Fixed points of endomorphisms of trace monoids

Emanuele Rodaro, Pedro V. Silva

Centro de Matemática, Faculdade de Ciências, Universidade do Porto, R. Campo Alegre 687, 4169-007 Porto, Portugal
e-mail: emanuele.rodaro@fc.up.pt, pvsilva@fc.up.pt

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ABSTRACT

It is proved that the fixed point submonoid and the periodic point submonoid of a trace monoid endomorphism are always finitely generated. Considering the Foata normal form metric on trace monoids and uniformly continuous endomorphisms, a finiteness theorem is proved for the infinite fixed points of the continuous extension to real traces.

1 Introduction

In [4], Cassaigne and the second author studied finiteness conditions for the infinite fixed points of (uniformly continuous) endomorphisms of monoids defined by special confluent rewriting systems, extending results known for free monoids [11]. This line of reasearch was pursued by the second author in subsequent papers [13, 14].

Similar problems were considered by the second author for virtually free groups in [15] (see also [17]).

Recently, in [12], Sykiotis and the authors showed that, given a graph group G, the subgroup of fixed points is finitely generated for every endomorphism of G if and only if G is a free product of free abelian groups.

Motivated by these papers, we perform here a similar study for trace monoids, both at finite and infinite level. We remark that trace monoids are one of the most important models for parallel computing in theoretical computer science [6, 7]. In our study, the infinite fixed points are taken among real traces, the completion of a trace monoid for the Foata normal form metric, introduzed by Bonizzoni, Mauri and Pighizzini [2].

2 Preliminaries

Given a monoid M, we denote by $\operatorname{End} M$ the endomorphism monoid of M. Given $\varphi \in \operatorname{End} M$, we say that $x \in M$ is a *fixed point* of φ if $x\varphi = x$. If $x\varphi^n = x$ for some $n \geq 1$, we say that x is a *periodic point* of φ. Let $\operatorname{Fix}\varphi$ (respectively $\operatorname{Per}\varphi$) denote the set of all fixed points (respectively periodic points) of φ. Clearly,

$$\operatorname{Per}\varphi = \bigcup_{n\geq 1} \operatorname{Fix}\varphi^n.$$

Given $X \subseteq M$, the *star* operator $X \mapsto X^*$ defines the submonoid of M generated by X. We say that $X \subseteq M$ is *rational* if X can be obtained from finite subsets of M by applying finitely many times the operators union, product and star. For more details on rational subsets, namely connections with finite automata, see [1].

Given an alphabet A, the *free monoid* on A is denoted as usual by A^*. A (finite) *independence alphabet* is an ordered pair of the form (A, I), where A is a (finite) set and I is a symmetric anti-reflexive relation on A. We can view (A, I) as an undirected graph without loops or multiple edges, denoted by $\Gamma(A, I)$, by taking A as the vertex set and I as the edge set. Conversely, every such graph determines an independence alphabet.

Let $\rho(A, I)$ denote the congruence on A^* generated by the the relation

$$\{(ab, ba) \mid (a, b) \in I\}. \tag{1}$$

The *trace monoid* $M(A, I)$ is the quotient $A^*/\rho(A, I)$, i.e the monoid defined by the monoid presentation

$$\langle A \mid ab = ba \ ((a, b) \in I)\rangle.$$

Such monoids are also known as *free partially commutative monoids*. Fore details on trace monoids, the reader is referred to [6, 7].

The elements of $M(A, I)$ are often described through the *Foata normal form* (FNF), defined as follows. We say that $B \subseteq A$ is a *clique* if $B \neq \emptyset$ and the full subgraph of $\Gamma(A, I)$ induced by B is complete. Let then $w_B \in M(A, I)$ be the product of all the letters in B. Note that, since the letters of B commmute with each other in $M(A, I)$, we do not need to specify any multiplication sequence. A FNF is either the empty word or a product $w_{B_1} \dots w_{B_k}$ such that:

- B_i is a clique for $i = 1, \dots, k$;
- for all $i = 2, \dots, k$ and $a \in B_i$, there exists some $b \in B_{i-1}$ such that $(a, b) \notin I$.

Then each element of $M(A, I)$ admits a unique representation in FNF [3].

Given $u \in M(A, I)$ and $a \in A$, we denote by $|u|_a$ the number of occurrences of a in u. Given $B \subseteq A$, let $\pi_B \in \operatorname{End} M$ keep the letters of B and remove the others. A well-known result (see [7]) states that, for all $u, v \in A^*$,

$$u = v \text{ in } M(A, I) \quad \Leftrightarrow \quad \forall (a, b) \in (A \times A) \setminus I \quad u\pi_{a,b} = v\pi_{a,b}. \tag{2}$$

This leads to an embedding of $M(A, I)$ into a direct product of free monoids. It follows that $M(A, I)$ is a cancellative monoid.

3 The submonoids of fixed and periodic points

Theorem 3.1 *Let (A, I) be a finite independence alphabet. Then* $\mathrm{Fix}\,\varphi$ *is finitely generated for every* $\varphi \in \mathrm{End}\, M(A, I)$.

Proof. Write $M = M(A, I)$. We use induction on $|A|$. The case $|A| = 0$ being trivial, assume that $|A| > 0$ and the result holds for smaller alphabets. We consider three cases.

Case I: $\varphi|_A$ is a permutation.

We start showing that

$$(a, b) \in I \Leftrightarrow (a\varphi, b\varphi) \in I \tag{3}$$

holds for all $a, b \in A$. Indeed, the direct implication follows from φ being an endomorphism since $ab = ba$ yields $(a\varphi)(b\varphi) = (b\varphi)(a\varphi)$ in M. The converse implication follows from the equality

$$\begin{aligned} &|\{(a, b) \in A \times A \mid ab = ba \text{ in } M\}| \\ &\qquad = |\{(a, b) \in A \times A \mid (a\varphi)(b\varphi) = (b\varphi)(a\varphi) \text{ in } M\}| \end{aligned}$$

combined with the direct implication. Thus (3) holds.

