A class of polynomials and discrete transformations associated with the Kontorovich- Lebedev operators

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Abstract

We consider a class of polynomials related to the kernel $K_{i\tau}(x)$ of the Kontorovich-Lebedev transformation. Algebraic and differential properties are investigated and integral representations are derived. We draw a parallel and establish a relationship with Bernoulli's and Euler's numbers and polynomials. Finally, as an application we invert a discrete transformation with the introduced polynomials as the kernel, basing on a decomposition of Taylor's series in terms of the Kontorovich-Lebedev operator.

Keywords: *Polynomials, Taylor series, Bernoulli and Euler numbers and polynomials, Kontorovich-Lebedev transform, modified Bessel function, Fourier transform*

AMS subject classification: 11B68, 12E10, 33C10, 44A15, 44A45

1 Introduction

As it is known (see [1, Vol. II]), the modified Bessel function of the second kind $K_{i\tau}(x)$ of the argument $x > 0$ and the pure imaginary subscript $i\tau$ can be defined by the cosine Fourier transform

$$K_{i\tau}(x) = \int_0^\infty e^{-x\cosh u} \cos \tau u \, du, \ x > 0, \tag{1.1}$$

and reciprocally by the inversion formula we immediately obtain

$$e^{-x\cosh u} = \frac{2}{\pi} \int_0^\infty K_{i\tau}(x) \cos \tau u \, d\tau. \tag{1.2}$$

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This function is real-valued and represents a kernel of the Kontorovich-Lebedev integral expansion [2], [5], [6]

$$f(\tau) = \frac{2}{\pi^2} \sinh \pi\tau \int_0^\infty \frac{K_{i\tau}(x)}{x} \int_0^\infty y K_{iy}(x) f(y) dy dx, \tag{1.3}$$

which generates the reciprocal Kontorovich-Lebedev operators

$$G(x) = \int_0^\infty y K_{iy}(x) f(y) dy,$$

$$f(\tau) = \frac{2}{\pi^2} \sinh \pi\tau \int_0^\infty \frac{K_{i\tau}(x)}{x} G(x) dx.$$

Moreover, it is an eigenfunction for the differential operator

$$\mathcal{A} \equiv x^2 - x\frac{d}{dx} x \frac{d}{dx}, \tag{1.4}$$

i.e. we have

$$\mathcal{A} K_{i\tau}(x) = \tau^2 \ K_{i\tau}(x). \tag{1.5}$$

Denoting by $\mathcal{A}^n, \ n \in \mathbb{N}, \ \mathcal{A}^0 = I$ the n-th iteration of the operator (1.4) we find from (1.5)

$$\mathcal{A}^n K_{i\tau}(x) = \tau^{2n} \ K_{i\tau}(x), \ n \in \mathbb{N}. \tag{1.6}$$

The modified Bessel function has the asymptotic behaviour with respect to x [1, Vol. II], [6]

$$K_\nu(x) = \left(\frac{\pi}{2x}\right)^{1/2} e^{-x}[1 + O(1/x)], \qquad x \to +\infty, \tag{1.7}$$

$$K_\nu(x) = O\left(x^{-|\text{Re}\nu|}\right), \ x \to 0, \tag{1.8}$$

$$K_0(x) = O(-\log x), \ x \to 0 \tag{1.9}$$

and with respect to the index $\nu = i\tau$

$$K_{i\tau}(x) = O\left(\frac{e^{-\pi\tau/2}}{\sqrt{\tau}}\right), \ \tau \to +\infty. \tag{1.10}$$

It has the following inequality

$$\left|\frac{\partial^m K_{i\tau}(x)}{\partial x^m}\right| < 2 \ e^{-\delta\tau} K_m(x \cos \delta), \ x > 0, \ \tau > 0, \ m = 0, 1, \ldots \tag{1.11}$$

for each $\delta \in \left[0; \frac{\pi}{2}\right)$.

Returning to (1.2) let us differentiate the integral $2n$ times with respect to u. Hence via the absolute and uniform convergence by $u \geq 0$ we come out with the formula

$$\frac{\partial^{2n} e^{-x\cosh u}}{\partial u^{2n}} = (-1)^n \frac{2}{\pi} \int_0^\infty K_{i\tau}(x)\tau^{2n} \cos \tau u \, d\tau, \; x > 0. \tag{1.12}$$

Therefore it yields

$$\lim_{u\to 0} \frac{\partial^{2n} e^{-x\cosh u}}{\partial u^{2n}} = (-1)^n \frac{2}{\pi} \int_0^\infty K_{i\tau}(x)\tau^{2n} \, d\tau. \tag{1.13}$$

Multiplying (1.13) by e^x it gives

$$\lim_{u\to 0} \frac{\partial^{2n} e^{-2x\sinh^2(u/2)}}{\partial u^{2n}} = (-1)^n \frac{2}{\pi} e^x \int_0^\infty K_{i\tau}(x)\tau^{2n} \, d\tau. \tag{1.14}$$

We denote the right-hand side of (1.14) by $p_n(x)$ and we will show in the next section, that it represents a polynomial of degree $n \in \mathbb{N}_0$. It is not difficult to see from (1.2) and (1.14) that, for instance,

$$p_0(x) = 1, \quad p_1(x) = -x.$$

We will study various algebraic and differential properties of this class of polynomials and will find its relationship with the Bernoulli and Euler numbers and polynomials [1, Vol. I], [3]. As an application of these results we will invert the following discrete transformation

$$f(x) = e^{-x} \sum_{n=1}^{\infty} c_n p_n(x), \quad x \geq 0, \tag{1.15}$$

i.e. we will prove an inversion formula for coefficients c_n from the weighted l_1-space of sequences. To do this, we will use an expansion of the corresponding class of Taylor's series in terms of the Kontorovich-Lebedev operator.

