Rank Function on Lagrangian Matroids

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Abstract

We define rank function for Lagrangian matroids in a natural way and characterize Lagrangian matroids in terms of rank axioms.

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1 Introduction

Sympletic matroids are an important subclass of Coxeter matroids introduced by Gelfand and Serganova in [6] in order to study stratification on compact homogenous manifolds. They include ordinary matroids as a subclass and, as them, they can be characterized by its set of basis (see [2]). As in ordinary matroids, independent sets can be defined as the subsets of basis and, for the purposes of this paper, we will use a characterization of sympletic matroids obtained by Chow [5], using an augmentation property that generalizes the one from matroid theory.

Let $I = \{1, \ldots, n\}$, $I^* = \{1^*, \ldots, n^*\}$, and $J = I \cup I^*$, where the union is disjoint. An involution $*$ is defined on J by setting $(i^*)^* = i$, for $i^* \in I^*$, and extended it to sets in the obvious way. A set $A \subset J$ is said to be *admissible* if $A \cap A^* = \emptyset$, and we write J_k for the collection of admissible k-subsets of J.

A family $\mathcal{I}$ of admissible subsets of J is the collection of *indepenedent sets* of a sympletic matroid if and only if it is subset-closed and it has the following property:

$\mathcal{II}$ If X and Y are members of $\mathcal{I}$ such that $|X| < |Y|$ then either:

- there exists $y \in Y \setminus X$ such that $X \cup \{y\} \in \mathcal{I}$, or
- $X \cup Y$ is not admissible , and there exists $x \notin X \cup Y$ such that both $X \cup \{x\} \in \mathcal{I}$ and $(X \setminus Y^*) \cup \{x^*\} \in \mathcal{I}$.

Defining a sympletic matroid in terms of independents set we can recover the basis set taking the maximal (for inclusion) independent sets. The number of elements of any base is called the *rank* of the matroid and a sympletic matroid of maximal rank, n, is called *Lagrangian*. Ordinary matroids can be included in this special class of sympletic matroids that also verifies a symmetric exchange property [3].

Characterizations of Lagrangian matroids in terms of basis exchange [3] and in terms of circuits properties [1] show that they have many of the classical matroids behaviour. We seek for analogous properties for general sympletic matroids Behind their characterization using indepenendent sets given above also the one in terms of greedy algorithm (see [2]) points to it.

So, we expect that the characterization obtained here for Lagrangian matroids, in terms of rank function, can be somehow generalized at least for other classes of sympletic matroids, as the one coming from graphs constructed by Chow in [5].

2 Results

Let M be a sympletic matroid over $J = [n] \cup [n^*]$ and $\mathcal{I}$ be the set of independents for M and $\mathcal{A} = \mathcal{A}(J)$ be the set of admissible subsets of J. The following function defined on $\mathcal{P}(J) = \{X : X \subset J\}$, is called the *rank* function on M:

$r : \mathcal{P}(J) \longrightarrow \Re_0^+$, where $r(X) = max_{I \in \mathcal{I}} |X \cap I|$, for all $X \subset J$.

As independent sets are admissible, $forallX \in \mathcal{P}(J)$, $r(X) = r(A)$, for some $A \in \mathcal{A}$ and $A \subset X$; besides, $r(X) = max_{B \in \mathcal{B}}|X \cap B|$, where $\mathcal{B}$ is the set of basis, because, for any independent set there is always a base containing it. Finally, $r(I) = |I|$ for any independent set.

Lemma 2.1 *If M is a Lagrangian matroid and r is the rank function on M, then r verifies the following properties:*

1. $r(\emptyset) = 0$
2. $\forall X \in P(J), \forall x \in J \quad r(X) \leq r(X+x) \leq r(X)+1$
3. $\forall A \in \mathcal{A}, \forall x \in J$
   $(r(A) = |A|$ *and* $x \notin A) \Rightarrow (x^* \in A$ *or* $r(A+x) = r(A)+1$ *or* $r(A+x^*) = r(A)+1)$
4. $\forall A \in \mathcal{A}\ \forall x, y \in J$

$$\begin{array}{c} r(A) = r(A+x) = r(A+y) \\ \text{and} \\ r(A+x+y) = r(A)+1 \end{array} \quad \Rightarrow \quad \begin{array}{c} r(A+x^*+y^*) = r(A)+1 \\ \text{and} \\ |A \cap \{x^*, y^*\}| = 1 \end{array}$$

Proof: We observe that r trivially satisfies the conditions (1), (2). For condition (3) we need to notice that if I is a maximal independent set (a base) of a Lagrangian matroid and $x \notin I$, then $x^* \in I$. It only remains for us to justifie the last property.

Let us assume that $S \in \mathcal{A}$ verifies $r(S) = r(S+x) = r(S+y)$ and $r(S+x+y) = r(S)+1$.

Let $I, I' \in \mathcal{I}$ be such that $r(S) = |S \cap I|$ and $r(S+x+y) = |(S+x+y) \cap I'|$ and let $X = S \cap I$, $Y = (S+x+y) \cap I'$ and $Y_1 = S \cap I'$.

Using the hypothesis one can see that neither x, nor y belong to S and must both belong to I' and so $|Y_1| = |Y| - 2 (= |X| - 1)$; besides $Y_1 \cup X \subset S$ so $Y_1 \cup X$ is admissible. Using $\mathcal{I}_2$ we know that $\exists t \in X - Y_1$ such that $Y_1 + t$ is an independent set of the matroid M.