Let $\mathcal{B}$ be the set of all cliques B such that $B\varphi = B$. We claim that

$$\mathrm{Fix}\,\varphi = \{w_B \mid B \in \mathcal{B}\}^*. \tag{4}$$

It is immediate that $w_B \in \mathrm{Fix}\,\varphi$ for every $B \in \mathcal{B}$. Hence $\{w_B \mid B \in \mathcal{B}\}^* \subseteq \mathrm{Fix}\,\varphi$. Conversely, let $u \in \mathrm{Fix}\,\varphi$. Assuming that $u \neq 1$, let $u = w_{B_1} \dots w_{B_k}$ be its FNF. Then $u\varphi = (w_{B_1}\varphi) \dots (w_{B_k}\varphi)$. By (3), $(w_{B_1}\varphi) \dots (w_{B_k}\varphi)$ must be a FNF as well. Since $u\varphi = u$, and the representation in FNF is unique, we get $w_{B_i} = w_{B_i}\varphi$ for $i = 1, \dots, k$. Hence $B_1, \dots, B_k \in \mathcal{B}$ and so $u \in \{w_B \mid B \in \mathcal{B}\}^*$. Thus (4) holds and so $\mathrm{Fix}\,\varphi$ is finitely generated in this case.

Case II: $1 \notin A\varphi$.

Write

$$A_0 = A \cap \mathrm{Per}\,\varphi$$

and let Λ be the directed graph with vertices in A_0 and edges $a \longrightarrow b$ whenever $b = a\varphi$. It is easy to see that A_0 consists of all the vertices occurring in some cycle of Λ. Let $p = |A_0|!$. Since p is a multiple of the length of any cycle in Λ, it follows that

$$A_0 = A \cap \mathrm{Fix}\,\varphi^p. \tag{5}$$

We claim next that

$$A_0 = A \cap A\varphi^p. \tag{6}$$

Indeed, $A_0 \subseteq A \cap A\varphi^p$ follows from (5). Conversely, let $a \in A \cap A\varphi^p$. Since $1 \notin A\varphi$, there exists some path $b \longrightarrow a$ of length p in Λ and so a lies in some cycle of Λ. Thus $a \in A_0$ and so (6) holds.

Next we show that

$$A_0\varphi = A_0. \tag{7}$$

Let $a \in A_0$. By (6), we have $a\varphi = (a\varphi)\varphi^p$ and so $a\varphi \in A_0$. Hence $A_0\varphi \subseteq A_0$. But now $a\varphi^{p-1} \in A_0$ and $a = (a\varphi^{p-1})\varphi$ yield $A_0 \subseteq A_0\varphi$ and so (7) holds.

Let M' be the (trace) submonoid of M generated by A_0. In view of (7), φ restricts to some endomorphism φ' of M'. We show that

$$\mathrm{Fix}\,\varphi = \mathrm{Fix}\,\varphi'. \tag{8}$$

Let $u = a_1 \dots a_k \in \mathrm{Fix}\,\varphi$ with $a_1, \dots, a_k \in A$. Then

$$a_1 \dots a_k = u = u\varphi^p = (a_1\varphi^p) \dots (a_k\varphi^p)$$

and $1 \notin A\varphi$ yields $\{a_1, \dots, a_k\} = \{a_1\varphi^p, \dots, a_k\varphi^p\}$. By (6), we get $a_1, \dots, a_k \in A_0$ and so $u \in \mathrm{Fix}\,\varphi'$. The converse inclusion holds trivially, hence $\mathrm{Fix}\,\varphi = \mathrm{Fix}\,\varphi'$.

If $A_0 = A$, then $\mathrm{Fix}\,\varphi$ is finitely generated by Case I, hence we may assume that $A_0 \subset A$. By the induction hypothesis, $\mathrm{Fix}\,\varphi'$ is finitely generated, and so is $\mathrm{Fix}\,\varphi$.

Case III: $1 \in A\varphi$.

Write

$$A_1 = A \cap 1\varphi^{-1}, \quad A_2 = A \setminus A_1.$$

Let M'' be the (trace) submonoid of M generated by A_2. Write $\pi = \pi_{A_2}$ and let $\varphi'' = (\varphi\pi)|_{M''}$. Clearly, $\varphi'' \in \mathrm{End}\,M''$ and $A_1 \neq \emptyset$ implies $|A_2| < |A|$. By the induction hypothesis, $\mathrm{Fix}\,\varphi''$ is finitely generated. We claim that

$$\mathrm{Fix}\,\varphi = (\mathrm{Fix}\,\varphi'')\varphi. \tag{9}$$

Let $u \in \mathrm{Fix}\,\varphi$. We may factor $u = u_0a_1u_1 \dots a_ku_k$ with $a_1, \dots, a_k \in A_2$ and $u_0, \dots, u_k \in A_1^*$. It follows that $u = u\varphi = (a_1a_2 \dots a_k)\varphi$. Now $a_1a_2 \dots a_k \in M''$ and

$$(a_1a_2 \dots a_k)\varphi'' = (a_1a_2 \dots a_k)\varphi\pi = u\pi = (u_0a_1u_1 \dots a_ku_k)\pi = a_1a_2 \dots a_k,$$

hence $a_1a_2 \dots a_k \in \mathrm{Fix}\,\varphi''$ and so $u = (a_1a_2 \dots a_k)\varphi \in (\mathrm{Fix}\,\varphi'')\varphi$. Thus $\mathrm{Fix}\,\varphi \subseteq (\mathrm{Fix}\,\varphi'')\varphi$.