2 Algebraic and differential properties of polynomials $p_n(x)$

We begin, proving the representation of $p_n(x)$ in the form

$$p_n(x) = (-1)^n e^x \mathcal{A}^n \; e^{-x}, \; n \in \mathbb{N}_0. \tag{2.1}$$

To do this, we appeal to (1.2), (1.6) and use the absolute and uniform convergence of the integral

$$\int_0^\infty \frac{\partial^m K_{i\tau}(x)}{\partial x^m} \tau^{2n} \, d\tau, \; m = 0, 1, 2, \ldots, \tag{2.2}$$

with respect to $x \geq x_0 > 0$, which is easy to verify taking into account the inequality (1.11) with $\left(0; \frac{\pi}{2}\right)$ and continuity of all derivatives in (2.2) as functions of two variables. Hence we motivate an interchange of the order of the integral and operator $\mathcal{A}^n$ in (1.14)

$$p_n(x) = (-1)^n e^x \frac{2}{\pi} \int_0^\infty \mathcal{A}^n K_{i\tau}(x)\, d\tau$$

$$= (-1)^n e^x \frac{2}{\pi} \mathcal{A}^n \int_0^\infty K_{i\tau}(x)\, d\tau = (-1)^n e^x \mathcal{A}^n\, e^{-x},$$

which leads to (2.1). Furthermore, with the Schwarz inequality, integral representation of Bernoulli numbers $B_{4n}, n = 1, 2, \ldots$ [1, Vol. I], relation (2.16.51.8) [4, Vol. II] we derive the estimate

$$|p_n(x)| \leq \frac{2}{\pi} e^x \int_0^\infty |K_{i\tau}(x)| \tau^{2n}\, d\tau \leq \frac{2}{\pi} e^x \left(\int_0^\infty \tau \sinh\left(\frac{\pi\tau}{2}\right) K^2_{i\tau}(x)\, d\tau \right)^{1/2}$$

$$\times \left(\int_0^\infty \frac{\tau^{4n-1}}{\sinh(\pi\tau/2)} d\tau \right)^{1/2} = 2^{2n-3/4} (2^{4n} - 1)^{1/2} \sqrt{\frac{-B_{4n}}{\pi n}}$$

$$\times e^x \left(x K_1(x\sqrt{2}) \right)^{1/2}, \ x > 0. \tag{2.3}$$

A direct consequence of the relation (1.14) is a definition of the generating function for polynomials $p_n(x)$. Precisely we have the series expansion

$$e^{-2x\sinh^2(t/2)} = \sum_{n=0}^\infty \frac{p_n(x)}{(2n)!}\, t^{2n}. \tag{2.4}$$

Hence letting $x = 0$ in (2.4) we get

$$p_n(0) = 0, \quad n = 1, 2, \ldots. \tag{2.5}$$

Further, taking (2.1) and definition (1.4) of the operator $\mathcal{A}$ we easily derive the following differential recurrence relation for this system of polynomials

$$p_{n+1}(x) = x^2 p''_n(x) + x(1 - 2x) p'_n(x) - x p_n(x), \ n = 0, 1, 2, \ldots\ . \tag{2.6}$$

Hence it shows by the method of mathematical induction that p_n is indeed a polynomial of degree $\leq n$. In particular, we have

$$p_2(x) = 3x^2 - x, \quad p_3(x) = -15x^3 + 15x^2 - x.$$

But denoting by a_n the leading coefficient of

$$p_n(x) = \sum_{k=0}^n a_k x^k, \ n \in \mathbb{N}_0$$

we find its value appealing to equation (2.6). We obtain

$$a_n = (-1)^n (2n-1)!!, \quad n = 1, 2, \dots \quad . \tag{2.7}$$

So we get that p_n is in fact a polynomial of degree n. As we could see above, $a_0 = 0$ for all $p_n(x), n \in \mathbb{N}$. From (2.1) we conclude that all coefficients of $p_n(x)$ are real numbers. Therefore if z_0 is a zero of p_n, then $\bar{z}_0$ is a zero as well.

According to [3] the generalized Bernoulli polynomials are defined by the following generating function

$$\frac{t^a e^{xt}}{(e^t - 1)^a} = \sum_{k=0}^{\infty} B_k^{(a)}(x) \frac{t^k}{k!}, \quad |t| < 2\pi.$$

The generalized Bernoulli numbers are defined as $B_k^{(a)} = B_k^{(a)}(0)$. When $a = 1$ we arrive at the classical Bernoulli numbers and polynomials [1, Vol. I], [3]. We appeal to the following relation for the Bernoulli numbers [1, Vol. I]

$$-B_{4n} = \frac{2(4n)!}{(2\pi)^{4n}} \zeta(4n),$$

where $\zeta(z)$ denotes the Riemann zeta-function in order to estimate the behaviour of the sequence in the right-hand side of inequality (2.3) when $n \to \infty$. So with the use of Stirling's formula for factorials [1, Vol. I] we find

$$\frac{|p_n(x)|}{(2n)!} \leq \frac{2^{2n-3/4}(2^{4n}-1)^{1/2}}{(2n)!\sqrt{\pi n}} \sqrt{-B_{4n}}\; e^x \left(x K_1(x\sqrt{2}) \right)^{1/2}$$
$$= O\left(\left(\frac{4}{\pi} \right)^{2n} \right) \; e^x \left(x K_1(x\sqrt{2}) \right)^{1/2}, \; n \to \infty$$

and series (2.4) converges absolutely and uniformly on any compact set of $x \in [x_0,\ X_0] \subset \mathbb{R}_+$ and t from the interval $|t| \leq t_0 < \frac{\pi}{4}$. Let us differentiate series (2.4) with respect to x. The motivation comes via the absolute and uniform convergence of the series with a derivative of $p_n(x)$ because (see (1.11), (1.14))

$$\frac{|p'(x)|}{(2n)!} \leq \frac{1}{(2n)!} \left[|p(x)| + \frac{2}{\pi} e^x \int_0^\infty \left| \frac{\partial K_{i\tau}(x)}{\partial x} \right| \tau^{2n}\, d\tau \right]$$
$$< \frac{4e^x}{\pi \delta^{2n+1}} \left[K_0(x\cos\delta) + K_1(x\cos\delta) \right], \; \delta \in \left(0, \frac{\pi}{2}\right).$$

Consequently, series $\sum_{n=0}^{\infty} \frac{p_n'(x)}{(2n)!}\, t^{2n}$ converges absolutely and uniformly on any compact set of $x \in [x_0,\ X_0] \subset \mathbb{R}_+$ and t such that $|t| \leq t_0 < \delta \leq \frac{\pi}{4}$. Therefore the differentiation through in (2.4) with respect to x drives us at the equality

$$(1 - \cosh t)\; e^{-2x\sinh^2(t/2)} = \sum_{n=0}^{\infty} \frac{p_n'(x)}{(2n)!}\, t^{2n}. \tag{2.8}$$