As $t \in X \subset S$ and $Y_1 \subset S$, $Y_1 + t \subset S$ and $r(Y_1 + t) = |Y_1| + 1 = |X|$.

Let us consider the independent sets Y and $Y_1 + t$:

If $(Y_1 + t) \cup Y$ was admissible there will be $s \in Y - (Y_1 + t) = \{x, y\}$ such that $Y_1 + t + s$ is an independent set. But then $|Y_1 + t + s| = r(S) + 1$ and $(Y_1 + t + s) \subset S + x$ or $(Y_1 + t + s) \subset S + y$ contradicting the hypothesis.

So, being $(Y_1 + t) \cup Y$ not admissible, $t = x^*$ or $t = y*$ and therefore $r(S + x^* + y^*) = r(S) + 1$ and $|S \cap \{x^*, y^*\}| = 1$.

Theorem 2.1 *Let $J = [n] \cup [n^*]$ and let $r : P(J) \longrightarrow \Re_0^+$ verify the following:*

1. $r(\emptyset) = 0$
2. $\forall X \in \mathcal{P}(J)\ \forall x \in J \quad r(X) \le r(X+x) \le r(X)+1$
3. $\forall S \in \mathcal{A}\ \forall x \in J$
   ($r(S) = |S|$ and $x \notin S$) $\Rightarrow$ ($x^* \in S$ or $r(S+x) = r(S)+1$ or $r(S+x^*) = r(S)+1$)
4. $\forall S \in \mathcal{A}\ \forall x, y \in J$

$$\begin{array}{c} r(S) = r(S+x) = r(S+y) \\ \text{and} \\ r(S+x+y) = r(S)+1 \end{array} \quad \Rightarrow \quad \begin{array}{c} r(S+x^*+y^*) = r(S)+1 \\ \text{and} \\ |S \cap \{x^*, y^*\}| = 1 \end{array}$$

Then the set $\mathcal{I} = \{S \in \mathcal{A} : \ r(S) = |S|\}$ is the set of independent sets of a Lagrangian matroid.

Proof: Observe that, from (1) and (2), you can deduce that $\mathcal{I}$ is a subsetclosed family and that (3) garantees that there is an admissible set with n elements that belongs to $\mathcal{I}$.

Now, let X and Y be elements of $\mathcal{I}$ with $|X| < |Y|$: we want to show that either $\exists y \in Y \setminus X : r(X+y) = r(X)+1$ or $X \cup Y$ is not admissible and $\exists z \notin X \cup Y : r(X+z) = r(X)+1$, $r((X \setminus Y^*)+z^*) = r(X \setminus Y^*)+1$ and $X+z$ is admissible.

Let us start by assuming that $X \cup Y$ is admissible and that for all $y \in Y \setminus X$ $r(X+y) = r(X)$.

Since $r(X \cup Y) \ge r(Y) > r(X)$, $\sum = \{S \subset Y \setminus X : r(S \cup X) > r(X)\}$ is not empty and, as r verifies (2), $\sum_0 = \{S \subset Y \setminus X : r(S \cup X) = r(X)+1\}$ is not empty too. Let Y_0 be a minimal (for inclusion) element of $\sum_0$.

Under our hypothesis, $r(X+y) = r(X)$, $\forall y \in Y \setminus X$, and so $|Y_0| > 1$. Let y_0 and y_1 be distinct elements of Y_0 and $Z = X \cup Y_0 - \{y_0, y_1\}$. Now $r(X) = r(Z) = r(Z+y_0) = r(Z+y_1)$ and $r(Z+y_0+y_1) = r(X \cup Y_0) = r(X)+1$ and therefore, by (4), $r(Z+y_0^*+y_1^*) = r(Z)+1$ and $|Z \cap \{y_0^*, y_1^*\}| = 1$. Thus $Y_0^* \cap Z = Y_0^* \cap (X \cup Y_0) \neq \emptyset$ and so $Y^* \cap (X \cup Y) (= Y^* \cap X) \neq \emptyset$, which is absurd. Therefore $\exists y \in Y \setminus X$ $r(X+y) = r(X)+1$, i.e., $X+y \in \mathcal{I}$

Let us assume now that $X \cup Y$ is not admissible.

Consider the admissible set $T = Y \setminus (X \cup X^*)$; $T \neq \emptyset$ because $|Y| > |X|$ and X, Y are both admissible. Take $t \in T$:

If $r(X+t^*) = r(X)+1$ make $z = t^*$; $t^* \notin X \cup Y$ and, since $t \notin X \setminus Y^*$ and $t \in Y$, $r((X \setminus Y^*)+t) = r(X \setminus Y^*)+1$.

On the other hand, if $r(X+t^*) = r(X)$, as $t \notin X$ and $t^* \notin X^*$, condition (3) implies that $r(X+t) = r(X)+1$ and $t \in T \subset Y \setminus X$.

Observation: A Lagrangian matroid M is orthogonal if and only if r verifies the following condition:

$$\forall B_1, B_2 \in A(E)\ r(B_1) = r(B_2) = n \Rightarrow |B_1 \cap [n]| \equiv |B_2 \cap [n]|\ (mod\ 2).$$

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