Conversely, let $v = a_1a_2 \dots a_k \in \mathrm{Fix}\,\varphi''$, with $a_1, \dots, a_k \in A_2$. By checking directly on the generators, we get

$$\pi\varphi = \varphi. \tag{10}$$

Hence $v = v\varphi'' = v\varphi\pi$ yields $(v\varphi)\varphi = v\varphi\pi\varphi = v\varphi$ and so $v\varphi \in \mathrm{Fix}\,\varphi$. Thus $(\mathrm{Fix}\,\varphi'')\varphi \subseteq \mathrm{Fix}\,\varphi$ and so (9) holds. Since $\mathrm{Fix}\,\varphi''$ is finitely generated, then also $\mathrm{Fix}\,\varphi$ is finitely generated in this third and last case. □

This proof can be adapted to the case of periodic points:

Theorem 3.2 *Let (A, I) be a finite independence alphabet. Then* $\mathrm{Per}\,\varphi$ *is finitely generated for every $\varphi \in \mathrm{End}\,M(A, I)$.*

Proof. Write $M = M(A, I)$ and let $m = |A|!$. We show that

$$\mathrm{Per}\,\varphi = \mathrm{Fix}\,\varphi^m \tag{11}$$

by induction on $|A|$. Then the claim follows from Theorem 3.1.

The case $|A| = 0$ being trivial, assume that $|A| > 0$ and the result holds for smaller alphabets. We consider two cases.

Case I: $1 \notin A\varphi$.

We keep the notation introduced in Case II of the proof of Theorem 3.1. We may assume that $A_0 \subset A$, otherwise $\varphi|_A$ would be a permutation, and since the order of $\varphi|_A$ must divide the order of the symmetric group on A, which is m, we would get $(\varphi|_A)^m = 1$ and therefore $\varphi^m = 1$, yielding $\mathrm{Fix}\,\varphi^m = M = \mathrm{Per}\,\varphi$.

If we replace φ by φ^n, then A_0 remains the same in view of $\mathrm{Per}\,\varphi = \mathrm{Per}\,\varphi^n$ and so does M'. On the other hand, it follows from (7) that $\varphi^n|_{M'} = (\varphi|_{M'})^n = (\varphi')^n$, hence

$$\mathrm{Fix}\,\varphi^n = \mathrm{Fix}\,(\varphi')^n \tag{12}$$

by applying (8) to φ^n. By the induction hypothesis, we have $\mathrm{Per}\,\varphi' = \mathrm{Fix}\,(\varphi')^{|A_0|!}$. Since $|A_0|!$ divides m, we get

$$\mathrm{Per}\,\varphi' = \mathrm{Fix}\,(\varphi')^{|A_0|!} \subseteq \mathrm{Fix}\,(\varphi')^m \subseteq \mathrm{Per}\,\varphi'$$

and so $\mathrm{Per}\,\varphi' = \mathrm{Fix}\,(\varphi')^m$. Together with (12), this yields

$$\mathrm{Per}\,\varphi = \cup_{n\geq 1}\mathrm{Fix}\,\varphi^n = \cup_{n\geq 1}\mathrm{Fix}\,(\varphi')^n = \mathrm{Per}\,\varphi' = \mathrm{Fix}\,(\varphi')^m = \mathrm{Fix}\,\varphi^m$$

as required.

Case II: $1 \in A\varphi$.

We keep the notation introduced in Case III of the proof of Theorem 3.1.

Let $u \in \mathrm{Per}\,\varphi$, say $u \in \mathrm{Fix}\,\varphi^n$. We may factor $u = u_0a_1u_1\ldots a_ku_k$ with $a_1,\ldots,a_k \in A_2$ and $u_0,\ldots,u_k \in A_1^*$. It follows that $u = u\varphi^n = (a_1a_2\ldots a_k)\varphi^n$. Now $a_1a_2\ldots a_k \in M''$ and (10) yields $a_1a_2\ldots a_k = (a_1a_2\ldots a_k)\varphi^n\pi = (a_1a_2\ldots a_k)(\varphi\pi)^n$ and consequently $a_1a_2\ldots a_k \in \mathrm{Fix}\,(\varphi'')^n \subseteq \mathrm{Per}\,\varphi''$. By the induction hypothesis, we get $a_1a_2\ldots a_k \in \mathrm{Fix}\,(\varphi'')^m$ and so $a_1a_2\ldots a_k = (a_1a_2\ldots a_k)\varphi^m\pi$ in view of (10). Hence $u\varphi^m = v_0a_1v_1\ldots a_kv_k$ for some $v_0,\ldots,v_k \in A_1^*$. Thus

$$u\varphi^{2m} = u\varphi^m\pi\varphi^m = (v_0a_1v_1\ldots a_kv_k)\pi\varphi^m = (u_0a_1u_1\ldots a_ku_k)\pi\varphi^m = u\varphi^m$$

which together with $u\varphi^n = u$ yields

$$u = u\varphi^n = u\varphi^{2n} = \ldots = u\varphi^{nm} = u\varphi^{(n-1)m} = \ldots = u\varphi^m.$$

Therefore $\mathrm{Per}\,\varphi = \mathrm{Fix}\,\varphi^m$ and so (11) holds in all cases as required. □

Corollary 3.3 *Let (A,I) be a finite independence alphabet and let $\varphi \in \mathrm{End}\,M(A,I)$. Then we can effectively compute finite sets of generators for* $\mathrm{Fix}\,\varphi$ *and* $\mathrm{Per}\,\varphi$.

Proof. For $\mathrm{Fix}\,\varphi$, it suffices to remark that all the morphisms, subsets and submonoids appearing in the induction proof of Theorem 3.1 can be effectively computed, namely in connection with the key equalities (4), (8) and (9). The periodic case follows from the fixed point case and (11). □

4 Extension to real traces

In the late eighties, two ultrametrics were introduced for trace monoids. One of them, defined by Bonizzoni, Mauri and Pighizzini [2], is known as the *FNF metric*. Given $u, v \in M(A, I)$, let $u = w_{B_1} \dots w_{B_m}$ and $v = w_{C_1} \dots w_{C_n}$ denote their FNFs. We define

$$r(u,v) = \begin{cases} \max\{k \geq 0 \mid B_1 = C_1, \dots, B_k = C_k\} & \text{if } u \neq v \\ \infty & \text{otherwise} \end{cases}$$

The ultrametric d is defined by $d(u, v) = 2^{-r(u,v)}$, using the convention $2^{-\infty} = 0$.