Hence decomposing the left-hand side of (2.8) as a product of series and equating coefficients in front of t^{2n} we obtain the following recurrence relation

$$p'_n(x) = -\sum_{k=0}^{n-1} \binom{2n}{2k} p_k(x), \ n \in \mathbb{N}. \tag{2.9}$$

Putting $x = 0$ in (2.9) and using values $p_0(0) = 1, \ p_k(0) = 0, \ k \in \mathbb{N}$ we obtain $p'_n(0) = -1, \ n \in \mathbb{N}$. Analogously, a differentiation with respect to t in (2.4) gives the equality

$$-x \sinh t \ e^{-2x \sinh^2(t/2)} = \sum_{n=1}^{\infty} \frac{p_n(x)}{(2n-1)!} \ t^{2n-1}. \tag{2.10}$$

By the same manipulations we arrive at the relation

$$p_{n+1}(x) = -x \sum_{k=0}^{n} \binom{2n+1}{2k} p_k(x), \ n \in \mathbb{N}_0. \tag{2.11}$$

Differentiating through in (2.11), employing (2.9) and using simple relations for binomial coefficients we come out with the identity

$$x \sum_{k=0}^{n} \binom{2n+1}{2k} p'_k(x) = \sum_{k=1}^{n} \binom{2n+1}{2k-1} p_k(x). \tag{2.12}$$

Returning to (2.1) we have

$$p_{m+n}(x) = (-1)^{m+n} e^x \mathcal{A}^{m+n} \ e^{-x}, \ m, n \in \mathbb{N}_0.$$

On the other hand, with (1.6) and (1.14) we deduce

$$p_{m+n}(x) = (-1)^n e^x \mathcal{A}^n \ \left(e^{-x} p_m(x)\right) = (-1)^m e^x \mathcal{A}^m \ \left(e^{-x} p_n(x)\right)$$

and therefore

$$\mathcal{A}^n \ \left(e^{-x} p_m(x)\right) = (-1)^{n+m} \mathcal{A}^m \ \left(e^{-x} p_n(x)\right). \tag{2.13}$$

Taking $m \geq n$ we obtain from (2.13)

$$p_m(x) = (-1)^{m+n} e^x \mathcal{A}^{m-n} \ \left(e^{-x} p_n(x)\right).$$

In particular, it gives the relations

$$p_{2n}(x) = (-1)^n e^x \mathcal{A}^n \ \left(e^{-x} p_n(x)\right),$$

$$p_{2n+1}(x) = (-1)^{n+1} e^x \mathcal{A}^{n+1} \ \left(e^{-x} p_n(x)\right). \tag{2.14}$$

Now we will write the expression for $p_n(x)$ in an explicit form. Taking the generating function (2.4) we decompose it into the series to have the equality

$$\sum_{k=0}^{\infty} \frac{(-2x\sinh^2(t/2))^k}{k!} = \sum_{n=0}^{\infty} \frac{p_n(x)}{(2n)!}\, t^{2n}.$$

Let us introduce the sequence of functions $\varphi_k(t),\ k \in \mathbb{N}_0$

$$\varphi_k(t) = \begin{cases} t^{-2k}\sinh^{2k}(t/2), & \text{if } t \neq 0, \\ \frac{1}{4^k}, & \text{if } t = 0. \end{cases}$$

It is clear, that $\varphi_k(t)$ is entire for each k and be represented by the Maclaurin series

$$\varphi_k(t) = \sum_{m=0}^{\infty} \frac{\varphi_k^{(2m)}(0)}{(2m)!} t^{2m}.$$

Substituting into (2.14), its left-hand side becomes

$$\sum_{k=0}^{\infty} \frac{(-2xt^2)^k}{k!} \sum_{m=0}^{\infty} \frac{\varphi_k^{(2m)}(0)}{(2m)!} t^{2m} = \sum_{k=0}^{\infty} \frac{(-2xt^2)^k}{k!} \sum_{n=k}^{\infty} \frac{\varphi_k^{(2(n-k))}(0)}{(2(n-k))!} t^{2(n-k)}$$

$$= \sum_{n=0}^{\infty} t^{2n} \sum_{k=0}^{n} A_{n,k} \frac{(-2x)^k}{k!(2(n-k))!}, \quad A_{n,k} = \varphi_k^{(2(n-k))}(0).$$

Hence equating with the right-hand side of (2.14) and invoking simple relations for binomial coefficients, we come out with the representation for $p_n(x)$

$$p_n(x) = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{n} A_{n,k} \binom{2n}{2k} \Gamma\left(k + \frac{1}{2}\right) (-8x)^k,$$

where $\Gamma(z)$ is Euler's gamma-function [1, Vol. I].

3 A relationship with Bernoulli's and Euler's numbers and polynomials

We call again integral (1.14) and the Kontorovich-Lebedev integral expansion (1.3). It is not difficult to verify the validity conditions of this expansion for the function $f(\tau) = \tau^{2n-1},\ n \in \mathbb{N}$ (see [2], [5], [6]). Therefore we find the equality

$$\frac{\tau^{2n-1}}{\sinh \pi\tau} = \frac{(-1)^n}{\pi} \int_0^{\infty} e^{-x} K_{i\tau}(x) p_n(x) \frac{dx}{x},\ n \in \mathbb{N}. \tag{3.1}$$

Hence letting $\tau \to 0$ in (3.1), we obtain

$$\int_0^\infty e^{-x} K_0(x) p_n(x) \frac{dx}{x} = 0, \ n = 2, 3, \dots \ .$$

Integrating through in (3.1) with respect to τ over $\mathbb{R}_+$, we change the order of integration in its right-hand side by Fubini's theorem via the absolute and uniform convergence of the iterated integral (use (1.11)). Then after calculation of the inner integral via (1.2), we invoke the relation (see (2.3)) [1, Vol. I]

$$\int_0^\infty \frac{\tau^{2n-1}}{\sinh \pi\tau} d\tau = (-1)^{n+1} \frac{(2^{2n}-1) B_{2n}}{2n} \tag{3.2}$$

to get the following representation for even Bernoulli numbers B_{2n}

$$B_{2n} = \frac{n}{1-2^{2n}} \int_0^\infty e^{-2x} p_n(x) \frac{dx}{x}, \ n \in \mathbb{N}. \tag{3.3}$$

On the other hand, making use the Parseval equality for the Kontorovich-Lebedev transform [6] we find from (3.1)

$$\int_0^\infty e^{-2x} p_n^2(x) \frac{dx}{x} = 2 \int_0^\infty \frac{\tau^{4n-1}}{\sinh \pi\tau} d\tau. \tag{3.4}$$

