sum_{m=0}^N x_m\varphi(x_m)\int_\varepsilon^\infty I_{ix_m}(y)F(y)\frac{dy}{y}$$

$$= \lim_{N\to\infty}\left\langle f_x, \frac{1}{2\pi}\sum_{m=0}^N \tau_m\varphi(\tau_m)\int_\varepsilon^\infty I_{i\tau_m}(y)K_{ix}(y)\frac{dy}{y}\right\rangle. \tag{3.9}$$

But in the meantime we will establish that when $N\to\infty$

$$\frac{1}{2\pi}\sum_{m=0}^N \tau_m\varphi(\tau_m)\int_\varepsilon^\infty I_{i\tau_m}(y)K_{ix}(y)\frac{dy}{y} = \varphi_{N,\varepsilon}(x)$$

converges in $\mathcal{E}(\mathbb{R})$ to $\varphi_\varepsilon(x)$, which, in turn, is defined by (2.5). Indeed, calling proofs of Theorems 1, 2 we find (cf. (1.16))

$$\max_{0\le p\le r}\sup_{x\in[-x_0,x_0]}|D_x^p(\varphi_{N,\varepsilon}(x) - \varphi_\varepsilon(x))| = \max_{0\le p\le r}\sup_{x\in[-x_0,x_0]}\left|\int_{|\tau|>N} D_x^p\mathcal{K}_\varepsilon(\tau,x)\varphi(\tau)d\tau\right|$$

$$< C \int_{\varepsilon}^{\infty} \frac{dy}{y^{3/2}} \int_{|\tau|>N} \frac{d\tau}{|\tau|^{\alpha-1}} \to 0, \ N \to \infty, \ \alpha > 2.$$

Combining with (3.9) we get

$$\lim_{N\to\infty} \left\langle f_x, \frac{1}{2\pi} \sum_{m=0}^{N} \tau_m \varphi(\tau_m) \int_{\varepsilon}^{\infty} I_{i\tau_m}(y) K_{ix}(y) \frac{dy}{y} \right\rangle = \langle f, \varphi_\varepsilon \rangle .$$

Hence appealing to Theorem 1 we arrive at the representation (3.7). Theorem 3 is proved.

As a corollary this immediately yields the uniqueness property for the Kontorovich-Lebedev transform (3.1).

Corollary 1. *If* $F(y) = G(y), \ y > 0$, *where* F, G *are Kontorovich-Lebedev transforms of* f *and* g, *respectively, then* $f = g$ *in the sense of equality in* $\mathcal{E}'(\mathbb{R})$ *for all* φ *from Theorem 1.*

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S.B.Yakubovich
Department of Pure Mathematics,
Faculty of Sciences,

University of Porto,
Campo Alegre st., 687
4169-007 Porto
Portugal
E-Mail: syakubov@fc.up.pt