The other metric, defined by Kwiatkowska [10], is known as the *prefix metric*. Given $u, v \in M(A, I)$, we say that u is a prefix of v and write $u \leq_p v$ if $v = uw$ for some $w \in M(A, I)$. For every $n \in \mathbb{N}$, denote by $\mathrm{Pref}_n(v)$ the set of all prefixes of u of length n. We define, for all $u, v \in M(A, I)$,

$$r'(u, v) = \sup\{n \in \mathbb{N} \mid \mathrm{Pref}_n(u) = \mathrm{Pref}_n(v)\}$$

and $d'(u, v) = 2^{-r'(u,v)}$. It is well known that, for a finite dependence alphabet, these metrics are uniformly equivalent (i.e. the identity mappings between $(M(A, I), d)$ and $(M(A, I), d')$ are uniformly continuous), see e.g. [9]. We recall that a mapping $\varphi : (X_1, d_1) \to (X_2, d_2)$ between metric spaces is *uniformly continuous* if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x, y \in X_1 \ (d_1(x, y) < \delta \Rightarrow d_2(x\varphi, y\varphi) < \varepsilon).$$

An important particular case is given by *contractions*, i.e. $d_2(x\varphi, y\varphi) \leq d_1(x, y)$ for all $x, y \in X_1$.

In [10], Kwiatkowska also showed that the completion of $(M(A, I), d')$ is obtained by adding the *infinite real traces* and is a compact metric space. By standard topology results, this completion is up to homeomorphism the completion of $(M(A, I), d)$, hence we shall describe it with respect to the FNF metric, favoured in this paper.

Let $\partial M(A, I)$ consist of all infinite sequences of the form $w_{B_1} w_{B_2} \dots$ such that:

- B_i is a clique for every $i \geq 1$;
- for all $i \geq 2$ and $a \in B_i$, there exists some $b \in B_{i-1}$ such that $(a, b) \notin I$.

Note that $w_{B_1} \dots w_{B_n}$ is a Foata normal form for every $n \geq 1$. We say that $\hat{M}(A, I) = M(A, I) \cup \partial M(A, I)$ is the set of *real traces* on (A, I).

The metric d extends to $\hat{M}(A, I)$ in the obvious way, and it is easy to check that $(\hat{M}(A, I), d)$ is complete: given a Cauchy sequence $(u^{[n]})_n$ with $u^{[n]} = w_{B_{n1}} w_{B_{n2}} \dots$, it follows easily that each sequence $(w_{B_{nk}})_k$ is stationary with limit, say, w_{B_k}, and we get $w_{B_1} w_{B_2} \dots = \lim_{n \to \infty} u^{[n]}$. Since $w_{B_1} w_{B_2} \dots = \lim_{n \to \infty} w_{B_1} \dots w_{B_n}$, then $(\hat{M}(A, I), d)$ is indeed the completion of $(M(A, I), d)$ and therefore compact by the aforementioned Kwiatkowska's result. We may refer to $\partial M(A, I)$ as the *boundary* of $M(A, I)$.

It is well known that $M(A, I)$ acts continuously on the left of $\partial M(A, I)$. Indeed, it follows easily from standard topological arguments which we sketch here. We start by noting that

$$d(uv, uw) \leq d(v, w) \tag{13}$$

holds afor all $u, v, w \in M(A,I)$ (i.e. multiplication by a constant on the left constitutes a contraction of $(M(A,I), d)$). Since $(\hat{M}(A,I), d)$ is the completion of $(M(A,I), d)$, it follows easily that the action

$$\begin{array}{c} M(A,I) \times \partial M(A,I) \to \partial M(A,I) \\ (u, \lim_{n\to\infty} v_n) \mapsto \lim_{n\to\infty} uv_n \end{array}$$

is well defined, and in view of (13) it turns out to be continuous. We shall call this action *mixed product*.

We remark that the mixed product is left cancellative: if $uX = uX'$ with $u \in M(A,I)$ and $X, X' \in \hat{M}(A,I)$, then $X = X'$. Indeed, we can extend (2) to real traces the obvious way, and left cancellativity becomes a simple exercise.

Kwiatowska's approach, on the other hand, leads to a geometric description of the boundary reminiscent of the theory of hyperbolic groups:

Given a monoid M generated by a subset A, the *Cayley graph* $\mathrm{Cay}_A M$ has vertex set M and labelled directed edges of the form $m \xrightarrow{a} ma$ for all $m \in M$ and $a \in A$. Let $\pi : A^* \to M$ be the canonical epimorphism. We say that a word $u \in A^*$ is a *geodesic* of $\mathrm{Cay}_A M$ if $1 \xrightarrow{u} u\pi$ has shortest length among the paths connecting 1 to $u\pi$ in $\mathrm{Cay}_A M$. Note that geodesics $1 \longrightarrow u\pi$ need not to be unique! An infinite word $\alpha \in A^\omega$ is a *ray* of $\mathrm{Cay}_A M$ if every finite prefix of α is a geodesic. In the particular case of a trace monoid $M(A,I)$, when we take the canonical generating set A, all relations are length-preserving and so every finite (respectively infinite) word represents a geodesic (respectively a ray). The operators Pref_n can be extended in the obvious way to infinite words and an equivalence relation can be defined on the set of rays (in our case, A^ω) by

$$\alpha \equiv \beta \qquad \text{if } \mathrm{Pref}_n(\alpha) = \mathrm{Pref}_n(\beta) \text{ for every } n \geq 0.$$

Then the boundary of $M(A,I)$ can be viewed as the set of $\equiv$-equivalence classes of A^ω.

We introduce now a subclass of subsets of $\hat{M}(A,I)$ which generalizes the usual concept of rational subset of a monoid. We say that $Y \subseteq \hat{M}(A,I)$ is *mp-rational* if Y can be obtained from finite subsets of $\hat{M}(A,I)$ by applying finitely many times the operators union, product, star and mixed product. It follows easily that L is mp-rational if and only if $L = L_0 \cup (\cup_{i=1}^n L_i X_i)$ for some rational subsets $L_0, \ldots, L_n$ of $M(A,I)$ and $X_1, \ldots, X_n \in \partial M(A,I)$.