Consequently, combining with (3.2) we establish the following integral representation of Bernoulli numbers B_{4n}

$$B_{4n} = \frac{2n}{1-2^{4n}} \int_0^\infty e^{-2x} p_n^2(x) \frac{dx}{x}. \tag{3.5}$$

Moreover, a comparison with (3.3) gives us a surprising equality

$$\int_0^\infty e^{-2x} p_n^2(x) \frac{dx}{x} = \int_0^\infty e^{-2x} p_{2n}(x) \frac{dx}{x}, \ n \in \mathbb{N}. \tag{3.6}$$

But we will prove something more. Namely, we will show that

$$\int_0^\infty e^{-2x} p_n(x) p_m(x) \frac{dx}{x} = (-1)^{m+n} \int_0^\infty e^{-2x} p_{m+n}(x) \frac{dx}{x}, \ n, m \in \mathbb{N}_0, \ n^2 + m^2 \neq 0. \tag{3.7}$$

In order to proceed this, we suppose that for instance, $m \in \mathbb{N}, \ n \in \mathbb{N}_0$. Taking (2.1) we substitute it in the left-hand side of (3.7). Thus

$$\int_0^\infty e^{-2x} p_n(x) p_m(x) \frac{dx}{x} = (-1)^n \int_0^\infty e^{-x} p_m(x) \mathcal{A}^n e^{-x} \frac{dx}{x}. \tag{3.8}$$

Since $p_m(0) = 0,\ m \in \mathbb{N},\ \mathcal{A}^k\left(e^{-x}p_m(x)\right) = (-1)^k e^{-x} p_{k+m}(x),\ k \in \mathbb{N}_0$ (see (2.5), (2.13)), we appeal to the definition (1.4) of the operator $\mathcal{A}$ to apply it successively in the right-hand side of (3.8), that presumes to involve also an integration by parts and an elimination of the outintegrated terms. Hence we rewrite (3.8) as a chain of equalities

$$\int_0^\infty e^{-2x} p_n(x) p_m(x) \frac{dx}{x} = (-1)^{n+1} \int_0^\infty e^{-x} p_{m+1}(x) \mathcal{A}^{n-1} e^{-x} \frac{dx}{x}$$

$$= (-1)^{n+2} \int_0^\infty e^{-x} p_{m+2}(x) \mathcal{A}^{n-2} e^{-x} \frac{dx}{x} = \cdots = (-1)^{m+n} \int_0^\infty e^{-2x} p_{m+n}(x) \frac{dx}{x},$$

which yields finally (3.7). Moreover, it guarantees immediately

$$\int_0^\infty e^{-2x} p_n(x) p_m(x) \frac{dx}{x} = \int_0^\infty e^{-2x} p_{n-k}(x) p_{m+k}(x) \frac{dx}{x} \tag{3.9}$$

for any $k = 0, 1, \ldots,\ n$. Furthermore, with (3.3) we derive

$$B_{2(m+n)} = \frac{(-1)^{m+n}(m+n)}{1 - 2^{2(m+n)}} \int_0^\infty e^{-2x} p_{n-k}(x) p_{m+k}(x) \frac{dx}{x}. \tag{3.10}$$

A connection between Bernoulli's polynomials $B_{2n+1}(x)$ and polynomials $p_n(x)$ can be established on the following way. We use integral representation of $B_{2n+1}(-x/2)$ in [I, Vol. I]

$$B_{2n+1}\left(-\frac{x}{2}\right) = \frac{(-1)^n(2n+1)}{2^{2n+1}} \sin \pi x$$

$$\times \int_0^\infty \frac{\tau^{2n}}{\cosh \pi\tau - \cos \pi x} d\tau,\ -2 < \mathrm{Re}\ x < 0,\ n \in \mathbb{N}_0 \tag{3.11}$$

and relation (2.16.33.2) in [4, Vol. II] to find

$$\frac{1}{\cosh \pi\tau - \cos \pi x} = \frac{2}{\pi^2} \int_0^\infty K_{i\tau}(y) K_{x+1}(y) dy,\ -2 < \mathrm{Re}\ x < 0. \tag{3.12}$$

Substituting (3.12) into (3.11), we invert the order of integration easily motivating this by inequality (1.11). Then we make use (1.14) and come out with the following representation

$$B_{2n+1}\left(-\frac{x}{2}\right) = \frac{(2n+1)}{2^{2n+1}\pi} \sin \pi x \int_0^\infty K_{x+1}(y) e^{-y} p_n(y) dy. \tag{3.13}$$

Considering pure imaginary subscripts $x + 1 = i\tau$ we rewrite (3.13) in the form

$$B_{2n+1}\left(\frac{1 - i\tau}{2}\right) = \frac{2n+1}{2^{2n+1}\ \pi i}\ \sinh \pi\tau \int_0^\infty K_{i\tau}(y) e^{-y} p_n(y) dy.$$

Reciprocally, by the Kontorovich-Lebedev integral (1.3) we establish the representation of the polynomials $p_n(x)$ as the Kontorovich-Lebedev transform of the Bernoulli polynomials, namely

$$p_n(x) = -\frac{2^{2(n+1)}\ e^x}{(2n+1)\ \pi i}\int_0^\infty \tau\frac{K_{i\tau}(x)}{x}B_{2n+1}\left(\frac{1-i\tau}{2}\right)d\tau,\ n\in\mathbb{N}_0. \tag{3.14}$$

Hence applying again the Parseval equality for the Kontorovich-Lebedev transform [5], [6] we deduce the identity

$$\int_0^\infty \left|B_{2n+1}\left(\frac{1-i\tau}{2}\right)\right|^2 \frac{\tau}{\sinh \pi\tau}d\tau = \frac{(2n+1)^2}{2^{4n+3}}\int_0^\infty e^{-2x}p_n^2(x)\frac{dx}{x},\ n\in\mathbb{N}. \tag{3.15}$$

Moreover, with representation (3.5) we get the value of the integral in the left-hand side of (3.15). Indeed, we have the formula

$$\int_0^\infty \left|B_{2n+1}\left(\frac{1-i\tau}{2}\right)\right|^2 \frac{\tau}{\sinh \pi\tau}d\tau = \frac{(2n+1)^2(2^{-4n}-1)}{16n}B_{4n},\ n\in\mathbb{N}.$$