For every $u \in M(A,I)$, let $u\xi$ denote the *content* of u, i.e. the set of letters occurring in u. We define a symmetric relation $\sim_I$ on $M(A,I)$ by

$$u \sim_I v \qquad \text{if } u\xi \times v\xi \subseteq I.$$

Theorem 4.1 *Let (A,I) be a finite independence alphabet and let $\varphi \in \mathrm{End}\, M(A,I)$. Then the following conditions are equivalent:*

(i) φ is uniformly continuous with respect to d;

(ii) φ is a contraction with respect to d;

(iii) for all $a, b, c \in A$,

$$(c \leq_p b\varphi \ \wedge \ c \sim_I a\varphi) \ \Rightarrow \ (a,b) \in I. \tag{14}$$

Proof. (i) $\Rightarrow$ (iii). Suppose that (14) fails for some $a, b, c \in A$. For every $n \in \mathbb{N}$, let $u_n = a^n$ and $v_n = a^n b$. Since $(a, b) \notin I$, then $r(u_n, v_n) = n$ and so $d(u_n, v_n) = 2^{-n}$. On the other hand, it follows from $c \leq_p b\varphi$ and $c \sim_I a\varphi$ that $c \not\leq_p u_n\varphi$ and $c \leq_p v_n\varphi$, hence the FNFs of $u_n\varphi$ and $v_n\varphi$ differ at the first components. It follows that $r(u_n\varphi, v_n\varphi) = 0$ and so $d(u_n\varphi, v_n\varphi) = 1$. Therefore φ is not uniformly continuous.

(iii) $\Rightarrow$ (ii). Assume that (14) holds. We say that an occurrence of a letter in a word $u \in A^*$ has *height* k if it ends up in the kth component when we compute the FNF of u by applying the relations from (1). Note that we are not allowed to swap consecutive occurrences of the same letter!

Let $u = a_1 \dots a_n$ with $a_1, \dots, a_n \in A$ and assume that a_i has height k in u. We show that every occurrence of a letter in $a_i\varphi$ has height $\geq k$ in $(a_1\varphi)\dots(a_n\varphi)$. We use induction on k. The case $k = 1$ holding trivially, assume that $k > 1$ and the claim holds for $k - 1$. There exists some $j < i$ such that $(a_j, a_i) \notin I$ and a_j has height $k - 1$ in u. Consider an occurrence of a letter c in $a_i\varphi$. Suppose first that this occurrence has height 1 in $a_i\varphi$. We may write $c \leq_p a_i\varphi$. Since $(a_j, a_i) \notin I$, (14) yields $c \not\sim_I a_j\varphi$ and so the height of our occurrence of c is greater than the height of some occurrence of a letter in $a_j\varphi$, which is $\geq k - 1$ by the induction hypothesis. On the other hand, if our occurrence of c has height > 1 in $a_i\varphi$, the first letter in $a_i\varphi$ must have height $\geq k$ by the previous case, and we get height $> k$ for our occurrence. This completes the induction process and the proof of our claim.

It follows that

$$p \geq k \quad \Rightarrow \quad r((w_{B_1} \dots w_{B_k})\varphi, (w_{B_1} \dots w_{B_p})\varphi) \geq k \tag{15}$$

holds whenever $w_{B_1} \dots w_{B_p}$ is a FNF. Indeed, the freshly proven claim implies that every occurrence of a letter in $(w_{B_{k+1}} \dots w_{B_p})\varphi$ has height $> k$ in $(w_{B_1} \dots w_{B_k})\varphi(w_{B_{k+1}} \dots w_{B_p})\varphi = (w_{B_1} \dots w_{B_p})\varphi)$ and we get (15).

Now it follows easily that

$$r(u\varphi, v\varphi) \geq r(u, v) \tag{16}$$

holds for all $u, v \in M(A, I)$. Indeed, if $r(u, v) = k$, we can write $u = w_{B_1} \dots w_{B_k} w_{C_1} \dots w_{C_p}$ and $v = w_{B_1} \dots w_{B_k} w_{D_1} \dots w_{D_q}$ in FNF. By (15), we get $r((w_{B_1} \dots w_{B_k})\varphi, u\varphi) \geq k$ and $r((w_{B_1} \dots w_{B_k})\varphi, v\varphi) \geq k$, hence $r(u\varphi, v\varphi) \geq k$ and so (15) holds. Thus $d(u\varphi, v\varphi) \leq d(u, v)$ and so φ is a contraction.

(ii) $\Rightarrow$ (i). Trivial. $\square$

Corollary 4.2 *Let (A, I) be a finite independence alphabet and let $\varphi \in \operatorname{End} M(A, I)$. Then the following conditions are decidable:*

(i) φ is uniformly continuous with respect to d;

(ii) φ is a contraction with respect to d.

Proof. Immediate from Theorem 4.1, since condition (iii) is obviously decidable. $\square$

Assume now that $\varphi \in \mathrm{End}\, M(A,I)$ is uniformly continuous with respect to d. Since $(\hat{M}(A,I), d)$ is the completion of $(M(A,I), d)$, φ admits a unique continuous extension Φ to $(\hat{M}(A,I), d)$. By continuity, we must have

$$X\Phi = \lim_{n\to\infty} u_n\varphi \tag{17}$$

whenever $X \in \partial M(A,I)$ and $(u_n)_n$ is a sequence on $M(A,I)$ satisfying $X = \lim_{n\to\infty} u_n$.

Given $Y \subseteq M(A,I)$, let $\overline{Y}$ denote the topological closure of Y in $(\hat{M}(A,I), d)$. It is immediate that $\mathrm{Fix}\,\Phi$ is closed: if $X = \lim_{n\to\infty} X_n$ with every $X_n \in \mathrm{Fix}\,\Phi$, then $X\Phi = \lim_{n\to\infty} X_n\varphi = \lim_{n\to\infty} X_n = X$. It follows that $\overline{\mathrm{Fix}\,\varphi} \subseteq \overline{\mathrm{Fix}\,\Phi} = \mathrm{Fix}\,\Phi$.

Note that, for every $u \in M(A,I)$, the sequence $(u^n)_n$ is Cauchy and therefore convergent. We denote its limit by u^ω.

Our study of $\mathrm{Fix}\,\Phi$ starts with the case of free commutative monoids.

Lemma 4.3 *Let M be a free commutative monoid of finite rank and let $\varphi \in \mathrm{End}\, M$ be uniformly continuous with respect to d. Let $u \in M$ and*

$$Y_{\varphi,u} = \{X \in \hat{M} \mid u(X\Phi) = X\}.$$

Then $Y_{\varphi,u}$ is mp-rational.

Proof. Let A be the basis of M. We show that

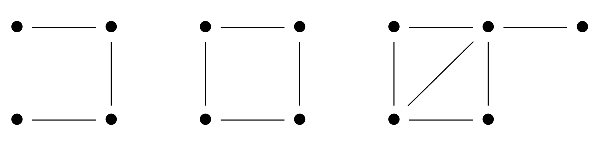
$$Y_{\varphi,u} = \bigcup_{B\subseteq A} L_B w_B^\omega \tag{18}$$

for some rational subsets L_B of $M\pi_{A\setminus B}$. Since $\hat{M} = \cup_{B\subseteq A}(A\setminus B)^* w_B^\omega$, it is just the rationality of the subsets L_B which is at stake. Clearly, a necessary condition for $L_B \neq \emptyset$ is $B\varphi\xi = B\xi$, hence we assume the latter. Assume that $A \setminus B = \{a_1, \ldots, a_k\}$. Consider the equation $u((a_1^{x_1} \ldots a_k^{x_k} w_B^\omega)\Phi) = a_1^{x_1} \ldots a_k^{x_k} w_B^\omega$ on the variables $x_1, \ldots, x_k \in \mathbb{N}$. The computation of L_B is done through the system of equations

$$|u|_{a_i} + x_1|a_1\varphi|_{a_i} + \ldots + x_k|a_k\varphi|_{a_i} = x_i \quad (i = 1, \ldots, k),$$

and linear diophantine systems such as this are known to have semilinear (therefore rational) sets of solutions [16]. Therefore (18) holds and so $\mathrm{Fix}\,\Phi$ is mp-rational. $\square$

We say that a graph is of *type T* if it has no full subgraphs of one of the following forms



Write $\Delta_A = \{(a,a) \mid a \in A\}$. The next result gives a complete solution for the case when $\Gamma(A,I)$ is of type T.

Theorem 4.4 *Let (A,I) be a finite independence alphabet such that $\Gamma(A,I)$ is of type T. Then the following conditions are equivalent:*

(i) for every $\varphi \in \mathrm{End}\, M(A,I)$, uniformly continuous with respect to d, there exists some mp-rational $Y \subseteq \hat{M}(A,I)$ such that $\mathrm{Fix}\,\Phi = \overline{\mathrm{Fix}\,\varphi} \cup Y$;

(ii) $I \cup \Delta_A$ *is transitive;*

(iii) $\Gamma(A, I)$ *is a disjoint union of complete graphs;*

(iv) $M(A, I)$ *is a free product of finitely many free commutative monoids of finite rank.*

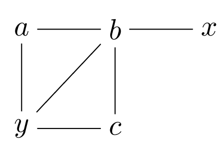
Proof. (i) $\Rightarrow$ (ii). Write $M = M(A, I)$. Suppose that $I \cup \Delta_A$ is not transitive. Then there exist distinct $a, b, c \in A$ such that $(a, b), (b, c) \in I$ but $(a, c) \notin I$. Let $\varphi \in \operatorname{End} M$ be defined by

$$x\varphi = \begin{cases} xb & \text{if } (x, b) \in I \text{ and } x \not\sim_I (ac) \\ x & \text{otherwise} \end{cases}$$

for $x \in A$.

To show that φ is well defined, i.e. $(x, y) \in I \Rightarrow (xy)\varphi = (yx)\varphi$, the only nontrivial case occurs when $x\varphi = bx$ and $y\varphi = y$. Suppose that $by \neq yb$ in M. We have $(x, b) \in I$ and from $x \not\sim_I (ac)$ we may assume without loss of generality that $xa \neq ax$ in M. Hence a — b — x — y is a 4-vertex subgraph of $\Gamma(A, I)$ and from $by \neq yb$ and $xa \neq ax$ it follows that the full subgraph induced by $\{a, b, x, y\}$ has one of the first two forbidden configurations, a contradiction. Thus $by = yb$ in M and it follows easily that φ is well defined.

To show that φ is uniformly continuous, let $x, y, z \in A$ satisfy $z \leq_p x\varphi$ and $z \sim_I y\varphi$. We must show that $(x, y) \in I$. Suppose not. The case $z = x$ leads to immediate contradiction, hence we may assume that $x\varphi = xb$ and $z = b$. Obviously, we may assume that $y\varphi = y$, and also $xa \neq ax$ in M. Now $b \sim_I y\varphi = y$ yields $y \sim_I (ac)$ and so $c \neq x$. It follows that $\Gamma(A, I)$ has a subgraph of the form



where all the vertices must be distinct. This cannot be a full subgraph since it is of the third forbidden type. But the only potential edge that has not been excluded yet is x — c, when a — y — c — x would be a full subgraph of the first forbidden type. Hence we reach a contradiction in both cases and so $(x, y) \in I$. Thus φ satisfies (14) and is therefore uniformly continuous.

It is immediate that $\operatorname{Fix}\varphi$ is generated by a subset of $A \setminus \{a, c\}$, hence none of the fixed points $V_n = ((ab)^n(bc)^n)^\omega$ belongs to $\overline{\operatorname{Fix}\varphi}$.