Let us establish a connection between $p_n(x)$ and Euler's numbers, which are defined as usual by the series [1, Vol. I]

$$\frac{1}{\cosh t} = \sum_{n=0}^\infty E_{2n}\frac{t^{2n}}{(2n)!},\ |t|<\frac{\pi}{2}. \tag{3.16}$$

To do this, the estimate from Section 2

$$e^{-x}\frac{|p_n(x)|}{(2n)!} \le A\ \left(\frac{4}{\pi}\right)^{2n}(xK_1(x\sqrt{2}))^{1/2},\ x>0$$

where $A>0$ is an absolute constant, applies to motivate the convergence of the following integral for $|t|\le t_0<\frac{\pi}{4}$

$$\int_0^\infty e^{-x}\sum_{n=0}^\infty |p_n(x)|\frac{|t|^{2n}}{(2n)!}\ dx \le A\sum_{n=0}^\infty\left(\frac{4|t|}{\pi}\right)^{2n}\int_0^\infty (xK_1(x\sqrt{2}))^{1/2}dx$$

$$\le A\sum_{n=0}^\infty\left(\frac{4|t_0|}{\pi}\right)^{2n}\int_0^\infty (xK_1(x\sqrt{2}))^{1/2}dx<\infty,$$

since the latter series and integral are convergent (see (1.7), (1.8)). Therefore, we can integrate through in the equality (2.4) with respect to x over $\mathbb{R}_+$ at least for $|t|\le t_0<\frac{\pi}{4}$, multiplying first both of its sides by e^{-x}. Consequently, we obtain

$$\frac{1}{\cosh t} = \sum_{n=0}^\infty\frac{t^{2n}}{(2n)!}\int_0^\infty e^{-x}p_n(x)dx,\ |t|\le t_0<\frac{\pi}{4}.$$

Combining with (3.16) we find the representation of Euler's numbers in terms of polynomials $p_n(x)$

$$E_{2n} = \int_0^\infty e^{-x} p_n(x) dx, \quad n \in \mathbb{N}_0. \tag{3.17}$$

Hence taking (2.11), we divide it by x and integrate through with respect to x over $\mathbb{R}_+$ to deduce the relation

$$-\int_0^\infty e^{-x} p_{n+1}(x) \frac{dx}{x} = \sum_{k=0}^{n} \binom{2n+1}{2k} E_{2k}.$$

Further, the basic relation for binomial coefficients

$$\binom{2n+1}{2k} = \binom{2n}{2k} + \binom{2n}{2k-1}$$

and recurrence formula for Euler's numbers [1, Vol. I]

$$\sum_{k=0}^{n} \binom{2n}{2k} E_{2k} = 0$$

yield the equality

$$\int_0^\infty e^{-x} p_{n+1}(x) \frac{dx}{x} = -\sum_{k=1}^{n} \binom{2n}{2k-1} E_{2k}.$$

Analogously, from identity (2.12) with the use of the integration by parts we derive

$$(n+1) \sum_{k=1}^{n} \binom{2n+1}{2k} \frac{E_{2k}}{n-k+1} = \sum_{k=1}^{n} \binom{2n+1}{2k} S_k,$$

where

$$S_k = \int_0^\infty e^{-x} x p_k(x) \ dx.$$

Further, we introduce some kind of Euler's type polynomials basing on the equality, which can be obtained from (2.4) by multiplication on e^{-x} and integration with respect to x over (x, ∞). Precisely, we get

$$\frac{1}{\cosh t} \ e^{-2x \sinh^2(t/2)} = \sum_{n=0}^{\infty} q_n(x) \frac{t^{2n}}{(2n)!}, \ |t| \leq t_0 < \frac{\pi}{4}, \tag{3.18}$$

where

$$q_n(x) = e^x \int_x^\infty e^{-x} p_n(x) dx = (-1)^n e^x \int_x^\infty \mathcal{A}^n e^{-x} \ dx. \tag{3.19}$$

Evidently from (3.17) we have $q_n(0) = E_{2n}$. Moreover, integrating by parts in (3.19), we find the representation of polynomials q_n

$$q_n(x) = \sum_{k=0}^{n} p_n^{(k)}(x),\ n \in \mathbb{N}_0, \tag{3.20}$$

where $p_n^{(k)}(x)$ is the k -th derivative of $p_n(x)$. For instance,

$$q_0(x) = 1, \quad q_1(x) = -(x+1),$$

$$q_2(x) = 3x^2 + 5x + 5, \quad q_3(x) = -15x^3 - 30x^2 - 61x - 61.$$

Making a product of series (3.16) and (2.4) in the left-hand side of (3.18), we equate coefficients in front of t^{2n} to obtain another representation of polynomials $q_n(x)$

$$q_n(x) = \sum_{k=0}^{n} E_{2k} \binom{2n}{2k} p_{n-k}(x). \tag{3.21}$$

From (3.20) we derive

$$q_n'(x) = \sum_{k=0}^{n} p_n^{(k+1)}(x) = \sum_{k=1}^{n} p_n^{(k)}(x) = q_n(x) - p_n(x).$$

Thus, $p_n(x) = q_n(x) - q_n'(x)$ and since $p_n(0) = 0, n \in \mathbb{N}$, we find $q_n'(0) = E_{2n}$. Further, differentiating again, we obtain

$$p_n'(x) = q_n'(x) - q_n''(x)$$

and the value $p_n'(0) = -1, n = 1, 2, \ldots$ (see above) implies

$$q_n''(0) = E_{2n} + 1.$$

Finally we note that the leading coefficients of polynomials $p_n,\ q_n$ are the same.

4 A discrete transformation and relations with the Kontorovich- Lebedev operators

Let us consider the discrete transformation (1.15). We will prove that this series converges absolutely and uniformly for all $x \geq 0$, if

$$\sum_{n=1}^{\infty} |c_n| (2n)! \left(\frac{4}{\pi}\right)^{2n} < \infty. \tag{4.1}$$

Therefore $f(x)$ will be bounded continuous on $[0,+\infty)$ and $f(0)=0$ since $p_n(0)=0,\ n\in\mathbb{N}$.

In fact, from the asymptotic properties and integral representation of the modified Bessel function $K_1(x)$ it is not difficult to establish the inequality $xK_1(x)<1,\ x>0$. In the limit case $x\to 0+$ we have $xK_1(x)\to 1$. Therefore (see the previous section) for all $x\ge 0$

$$e^{-x}|p_n(x)|<A\ (2n)!\left(\frac{4}{\pi}\right)^{2n}$$

where $A>0$ is an absolute constant, and since series (4.1) is convergent we establish the absolute and uniform convergence of series (1.15) for all $x\ge 0$. Thus its sum is a bounded continuous function on $[0,+\infty)$ vanishing in zero. Applying the operator of the Kontorovich-Lebedev transform to a partial sum $f_N(x)$ of (1.15) and using (2.1) we deduce

$$\sinh\pi\tau\int_0^\infty K_{i\tau}(x)f_N(x)\frac{dx}{x}=\sinh\pi\tau\int_0^\infty K_{i\tau}(x)e^{-x}\sum_{n=1}^N c_n\ p_n(x)\frac{dx}{x}$$

$$=\sinh\pi\tau\sum_{n=1}^N(-1)^n c_n\int_0^\infty K_{i\tau}(x)\left(\mathcal{A}^n\ e^{-x}\right)\frac{dx}{x},\ \tau>0. \tag{4.2}$$

Since $\mathcal{A}\ e^{-x}=xe^{-x}$ and due to relation (2.16.6.4) in [4, Vol. II]

$$\int_0^\infty K_{i\tau}(x)\ e^{-x}dx=\frac{\pi\tau}{\sinh\pi\tau},$$

the right-hand side of (4.2) can be written in the form

$$\sinh\pi\tau\sum_{n=1}^N(-1)^n c_n\int_0^\infty K_{i\tau}(x)\left(\mathcal{A}^n\ e^{-x}\right)\frac{dx}{x}=-\pi\tau c_1$$

$$+\sinh\pi\tau\sum_{n=2}^N(-1)^n c_n\int_0^\infty K_{i\tau}(x)\left(\mathcal{A}^n\ e^{-x}\right)\frac{dx}{x}. \tag{4.3}$$

But employing (1.4), (1.5) we take successfully the operator $\mathcal{A}$ to write the latter integral as follows

$$\int_0^\infty K_{i\tau}(x)\left(\mathcal{A}^n\ e^{-x}\right)\frac{dx}{x}=\int_0^\infty \mathcal{A}K_{i\tau}(x)\left(\mathcal{A}^{n-1}\ e^{-x}\right)\frac{dx}{x}$$

$$=\cdots=\int_0^\infty \mathcal{A}^{n-1}K_{i\tau}(x)\left(\mathcal{A}\ e^{-x}\right)\frac{dx}{x}=\tau^{2(n-1)}\int_0^\infty K_{i\tau}(x)e^{-x}dx=\frac{\pi\tau^{2n-1}}{\sinh\pi\tau}$$

for any $n = 2, 3, \ldots$. This is because via (2.13), (2.1) $\mathcal{A}^n\ e^{-x} = (-1)^{n-1}\mathcal{A}\ \left(e^{-x}p_{n-1}(x)\right)$ and

$$\int_0^\infty K_{i\tau}(x)\mathcal{A}\ \left(e^{-x}p_{n-1}(x)\right)\frac{dx}{x} = \tau^2\int_0^\infty K_{i\tau}(x)e^{-x}p_{n-1}(x)\frac{dx}{x},\ n = 2, 3, \ldots$$

since terms, which are out of integration, are vanished owing to the property $p_n(0) = 0, n = 1, 2, \ldots$ and asymptotic behavior of the modified Bessel functions (see (1.7), (1.8), (1.9)). Consequently, returning to (4.3), (4.2) we have

$$\sinh \pi\tau \int_0^\infty K_{i\tau}(x)e^{-x}\sum_{n=1}^N c_n\ p_n(x)\frac{dx}{x} = \pi\sum_{n=1}^N(-1)^n c_n\ \tau^{2n-1},\ \tau > 0. \tag{4.4}$$

We will motivate now the passage to the limit when $N \to \infty$ through equality (4.4). To do this, we will use (2.11) to write

$$\frac{p_n(x)}{x} = -\sum_{k=0}^{n-1}\binom{2n-1}{2k}p_k(x),\ n \in \mathbb{N},\ x > 0.$$

Hence substituting the latter equality into (4.4), we estimate the following integral (see above)

$$\int_0^\infty |K_{i\tau}(x)|e^{-x}\sum_{n=1}^N |c_n|\ \sum_{k=0}^{n-1}\binom{2n-1}{2k}|p_k(x)|\ dx$$

$$\leq A\int_0^\infty K_0(x)dx\sum_{n=1}^N |c_n|\ \sum_{k=0}^{n-1}\binom{2n-1}{2k}(2k)!\left(\frac{4}{\pi}\right)^{2k}$$

$$= A_1\sum_{n=1}^N |c_n|\ \sum_{k=0}^{n-1}\frac{(2n-1)!}{(2(n-k)-1)!}\left(\frac{4}{\pi}\right)^{2k} < A_2\sum_{n=1}^\infty |c_n|\ (2n)!\left(\frac{4}{\pi}\right)^{2n} < \infty, \tag{4.5}$$

where $A, A_1, A_2 > 0$ are constants. Therefore the passage to the limit when $N \to \infty$ is possible by virtue of the Lebesgue dominated convergence theorem, and taking into account (1.15) we obtain the equality

$$\sinh \pi\tau \int_0^\infty K_{i\tau}(x)f(x)\frac{dx}{x} = \pi\sum_{n=1}^\infty(-1)^n c_n\ \tau^{2n-1}, \tag{4.6}$$

where the Taylor series in the left-hand side of (4.6) is convergent for all $\tau \geq 0$. Hence

$$\pi(-1)^n c_n = \frac{1}{(2n-1)!}\lim_{\tau\to 0}\frac{d^{2n-1}}{d\tau^{2n-1}}\left[\sinh \pi\tau\int_0^\infty K_{i\tau}(x)f(x)\frac{dx}{x}\right]$$

$$= \frac{1}{(2n-1)!}\sum_{k=1}^{n}\binom{2n-1}{2k-1}\pi^{2k-1}\lim_{\tau\to 0}\frac{d^{2(n-k)}}{d\tau^{2(n-k)}}\int_0^\infty K_{i\tau}(x)f(x)\frac{dx}{x}. \tag{4.7}$$

Meanwhile, from (1.1) we find

$$\frac{d^{2m}}{d\tau^{2m}}K_{i\tau}(x) = (-1)^m\int_0^\infty e^{-x\cosh u}u^{2m}\cos\tau u\, du, \ m \in \mathbb{N}_0.$$