Suppose that $\operatorname{Fix}\Phi = \overline{\operatorname{Fix}\varphi} \cup L_0 \cup L_1X_1 \cup \ldots \cup L_mX_m$ for some rational subsets $L_0, \ldots, L_m$ of M and $X_1, \ldots, X_m \in \partial M$. Then there exist distinct fixed points V_n and V_k belonging to the same subset L_iX_i. But then both V_n and V_k must share (infinite) suffixes with X_i and therefore with each other by transitivity, a contradiction since $n \neq k$. Therefore condition (i) must fail.

(ii) $\Rightarrow$ (i). Assume that $I \cup \Delta_A$ is transitive and let $\varphi \in \operatorname{End} M$ be uniformly continuous. In view of Lemma 4.3, we may assume that M is noncommutative. We may also assume that φ is nontrivial. With these assumptions, we claim that $1 \notin A\varphi$. Indeed, suppose that $a\varphi = 1$ for some $a \in A$. Let $b \in A$. By (14), we must have $b\varphi = 1$ whenever $(a, b) \notin I$. Since M is noncommutative, $\Gamma(A, I)$ has at least two connected components, and so $c\varphi = 1$

for every vertex c which is not in the connected component of a. But now, replacing a by one of these c, we must have $d\varphi = 1$ for every vertex d in the connected component of a and so φ would be trivial. Thus $1 \notin A\varphi$.

Let $A = A_1 \cup \ldots \cup A_r$ be the decomposition of the vertex set of $\Gamma(A, I)$ in its connected components, and let M_j denote the free commutative monoid on A_j. Let

$$J = \{j \in \{1, \ldots, r\} \mid M_j\varphi \subseteq M_j\}.$$

For every $j \in J$, let $\varphi_j = \varphi|_{M_j}$. Then φ_j is a (uniformly continuous) endomorphism of M_j, and its continuous extension Φ_j to the completion $\hat{M}_j \subseteq \hat{M}$ is a restriction of Φ.

Let $P = J \times \{0, \ldots, 2^r - 1\}$. For every $(j, k) \in P$, define

$$C_{jk} = \bigcup_{i=1}^{r} \{\, B \subseteq A_i \mid w_B\varphi = w_B z_B \text{ for some } z_B \neq 1 \text{ and } z_B\varphi^k \in M_j\}$$

and let

$$D = \bigcup_{i=1}^{r} \{B \subseteq A_i \mid \exists \lim_{n\to\infty} w_B\varphi^n\}.$$

We prove that

$$\begin{aligned} \operatorname{Fix}\Phi = \overline{\operatorname{Fix}\varphi} \;\cup (&\bigcup_{(j,k)\in P} \bigcup_{B\in C_{jk}} (\operatorname{Fix}\varphi) w_B z_B (z_B\varphi)\ldots(z_B\varphi^{k-1}) Y_{\varphi_j, z_B\varphi^k}) \\ &\cup ((\operatorname{Fix}\varphi)\{\lim_{n\to\infty} w_B\varphi^n \mid B \in D\}). \end{aligned} \tag{19}$$

The opposite inclusion follows from straightforward checking, we consider only the case $X = uw_B z_B(z_B\varphi)\ldots(z_B\varphi^{k-1})X'$ with $(j, k) \in P$, $B \in C_{jk}$, $u \in \operatorname{Fix}\varphi$ and $X' \in Y_{\varphi_j, z_B\varphi^k}$. Then

$$X\Phi = (u\varphi)w_B z_B(z_B\varphi)\ldots(z_B\varphi^{k-1})(z_B\varphi^k)(X'\Phi) = uw_B z_B(z_B\varphi)\ldots(z_B\varphi^{k-1})X' = X$$

as required.

Now take $X \in \operatorname{Fix}\Phi$ and write $X = w_{B_1}w_{B_2}\ldots$ in FNF. Without loss of generality, we may assume that $w_{B_1} \notin \operatorname{Fix}\varphi$. By continuity and (15), we get $r(w_{B_1}\varphi, (w_{B_1}w_{B_2}\ldots)\Phi) \geq 1$ and so $r(w_{B_1}\varphi, w_{B_1}w_{B_2}\ldots) \geq 1$. Hence we may write $w_{B_1}\varphi = w_{B_1}v$ with $v \neq 1$.

For every $u \in M$, let $u\overline{\xi} = \{i \in \{1, \ldots, r\} \mid u\xi \cap A_i \neq \emptyset\}$. We claim that

$$u\overline{\xi} = v\overline{\xi} \;\Rightarrow\; u\varphi\overline{\xi} = v\varphi\overline{\xi} \tag{20}$$

holds for all $u, v \in M$. Indeed, suppose that $a \in A_i$ occurs in u. Then some $b \in A_i$ must occur in v. Since $(a\varphi)(b\varphi) = (b\varphi)(a\varphi)$, it follows easily (directly or by using Levi's Lemma [7]) that $a\varphi\overline{\xi} = b\varphi\overline{\xi}$. Hence $u\varphi\overline{\xi} \subseteq v\varphi\overline{\xi}$ and (20) follows from symmetry.

Next we define a directed graph Ω having as vertices the nonempty subsets of $\{1, \ldots, r\}$ and edges $R \longrightarrow S$ whenever $u\overline{\xi} = R$ implies $u\varphi\overline{\xi} = S$. By (20), and since $1 \notin A\varphi$, Ω is well defined. Note that each vertex of Ω has outdegree 1.