Furthermore, since (see (4.5))

$$\int_0^\infty \left|\frac{d^{2m}}{d\tau^{2m}}K_{i\tau}(x)f(x)\right|\frac{dx}{x} < B\int_0^\infty \left|\frac{d^{2m}}{d\tau^{2m}}K_{i\tau}(x)\right| dx$$

$$\leq B\int_0^\infty\int_0^\infty e^{-x\cosh u}u^{2m}du dx = B\int_0^\infty \frac{u^{2m}}{\cosh u}du < \infty, \ m \in \mathbb{N}_0,$$

where $B > 0$ is a constant, one can differentiate and pass to the limit under the integral sign in (4.7). As a result we obtain

$$c_n = \sum_{k=1}^{n}\frac{(-1)^k\pi^{2(k-1)}}{(2k-1)!(2(n-k))!}\int_0^\infty\int_0^\infty e^{-x\cosh u}u^{2(n-k)}f(x)\frac{dx}{x}du.$$

Thus denoting by $\hat{p}_n(x)$ the following even polynomial

$$\hat{p}_n(x) = \sum_{k=0}^{n-1}\frac{(-1)^{n-k}\pi^{2(n-k-1)}}{(2(n-k)-1)!(2k)!}x^{2k}$$

we have the final formula for coefficients c_n

$$c_n = \int_0^\infty S_n(x)f(x)\frac{dx}{x}, \ n \in \mathbb{N}, \tag{4.8}$$

where

$$S_n(x) = \int_0^\infty e^{-x\cosh u}\hat{p}_n(u)du, \ x > 0.$$

We summarize our results by the following

Theorem 1. *Let condition* (4.1) *take place. Then discrete transformation* (1.15) $f(x)$ *represents a bounded continuous function on* $[0, +\infty)$, $f(0) = 0$, *where the corresponding series converges absolutely and uniformly and coefficients* c_n, $n \in \mathbb{N}$ *are given by formula* (4.8)*. Besides, the Kontorovich-Lebedev operator of this function can be expanded by Taylor's series* (4.6)*, which is convergent for all* $\tau \geq 0$.

Corollary 1. *If* $f(x)$ *is decomposed by series* (1.15)*, where its coefficients satisfy condition* (4.1)*, then this expansion is unique.*

Proof. In fact, if it admits also the representation

$$f(x) = e^{-x} \sum_{n=1}^{\infty} a_n p_n(x),$$

then

$$0 = e^{-x} \sum_{n=1}^{\infty} (c_n - a_n) p_n(x),$$

and evidently, $c_n - a_n$ satisfies condition (4.1). Therefore, $c_n = a_n,\ n \in \mathbb{N}$ via (4.8). Corollary 1 is proved.

Corollary 2. *Under condition* (4.1) *series* (1.15) *converges in the mean square sense with respect to the norm of the space* $L_2\left(\mathbb{R}_+; \frac{dx}{x}\right)$ *to the same function* $f \in L_2\left(\mathbb{R}_+; \frac{dx}{x}\right)$.

Proof. Indeed, let us estimate the norm of the partial sums of series (1.15). Applying the Minkowski inequality, equality (3.5) and the representation of Bernoulli's numbers (see Section 2) we derive

$$||f_N||_{L_2\left(\mathbb{R}_+; \frac{dx}{x}\right)} = \left(\int_0^{\infty} e^{-2x} \left| \sum_{n=1}^{N} c_n p_n(x) \right|^2 \frac{dx}{x} \right)^{1/2} \le \sum_{n=1}^{N} |c_n| \left(\int_0^{\infty} e^{-2x} p_n^2(x) \frac{dx}{x} \right)^{1/2}$$

$$= \sum_{n=1}^{N} |c_n| \sqrt{\frac{1-2^{4n}}{2n} B_{4n}} < \text{const.} \sum_{n=1}^{N} |c_n| (2n)! \left(\frac{2}{\pi} \right)^{2n}$$

$$< \text{const.} \sum_{n=1}^{N} |c_n| (2n)! \left(\frac{4}{\pi} \right)^{2n}. \tag{4.9}$$

Thus $\{f_N\}$ is a Cauchy sequence in the space $L_2\left(\mathbb{R}_+; \frac{dx}{x}\right)$ because

$$||f_N - f_M||_{L_2\left(\mathbb{R}_+; \frac{dx}{x}\right)} < \text{const.} \sum_{n=M+1}^{N} |c_n| (2n)! \left(\frac{4}{\pi} \right)^{2n} \to 0,\ N > M,\ M \to \infty.$$

Since this sequence admits a pointwise convergence too, it asserts the same limit $f(x)$. Corollary 2 is proved.

Further, from (4.9) taking into account (3.3), (3.7) we have

$$||f_N||^2_{L_2\left(\mathbb{R}_+; \frac{dx}{x}\right)} = \sum_{n,m=1}^{N} (-1)^{n+m} c_n \overline{c_m} \, \frac{1 - 2^{2(n+m)}}{n+m} B_{2(n+m)}.$$

Passing here to the limit when $N \to \infty$, we obtain the Parseval type identity for discrete transformation (1.15)

$$\int_0^\infty |f(x)|^2 \frac{dx}{x} = \sum_{n,m=1}^\infty (-1)^{n+m} c_n \overline{c_m}\, \frac{1-2^{2(n+m)}}{n+m} B_{2(n+m)}, \tag{4.10}$$

where its right-hand side is evidently positive.

Denoting by $F_N(\tau)$ the odd polynomial in the right-hand side of (4.4), we see that it represents the Kontorovich-Lebedev transform of a partial sum f_N of series (1.15). From (1.14), (2.1) and (1.3) it follows the reciprocal formula for this partial sum

$$f_N(x) = \frac{2}{\pi^2}\int_0^\infty \tau K_{i\tau}(x) F_N(\tau) d\tau,\ N = 1, 2, \dots\ . \tag{4.11}$$

The reciprocity (4.11) follows from the uniqueness theorem for the Kontorovich-Lebedev transform [6], since $f_N \in L_1\left(\mathbb{R}_+; K_0(x)\frac{dx}{x}\right) \cap L_2\left(\mathbb{R}_+; \frac{dx}{x}\right)$ for each $N \in \mathbb{N}$. Now we will find sufficient conditions for the mean-square convergence with respect to the norm of the space $L_2\left(\mathbb{R}_+; \frac{dx}{x}\right)$ of a partial sum f_N to a given function f when $N \to \infty$. We have

Theorem 2. *Let $f(x)$ be a bounded continuous function on $[0, \infty)$ such that $f(x)/x$ is bounded for all $x > 0$. Let a sequence of polynomials $\{F_N(\tau)\}$ with coefficients* (4.7) *converge pointwisely for all $\tau > 0$ to a function $F(\tau)$. If $\{F_N(\tau)\}$ is a Cauchy sequence in the space $L_2\left(\mathbb{R}_+; \frac{\tau d\tau}{\sinh \pi\tau}\right)$, then the sequence of partial sums $\{f_N\}$ of series* (1.15) *converges to f in the mean square sense with respect to the norm of the space $L_2\left(\mathbb{R}_+; \frac{dx}{x}\right)$, i.e.*

$$||f - f_N||_{L_2\left(\mathbb{R}_+; \frac{dx}{x}\right)} \to 0,\ N \to \infty. \tag{4.12}$$

Finally, $f \in L_2\left(\mathbb{R}_+; \frac{dx}{x}\right)$ and

$$||f||_{L_2\left(\mathbb{R}_+; \frac{dx}{x}\right)} = \frac{\sqrt{2}}{\pi} ||F||_{L_2\left(\mathbb{R}_+; \frac{\tau d\tau}{\sinh \pi\tau}\right)}. \tag{4.13}$$

Proof. Since the sequence of Taylor polynomials $\{F_N(\tau)\}$ converges for every $\tau > 0$, we get that its limit $F(\tau)$ is equal to (see (4.4), (4.6), (4.7))

$$F(\tau) = \sinh \pi\tau \int_0^\infty K_{i\tau}(x) f(x) \frac{dx}{x},\ \tau > 0.$$

On the other hand, this sequence is a Cauchy one in the space $L_2\left(\mathbb{R}_+; \frac{\tau d\tau}{\sinh \pi\tau}\right)$. Thus it converges to the same function F, which belongs to $L_2\left(\mathbb{R}_+; \frac{\tau d\tau}{\sinh \pi\tau}\right)$. Moreover, the Parseval identity for the Kontorovich-Lebedev transform yields (see (4.11))

$$||f_N - f_M||_{L_2\left(\mathbb{R}_+; \frac{dx}{x}\right)} = \frac{\sqrt{2}}{\pi} ||F_N - F_M||_{L_2\left(\mathbb{R}_+; \frac{\tau d\tau}{\sinh \pi\tau}\right)} \to 0,\ M, N \to \infty. \tag{4.14}$$

Consequently, $\{f_N\}$ converges in the mean square sense with respect to the norm of the space $L_2\left(\mathbb{R}_+; \frac{dx}{x}\right)$ to some function $g(x) \in L_2\left(\mathbb{R}_+; \frac{dx}{x}\right)$. Our goal is to prove that $g = f$ for almost all $x > 0$. Indeed, (4.14) implies

$$||g||_{L_2\left(\mathbb{R}_+; \frac{dx}{x}\right)} = \frac{\sqrt{2}}{\pi} ||F||_{L_2\left(\mathbb{R}_+; \frac{\tau d\tau}{\sinh \pi\tau}\right)}.$$

Moreover, the Plancherel theorem for the Kontorovich-Lebedev operator [5], [6] says that for all $x > 0$

$$\int_0^x g(x)dx = \frac{2}{\pi^2} \int_0^\infty \tau \sinh \pi\tau \int_0^x K_{i\tau}(x)dx \int_0^\infty K_{i\tau}(y)f(y)\frac{dy}{y} d\tau, \tag{4.15}$$

where the integral with respect to τ in the right-hand side of (4.15) converges absolutely. Hence owing to Lebesgue's monotone convergence theorem we find

$$\int_0^x g(x)dx = \frac{2}{\pi^2} \lim_{\varepsilon \to 0+} \int_0^\infty \tau \sinh(\pi - \varepsilon)\tau \int_0^x K_{i\tau}(x)dx \int_0^\infty K_{i\tau}(y)f(y)\frac{dy}{y} d\tau. \tag{4.16}$$

Now Fubini's theorem and conditions on f of the present theorem allow to change the order of integration in the right-hand side of (4.16) and calculate the inner integral by τ employing relation (2.16.51.8) in [4, Vol. II]. Thus after simple substitutions we arrive at the representation

$$\int_0^x g(x)dx = \frac{1}{\pi} \lim_{\varepsilon \to 0+} \int_0^x x \sin \varepsilon \int_{-\cot \varepsilon}^\infty \frac{K_1\left(x \sin \varepsilon \sqrt{t^2+1}\right)}{\sqrt{t^2+1}}$$
$$\times f\left(x\left(\cos \varepsilon + t \sin \varepsilon\right)\right) dt dx. \tag{4.17}$$

But $f(x)$ is bounded continuous for $x \in [0, \infty)$ and

$$x \sin \varepsilon \sqrt{t^2+1} K_1\left(x \sin \varepsilon \sqrt{t^2+1}\right) < 1, \ \ x, t > 0, \varepsilon \in (0, \pi)$$

(see above the inequality $yK_1(y) < 1, \ y > 0$). Therefore the estimate

$$x \sin \varepsilon \int_{-\cot \varepsilon}^\infty \frac{K_1\left(x \sin \varepsilon \sqrt{t^2+1}\right)}{\sqrt{t^2+1}} |f\left(x\left(\cos \varepsilon + t \sin \varepsilon\right)\right)| dt$$
$$< \text{const.} \int_{-\infty}^\infty \frac{dt}{t^2+1} = \text{const.}$$

and the dominated convergence theorem permit the passage to the limit under the integral with respect to x. Further, appealing to results in [7] we prove that for all $x \geq 0$

$$\frac{1}{\pi} \lim_{\varepsilon \to 0+} x \sin \varepsilon \int_{-\cot \varepsilon}^\infty \frac{K_1\left(x \sin \varepsilon \sqrt{t^2+1}\right)}{\sqrt{t^2+1}} f\left(x\left(\cos \varepsilon + t \sin \varepsilon\right)\right) dt = f(x).$$

Thus from (4.16)

$$\int_0^x g(x)dx = \int_0^x f(x)dx$$

and by differentiation we conclude $g = f$ for almost all $x \geq 0$. This gives (4.12), (4.13) and concludes the proof of Theorem 2.

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