Suppose first that there exists some $k \geq 0$ and $j \in \{1, \ldots, r\}$ such that

$$\{j\} = v\varphi^k\overline{\xi} = v\varphi^{k+1}\overline{\xi} = \ldots \tag{21}$$

It follows that $j \in J$. Moreover, we may assume that $k < 2^r$ because after following $2^r - 1$ edges in Ω we must be in a cycle, and starting at $v\overline{\xi}$ the cycle is necessarily trivial. Hence $(j,k) \in P$, $B \in C_{jk}$ and we may write $z_{B_1} = v$. Now, for every $\ell \geq k$ we get

$$w_{B_1}\varphi^\ell = (w_{B_1}z_{B_1})\varphi^{\ell-1} = (w_{B_1}z_{B_1}(z_{B_1}\varphi))\varphi^{\ell-2} = \ldots = w_{B_1}z_{B_1}(z_{B_1}\varphi)\ldots(z_{B_1}\varphi^{\ell-1}),$$

and so

$$w_{B_1}w_{B_2}\ldots = (w_{B_1}w_{B_2}\ldots)\Phi^\ell = w_{B_1}z_{B_1}(z_{B_1}\varphi)\ldots(z_{B_1}\varphi^{\ell-1})((w_{B_2}w_{B_3}\ldots)\Phi^\ell). \tag{22}$$

Hence

$$\begin{aligned} w_{B_1}z_{B_1}(z_{B_1}\varphi)\ldots(z_{B_1}\varphi^{\ell-1})((w_{B_2}w_{B_3}\ldots)\Phi^\ell) &= w_{B_1}w_{B_2}\ldots \\ &= w_{B_1}z_{B_1}(z_{B_1}\varphi)\ldots(z_{B_1}\varphi^{\ell})((w_{B_2}w_{B_3}\ldots)\Phi^{\ell+1}), \end{aligned}$$

yielding by left cancellativity

$$(w_{B_2}w_{B_3}\ldots)\Phi^\ell = (z_{B_1}\varphi^\ell)((w_{B_2}w_{B_3}\ldots)\Phi^{\ell+1}). \tag{23}$$

Iterating (23) for $\ell = k, k+1, \ldots, k'$, we get

$$(w_{B_2}w_{B_3}\ldots)\Phi^k = (z_{B_1}\varphi^k)(z_{B_1}\varphi^{k+1})\ldots(z_{B_1}\varphi^{k'})((w_{B_2}w_{B_3}\ldots)\Phi^{k'+1}).$$

Since $1 \notin A\varphi$, it follows easily that

$$(w_{B_2}w_{B_3}\ldots)\Phi^k = (z_{B_1}\varphi^k)(z_{B_1}\varphi^{k+1})\ldots$$

and so (21) yields $(w_{B_2}w_{B_3}\ldots)\Phi^k \in M_j$. Considering (23) for $\ell = k$, we get $(w_{B_2}w_{B_3}\ldots)\Phi^k \in Y_{\varphi_j, z_{B_1}\varphi^k}$. Together with (22), this implies that X belongs to the right hand side of (19).

Thus we may assume that the sequence $(v\varphi^n\overline{\xi})_n$ never stabilizes on a singular set. For every $n \geq 1$, write $v_n = w_{B_1}\varphi^n = w_{B_1}v(v\varphi)\ldots(v\varphi^{n-1})$. It follows that the number of alternating connected components in the sequence $(v_n)_n$ increases unboundedly. Since $v_n \leq v_{n+1}$ for every n, it follows easily that $(v_n)_n$ is a Cauchy sequence and therefore convergent in $\hat{M}$. Thus $B_1 \in D$. Now

$$X = w_{B_1}w_{B_2}\ldots = (w_{B_1}w_{B_2}\ldots)\Phi^n = v_n((w_{B_2}w_{B_3}\ldots)\Phi^n)$$

yields $X = \lim_{n\to\infty} v_n((w_{B_2}w_{B_3}\ldots)\Phi^n)$. Since the number of alternating connected components in $(v_n)_n$ increases unboundedly, we immediately get $X = \lim_{n\to\infty} v_n = \lim_{n\to\infty} w_{B_1}\varphi^n$ and so (19) holds.

In view of Theorem 3.1 and Lemma 4.3, it follows that $\mathrm{Fix}\,\Phi$ is the union of $\overline{\mathrm{Fix}\,\varphi}$ with an mp-rational subset of $\hat{M}$.

(ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). Immediate. $\square$

Corollary 4.5 *Let M be a free product of finitely many commutative monoids of finite rank and let $\varphi \in \mathrm{End}\,M$ be uniformly continuous with respect to d. If $\mathrm{Fix}\,\varphi$ is finite, so is $\mathrm{Fix}\,\Phi$.*

Proof. First we note that if $\mathrm{Fix}\,\varphi$ is finite then $\mathrm{Fix}\,\varphi = \{1\}$ and so $\overline{\mathrm{Fix}\,\varphi} = \{1\}$. In view of (19), it suffices to show that every $Y_{\varphi_j,u}$ in the conditions of Lemma 4.3 is finite. Write $Y_{\varphi_j,u} = \bigcup_{B\subseteq A_j} L_B w_B^\omega$ as in (18) and suppose that L_B is infinite for some $B \subseteq A_j$. Assume that $A_j \setminus B = \{a_1, \ldots, a_k\}$. Then the proof of Lemma 4.3 shows that the system of equations

$$|u|_{a_i} + x_1|a_1\varphi_j|_{a_i} + \ldots + x_k|a_k\varphi_j|_{a_i} = x_i \quad (i = 1, \ldots, k)$$

has infinitely many solutions $(x_1, \ldots, x_k) \in \mathbb{N}^k$. By Dickson's Lemma [5], there exist some distinct solutions $(x_1, \ldots, x_k), (y_1, \ldots, y_k)$ such that $x_1 \geq y_1, \ldots, x_k \geq y_k$. It follows that

$$(x_1 - y_1)|a_1\varphi'|_{a_i} + \ldots + (x_k - y_k)|a_k\varphi'|_{a_i} = x_i - y_i \quad (i = 1, \ldots, k)$$

and so by the proof of Lemma 4.3 we get $\mathrm{Fix}\,\varphi_j \neq \{1\}$ and so $\mathrm{Fix}\,\varphi \neq \{1\}$, a contradiction. Thus $Y_{\varphi_j,u}$ is always finite and so is $\mathrm{Fix}\,\Phi$. □

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